REALISATION OF BENDING MEASURED LAMINATIONS BY KLEINIAN SURFACE GROUPS

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Abstract. For geometrically finite Kleinian surface groups, Bonahon and Otal proved the existence part, and partly the uniqueness part of the bending lamination conjecture. In this paper, we generalise the existence part to general Kleinian surface groups including geometrically infinite ones. Along the way, we also prove the compactness of the set of Kleinian surface groups realising an arbitrarily fixed data of bending laminations and ending laminations. Our proof is independent of that of Bonahon and Otal.

1. Introduction

The simultaneous uniformisation theorem by Bers [2] gives a parametrisation of the quasi-Fuchsian space for a closed oriented surface S by the product of two Teichmüller spaces $\mathcal{T}(S) \times \mathcal{T}(\bar{S})$, where $\bar{S}$ denotes S with its orientation reversed. This was generalised by the work of Kra, Maskit, Marden and Sullivan, which shows that for any Kleinian surface group, or more generally for any freely indecomposable Kleinian group G, its quasi-conformal deformation space is parametrised by $\mathcal{T}(\Omega_G/G)$, where Ω_G is the region of discontinuity of G in the Riemann sphere.

In his lecture notes [24], Thurston considered the convex core of the quotient hyperbolic 3-manifold $\mathbb{H}^3/G$ for a Kleinian (surface) group G. He noticed that the boundary of the convex core has two pieces of information: the hyperbolic structure and the bending lamination. In contrast to the previous work of Bers et al, these are obtained by just considering the quotient hyperbolic manifolds, without looking at structures at infinity. He seems to have conjectured that both hyperbolic structures and bending laminations on the boundaries serve as other kinds of parametrisation of the quasi-Fuchsian space, or more generally, the quasi-conformal deformation space of a freely indecomposable Kleinian group.

As for the first of these two, the hyperbolic structures on the boundaries of convex cores, Sullivan's lemma (see e.g. [9]) shows that they are within universally bounded distance in the corresponding Teichmüller spaces from the conformal structures at infinity on Ω_G/G. Nevertheless, it is still unknown if they really give a parametrisation of the deformation space.

As for bending laminations, Bonahon-Otal showed in [4] that every pair of measured laminations on a closed orientable surface S, without homotopic components and without compact leaves with weight larger than or equal to π, can be realised as bending laminations of a quasi-Fuchsian group corresponding to S. In particular, when both of the measured laminations are weighted multi-curves, it was proved that the realising quasi-Fuchsian group is unique (up to conjugation). They also showed the same result for quasi-conformal deformation spaces of general freely indecomposable geometrically finite groups. Their result was generalised to freely decomposable Kleinian groups by Lecuire [12].

In this paper, we prove a generalisation of the existence part of this result by Bonahon-Otal to general Kleinian surface groups including geometrically infinite ones (Theorem 3.1-(1)), by adding ending laminations to the data. We shall furthermore prove the compactness of the set of representations (up to conjugacy) realising given bending laminations and ending laminations (Theorem 3.1-(2)). The proofs of both are different from and independent of the results by Bonahon-Otal. On the other hand, we do not have the partial uniqueness result as was given by Bonahon-Otal, for we cannot invoke the theory of cone manifold deformation, whose generalisation to the case of geometrically infinite groups does not exist for the moment.

We shall prove the existence part and the compactness by showing the properness of the following composition of maps in Theorem 3.3. For a Kleinian surface group G, the theory of Bers-Kra-Maskit-Marden-Sullivan gives a parametrisation of the quasi-conformal deformation space $q\colon \mathcal{T}(\Omega_G/G) \to \mathsf{QH}(G)$, where Ω_G denotes the region of discontinuity of G, $\mathcal{T}(\Omega_G/G)$ the Teichmüller space of the Riemann surface Ω_G/G, and $\mathsf{QH}(G)$ the space of quasi-conformal deformations of G modulo conjugacy. Sending each Kleinian group to its bending lamination, we get a map $b\colon \mathsf{QH}(G) \to \mathcal{ML}(\Omega_G/G)$, where $\mathcal{ML}$ denotes the space of measured laminations. Let $D \subset \mathcal{ML}(\Omega_G/G)$ be the set of measured laminations evidently unrealisable, whose exact definition is given in Theorem 3.1. In this setting, Theorem 3.3 states that $q \circ b$ is a proper, degree-1 map to $\mathcal{ML}(\Omega_G/G) \setminus D$. This in particular says that $q \circ b$ is surjective to $\mathcal{ML}(\Omega_G/G) \setminus D$, and hence we obtain the existence part of the main result. The compactness part is derived from the properness of the map.

Our proof of Theorem 3.3, is divided into two parts: we shall first show the properness of the map $b \circ q$ in Section 5, and then that $b \circ q$ has degree 1 in Section 6. In the first part, relying on the analysis of geometric limits, as was given in Ohshika-Soma [20] and Ohshika [19], we shall show that any sequence going to infinity in $\mathcal{T}(\Omega_G/G)$ has image under $b \circ q$ which cannot stay in a compact set disjoint from D. In the second part, we shall show that $q \circ b$ can be properly homotoped to a local degree-1 map which is constructed using the earthquake map. Since the degree is invariant under a proper homotopy, this implies that $b \circ q$ has also degree 1.

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2. Preliminaries

2.1. **Basics of Kleinian groups.** A *Kleinian group* is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$. In this paper, we only consider Kleinian groups isomorphic to the fundamental groups of closed orientable surfaces of genus greater than 1, which we call *Kleinian surface groups*. A Kleinian group acts on the Riemann sphere $\hat{\mathbb{C}}$ by linear fractional transformations and on the hyperbolic space $\mathbb{H}^3$ by orientation-preserving isometries. By considering the Poincaré model of $\mathbb{H}^3$, the Riemann sphere $\hat{\mathbb{C}}$ is regarded as the sphere at infinity of $\mathbb{H}^3$. The action on $\hat{\mathbb{C}}$ is a continuous extension of the action on $\mathbb{H}^3$ if we regard $\hat{\mathbb{C}}$ as the points at infinity in this way. For a Kleinian group G, its *limit set* Λ_G is the closure of the set of fixed points of non-trivial elements of G. The complement of Λ_G in $\hat{\mathbb{C}}$ is called the *region of discontinuity* of G, and is denoted by Ω_G.

The smallest convex subset of $\mathbb{H}^3$ containing all geodesics both of whose endpoints at infinity lie on Λ_G is called the *Nielsen convex hull* and is denoted by H_G. Since H_G is a closed convex subset invariant under G, its quotient H_G/G is a closed convex subset of $\mathbb{H}^3/G$, which is a 3-submanifold except for the case when G is Fuchsian. The quotient H_G/G is called the convex core of $\mathbb{H}^3/G$ and is denoted by $C(\mathbb{H}^3/G)$. The Kleinian group G is said to be *geometrically finite* if $C(\mathbb{H}^3/G)$ has finite volume.

2.2. **Geodesic and measured laminations.** In this section, we consider an orientable surface S which may have punctures but has no boundary. We fix a complete hyperbolic metric on S which makes punctures cusps. A *geodesic lamination* λ on S is a closed subset consisting of disjoint simple geodesics which do not tend to cusps. A geodesic constituting λ is called a *leaf*. A geodesic lamination is said to be *minimal* when it does not have a non-empty proper sublamination. Any geodesic lamination is decomposed into disjoint finitely many minimal sublaminations, which we call *minimal components*, and isolated leaves spiralling around minimal components. We say that a geodesic lamination is *arational* when every component of its complement is either simply connected or an annulus containing a cusp.

A *measured lamination* is a geodesic lamination equipped with a transverse invariant measure. The support of a measured lamination is a geodesic lamination having the property that the entire lamination coincides with the union of its minimal components. We denote the support of a measured lamination λ by $|\lambda|$. Conversely, a geodesic lamination with this property always supports a transverse invariant measure. We always assume that the support of a measured lamination is the entire lamination.

For two measured laminations λ and μ, their intersection number $\iota(\lambda, \mu)$ is defined to be the integral of the product of the transverse measures of λ and μ over the surface S. In particular, when c is a simple closed curve, we regard c as having the unit Dirac transverse measure and define $\iota(\lambda, c)$ as such. For two geodesic laminations λ and λ' on a hyperbolic surface S, and a point $p \in \lambda \cap \lambda'$, we can consider the angle formed by λ and λ' at p taking the value in $[0, \pi/2]$, which we denote by $\angle_p(\lambda, \lambda')$. We define the angle between λ and λ' to be $\sup_{p \in \lambda \cap \lambda'} \angle_p(\lambda, \lambda')$ and denote it by $\angle_S(\lambda, \lambda')$.

A measured lamination λ is said to be *uniquely ergodic* if the transverse measure of λ is a unique transverse measure on its support up to scaling. A pair of geodesic (or measured) laminations λ_1 and λ_2 is said to *fill up* S if every geodesic lamination μ on S intersects λ_1 or λ_2 transversely.

For a measured lamination λ or a geodesic lamination supporting a measured lamination on S, its *minimal supporting surface* $S(\lambda)$ is an incompressible compact subsurface containing λ and is minimal with respect to the inclusion, which is unique up to isotopy.

Geodesic laminations and measured laminations defined above depend on the hyperbolic metric given on S. Still for two complete hyperbolic metrics m, n on S, and a geodesic lamination λ on (S, m), there is a unique geodesic lamination λ' on (S, n) which is isotopic to λ. By identifying λ and λ' as above, we can talk about geodesic laminations and measured laminations without specifying a hyperbolic metric. We note the intersection number does not depend on the choice of a hyperbolic metric whereas the angle does depend on it.

Thurston proved that the space of measured laminations with the weak topology with respect to the transverse measures is homeomorphic to the Euclidean space of dimension $6g - 6 + 2b$, where g is the genus and b is the number of punctures of S. We denote this space by $\mathcal{ML}(S)$ and call it the *measured lamination space* of S. There is a PL local chart of $\mathcal{ML}(S)$, which can be constructed using train tracks as in the next section.

2.3. **Train tracks.** We shall define basic terms on train tracks in this subsection. We refer the reader to Penner-Harer [22] for a more detailed account.

A *train track* τ on S is a C^1-graph (i.e. a graph whose edges are C^1-arcs and tangent to each other at vertices) embedded in S whose edges are called *branches* and whose vertices are called *switches*, such that no component of $S \setminus \tau$ is a disc with one corner or an annulus with C^1-smooth boundary. A *weight system* ω on a train track τ is a system of non-negative numbers, called *weights*, given on branches of τ such that at each switch the sum of the weights on the incoming branches coincides with the sum of the weights on the outgoing branches.

A geodesic lamination λ is said to be *carried by* a train track τ, when it can be regularly homotoped to an immersion in τ. Any geodesic lamination has a train track carrying it. In particular, if a measured lamination λ is carried by a train track τ, it induces a weight system on τ, by defining the weight of a branch to be the total transverse measure of the leaves lying there (after a regular homotopy). We denote this weight system induced from λ by $w(\lambda)$. Conversely, for any weight system ω on a train track, we can construct a measured lamination λ such that $w(\lambda) = \omega$.

A train track τ is said to be *recurrent* if it has a weight system which takes only positive values, and *transversely recurrent* if for each branch b of τ, there is a simple closed curve intersecting τ essentially (i.e. without cobounding a bigon) and transversely with non-empty intersection with b. Train tracks which are both recurrent and transversely recurrent are called *bi-recurrent.* Every measured lamination is carried by a bi-recurrent train track. For a bi-recurrent train track τ, the set of measured laminations inducing weight systems with positive values on τ forms an open set in $\mathcal{ML}(S)$, which we denote by $U(\tau)$. For an arational measured lamination λ, the open sets $U(\tau)$ for all bi-recurrent train tracks τ carrying λ form a base of neighbourhoods of λ in $\mathcal{ML}(S)$.

2.4. **Pleated surfaces.** Let M be a hyperbolic 3-manifold, and F an orientable surface, which we assume to be either closed or the interior of a compact surface. A *pleated surface* $f\colon (F, m) \to M$, where m is a complete hyperbolic metric on F, is a continuous map taking each cusp of (F, m) to a cusp of M such that for every point $x \in F$, there is at least one geodesic segment containing x in its interior which is mapped isometrically to a geodesic segment in M by f. The set of points on F at which only one direction is mapped geodesically constitutes a geodesic lamination μ on (F, m). We call μ the *pleating locus* of the pleated surface f. More generally, if a geodesic (or a measured) lamination λ is mapped geodesically by a pleated surface f, we say that f *realises* λ.

The boundary component of the convex core of a hyperbolic 3-manifold is an example of pleated surface. It has moreover a special property that the surface is bent only in one direction. The pleating locus of such a surface has a transverse measure coming from bending angles, and is called the *bending lamination* when it is regarded as a measured lamination.

2.5. **Ending laminations.** By Bonahon's tameness theorem [3], it is known that for any faithful discrete representation $\phi\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$, there is an orientation-preserving homeomorphism $\Phi\colon S \times (0,1) \to \mathbb{H}^3/\phi(\pi_1(S))$ which induces ϕ between their fundamental groups.

For a hyperbolic 3-manifold $M = \mathbb{H}^3/\phi(\pi_1(S))$, its *non-cuspidal part*, denoted by M_0, is the complement of ϵ-thin cusp neighbourhoods for some fixed positive number ϵ smaller than the three-dimensional Margulis constant. The boundary of M_0 consists of incompressible open annuli. (In the case of general Kleinian groups, incompressible tori may appear. We do not have such components since we only deal with Kleinian surface groups.) By the relative core theorem ([23, 13]), there is a compact submanifold C of M_0, called a *relative compact core* of M_0, such that the inclusion is a homotopy equivalence and $C \cap \partial M_0$ is the union of core annuli of the components of ∂M_0. A core curve of each component of $C \cap \partial M_0$ represents a generator of a maximal parabolic subgroup of $\phi(\pi_1(S))$. We call these curves *parabolic curves*. A relative compact core C of M_0 is always homeomorphic to $S \times [0,1]$ in our setting. We fix orientations on S and $\mathbb{H}^3$, and assume the identification of $S \times [0,1]$ with C to preserve the orientations.

An *end* of M_0 is an inverse limit (with respect to the inclusion) of complementary components of compact sets in M_0. Each component U of $M_0 \setminus C$ contains a unique end e of M_0, and also its closure contains a unique component Σ of $\mathrm{Fr}_{M_0} C$, where Fr_{M_0} denotes the frontier as a subspace of M_0. In this situation, we say that Σ *faces* the end e. The end e is said to be *geometrically finite* when it has a neighbourhood disjoint from any closed geodesic, and otherwise *geometrically infinite*. If e is geometrically finite, there is a boundary component F of the convex core $C(M)$ such that $F \cap M_0$ is isotopic to Σ. This component F in turn corresponds to a component of Ω_G/G which is regarded as lying at infinity.

When e is geometrically infinite, it was proved in [3] that there is a sequence of simple closed curves c_i on Σ which are homotopic in $U \cup \Sigma$ to closed geodesics c_i^* tending to the end. Such an end is called *simply degenerate*. Regarding c_i as a geodesic lamination on Σ, after fixing any hyperbolic metric on Σ, we consider the Hausdorff limit c_∞ of c_i, which is a geodesic lamination. It was shown by Thurston [24] and Bonahon [3] that c_∞ has only one minimal component λ, which is called the *ending lamination* of e, and that $S(\lambda) = \Sigma$. The geodesic lamination λ is the support of a measured lamination which is a limit of $\{r_i c_i\}$ in the space of measured laminations, where r_i is a positive scalar.

The notion of ending lamination was first introduced by Thurston using pleated surfaces as follows. Let Σ be a subsurface of S as above. If the end e facing Σ is geometrically infinite, there is a sequence of pleated surfaces $\{f_i\}$ homotopic to the inclusion of Σ which tends to e. For instance, in the setting of the preceding paragraph, pleated surfaces realising the simple closed curves c_i are such pleated surfaces. Thurston considered the Hausdorff limit of the geodesic laminations realised by such pleated surfaces, and proved that the limit has only one minimal component, which is defined to be the ending lamination of e. He also showed that the ending lamination thus defined does not depend on the choice of pleated surfaces.

Recall that the relative compact core C is identified with $S \times [0,1]$. When a parabolic curve lies on $S_+ = S \times \{1\}$ (resp. $S_- = S \times \{0\}$), we call it an *upper* (resp. a *lower*) parabolic curve. When the end e is above C, i.e., when Σ lies on $S \times \{1\}$ (resp. $S \times \{0\}$), we say that the ending lamination λ is an upper (resp. a lower) ending lamination. It was also proved in [24] and [3] that for each upper (resp. lower) ending lamination λ of $\mathbb{H}^3/\phi(\pi_1(S))$, each boundary component of $S(\lambda)$ is an upper (resp. lower) parabolic curve. We call the union of the parabolic curve and the ending laminations regarded as lying on $S_- \sqcup S_+$ the *qi(quasi-isometric)-end invariant* of ϕ (or $\mathbb{H}^3/\phi(\pi_1(S))$). In particular, the union of those lying on S_+ (resp. S_-) is called the upper (resp. lower) qi-end invariant.

2.6. **Deformation spaces.** The space of faithful discrete representations of $\pi_1(S)$ into $\mathrm{PSL}_2(\mathbb{C})$ modulo conjugacy is denoted by $\mathsf{AH}(S)$. We endow $\mathsf{AH}(S)$ with the topology induced from the representation space. Although each element of $\mathsf{AH}(S)$ is a conjugacy class of representations, by abusing notation, we denote it by its representative.

The interior of $\mathsf{AH}(S)$ is known to be the quasi-Fuchsian space $\mathsf{QF}(S)$. When $\phi \in \mathsf{AH}(S)$ is quasi-Fuchsian, letting G be $\phi(\pi_1(S))$, the conformal structure on Ω_G/G induces a marked conformal structures at infinity, on $S \times \{0\}$ and $S \times \{1\}$. We note that the identification of $S \times \{0\}$ to a component of Ω_G/G is orientation-preserving, but that of $S \times \{1\}$ is orientation-reversing. We use the symbol $\mathcal{T}(\bar{S})$ to denote the Teichmüller space of S with its orientation reversed. Then the conformal structure on Ω_G/G determines a point in $\mathcal{T}(S) \times \mathcal{T}(\bar{S})$. Bers showed that this identification of the conformal structure on Ω_G/G and a point in $\mathcal{T}(S) \times \mathcal{T}(\bar{S})$ gives a parametrisation of $\mathsf{QF}(S)$, which we denote by $q\colon \mathcal{T}(S) \times \mathcal{T}(\bar{S}) \to \mathsf{QF}(S)$.

For a general point $\phi \in \mathsf{AH}(S)$ and $G = \phi(\pi_1(S))$, let λ_- and λ_+ be the lower and upper qi-end invariants of $\mathbb{H}^3/\phi(\pi_1(S))$. We recall that λ_- (resp. λ_+) has the property that for any component λ of λ_- (resp. λ_+), every boundary component of $S(\lambda)$ is contained in λ_- (resp. λ_+). The quotient of the region of discontinuity Ω_G/G is identified with the disjoint union of $S_- \setminus S(\lambda_-)$ and $S_+ \setminus S(\lambda_+)$, which we denote by Σ_- and Σ_+. The ending lamination theorem, proved by Minsky and Brock-Canary-Minsky [14, 7], shows that any Kleinian group in $\mathsf{AH}(S)$ having λ_- and λ_+ as lower and upper qi-end invariants is a quasi-conformal deformation of G. Therefore, we denote the quasi-conformal deformation space of G by $\mathsf{QH}_{\lambda_-,\lambda_+}$. The theory of Bers-Kra-Maskit-Marden-Sullivan shows that the conformal structures at infinity give a parametrisation $q\colon \mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+) \to \mathsf{QH}_{\lambda_-,\lambda_+}$.

2.7. **Geometric limits.** For a sequence of Kleinian groups $\{G_i\}$, we say that $\{G_i\}$ converges to a Kleinian group Γ *geometrically* if (i) every element $\gamma \in \Gamma$ is a limit of some sequence $\{g_i \in G_i\}$, and (ii) for every convergent subsequence $\{g_{i_j} \in G_{i_j}\}$, its limit lies in Γ. The geometric convergence is equivalent to the pointed Gromov-Hausdorff convergence of the corresponding hyperbolic 3-manifolds: fixing a basepoint $x \in \mathbb{H}^3$ and letting x_i and x_∞ be the projections of x to $\mathbb{H}^3/G_i$ and $\mathbb{H}^3/\Gamma$ respectively, the sequence of pointed hyperbolic 3-manifolds $\{(\mathbb{H}^3/G_i, x_i)\}$ converges to $(\mathbb{H}^3/\Gamma, x_\infty)$ in the sense of Gromov-Hausdorff if and only if $\{G_i\}$ converges to Γ geometrically. Due to this fact, we also refer to Gromov-Hausdorff limits as geometric limits. The compactness of Gromov-Hausdorff topology shows that every sequence of non-elementary Kleinian groups has a geometric limit after passing to a subsequence. We recall that, by definition, if $\{(\mathbb{H}^3/G_i, x_i)\}$ converges to $(\mathbb{H}^3/\Gamma, x_\infty)$ in the sense of Gromov-Hausdorff, then there is a K_i-bi-Lipschitz diffeomorphism, which is called an *approximate isometry* between the R_i-ball around x_i and the K_iR_i-ball around x_∞, with $K_i \longrightarrow 1$ and $R_i \longrightarrow \infty$.

Suppose that a sequence $\{\phi_i\}$ in $\mathsf{AH}(S)$ converges to $\psi \in \mathsf{AH}(S)$. In this situation, we always assume that we take representatives so that $\{\phi_i\}$ converges to ψ as genuine representations from $\pi_1(S)$ into $\mathrm{PSL}_2(\mathbb{C})$. Then, passing to a subsequence, $\{\phi_i(\pi_1(S))\}$ converges to some Kleinian group Γ. From the definition of geometric limits, it is easy to see that Γ contains $\psi(\pi_1(S))$. If $\Gamma = \psi(\pi_1(S))$, we say that $\{\phi_i\}$ converges to ψ *strongly*.

In Ohshika-Soma [20] a classification of geometric limits of Kleinian surface groups was given. An alternative description was also given in [19]. We shall now review some of the results there which will be used in the proof of the main theorem, in particular, in Section 5.2.

Let $\{\phi_i\}$ be a sequence in $\mathsf{AH}(S)$, and suppose that $\{\phi_i(\pi_1(S))\}$ converges geometrically to a Kleinian group Γ. The following is a paraphrase of a part of Theorem A in [20].

Theorem 2.1. *The non-cuspidal part $(\mathbb{H}^3/\Gamma)_0$ is topologically embedded in $S \times (0,1)$ in such a way that the following hold.*

(a) Every end of $(\mathbb{H}^3/\Gamma)_0$ is mapped to a horizontal surface $\Sigma \times \{t\}$, where Σ is an incompressible subsurface of S and t lies in $[0,1]$.

(b) Every geometrically finite end of $(\mathbb{H}^3/\Gamma)_0$ lies on either $S\times\{0\}$ or $S\times\{1\}$.

(c) Each boundary component of the convex core of $(\mathbb{H}^3/\Gamma)_0$ is a Gromov-Hausdorff limit of boundary components of convex cores of $(\mathbb{H}^3/\phi_i(\pi_1(S))_0$ with some base points.

(d) Every geometrically infinite end is either simply degenerate or an accumulation set of countably many torus cusps or simply degenerate ends or both.

Identifying $(\mathbb{H}^3/\Gamma)_0$ with the image of its embedding in $S \times (0,1)$ as above, we can talk about the horizontal direction and the vertical direction in $(\mathbb{H}^3/\Gamma)_0$.

The following lemma, which is [19, Lemma 4.13] and also can be found in [6, §3], will be used in Section 5.2.

Lemma 2.2. *In the setting as above, suppose moreover that $\{\phi_i\}$ converges to $\psi \in \mathsf{AH}(S)$. Then the image of the inclusion of $\psi(\pi_1(S))$ into $(\mathbb{H}^3/\Gamma)_0$ is represented by an immersion $f_\infty \colon S \to (\mathbb{H}^3/\Gamma)_0$ which is horizontal except for disjoint annuli in S whose images wrap around torus cusps.*

An immersion f_∞ as above is called an *algebraic locus*. We note that for an approximate isometry ρ_i from $\mathbb{H}^3/\phi_i(\pi_1(S))$ to $\mathbb{H}^3/\Gamma$, the composition $\rho_i^{-1}\circ f_\infty$ induces the same isomorphism as ϕ_i between the fundamental groups for sufficiently large i.

By Theorem 2.1 and Lemma 2.2, we can show the following.

Corollary 2.3. *Suppose that $\{\phi_i\}$ in $\mathsf{AH}(S)$ converges to $\psi \in \mathsf{AH}(S)$. If $(\mathbb{H}^3/\Gamma)_0$ does not have a torus boundary component and every geometrically infinite end is homotopic into an algebraic locus, then $\{\phi_i\}$ converges to ψ strongly.*

Proof. Since we assumed that $(\mathbb{H}^3/\Gamma)_0$ does not have a torus boundary component, the algebraic locus f_∞ in Lemma 2.2 cannot wrap around a boundary component, and hence is a horizontal surface. By assumption, every geometrically infinite end can be lifted to the algebraic limit, and hence simply degenerate. This implies that in the embedded image of $(\mathbb{H}^3/\Gamma)_0$ in $S \times (0,1)$, there is no other end on the side farther from $f_\infty(S)$ of each geometrically infinite end. Therefore, we can isotope the embedding so that every geometrically infinite end lies on $S \times \{0\} \cup S \times \{1\}$.

By Theorem 2.1-(a), every geometrically finite also lies on $\Sigma \times \{0,1\}$. Each boundary component of $(\mathbb{H}^3/\Gamma)_0$, which is an open annulus, has both ends on the same level, either on $S \times \{0\}$ or $S \times \{1\}$, for it cannot pass through the algebraic locus. This means that $(\mathbb{H}^3/\Gamma)_0$ coincides with the complement of finitely many half solid tori lying on a neighbourhood of $S \times \{0\}$ or $S \times \{1\}$. Therefore it is homeomorphic to $S \times (0,1)$, and hence $\psi(\pi_1(S)) = \Gamma$. □

As explained in [20, 19], if a sequence $\{\phi_i\}$ in $\mathsf{AH}(S)$ converges geometrically to a Kleinian group Γ, a geometric limit of (uniform) bi-Lipschitz model manifolds of $(\mathbb{H}^3/\phi_i(S))_0$ (due to Minsky [14]) serves as a bi-Lipschitz model manifold of $(\mathbb{H}^3/\Gamma)_0$. Suppose that $\mathbb{H}^3/\phi_i(\pi_1(S))$ has a lower qi-end invariant λ_- and an upper qi-end invariant λ_+. Let Σ_- and Σ_+ be $S \setminus S(\lambda_-)$ and $S \setminus S(\lambda_+)$ respectively, and $\mathbf{m}_i^-$ and $\mathbf{m}_i^+$ the structures at infinity of $\mathbb{H}^3/\phi_i(\pi_1(S))$ on Σ_- and Σ_+ respectively. Let P_i^- and P_i^+ be shortest pants decompositions (with respect to the hyperbolic length) of $(\Sigma_-, \mathbf{m}_i^-)$ and $(\Sigma_+, \mathbf{m}_i^+)$ respectively. By adding a shortest transversal simple closed curve to each component of P_i^- (resp.P_i^+) disjoint from all the other components, we get shortest markings M_i^- (resp. M_i^+). Minsky's model manifold is constructed from the hierarchy of tight geodesics connecting the generalised markings $M_i^- \cup \lambda_-$ and $M_i^+ \cup \lambda_+$.

In this setting, we have the following lemmas. We recall that the geometric convergence of $\{\phi_i(\pi_1(S))\}$ to Γ is equivalent to the geometric convergence of $(\mathbb{H}^3/\phi_i(\pi_1(S)), x_i)$ to $(\mathbb{H}^3/\Gamma, x_\infty)$.

Lemma 2.4. *Suppose that $\{\phi_i(\pi_1(S))\}$ converges geometrically to Γ. Let l be a non-contractible simple closed curve on S. We assume that the translation length of $\phi_i(l)$ is bounded from above as $i \longrightarrow \infty$. Let ϵ be a positive constant less than the three-dimensional Margulis constant. Let V_i be the ϵ-Margulis tube around the closed geodesic representing $\phi_i(l)$.*

(1) If V_i converges geometrically to a torus cusp neighbourhood in $\mathbb{H}^3/\Gamma$ as $i \longrightarrow \infty$, then both $\mathrm{length}_{\mathbf{m}_i^+}(l)$ *and* $\mathrm{length}_{\mathbf{m}_i^-}(l)$ *(if one or both of them are defined) are bounded from below by a positive constant, and $d_{A(l)}(M_i^- \cup \lambda_-, M_i^+ \cup \lambda_+)$ goes to ∞. Here $A(l)$ denotes an annular neighbourhood of l and $d_{A(l)}$ the distance in the curve complex of $A(l)$. Conversely if both* $\mathrm{length}_{\mathbf{m}_i^+}(l)$ *and* $\mathrm{length}_{\mathbf{m}_i^-}(l)$ *are bounded from below by a positive constant, the distance from x_i to V_i is bounded as $i \longrightarrow \infty$, and $d_{A(l)}(M_i^- \cup \lambda_-, M_i^+ \cup \lambda_+)$ goes to ∞,*

then V_i converges geometrically to either a torus cusp neighbourhood or a $\mathbb{Z}$-cusp attached to a geometrically infinite end in $\mathbb{H}^3/\Gamma$.

(2) *Suppose moreover that $\{\phi_i\}$ converges in $\mathsf{AH}(S)$, and let k be a simple closed curve intersecting l essentially. If $U_i(l)$ converges geometrically to a torus cusp in $\mathbb{H}^3/\Gamma$ lying above (resp. below) an algebraic locus as $i \longrightarrow \infty$, then $d_{A(l)}(k, M_i^+ \cup \lambda_+)$ (resp. $d_{A(l)}(k, M_i^- \cup \lambda_-)$) goes to ∞. In the case when an algebraic locus wraps around a torus cusp, we regard the cusp as lying both above and below the locus. Conversely, if $d_{A(l)}(k, M_i^+ \cup \lambda_+)$ (resp. $d_{A(l)}(k, M_i^- \cup \lambda_-)$) goes to ∞, then V_i converges geometrically to either a torus cusp neighbourhood or a $\mathbb{Z}$-cusp neighbourhood attached to a geometrically infinite end.*

Proof. Recall that the model manifold of $(\mathbb{H}^3/\Gamma)_0$ can be taken to be a geometric limit of models constructed from hierarchies of tight geodesics. If $\mathrm{length}_{\mathbf{m}_i^-}(l)$ or $\mathrm{length}_{\mathbf{m}_i^+}(l)$ goes to 0 after passing to a subsequence, then the corresponding geometrically finite block of the model manifold of $(\mathbb{H}^3/\phi_i(\pi_1(S)))_0$ splits along a simple closed curve representing l as $i \longrightarrow \infty$. This shows that in this case, the geometric limit of V_i must be a $\mathbb{Z}$-cusp if it is contained in $\mathbb{H}^3/\Gamma$.

Now suppose that $\mathrm{length}_{\mathbf{m}_i^-}(l)$ and $\mathrm{length}_{\mathbf{m}_i^+}(l)$ are bounded from below by a positive constant if one or both of them are defined. Then by [14, Lemma 9.4], we see that V_i converges geometrically to a torus cusp neighbourhood if and only if $d_{A(l)}(M_i^- \cup \lambda_-, M_i^+ \cup \lambda_+)$ goes to ∞. Thus we are done for the part (1).

A proof of the part (2) can be found in [19, Proof of Theorem 5.2]. We summarise its argument here. Note that V_i converges to either a torus cusp or a $\mathbb{Z}$-cusp in $\mathbb{H}^3/\Gamma$ since we assumed that the distance from x_i to V_i is bounded as $i \longrightarrow \infty$. Consider the situation where V_i converges to a torus cusp neighbourhood V_∞. First suppose that the algebraic locus does not wrap around V_∞ and that V_∞ lies above an algebraic locus. Recall that Minsky's model manifold is constructed from a hierarchy H_i of tight geodesics in the curve complex of (subsurfaces of) S. If the Margulis tube V_i converges to a torus cusp neighbourhood, then the annular neighbourhood $A(l)$ supports a geodesic g_i in H_i whose length goes to ∞. The last vertex of g_i is within bounded distance from $\pi_{A(l)}(M_i^+ \cup \lambda_+)$ as $i \longrightarrow \infty$, where $\pi_{A(l)}$ denotes the projection between the curve complexes $\mathcal{CC}(S)$ and $\mathcal{CC}(A(l))$ induced by restricting curves to $A(l)$. On the other hand, since the torus cusp lies above an algebraic locus, the initial vertex of g_i is within bounded distance from $\pi_{A(l)}(k)$ as $i \longrightarrow \infty$. Thus, we have $d_{A(l)}(k, M_i^+ \cup \lambda_+) \longrightarrow \infty$. The same argument works also in the case when V_∞ lies below an algebraic locus. (See [19, Claim 5.3] for more details.)

Next suppose that an algebraic locus wraps n-times around V_∞ with $n \neq 0$. Then there is a sequence of integers $r(i)$ whose absolute values go to ∞ such that the initial vertex of h_i is within bounded distance from $\pi_{A(l)}(\tau_l^{nr(i)}(k))$ and the last vertex is within bounded distance from $\pi_{A(l)}(\tau_l^{(n+1)r(i)}(k))$, where τ_l denotes the Dehn twist around l. The initial vertex of h_i is also within bounded distance from $\pi_{A(l)}(M_i^- \cup \lambda_-)$, and the last vertex of h_i is also within bounded distance from $\pi_{A(l)}(M_i^+ \cup \lambda_+)$. Therefore both $d_{A(l)}(k, M_i^- \cup \lambda_-)$ and $d_{A(l)}(k, M_i^+ \cup \lambda_+)$ go to ∞ in this case.

Finally, suppose that $d_{A(l)}(k, M_i^- \cup \lambda_-)$ goes to ∞. Then by [14, Lemma 9.4] again, we see that the boundary of the Margulis tube ∂V_i either converges to the boundary of a torus cusp neighbourhood or diverges to give rise to an open annulus whose end is attached to a lower end of $(\mathbb{H}^3/\Gamma)_0$. Since $d_{A(l)}(k, M_i^- \cup \lambda_-) \longrightarrow \infty$, the curve l cannot be contained in the shortest pants decomposition M_i. Therefore, the upper geometrically finite block of the model manifold of $(\mathbb{H}^3/\phi_i(\pi_1(S)))_0$ cannot split along l, and hence neither end of the geometric limit of ∂V_i cannot tend to $S \times \{0\}$. It follows that the end of $(\mathbb{H}^3/\Gamma)_0$ to which an end of the geometric limit of ∂V_i tends must be geometrically infinite by Theorem 2.1-(b). Thus we have shown that the second part of (2) holds. □

Lemma 2.5. *Suppose that $\{\phi_i\}$ converges in $\mathsf{AH}(S)$ to ψ. Let Σ be an open incompressible subsurface of S, with negative Euler characteristic. Then, $\{P_i^+|\Sigma\}$ (resp. $\{P_i^-|\Sigma\}$) converges (in the Hausdorff topology) to a geodesic lamination containing a minimal component λ whose minimal supporting surface is Σ if and only if there is an upper (resp. a lower) simply degenerate end of $(\mathbb{H}^3/\psi(\pi_1(S)))_0$ whose ending lamination is λ.*

Proof. This is contained in [19, Proposition 4.18, Theorem 5.2] and their generalisations explained in [19, §10], or alternatively in Brock-Bromberg-Canary-Lecuire [6, Theorem 1.2]. □

3. Main theorems

The purpose of this paper is to prove the following theorem, which is a generalisation of the existence part of the theorem of Bonahon-Otal [4] to general, possibly geometrically infinite Kleinian surface groups. We do not have the uniqueness part of Bohaon-Otal's theorem (in the case when the bending laminations are multi-curves), but instead have compactness for the set of Kleinian groups realising given data of qi-end invariants and bending laminations.

Theorem 3.1. *Let S be a closed oriented surface of genus greater than 1, and S_-, S_+ two copies of S, where S_+ has the same orientation as S whereas S_- has the opposite orientation.*

— *Let λ_- and λ_+ be (possibly empty) geodesic laminations without non-compact isolated leaves on S_- and S_+ respectively, such that for every component λ of λ_- (resp. λ_+), each boundary component of the minimal supporting surface $S(\lambda)$ is isotopic to a closed geodesic contained in λ_- (resp. λ_+).*

— *Let μ_- and μ_+ be measured laminations on S_- and S_+ such that*
  (a) $\lambda_- \cap \mu_- = \emptyset$ *and* $\lambda_+ \cap \mu_+ = \emptyset$*;*
  (b) *neither μ_- nor μ_+ contains a compact leaf of weight larger than or equal to π; and*
  (c) $\lambda_- \sqcup \mu_-$ *and* $\lambda_+ \sqcup \mu_+$ *fill up S if we identify both S_- and S_+ with S.*

Then the following hold.

(1) *There is $\phi \in \mathsf{AH}(S)$ such that*
  (i) *the hyperbolic 3-manifold $\mathbb{H}^3/\phi(\pi_1(S))$ has λ_- as its lower qi-end invariant and λ_+ as its upper qi-end invariant, and*
  (ii) *the hyperbolic 3-manifold $\mathbb{H}^3/\phi(\pi_1(S))$ realises μ_- and μ_+ as the bending laminations on the lower and the upper boundaries respectively of its convex core $C(\mathbb{H}^3/\phi(\pi_1(S)))$.*

(2) *The set of all $\rho \in \mathsf{AH}(S)$ satisfying the condition* (1) *is a compact subset of $\mathsf{QH}_{\lambda_-,\lambda_+}$.*

Remark 3.2. In the theorem above, the existence of μ_-, μ_+ satisfying (c) imposes on λ_-, λ_+ the condition that they share no minimal component.

This theorem is an immediate consequence of following Theorem 3.3, which states something a bit stronger.

Let λ_- and λ_+ be geodesic laminations on S_- and S_+ as in Theorem 3.1, which share no minimal component. Let Σ_- and Σ_+ be the complements $S_- \setminus S(\lambda_-)$ and $S_+ \setminus S(\lambda_+)$ respectively. As we explained in Section 2.6, we have a parametrisation $q\colon \mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+) \to \mathsf{QH}_{\lambda_-,\lambda_+}$. Let $b\colon \mathsf{QH}_{\lambda_-,\lambda_+} \to \mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+)$ be the map taking $\rho \in \mathsf{QH}_{\lambda_-,\lambda_+}$ to the bending lamination of $C(\mathbb{H}^3/\rho(\pi_1(S)))$.

Theorem 3.3. *Let D be the subset of $\mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+)$ consisting of measured laminations which do not satisfy at least one of the conditions (b) and (c) in Theorem 3.1. Then the map $b \circ q\colon \mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+) \to \mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+) \setminus D$ is a proper, degree-1 map.*

4. Bending laminations

We present some lemmas which will be useful for the proofs of our main theorems.

Lemma 4.1. *For every $K > 0$, there is a positive constant $L(K)$ depending only on K which goes to 0 as $K \longrightarrow 0$ with the following property. For every $\phi \in \mathsf{AH}(S)$, letting $M = \mathbb{H}^3/\phi(\pi_1(S))$, if Σ is a boundary component of $C(M)$ with bending lamination λ and c is a simple closed curve on Σ with length less than K, then $\iota(c, \lambda) < L(K)$.*

Proof. This is a direct consequence of the boundedness of the average bending measure proved by Bridgeman [5]. $\square$

As a consequence of this lemma, we have the following.

Corollary 4.2. *Consider the situation in Theorem 3.1, and let* Σ *be a component of* $\Sigma_- \sqcup \Sigma_+$*. Let* $\{\mathbf{g}_i\}$ *be a sequence in* $\mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+)$*. Let* γ_i *be a shortest pants decomposition of* $\mathbf{g}_i|\Sigma$*. Then the sequence* $\{\iota(b \circ q(\mathbf{g}_i), \gamma_i)\}$ *is bounded.*

Proof. By Bers's lemma, there is a constant C depending only on S such that each component of the shortest pants decomposition of $(\Sigma, \mathbf{g}_i)$ has length less than C. Therefore, by Lemma 4.1, $\iota(b \circ q(\mathbf{g}_i), \gamma_i)$ is bounded as $i \longrightarrow \infty$. □

Lemma 4.3. *For any* $\epsilon > 0$*, there is a positive constant* $\delta > 0$ *depending only on* ϵ *satisfying the following.*

Let $\{\phi_i\}$ *be a sequence in* $\mathsf{QH}_{\lambda_-,\lambda_+}$*, and* Σ *a component of* $\Sigma_- \sqcup \Sigma_+$*. Let* μ_i *be the restriction of the bending lamination of the convex core of* $\mathbb{H}^3/\phi_i(\pi_1(S))$ *to (the component corresponding to)* Σ*. We denote the hyperbolic metric on* Σ *as a boundary component of the convex core* $C(\mathbb{H}^3/\phi_i(\pi_1(M)))$ *by* m_i'*.*

Suppose

- *(i) that* $\{\mu_i\}$ *converges to a measured lamination* μ_∞*,*
- *(ii) that* μ_∞ *is decomposed into disjoint (possibly empty) sublaminations* μ^1_∞ *and* μ^2_∞*,*
- *(iii) and that* c *is a simple closed curve on* Σ *such that* $\iota(c, \mu^1_\infty) < \delta$ *and* $\angle_{(\Sigma, m_i')}(c, \mu^2_\infty) < \delta$ *for large* i*.*

Let c_i *be the closed geodesic on* Σ *with respect to* m_i' *freely homotopic to* c*, and* c_i^* *the closed geodesic in* $\mathbb{H}^3/\phi_i(\pi_1(S))$ *freely homotopic to* $\phi_i(c)$*.*

Then we have

$$1 \leq \frac{\mathrm{length}_{(\Sigma, m_i')}(c_i)}{\mathrm{length}_{\mathbb{H}^3/\phi_i(\pi_1(S))}(c_i^*)} \leq 1 + \epsilon.$$

Proof. Since the length of c_i^* is less than or equal to that of c_i, the first inequality obviously holds.

Since the Hausdorff limit of the support $|\mu_i|$ is a geodesic lamination containing $\mu_\infty = \mu^1_\infty \sqcup \mu^2_\infty$, for sufficiently large i, we can decompose c_i into two parts c_i^2 and c_i^1 such that $\angle_{(\Sigma, m_i')}(c_i^2, \mu_i) < 2\delta$ and $\iota(c_i^1, \mu_i) < 2\delta$. By applying [1, Proposition 4.1] for c_i^2 and [8, Corollary 4.6] for c_i^1, we obtain the second inequality. □

5. PROPERNESS

In this section, we shall prove the properness of $b \circ q$ in Theorem 3.3. The argument is by contradiction. Suppose that $b \circ q$ is not proper. Then, there is a sequence $\{(\mathbf{m}_i^-, \mathbf{m}_i^+)\} \subset \mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+)$ without a convergent subsequence such that $\{b \circ q(\mathbf{m}_i^-, \mathbf{m}_i^+)\}$ converges in $\mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+) \setminus D$ to a measured lamination ν on $\Sigma_- \sqcup \Sigma_+$.

We are going to analyse a geometric limit of $\{\mathbb{H}^3/q(\mathbf{m}_i^-, \mathbf{m}_i^+)\}$, making use of results in Ohshika-Soma [20] and Ohshika [19]. Let $\phi_i \colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ be a representative of $q(\mathbf{m}_i^-, \mathbf{m}_i^+)$ for each i. We take the ϕ_i to converge to some $\psi \in \mathsf{AH}(S)$ if $\{q(\mathbf{m}_i^-, \mathbf{m}_i^+)\}$ converges in $\mathsf{AH}(S)$.

We divide our argument into the following three cases:

(a) The case when $\{q(\mathbf{m}_i^-, \mathbf{m}_i^+)\}$ converges in $\mathsf{AH}(S)$ strongly after passing to a subsequence.

(b) The case when $\{q(\mathbf{m}_i^-, \mathbf{m}_i^+)\}$ converges in $\mathsf{AH}(S)$ after passing to a subsequence, but not strongly.

(c) The case when $\{q(\mathbf{m}_i^-, \mathbf{m}_i^+)\}$ diverges in $\mathsf{AH}(S)$ (even after passing to a subsequence).

In the cases (a) and (b), we put a basepoint $\tilde{x}$ in $\mathbb{H}^3$, and by projecting it to $\mathbb{H}^3/\phi_i(\pi_1(S))$, we get a basepoint x_i. Taking the Gromov-Hausdorff limit of $(\mathbb{H}^3/\phi_i(\pi_1(S)), x_i)$, we obtain a geometric limit (M_∞, x_∞) which is covered by the algebraic limit $\mathbb{H}^3/\psi(\pi_1(S))$.

Now we start to show that in each of the three cases, we get a contradiction.

5.1. **Case (a).** In this case, $M_\infty = \mathbb{H}^3/\psi(\pi_1(S))$. There are three possibilities for ψ:

(a)-(1) The case when ψ is also contained in $\mathsf{QH}_{\lambda_-,\lambda_+}$.

(a)-(2) The case when $(\mathbb{H}^3/\psi(\pi_1(S))_0$ has a 'new simply degenerate end' not corresponding to a simply degenerate end of $(\mathbb{H}^3/\phi_i(\pi_1(S)))_0$.

(a)-(3) The case when $\mathbb{H}^3/\psi(\pi_1(S))$ does not have a new simply degenerate end, but has a 'new $\mathbb{Z}$-cusp' not corresponding to a $\mathbb{Z}$-cusp of $(\mathbb{H}^3/\phi_i(\pi_1(S)))_0$.

Since we are assuming that $\{(\mathbf{m}_i^-, \mathbf{m}_i^+)\}$ has no convergent subsequence, we can exclude the case (a)-(1).

We first assume that the condition (a)-(2) holds. Let $\Psi \colon S \times (0,1) \to \mathbb{H}^3/\psi(\pi_1(S))$ be an orientation-preserving homeomorphism inducing ψ between the fundamental groups. For brevity of description, we now assume that a new simply degenerate end is lower. The case when the new simply degenerate end is upper can be dealt with in the same way.

Let γ_i be a shortest pants decomposition of $(\Sigma_-, \mathbf{m}_i^-)$. Let γ_∞ be the Hausdorff limit of the γ_i regarded as geodesic laminations. Since we assumed that there is a new lower simply degenerate end, by Lemma 2.5, there is a minimal component ℓ of γ_∞ which is the ending lamination of such an end. Let Σ be the minimal supporting surface of ℓ. Let λ be a measured lamination supported on ℓ. Let $\delta > 0$ be the positive constant given in Lemma 4.3 for $\epsilon = 1$. Now, take an essential simple closed curve d on Σ such that $\iota(\nu, d) < \delta$. Since we have $\iota(\lambda, d) > 0$ from the fact that $\Sigma = S(\lambda)$, and γ_i converges to γ_∞ containing $\ell = |\lambda|$ in the Hausdorff topology, the intersection number $\iota(\gamma_i, d)$ goes to ∞, which implies the length of d with respect to $\mathbf{m}_i^-$ also goes to ∞, for γ_i is a shortest pants decomposition. On the other hand, $\{b \circ q(\mathbf{m}_i^-, \mathbf{m}_i^+)\}$ converges to ν by our assumption. Then, by setting μ_∞^1 in the statement of Lemma 4.3 to be ν and μ_∞^2 to be empty, it follows that the translation length of $\phi_i(d)$ goes to ∞ as $i \longrightarrow \infty$, contradicting the assumption that $\{\phi_i\}$ converges.

We now assume the condition (a)-(3) that $\mathbb{H}^3/\psi(\pi_1(S))$ has a new $\mathbb{Z}$-cusp, represented by a non-contractible, non-peripheral simple closed curve c on $\Sigma_- \sqcup \Sigma_+$, but does not have a new simply degenerate end. In the same way as in the preceding paragraph, we can assume that c lies on Σ_-. Since we assumed that there is no new simply degenerate end, if the $\mathbb{Z}$-cusp represented by $\psi(c)$ touches a simply degenerate end, it corresponds to a simply degenerate end of $(\mathbb{H}^3/\phi_i(\pi_1(S))_0$, which implies that $\phi_i(c)$ is parabolic, contradicting our assumption that c is a new parabolic curve. Therefore, the lower $\mathbb{Z}$-cusp represented by c has geometrically finite ends on its both sides, which may coincide. Therefore, the lower boundary of the convex core of $\mathbb{H}^3/\psi(\pi_1(S))$ has bending angle π along c. Since we assumed that $\{\phi_i\}$ converges to ψ strongly, the convex core of $\mathbb{H}^3/\phi_i(\pi_1(S))$ converges to that of $\mathbb{H}^3/\psi(\pi_1(S))$ (see [17, Proof of Theorem 5]), and hence the bending angle along $c \subset \Sigma_-$ of the convex core of $\mathbb{H}^3/\phi_i(\pi_1(S))$ converges to π. This means that $\{b \circ q(\mathbf{m}_i^-, \mathbf{m}_i^+)\}$ tends to a point in D, contradicting our assumption. Thus, in every sub-case of the case (a), we have obtained a contradiction.

We state what we have proved in the last paragraph as a lemma to refer to it in the following section.

Lemma 5.1. *Suppose that $\{\phi_i\}$ converges to ψ strongly, and that $\mathbb{H}^3/\psi(\pi_1(S))$ has a $\mathbb{Z}$-cusp whose core curve is a simple closed curve c on a component Σ of $\Sigma_- \sqcup \Sigma_+$. Suppose furthermore that the $\mathbb{Z}$-cusp has geometrically finite ends on its both sides. Then the bending angle on the boundary of the convex core of $\mathbb{H}^3/\phi_i(\pi_1(S))$ along c converges to π.*

5.2. **Case (b).** By Corollary 2.3, if $\{q(\mathbf{m}_i^-, \mathbf{m}_i^+)\}$ converges algebraically but not strongly, then the geometric limit either has a $\mathbb{Z} \times \mathbb{Z}$-cusp or a new simply degenerate end. We note that the following arguments only use the assumption that ν satisfies the condition (c) of Theorem 3.1. Later in §6, we shall use the arguments again in the case when ν satisfies (c) but not (b) of Theorem 3.1.

By Theorem 2.1, the geometric limit M_∞ is topologically embedded in $S \times (0,1)$. We regard M_∞ as embedded in $S \times (0,1)$ from now on, and we call the direction of $S \times \{t\}$ *horizontal.* By Lemma 2.2, there is an algebraic locus $f_\infty \colon S \to M_\infty$ which can be lifted to an immersion into $\mathbb{H}^3/\psi(\pi_1(S))$ inducing ψ between the fundamental groups such that $f_\infty(S)$ is horizontal except for the part where it goes around a torus cusp. We recall that the pull-back of f_∞ to $\mathbb{H}^3/\phi_i(\pi_1(S))$ by an approximate isometry between $\mathbb{H}^3/\phi_i(\pi_1(S))$ and M_∞ induces the isomorphism ϕ_i between the fundamental groups.

A geometrically infinite end or a torus cusp is situated either above $f_\infty(S)$ or below $f_\infty(S)$, except for the case of a torus cusp around which $f_\infty(S)$ goes. When $f_\infty(S)$ goes around a torus cusp T, we regard T as being situated both above and below $f_\infty(S)$. We say that a simply degenerate end or a torus cusp is *nearest to* $f_\infty(S)$ when pleated surfaces tending to the end can be homotoped into $f_\infty(S)$ in M_∞ for a simply degenerate end, and when a longitude can be homotoped into $f_\infty(S)$ for a torus end. Since a wild geometrically infinite end is an accumulation of simply degenerate ends, it cannot be nearest to $f_\infty(S)$.

Claim 5.2. *Unless $\{\phi_i\}$ converges to ψ which has no new simply degenerate ends, there is either a new simply degenerate end or a torus cusp, which is nearest to $f_\infty(S)$. If a simply degenerate end is nearest to $f_\infty(S)$, then it can be lifted to the algebraic limit.*

Proof. Since M_∞ is embedded in $S \times (0,1)$, a simply degenerate end or a longitude of a torus end can be vertically homotoped into $f_\infty(S)$ unless there is another simply degenerate end or a torus end which impedes this. Since ends or torus cusps can accumulate only into a horizontal surface containing an end, there cannot be ends which accumulate into $f_\infty(S)$. Therefore, there must be a simply degenerate end or a torus cusp which can be homotoped into $f_\infty(S)$ without being obstructed by other ends. This shows the first statement.

If a simply degenerate end is nearest to $f_\infty(S)$, it is homotopic into $f_\infty(S)$, and hence can be lifted to the algebraic limit. $\square$

5.2.1. *Nearest new simply degenerate end.* Suppose that there is a new simply degenerate end E which is nearest to $f_\infty(S)$. We assume that E is situated above $f_\infty(S)$. The case when E is situated below $f_\infty(S)$ can be dealt with in the same way just by turning everything upside down. The end E has a neighbourhood of the form $\Sigma \times (s,t)$ for an incompressible subsurface Σ of S, where $\Sigma \times \{t\}$ corresponds to the end E (Figure 1). Let δ be the constant given in Lemma 4.3 for $\epsilon = 1$. Take a simple closed curve d contained in Σ satisfying $\iota(d,\nu) < \delta$. Let λ be a measured lamination on Σ whose support is the ending lamination of E. Take a shortest pants decomposition C_i of $(\Sigma_+, \mathbf{m}_i^+)$. Then by Lemma 2.5, we see that $C_i|\Sigma$ converges in the Hausdorff topology to a geodesic lamination whose only minimal component is $|\lambda|$. Since $\iota(d,\lambda) > 0$, we have $\iota(C_i, d) \longrightarrow \infty$ and see that the length of d with respect to $\mathbf{m}_i^+$ goes to ∞. On the other hand, $\{\iota(d, b \circ q(\mathbf{m}_i^-, \mathbf{m}_i^+))\}$ is bounded since $b \circ q(\mathbf{m}_i^-, \mathbf{m}_i^+)$ converges to ν. We set μ_∞^1 in the statement of Lemma 4.3 to be this ν and μ_∞^2 to be empty. Then we see that the translation length of $\phi_i(d)$ goes to ∞, contradicting our assumption that $\{\phi_i\}$ converges.

5.2.2. *Nearest torus cusp.* Let T be such a nearest torus cusp. Again, the case when there is a nearest torus cusp T below $f_\infty(S)$ can also be dealt

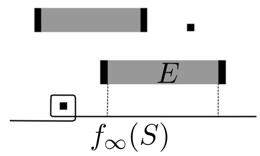

FIGURE 1. A nearest end above is a simply degenerate end

with in the same way, by turning everything upside down. Let l be a simple closed curve on Σ_+ whose image under f_∞ is homotopic to a longitude of T.

Let $\delta > 0$ be the constant given in Lemma 4.3 for $\epsilon = 1$, and take a simple closed curve d on Σ_+ with $\iota(d, l) > 0$ and $\iota(d, \nu \setminus l) < \delta$. Since l is homotopic to a longitude of a torus cusp in M_∞, by Lemma 2.4-(1), the length of l with respect to $\mathbf{m}_i^+$ is bounded from below by a positive constant. Consider the shortest pants decomposition P_i of $(\Sigma_+, \mathbf{m}_i^+)$, and extend it to a shortest marking M_i of Σ_+. Then by Lemma 2.4-(2), if we choose a simple closed curve k on S intersecting l essentially, we have $d_{A(l)}(M_i, k) \longrightarrow \infty$. This means that unless P_i contains l for all i after passing to a subsequence, there is a component a_i of P_i which spirals around l more and more as $i \longrightarrow \infty$. Then, since $\iota(P_i, d) \geq \iota(a_i, d)$ and the right hand side goes to ∞, the length of d with respect to $\mathbf{m}_i^+$ goes to ∞.

In the case when P_i contains l for all i, there is a curve t_i in M_i with $\iota(l, t_i) > 0$ which is shortest among such curves. Since we are assuming that the length of l with respect to $\mathbf{m}_i^+$ is bounded away from 0, the length of t_i with respect to $\mathbf{m}_i^+$ is bounded above. Then by Lemma 2.4-(2), we see that t_i spirals around l more and more as $i \longrightarrow \infty$. It follows that $d_{A(l)}(t_i, d)$ goes to ∞. Since t_i has bounded length and the length of l with respect $\mathbf{m}_i^+$ does not go to 0, we see that the length of d with respect to $\mathbf{m}_i^+$ goes to ∞ also in this case.

By the observation above, the closed geodesic representing d in $(\Sigma_+, \mathbf{m}_i^+)$ spirals around l more and more as $i \longrightarrow \infty$. This implies that $\angle_{(\Sigma_+, \mathbf{m}_i^+)}(d, l)$ goes to 0. We set μ_∞^1 to be $\nu \setminus l$ and μ_∞^2 to be l with a positive weight given by ν if l is contained in $|\nu|$, and μ_∞^1 to be ν and μ_∞^2 to be empty otherwise. Then we apply Lemma 4.3, and see that the translation length of $\phi_i(d)$ goes to ∞, contradicting our assumption that $\{\phi_i\}$ converges.

5.3. **Case (c).** Suppose that $\{q(\mathbf{m}_i^-, \mathbf{m}_i^+)\}$ does not converge algebraically. By considering efficient pleated surfaces as in Thurston [26], the following can be proved. See Ohshika [18, Theorem 3.1] for a complete proof.

Lemma 5.3. *There is a vertical codimension-1 lamination L properly embedded in $S \times [0, 1]$, which is disjoint from both λ_- and λ_+, such that for any sequence of weighted simple closed curves $s_i\gamma_i$ on S converging to a measured lamination μ, if $\iota(L, \mu) > 0$, then $s_i\mathrm{length}(\phi_i(\gamma_i)) \longrightarrow \infty$, where* $\mathrm{length}(\phi_i(\gamma_i))$ *denotes the length of the closed geodesic representing $\phi_i(\gamma_i)$ in* $\mathbb{H}^3/\phi_i(\pi_1(S))$*. Indeed, there is a pleated surface called an 'efficient' pleated* *surface in $\mathbb{H}^3/\phi_i(\pi_1(S))$ on which the growth of the length of every measured lamination is comparable to that in $\mathbb{H}^3/\phi_i(\pi_1(S))$.*

By the condition (c) of Theorem 3.1, which ν must satisfy, there is a component ν_0 of ν intersecting a component L_0 of L essentially. Let Σ be a component of $\Sigma_- \sqcup \Sigma_+$ containing ν_0, and take a shortest pants decomposition P_i of $(\Sigma, (\mathbf{m}_i^-, \mathbf{m}_i^+)|\Sigma)$. Since P_i is a shortest pants decomposition, by Sullivan's lemma or [15, Proposition 2.1], $\{\mathrm{length}(\phi_i(P_i))\}$ is bounded. By Lemma 5.3 we can apply the proof of [19, Lemma 5.7] using the length in $\mathbb{H}^3/\phi_i(\pi_1(S))$ instead of $(\mathbf{m}_i^-, \mathbf{m}_i^+)|\Sigma$, we see that the Hausdorff limit of P_i contains $L_0 \cap \Sigma$. This implies in turn that there is a sequence of positive numbers r_i tending to 0 and a component P_i' of P_i such that $r_i P_i'$ converges to a measured lamination containing $L_0 \cap \Sigma$ on $(\Sigma, (\mathbf{m}_i^-, \mathbf{m}_i^+)|\Sigma)$ unless $L_0 \cap \Sigma$ coincides with P_i' for every sufficiently large i. Even in the latter case, unless the length of $L_0 \cap \Sigma$ with respect to $(\mathbf{m}_i^-, \mathbf{m}_i^+)|\Sigma$ goes to 0, by letting P_i'' be the shortest simple closed curve transverse to P_i', we can find s_i going to 0 such that $s_i P_i'' \longrightarrow L_0 \cap \Sigma$. Since the length of P_i'' with respect to $(\mathbf{m}_i^-, \mathbf{m}_i^+)|\Sigma$ is bounded, abusing the symbols, we denote $s_i P_i''$ also by $r_i P_i'$ in this case. The properties which we shall use below are that $r_i \longrightarrow 0$ and that $\{\mathrm{length}_{(\mathbf{m}_i^-, \mathbf{m}_i^+)|\Sigma}(P_i)\}$ is bounded.

If the length of $L_0 \cap \Sigma$ with respect to $(\mathbf{m}_i^-, \mathbf{m}_i^+)|\Sigma$ goes to 0, the efficient pleated surface is pinched along $L_0 \cap \Sigma$. This implies that $\mathbb{H}^3/\phi_i(\pi_1(S))$ is also pinched along L_0, and hence $(\Sigma, (\mathbf{m}_i^-, \mathbf{m}_i^+)|\Sigma)$ is also pinched along $L_0 \cap \Sigma$. Since ν_0 intersects L_0, this contradicts Corollary 4.2.

It remains to consider the case when $\{r_i P_i\}$ converges to $L_0 \cap \Sigma$ with $r_i \longrightarrow 0$. Since $\iota(\nu_0, L_0) > 0$ and r_i goes to 0, we see that $\iota(P_i', \nu_0) \longrightarrow \infty$. This contradicts Corollary 4.2, for $\{\mathrm{length}_{(\mathbf{m}_i^-, \mathbf{m}_i^+)|\Sigma}(P_i)\}$ is bounded. Thus we have completed the proof of the case (c).

We note that we did not use the assumption (b) of Theorem 3.1. We state what we have proved in the case (c) as a lemme for later use.

Lemma 5.4. *Suppose that $\{b \circ q(\mathbf{m}_i^-, \mathbf{m}_i^+)\}$ converges to a measured lamination ν satisfying the assumption (c) of Theorem 3.1. Then $\{q(\mathbf{m}_i^-, \mathbf{m}_i^+)\}$ converges in $\mathsf{AH}(S)$.*

6. Homotopy to a degree-1 map

In this section, we shall prove that $b \circ q$ is a degree-1 map to $\mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+) \setminus D$ by constructing a homotopy in the one-point compactification of $\mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+) \setminus D$ between the map induced from $b \circ q$ and a degree-1 map. First, we shall define compactification where a homotopy will take place.

Definition 6.1. We let $\widehat{\mathrm{T}}$ be the one-point compactification of $\mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+)$, and $\widehat{\mathrm{ML}}$ the one-point compactification of $\mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+) \setminus D$. Since $b \circ q$ is proper, it induces a continuous map $\widehat{b \circ q} \colon \widehat{\mathrm{T}} \to \widehat{\mathrm{ML}}$.

The following is immediate from the definition of one-point compactification.

Lemma 6.2. *The map $b \circ q$ has degree 1 if and only if $\widehat{b \circ q}$ has degree 1.*

Therefore, we have only to show that $\widehat{b \circ q}$ has degree 1. For that, we shall construct a homotopy in an open set from $\widehat{b \circ q}$ to a locally degree-1 map. To construct the latter map, we shall make use of the following homeomorphism derived from the earthquake introduced by Thurston (see Thurston [25] and Kerckhoff [11]).

Definition 6.3. Fix $g_- \in \mathcal{T}(\Sigma_-)$ and $g_+ \in \mathcal{T}(\Sigma_+)$. For $j = -, +$, we let $E_j \colon \mathcal{ML}(\Sigma_j) \to \mathcal{T}(\Sigma_j)$ be the left earthquake map, that is, a homeomorphism sending a measured lamination $\lambda \in \mathcal{ML}(\Sigma_j)$ to the marked hyperbolic structure obtained by the left earthquake along λ on (Σ_j, g_j). Then, we have a homeomorphism $E_- \times E_+ \colon \mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+) \to \mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+)$. We define $\mathcal{E} \colon \mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+) \to \mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+)$ to be the inverse of $E_- \times E_+$. Slightly abusing notation, we denote the inverse of E_j defined on $\mathcal{T}(\Sigma_j)$ also by $\mathcal{E} \colon \mathcal{T}(\Sigma_j) \to \mathcal{ML}(\Sigma_j)$.

To construct a homotopy, we shall first define its support, which will be done by using an open neighbourhood of a point contained in the 'corner' of the product of the Thurston compactifications of the components of $\mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+)$. We now describe it more concretely. Let $\Sigma_1, \ldots, \Sigma_n$ be the components of $\Sigma_- \sqcup \Sigma_+$ that are not thrice-punctured spheres. We compactify each $\mathcal{T}(\Sigma_j)$ by attaching $\mathcal{PML}(\Sigma_j)$ as its boundary. We call their product $\prod_{j=1}^n (\mathcal{T}(\Sigma_j) \cup \mathcal{PML}(\Sigma_j))$ the *Thurston compactification product* of $\mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+)$ and denote its boundary by $\mathcal{PML}$. A point of $\mathcal{PML}$ has a form $(x_j)_{j=1}^n$, where x_j is either $\mathcal{T}(\Sigma_j)$ or $\mathcal{PML}(\Sigma_j)$ and at least one of the x_j is contained in the boundary $\mathcal{PML}(\Sigma_j)$. We put the product topology on the compactification. We call the subset $\prod_{j=1}^n \mathcal{PML}(\Sigma_j)$ of the boundary the *corner* and denote it by $\mathcal{PML}_c$. A sequence $\{\mathbf{m}_i\}$ of $\mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+)$ converges to a point in $\mathcal{PML}_c$ after passing to a subsequence if $\{\mathbf{m}_i|\Sigma_j\}$ diverges for every component Σ_j of $\Sigma_- \sqcup \Sigma_+$ that is not a thrice-punctured sphere.

Definition 6.4. Let Λ be a point in $\mathcal{ML}(\Sigma_- \sqcup \Sigma_+)$ not contained in D such that, for each component Σ_j of $\Sigma_- \sqcup \Sigma_+$ that is not thrice-punctured sphere, the restriction $\Lambda \cap \Sigma_j$ is an arational uniquely ergodic measured lamination. By setting its j-th coordinate to be $[\Lambda|\Sigma_j]$, we define $[\Lambda] \in \mathcal{PML}_c$.

We note that by the arationality, the condition that Λ is not contained in D is equivalent to, when $\lambda_- = \lambda_+ = \emptyset$, the condition that no two components of Λ are homotopic, and otherwise, the condition that no component of the support of Λ is homotopic to an ending lamination.

Definition 6.5. Let τ be a bi-recurrent train track on $\Sigma_- \sqcup \Sigma_+$ carrying Λ by a weight system ω in such a way that ω takes a positive value on every branch of τ. We call an arc connecting two measured laminations μ_1, μ_2 carried by τ a *segment* when it is a linear path with regard to the weight system. We note that this notion is independent of the choice of τ since the transition function between two weight systems is linear.

For two measured laminations λ_1, λ_2 carried by τ with weight systems ω_1, ω_2 respectively, we define $d_\tau(\lambda_1, \lambda_2)$ to be the sum of the differences of the weights of ω_1 and ω_2 on the branches of τ.

From now on until the end of this section, we fix a train track τ as above.

In the same way as we did for $\mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+)$, for each component Σ_j ($j = 1, \dots, n$) of $\Sigma_- \sqcup \Sigma_+$, we consider the ray compactification of $\mathcal{ML}(\Sigma_j)$ and regard $\mathcal{PML}(\Sigma_j)$ as its boundary at infinity, and define the *ray compactification* of $\mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+)$ to be $\prod_{j=1}^n (\mathcal{ML}(\Sigma_j) \cup \mathcal{PML}(\Sigma_j))$, whose boundary we denote by $\mathcal{PML}^r$.

As in the case of Teichmüller space, we call $\mathcal{PML}_c = \prod_{j=1}^n \mathcal{PML}(\Sigma_j)$ the corner also in this ray compactification, and we see that a sequence $\{\lambda_i\}$ in $\mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+)$ converges to a point in $\mathcal{PML}^r$ after passing to a subsequence if $\{\lambda_i | \Sigma_j\}$ diverges for every $j = 1, \dots, n$.

Definition 6.6. We call a subset U of $\mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+)$ a *truncated cone* if U consists of all measured laminations λ that satisfy the following conditions.

(a) The train track τ carries λ.

(b) Each weight of $w(\lambda)$ is greater than a fixed positive constant K. (Recall from Definition 6.5 that we denote the weight system on τ corresponding to λ by $w(\lambda)$.)

We say that a truncated cone U is a *truncated cone neighbourhood* of $[\Lambda]$ (supported on τ) when there is a neighbourhood V of $[\Lambda]$ in the boundary at infinity $\mathcal{PML}^r$ such that the closure of U in the ray compactification $(\mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+))$ coincides with V.

The positive weight systems on τ form an open set in $\mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+)$ since τ is bi-recurrent, and its ray compactification contains $[\Lambda]$ since each $\Lambda | \Sigma_j$ is arational and uniquely ergodic. Therefore we see that (the ray compactifications of) the truncated cone neighbourhoods form a basis of neighbourhoods of $[\Lambda]$ in the ray compactification $(\mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+)) \cup \mathcal{PML}^r$.

We define $\hat{\mathcal{E}} \colon \mathcal{PML} \to \mathcal{PML}^r$ to be a map such that for $(x_j) \in \mathcal{PML} = \prod_{j=1}^n (\mathcal{T}(\Sigma_j) \cup \mathcal{PML}(\Sigma_j))$, the k-th coordinate of $\hat{\mathcal{E}}((x_j))$ is $\mathcal{E}(x_k)$ if x_k lies in $\mathcal{T}(\Sigma_k)$, and x_k itself if x_k lies in $\mathcal{PML}(\Sigma_k)$. Then the following is a well-known property of the earthquake map. (See Papadopoulos [21] for instance.)

Lemma 6.7. *The map $\mathcal{E} \colon \mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+) \to \mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+)$ is a homeomorphism which can be extended continuously to $\hat{\mathcal{E}}$ on $\mathcal{PML}$. In particular, the extension is the identity on the corner $\mathcal{PML}_c$.*

We now show some lemmas and their corollaries which will be used in the main step of the proof of Theorem 3.3.

Lemma 6.8. *Let* $[\Lambda]$ *be a projective lamination in* $\mathcal{PML}_c$ *as in Definition 6.4. Let* $\{\mathbf{m}_i\}$ *be a sequence in* $\mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+)$ *converging to* $[\Lambda]$ *in the Thurston compactification product. Then* $\{b \circ q(\mathbf{m}_i)\}$ *also converges to* $[\Lambda]$ *in the ray compactification* $(\mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+)) \cup \mathcal{PML}^r$.

Proof. Let Σ_j be a component of $\Sigma_- \sqcup \Sigma_+$. Since $\{\mathbf{m}_i\}$ converges to $[\Lambda]$ contained in the corner, the restriction $\{\mathbf{m}_i|\Sigma_j\}$ converges in the Thurston compactification to $[\Lambda|\Sigma_j] \in \mathcal{PML}(\Sigma_j)$. Let k_i be a shortest simple closed geodesic in $(\Sigma_j, \mathbf{m}_i|\Sigma_j)$. Since $\Lambda|\Sigma_j$ is arational and uniquely ergodic, there is a sequence of positive numbers $\{s_i\}$ going to 0 such that $\{s_i k_i\}$ converges to the measured lamination $\Lambda|\Sigma_j$. By Corollary 4.2, $\{\iota(b \circ q(\mathbf{m}_i), k_i)\}$ is bounded. By the continuity of the intersection number combined with the arationality and the unique ergodicity of $\Lambda|\Sigma_j$, we see that either $\{b \circ q(\mathbf{m}_i)\}$ converges to $s(\Lambda|\Sigma_j)$ for some positive scalar s, or $\{b \circ q(\mathbf{m}_i)\}$ converges in the ray compactification to the point at infinity $[\Lambda|\Sigma_j] \in \mathcal{PML}(\Sigma_j)$.

It remains to show that the former case cannot happen. Suppose, seeking a contradiction, that $\{b \circ q(\mathbf{m}_i)\}$ converges to $s(\Lambda|\Sigma_j)$. Then, by Lemma 5.4, we see that $\{q(\mathbf{m}_i)|\pi_1(\Sigma_j)\}$ converges algebraically. Let the constant $\delta > 0$ given in Lemma 4.3 for $\epsilon = 1$. We then take a simple closed curve c on Σ_j approximating $|\Lambda|\Sigma_j|$ such that $\iota(c, s\Lambda) < \delta$. Since $\{\mathbf{m}_i|\Sigma_j\}$ converges to $[\Lambda|\Sigma_j]$ in the Thurston compactification, we see that $\mathrm{length}_{\mathbf{m}_i}(c)$ goes to ∞ as $i \longrightarrow \infty$. By Lemma 4.3, this implies that $\mathrm{length}_{q(\mathbf{m}_i)}(c)$ also goes to ∞, contradicting the fact that $\{q(\mathbf{m}_i)|\pi_1(\Sigma_j)\}$ converges.

Thus we have shown that $\{b \circ q(\mathbf{m}_i)|\Sigma_j\}$ converges to $[\Lambda|\Sigma_j]$ for every $j = 1, \ldots, n$, and hence $\{b \circ q(\mathbf{m}_i)\}$ converges to $[\Lambda]$ in the ray compactification. □

The lemma implies the following corollary.

Corollary 6.9. *Let* $[\Lambda]$ *be a projective measured lamination in* $\mathcal{PML}_c$ *as given in Definition 6.4. For any truncated cone neighbourhood* U *of* $[\Lambda]$*, there is a neighbourhood* V *of* $[\Lambda]$ *in the Thurston compactification product of* $\mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+)$ *such that* $b \circ q(V \cap (\mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+)))$ *is contained in* U.

Proof. Consider a sequence $\{\mathbf{m}_i\}$ in $\mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+)$ converging to $[\Lambda]$ in the Thurston compactification product. Then by Lemma 6.8, for any given truncated cone neighbourhood U, the sequence $\{b \circ q(\mathbf{m}_i)\}$ is contained in U for sufficiently large i. This implies the existence of a neighbourhood of $[\Lambda]$ as desired. □

In a more general case where we do not assume that the limit of $\{\mathbf{m}_i\}$ is $[\Lambda]$, we have the following.

Lemma 6.10. *Let* $\{\mathbf{m}_i\}$ *be a sequence in* $\mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+)$ *which converges to a point* $(x_j) \in \mathcal{PML}$ *in the Thurston compactification product. Suppose moreover that* $\{b \circ q(\mathbf{m}_i)\}$ *converges to a point* (y_j) *in the ray compactification. For each* x_j *that lies on the boundary at infinity* $\mathcal{PML}(\Sigma_j)$*, let* μ_j *be a measured lamination with* $[\mu_j] = x_j$*. Then* y_j *lies either in* $\mathcal{ML}(\Sigma_j)$ *and*

$\iota(y_j, \mu_j) = 0$ or on the boundary at infinity $\mathcal{PML}(\Sigma_j)$ and is represented by a measured lamination ν_j with $\iota(\mu_j, \nu_j) = 0$. Moreover, for every j such that y_j lies in $\mathcal{PML}(\Sigma_j)$, the point x_j also lies in $\mathcal{PML}(\Sigma_j)$.

Proof. Suppose that $x_j = [\mu_j]$ lies in $\mathcal{PML}(\Sigma_j)$. Let μ_j^0 be a connected component of μ_j. If μ_j^0 is a simple closed curve, either (a) $\text{length}_{\mathbf{m}_i}(\mu_j^0) \to 0$ or (b) there are simple closed curves d_i on Σ_j with bounded $\text{length}_{\mathbf{m}_i}(d_i)$ and positive numbers $r_i \longrightarrow 0$ such that $\{r_i d_i\}$ converges to a measured lamination $\hat{\mu}_j$ containing μ_j^0. The latter curve d_i can be chosen to be a shortest simple closed curve on Σ_j with respect to $\mathbf{m}_i$ intersecting μ_j^0. Moreover, if μ_j^0 is not a simple closed curve, the condition (b) always holds.

In the case (a), Lemma 4.1 implies that $\iota(\mu_j^0, b \circ q(\mathbf{m}_i)|\Sigma_j) \longrightarrow 0$, which implies that $\iota(\mu_j^0, y_j) = 0$ when y_j lies in $\mathcal{ML}(\Sigma_j)$ and $\iota(\mu_j^0, \nu_j) = 0$ when $x_j = [\nu_j]$ lies on the boundary at infinity, by the continuity of the intersection number. In the case (b), Lemma 4.1 implies that $\{\iota(d_i, b \circ q(\mathbf{m}_i))\}$ is bounded. Since r_i tends to 0, by the continuity of the intersection number, we have $\iota(\hat{\mu}_j, y_j) = 0$, and hence $\iota(\mu_j^0, y_j) = 0$ when y_j lies in $\mathcal{ML}(\Sigma_j)$, and in the same way, $\iota(\mu_j^0, \nu_j) = 0$ when $x_j = [\nu_j]$ lies on the boundary at infinity. Thus, we have $\iota(\mu_j^0, y_j) = 0$ or $\iota(\mu_j^0, \nu_j) = 0$ for every connected component μ_j^0 of μ_j, which implies that $\iota(\mu_j, y_j) = 0$ or $\iota(\mu_j, \nu_j) = 0$.

To show the last statement, suppose that x_j lies in $\mathcal{T}(\Sigma_j)$. Then, by Lemma 5.3, we see that $\{q(\mathbf{m}_i)|\pi_1(\Sigma_j)\}$ converges. Since $\{\mathbf{m}_i|\Sigma_j\}$ converges by assumption, the boundary component Σ_j^i of the convex core $C(\mathbb{H}^3/q(\mathbf{m}_i))$ corresponding to Σ_j converges geometrically to a boundary component Σ_j^∞ of the convex core of the geometric limit, which is homotopic to an algebraic locus of Σ_j (see the argument of [16, p.103]). Then the j-th component of $b \circ q(\mathbf{m}_j)$ converges to the bending lamination of Σ_j^∞, and hence y_j must be inside $\mathcal{ML}(\Sigma_j)$. This shows that if y_j lies on the boundary at infinity, then so does x_j. □

By a similar argument, we can also show the following proposition.

Proposition 6.11. *Let $[\Lambda]$ be a projective lamination as in Definition 6.4. Let $\{\mathbf{m}_i\}$ be a sequence in $\mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+)$ such that $\{b \circ q(\mathbf{m}_i)\}$ converges to a point at infinity represented by $[\Lambda]$ in the ray compactification. Then $\{\mathbf{m}_i\}$ converges to $[\Lambda]$ in the Thurston compactification product of $\mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+)$.*

Proof. We have only to show that any subsequence of $\{\mathbf{m}_i\}$ has a subsequence converging to $[\Lambda]$ in the Thurston compactification product. Passing to a subsequence, we can assume that $\{\mathbf{m}_i\}$ converges to either a point $\mathbf{n}$ in $\mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+)$ or a projective lamination $(y_j) \in \mathcal{PML}$ in the Thurston compactification product. In the former case, by the continuity of the function b due to [10], $\{b \circ q(\mathbf{m}_i)\}$ converges to $b \circ q(\mathbf{n})$, contradicting our assumption.

Suppose that $\{\mathbf{m}_i\}$ converges to (y_j) in the Thurston compactification product. Since $[\Lambda]$ lies in the corner, and $\Lambda \cap \Sigma_j$ is arational and uniquely ergodic for every j, by Lemma 6.10, we have we have $y_j = \Lambda \cap \Sigma_j$, and hence $(y_j) = [\Lambda]$. This completes the proof. $\square$

Since truncated cone neighbourhoods form a basis of neighbourhoods of $[\Lambda]$ as remarked before, we have the following corollary.

Corollary 6.12. *Let V be a neighbourhood of $[\Lambda]$ in the Thurston compactification product of $\mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+)$ for $[\Lambda]$ as in Definition 6.4. Then, there is a truncated cone neighbourhood U of $[\Lambda]$ in the ray compactification such that $(b \circ q)^{-1}(U)$ is contained in $V \cap (\mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+))$.*

By combining these results, we obtain the following technical proposition, which constitutes an essential step for our construction of a homotopy.

Proposition 6.13. *Let Λ be a measured lamination given in Definition 6.4. Then there are three nested truncated cone neighbourhoods $U_0 \subset U_1 \subset U_2$ which satisfy the following.*

1. *(1) Every measured lamination in U_2 satisfies the condition (c) of Theorem 3.1 (with λ_- and λ_+).*
2. *(2) Let V_1 be $(b \circ q)^{-1}(U_1)$. Then there is an open set V_2 containing the closure $\bar{V}_1$ such that both $b \circ q(V_2)$ and $\mathcal{E}(V_2)$ are contained in U_2.*
3. *(3) Neither $\mathcal{E}$ nor $b \circ q$ maps a point outside V_1 into U_0.*
4. *(4) For any point $m \in V_2 \setminus V_1$, the segment connecting $b \circ q(m)$ and $\mathcal{E}(m)$ is disjoint from U_0.*

Proof. If a measured lamination on $\Sigma_- \sqcup \Sigma_+$ does not satisfy the condition (c) of Theorem 3.1, then either it has a component homotopic to a component of λ_- or λ_+, or it has two components which are homotopic in $S \times [0,1]$. Since Λ is arational and is not contained in D, every sufficiently small truncated cone neighbourhood of $[\Lambda]$ contains no such measured laminations. Therefore, by choosing a truncated cone neighbourhood U_2 to be sufficiently small, the condition (1) is satisfied.

By Lemma 6.7 and Corollary 6.9, there is a neighbourhood V_2 of $[\Lambda]$ in the Thurston compactification product such that both $\mathcal{E}(V_2)$ and $b \circ q(V_2)$ are contained in U_2. By Lemma 6.7 and Corollary 6.12, we can take a neighbourhood U_1' of $[\Lambda]$ such that both $\mathcal{E}^{-1}(U_1')$ and $(b \circ q)^{-1}(U_1')$ are contained in V_2. Again by Lemma 6.7 and Corollaries 6.9 and 6.12, we can take a neighbourhood V_1' of $[\Lambda]$ in the Thurston compactification product such that $\overline{V_1'}$ is contained in $\mathcal{E}^{-1}(U_1') \cap (b \circ q)^{-1}(U_1')$, and a truncated cone neighbourhood U_1 of $[\Lambda]$ in the ray compactification such that both $\mathcal{E}^{-1}(U_1)$ and $(b \circ q)^{-1}(U_1)$ are contained in V_1'. These U_1, U_2, V_2 and $V_1 = (b \circ q)^{-1}(U_1)$ satisfy the condition (2).

Now, we shall show that we can take a truncated cone neighbourhood U_0 of $[\Lambda]$ in the ray compactification satisfying (3) and (4). Let $\{\mathbf{m}_i\}$ be an arbitrary sequence in $\mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+)$ lying outside V_1. Since V_1 is a neighbourhood of $[\Lambda]$ in the Thurston compactification product, Lemma 6.7

implies that $\{\mathcal{E}(\mathbf{m}_i)\}$ cannot converge to $[\Lambda]$ in the ray compactification. Similarly, by Proposition 6.11, we see that $\{b \circ q(\mathbf{m}_i)\}$ cannot converge to $[\Lambda]$ in the ray compactification either. Therefore, if we take U_0 to be a sufficiently small truncated cone neighbourhood of $[\Lambda]$, then the condition (3) is satisfied.

We next show that U_0 can be taken to satisfy the condition (4). Seeking a contradiction, let $\{\mathbf{m}_i\}$ be a sequence in $V_2 \setminus V_1$, and λ_i a point on the segment connecting $b \circ q(\mathbf{m}_i)$ and $\mathcal{E}(\mathbf{m}_i)$ such that $\{\lambda_i\}$ converges to $[\Lambda]$ in the ray compactification. Taking a subsequence, we can assume that either $\{\mathbf{m}_i\}$ converges to a point $\mathbf{n} \in \mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+)$ or does not have a convergent sequence inside $\mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+)$, and converges to a point $(w_j) \in \mathcal{PML}$ in the boundary of the Thurston compactification product. The boundary point $(w_j) \in \mathcal{PML}$ in the latter case is distinct from $[\Lambda]$ since $\{\mathbf{m}_i\}$ lies outside V_1. In the former case, passing to a subsequence, $\{\lambda_i\}$ converges to a point inside $\mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+)$ which lies on a segment connecting $\mathcal{E}(\mathbf{n})$ and $b \circ q(\mathbf{n})$, contradicting our assumption. In the latter case, by Lemma 6.7, we see that $\{\mathcal{E}(\mathbf{m}_i)\}$ converges to $(w'_j) \in \mathcal{PML}$ such that $w'_j = \mathcal{E}(w_j)$ if $w_j \in \mathcal{T}(\Sigma_j)$ and $w'_j = w_j$ otherwise. The sequence $\{b \circ q(\mathbf{m}_i)\}$ converges to a point (y_j) in the ray compactification passing to a subsequence. If there is j such that w_j lies in $\mathcal{T}(\Sigma_j)$, then y_j must lie in $\mathcal{ML}(\Sigma_j)$. Then the limit of λ_j also lies in $\mathcal{ML}(\Sigma_j)$ and contradicts the assumption that $[\Lambda]$ is a corner point. Therefore, we have $w_j \in \mathcal{PML}(\Sigma_j)$ and $(w'_j) = (w_j)$.

Let (μ_j) be a point in $\prod_{j=1}^n \mathcal{ML}(\Sigma_j)$ with $w_j = [\mu_j] \in \mathcal{PML}(\Sigma_j)$. By Lemma 6.10, we have $\iota(y_j, \mu_j) = 0$ if y_j lies inside $\mathcal{ML}(\Sigma_j)$, and y_i is represented by $\mu'_j \in \mathcal{ML}(\Sigma_j)$ with $\iota(\mu_j, \mu'_j) = 0$ if y_j lies in $\mathcal{PML}(\Sigma_j)$. Recall that λ_i lies on the segment between $\mathcal{E}(\mathbf{m}_i)$ and $b \circ q(\mathbf{m}_i)$. Therefore $\{\lambda_i\}$ converges to a point (z_j) in the ray compactification with the condition that for each $j = 1, \ldots, n$, the coordinate z_j is represented by (or coincides with if $z_j \in \mathcal{ML}(\Sigma_j)$) a 'weighted sum' in the following sense of μ_j and either y_j or μ'_j. Since $\iota(\mu_j, y_j) = 0$ if $y_j \in \mathcal{ML}(\Sigma_j)$ and $\iota(\mu_j, \mu'_j) = 0$ if $y_i \in \mathcal{PML}(\Sigma_j)$, there is no transverse intersection between μ_j and y_i or μ'_j. The weighted sum above is obtained by giving the union of the supports of μ_i and y_i a transverse measure which is a weighted sum of the one coming from μ_j and the one coming from y_j or μ'_j. It follows that we have $\iota(\mu_j, z_j) = 0$ if z_j lies inside $\mathcal{ML}(\Sigma_j)$, and that z_j is represented by $\nu_j \in \mathcal{ML}(\Sigma_j)$ with $\iota(\nu_j, \mu_j) = 0$ if z_j lies in $\mathcal{PML}(\Sigma_j)$. Since $([\mu_j]) \neq [\Lambda] = ([\Lambda|\Sigma_j])$ and $\Lambda|\Sigma_j$ is arational and uniquely ergodic, there is j such that $\iota(\Lambda|\Sigma_j, \mu_j) > 0$. Using the fact that $\Lambda|\Sigma_j$ is arational, this implies that $\iota(\nu_j, \Lambda|\Sigma_j) > 0$ if z_j lies in $\mathcal{PML}(\Sigma_j)$, which implies $z_j \neq [\Lambda|\Sigma_j]$. If z_j lies in $\mathcal{ML}(\Sigma_j)$, we obviously have $z_j \neq [\Lambda|\Sigma_j]$. Therefore in either case, we have $(z_j) \neq [\Lambda]$, and we are led to a contradiction. Thus we have shown that by taking U_0 to be sufficiently small, the condition (4) also holds. This completes the proof. □

To define a locally degree-1 map $\widehat{F}\colon \widehat{\mathrm{T}} \to \widehat{\mathrm{ML}}$, and a homotopy from $\widehat{b \circ q}$ to $\widehat{F}$, we shall first define a map $F\colon \mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+) \to \mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+)$ which induces $\widehat{F}$.

Definition 6.14. Let $U_0, U_1, U_2 \subset \mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+)$ and $V_1, V_2 \subset \mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+)$ be open sets given in Proposition 6.13, and let τ be the train track given in Definition 6.5. Let $F\colon \mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+) \to \mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+)$ be a continuous map defined as follows:

(i) For x outside V_2, we define $F(x) = b \circ q(x)$.

(ii) For $x \in V_1$, we define $F(x) = \mathcal{E}(x)$.

(iii) For $x \in V_2 \setminus V_1$, letting $t(x)$ be $\dfrac{d_\tau(x, \overline{V_1})}{d_\tau(x, \overline{V_1}) + d_\tau(x, \overline{V_2^c})}$, we define $F(x)$ to be the point dividing the segment connecting $\mathcal{E}(x)$ and $b \circ q(x)$ internally by $t(x) : 1 - t(x)$ in the weight system coordinates of τ.

Lemma 6.15. *For the map F defined above, there is no sequence in $\mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+)$ diverging to infinity, whose image under F has a subsequence converging in $\mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+) \setminus D$.*

Proof. Since both $\mathcal{E}$ and $b \circ q$ are proper, $\mathcal{E}$ as a map to $\mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+)$ and $b \circ q$ as a map to $\mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+) \setminus D$, we have only to consider the case when $\{\mathbf{m}_i\}$ lies in $V_2 \setminus V_1$. Let $\{\mathbf{m}_i\}$ be a sequence in $V_2 \setminus V_1$ which does not have a convergent subsequence. We can assume that it converges in the Thurston compactification product to a point (y_j) in $\mathcal{PML}$, passing to a subsequence. We need to show that $\{F(\mathbf{m}_i)\}$ does not have a convergent subsequence in $\mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+) \setminus D$.

Recall from Proposition 6.13 that both $\{\mathcal{E}(\mathbf{m}_i)\}$ and $\{b \circ q(\mathbf{m}_i)\}$ lie in U_2. Since $\mathcal{E}$ is a proper map to $\mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+)$, the sequence $\{\mathcal{E}(\mathbf{m}_i)\}$ diverges to infinity, necessarily within U_2. If $\{b \circ q(\mathbf{m}_i)\}$ also diverges to infinity (within U_2), the segment connecting $\mathcal{E}(\mathbf{m}_i)$ and $b \circ q(\mathbf{m}_i)$ also diverges to infinity within U_2 as $i \longrightarrow \infty$, and we are done. It remains to deal with the case when $\{b \circ q(\mathbf{m}_i)\}$ converges to a point ν in D after passing to a subsequence. By the part (1) of Proposition 6.13, ν must satisfy the condition (c) of Theorem 3.1, and hence contains a compact leaf with weight larger than or equal to π. We denote the union of all such components of ν by ν_0. Let Σ_k be a component of $\Sigma_- \sqcup \Sigma_+$ containing a component of ν_0. Then y_k lies in $\mathcal{PML}(\Sigma_k)$ since otherwise the component of the convex core boundary corresponding to Σ_k converges geometrically without giving rise to a new parabolic curve, as argued in the previous section.

We first remark the following, which was just a restatement of Lemma 5.4.

Claim 6.16. *Let $\phi_i\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ be a representation corresponding to $q(\mathbf{m}_i)$. Then, $\{\phi_i\}$ converges to some $\psi \in \mathsf{AH}(S)$ as $i \longrightarrow \infty$.*

Let P_i be a shortest pants decomposition of $(\Sigma_- \sqcup \Sigma_+, \mathbf{m}_i)$. Let P_∞ be the Hausdorff limit of $\{P_i\}$ (after passing to a subsequence), which is a geodesic lamination. Invoking an argument which we used in the preceding section, we can show the following.

Claim 6.17. *Every minimal component of P_∞ is a simple closed curve. The geometric limit M_∞ has neither a new geometrically infinite end (i.e. one not corresponding to that of $(\mathbb{H}^3/\phi_i(\pi_1(S))_0)$ nor a torus cusp.*

Proof. Suppose, seeking a contradiction, that P_∞ has a minimal component ϱ which is not a simple closed curve. Let $S(\varrho) \subset S$ be the minimal supporting surface of ϱ. Then M_∞ must have a new geometrically infinite end which lifts to the algebraic limit. (This follows from the fact in Minsky's model of $\mathbb{H}^3/\phi_i(\pi_1(S))$, the subsurface$S(\varrho)$ must support a tight geodesic whose length goes to ∞ in this case.)

Now, we put a basepoint on the algebraic locus $f_\infty(S)$, and consider a geometric limit M_∞ containing the algebraic limit of $\{\phi_i\}$. As in Section 5.2.1, by taking a non-peripheral simple closed curve d on $S(\varrho)$ with $\iota(d, \varrho) < \delta$ for the constant δ given in Lemma 4.3 with $\epsilon = 1$, we are led to a contradiction. Thus we have shown that every minimal component of P_∞ is a simple closed curve. This argument also shows that M_∞ cannot have a nearest simply degenerate end, for such an end can be lifted to the algebraic limit by Claim 5.2.

Next suppose that M_∞ has a torus cusp. Since there is no nearest simply degenerate end for M_∞, if there are torus cusps, we can take a nearest one by Claim 5.2. By repeating the arguments of Section 5.2.2, which can be applied also to our situation as remarked there, we get a contradiction.

Since M_∞ has neither a nearest simply degenerate end nor a nearest torus cusp, by Claim 5.2, we see that M_∞ does not have a new geometrically infinite end. □

We define a subset P^0_∞ of P_∞ to be the subset consisting of simple closed curves whose lengths with respect to $\mathbf{m}_i$ go to 0 as $i \longrightarrow \infty$.

Since $\{\phi_i\}$ converges to $\psi \in \mathsf{AH}(S)$ as mentioned above, by Corollary 2.3, the claim above implies that the convergence is strong.

Claim 6.18. *The multi-curve P^0_∞ is contained in ν_0.*

Proof. Let c be a component of P^0_∞. We regard c as lying on a compact core of $\mathbb{H}^3/\psi(\pi_1(S)))$. Then $\text{length}_{\mathbf{m}_i}(c)$ goes to 0 by the definition of P^0_∞, and hence it represents a core curve of a $\mathbb{Z}$-cusp neighbourhood lying outside the compact core. Since $\{\phi_i\}$ converges to ψ strongly and M_∞ does not have a new geometrically infinite end by Claim 6.17, the $\mathbb{Z}$-cusp has geometrically finite ends on its both sides. We can apply Lemma 5.1 to see that the bending angle along c converges to π, and hence c is contained in ν_0. □

We can further see the following.

Claim 6.19. *Let C be a minimal component of P_∞ not contained in P^0_∞. If P_∞ does not contain any other minimal component homotopic to C in $S \times [0, 1]$, then the twisting parameter along C of $\mathbf{m}_i$ is bounded as $i \longrightarrow \infty$.*

If P_∞ has two distinct minimal components C and C' homotopic to each other in $S \times [0,1]$, then the difference of the twisting parameters of $\mathbf{m}_i$ along C and C' is bounded as $i \longrightarrow \infty$.

Proof. Suppose first that there is no other component in P_∞ homotopic to C in $S \times I$. If the twisting parameter of $\mathbf{m}_i$ along C goes to ∞ after passing to a subsequence, then by the part (1) of Lemma 2.4, the geometric limit M_∞ has a corresponding torus cusp. This contradicts Claim 6.17. Also in the latter case when C is homotopic to C' in $S \times I$, by the same argument as above involving Lemma 2.4, we see that if the difference of twisting parameters of $\mathbf{m}_i$ goes to ∞, then M_∞ must have a torus cusp, contradicting Claim 6.17. □

This claim implies the following.

Claim 6.20. *The projective lamination y_k is supported in P^0_∞.*

Proof. By the definition of the topology of the Thurston compactification product, we see that $|y_k|$ is contained in P_∞. By Claim 6.17, every minimal component of P_∞ is a simple closed curve. If $|y_j|$ has a component C contained in P_∞ but not in P^0_∞, then the twist parameter of $\mathbf{m}_i$ around C must go to either ∞ or $-\infty$. This contradicts Claim 6.19 unless there is another component of P_∞ homotopic to C in $S \times I$. If there is such a component C', then the support of some coordinate y_l of (y_j) must contain C' as a component by Claim 6.19 again. On the other hand, (y_j) is contained in U_2, and hence cannot have two components of projective laminations homotopic to each other in $S \times I$. This is a contradiction. □

Having proved these claims, we can now complete the proof of Lemma 6.15. Since $|y_k|$ is contained in P^0_∞ by Claim 6.20, it lies in ν_0 by Claim 6.18. Therefore the segment connecting $\mathcal{E}(\mathbf{m}_i)$ and $b \circ q(\mathbf{m}_i)$ converges uniformly on any compact set to a ray entirely lying in D. This completes the proof. □

Now we can define $\widehat{F}$ which we mentioned before. By Lemma 6.15, $F\colon \mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+) \to \mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+)$ induces a unique continuous map $\widehat{F}\colon \widehat{\mathrm{T}} \to \widehat{\mathrm{ML}}$ similarly to Definition 6.1. We shall show that $\widehat{F}$ has degree 1. We use the symbols $\widehat{\mathcal{E}}\colon \widehat{\mathrm{T}} \to \widehat{\mathrm{ML}}$ to denote the map induced by $\mathcal{E}$.

Proposition 6.21. *The map $\widehat{F}\colon \widehat{\mathrm{T}} \to \widehat{\mathrm{ML}}$ has degree* 1.

Proof. Let $\widehat{U}_0$ be the open set in $\widehat{\mathrm{ML}}$ corresponding to $U_0 \setminus D$, where U_0 is the truncated cone neighbourhood in Proposition 6.13. We shall show that the restriction of $\widehat{F}$ to $\widehat{F}^{-1}(\widehat{U}_0)$ has degree 1, which immediately implies that $\widehat{F}$ has degree 1.

Since $F|\mathcal{E}^{-1}(U_0)$ coincides with $\mathcal{E}|\mathcal{E}^{-1}(U_0)$, it is a homeomorphism to its image. We have only to show that there is no point outside $\mathcal{E}^{-1}(U_0)$ which is mapped into U_0. Since $F|V_1$ coincides with $\mathcal{E}|V_1$, there are no points in

$V_1 \setminus \mathcal{E}^{-1}(U_0)$ mapped into U_0 by F. On the other hand, the conditions (3) and (4) in Proposition 6.13 guarantee that no points outside V_1 are mapped into U_0 by F. This completes the proof. □

The next step is to show that there is a homotopy between $\widehat{b \circ q}$ and $\widehat{F}$. We shall first construct a homotopy between $b \circ q$ and F.

We define a homotopy $H\colon \mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+) \times [0,1] \to \mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+)$ from $b \circ q$ to F as follows.

(1) For any x outside V_2 and any $s \in [0,1]$, $H(x,s)$ is defined to be $b \circ q(x)$.

(2) For any x in V_2 and any $s \in [0,1]$, $H(x,s)$ is defined to be the point dividing the segment connecting $b \circ q(x)$ and $F(x)$ internally by $s : 1-s$.

Lemma 6.22. *For any sequence $\{t_i \in [0,1]\}$ and any $\{\mathbf{m}_i\} \subset \mathcal{T}(\Sigma_-) \times \mathcal{T}(\Sigma_+)$ diverging to infinity, $\{H(\mathbf{m}_i, t_i)\}$ cannot have a subsequence converging in $\mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+) \setminus D$ as $i \longrightarrow \infty$.*

Proof. The proof is quite similar to that of Lemma 6.15. We can assume that $\{\mathbf{m}_i\}$ either lies outside V_2, in $V_2 \setminus V_1$, or in V_1, passing to a subsequence. In the case when $\{\mathbf{m}_i\}$ lies outside V_2, the statement follows immediately from the properness of $b \circ q$ as a map to $\mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+) \setminus D$. In the case when $\{\mathbf{m}_i\}$ lies in either $V_2 \setminus V_1$ or V_1, the point $H(\mathbf{m}_i, t_i)$ lies in the segment connecting $b \circ q(\mathbf{m}_i)$ and $\mathcal{E}(\mathbf{m}_i)$. By the same argument as in the proof of Lemma 6.15, such a segment converges to a ray on D as $i \longrightarrow \infty$. This shows that $\{H(\mathbf{m}_i, t_i)\}$ cannot have a subsequence converging in $\mathcal{ML}(\Sigma_-) \times \mathcal{ML}(\Sigma_+) \setminus D$. □

This lemma immediately implies the following.

Corollary 6.23. *The homotopy H induces a homotopy $\widehat{H}\colon \widehat{\mathrm{T}} \to \widehat{\mathrm{ML}}$ between $\widehat{b \circ q}$ to $\widehat{\mathcal{E}}$.*

Combining this corollary with Proposition 6.21, we conclude that $\widehat{b \circ q}$ has degree 1, which in turn implies that $b \circ q$ has degree 1. This completes the proof of the latter half of Theorem 3.3.

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