ON THURSTON'S PARAMETERIZATION OF $\mathbb{C}P^1$-STRUCTURES

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Abstract. Thurston established a correspondence between $\mathbb{C}P^1$-structures (complex projective structures) and equivariant pleated surfaces in the hyperbolic-three space $\mathbb{H}^3$, in order to give a parameterization of the deformation space of $\mathbb{C}P^1$-structures. In this note, we summarize Thurston's parametrization of $\mathbb{C}P^1$-structures, based on [15] and [17], giving an outline and the key points of its construction.

In addition we give independent proofs for the following well-known theorems on $\mathbb{C}P^1$-structures by means of pleated surfaces given by the parameterization. (1) Goldman's Theorem on $\mathbb{C}P^1$-structures with quasi-Fuchsian holonomy. (2) The path lifting property of developing maps in the domain of discontinuities in $\mathbb{C}P^1$.

Contents

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1. Introduction

Let P be the space of all (marked) $\mathbb{C}\mathrm{P}^1$-structures on a closed oriented surface S of genus at least two (§2). Thurston gave the following parameterization of P, using pleated surfaces in the hyperbolic three-space $\mathbb{H}^3$.

Theorem A. *(Thurston,* [15] [17] *)*

$$\mathsf{P} \cong \mathsf{ML} \times \mathsf{T},$$

where ML *is the space of measured laminations on* S *and* T *is the space of all (marked) hyperbolic structures on* S*.*

In §4, we outline this correspondence, in part, giving more details, following the work of Kulkarni and Pinkall [17]. A hyperbolic structure on S is in particular a $\mathbb{C}\mathrm{P}^1$ structure, and its holonomy is a discrete and faithful representation of $\pi_1(S)$ into $\mathrm{PSL}(2, \mathbb{R})$, called a *Fuchsian representation*. One holonomy representation of a $\mathbb{C}\mathrm{P}^1$-structure on S corresponds to countably many different $\mathbb{C}\mathrm{P}^1$-structures on S. Indeed, there is an operation called 2π-grafting (or simply grafting) which transforms a $\mathbb{C}\mathrm{P}^1$-structure to a new $\mathbb{C}\mathrm{P}^1$-structure, preserving its holonomy representations. The following theorem of Goldman characterizes all $\mathbb{C}\mathrm{P}^1$-structures with fixed Fuchsian holonomy.

Theorem B ([12])**.** *Every* $\mathbb{C}\mathrm{P}^1$*-structure* C *on* S *with Fuchsian holonomy* ρ *is obtained by grafting the hyperbolic structure* τ *along a unique multiloop* M*.*

Goldman actually proved the theorem for more general quasi-Fuchsian groups, although the proof is immediately reduced to the case of Fuchsian representations by a quasiconformal map of $\mathbb{C}\mathrm{P}^1$. Let C be a $\mathbb{C}\mathrm{P}^1$-structure with Fuchsian holonomy $\pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$. Then, by Theorem B, C corresponds to (τ, M), where τ is the hyperbolic structure $\mathbb{H}^2/\mathrm{Im}\rho$ and each loop of M has a 2π-multiple weight.

For a subgroup $\Gamma \subset \mathrm{PSL}(2, \mathbb{C})$, the *limit set* of Γ is the set of the accumulation points of a Γ-orbit in $\mathbb{C}\mathrm{P}^1$, and the domain of discontinuity is the complement of the limit set in $\mathbb{C}\mathrm{P}^1$. In §5, we give an alternative proof of Theorem B, directly using pleated surfaces given by the Thurston parameters.

The following Theorem is a technical part of the proof of Theorem B, which was originally missing.

Theorem C ([7], see also §14.4.1. in [11])**.** *Let* (f, ρ) *be a developing pair of a* $\mathbb{C}\mathrm{P}^1$*-structure on* S*. Let* Ω *be the domain of discontinuity of* $\mathrm{Im}\rho$*. Then, for each connected component* U *of* $f^{-1}(\Omega)$*, the restriction of* f *to* U *is a covering map onto its image.*

Note that as developing maps are local homeomorphisms, Theorem C is equivalent to saying that f has the path lifting property in the domain of discontinuity of Imρ.

We also give an alternative proof of Theorem C in §6, using Thurston's parametrization.

Theorem B states that given two $\mathbb{C}\mathrm{P}^1$-structures C_1 and C_2 with Fuchsian holonomy, C_1 can be transformed into C_2, via the hyperbolic structure, by a composition of an inverse-grafting and a grafting (where an inverse grafting is the opposite of grafting which remove a cylinder for 2π-grafting). The following question due to Gallo, Kapovich, and Marden remains open.

Conjecture 1.1 (§12.1 in [10]). *Given two $\mathbb{C}\mathrm{P}^1$-structures C_1, C_2 on S with fixed holonomy $\pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$, there is a composition of grafts and inverses of grafts which transforms C_1 into C_2.*

Although [10] stated this conjecture in the form of a question, we state it more positively since it has been solved affirmatively for generic holonomy representations, namely, for purely loxodromic representations [3, 4]. (For Schottky representations, see [2].) There is also a version of this question for branched $\mathbb{C}\mathrm{P}^1$-structures (Problem 12:1:2 in [10]); see [5] [19] for some progress in the case of branched $\mathbb{C}\mathrm{P}^1$-structures.

Recently, Gupta and Mj [13] gave a generalization of Theorem A to certain $\mathbb{C}\mathrm{P}^1$-structures on a surface with punctures (namely, $\mathbb{C}\mathrm{P}^1$-structures which corresponds to compact Riemann surfaces with meromorphic quadratic differentials whose poles are of order at least three); see also [1].

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2. $\mathbb{C}\mathrm{P}^1$-structures on surfaces

General references for $\mathbb{C}\mathrm{P}^1$-structures are can be found in, for example, [8, 16].

A $\mathbb{C}\mathrm{P}^1$-*structure* on S is a $(\mathbb{C}\mathrm{P}^1, \mathrm{PSL}(2,\mathbb{C}))$-structure, i.e. a maximal atlas of charts embedding open subsets of S onto open subsets of $\mathbb{C}\mathrm{P}^1$ such that their transition maps are in $\mathrm{PSL}(2,\mathbb{C})$. Let $\tilde{S}$ be the universal cover of S, which is topologically an open disk. Then, equivalently, a $\mathbb{C}\mathrm{P}^1$-structure on S is defined as a pair (f, ρ) consisting of

- a local homeomorphism $f\colon \tilde{S} \to \mathbb{C}\mathrm{P}^1$ (*developing map*) and
- a homomorphism $\rho\colon \pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$ (*holonomy representation*)

such that f is ρ-equivariant (i.e. $f\alpha = \rho(\alpha)f$ for all $\alpha \in \pi_1(S)$). This pair(f, ρ) is called the *developing pair* of C, and (f, ρ) is, by definition, equivalent to $(\gamma f, \gamma\rho\gamma^{-1})$ for all $\gamma \in \mathrm{PSL}(2, \mathbb{C})$. Due to the equivariant condition, we do not usually need to distinguish between an element of $\pi_1(S)$ and its free homotopy class. Let P be the deformation space of all $\mathbb{C}\mathrm{P}^1$-structures on S; then P has a natural topology, given by the open-compact topology on the developing maps $f\colon \tilde{S} \to \mathbb{C}\mathrm{P}^1$.

Notice that hyperbolic structures are, in particular, $\mathbb{C}\mathrm{P}^1$-structures, as $\mathbb{H}^2$ is the upper half-plane in $\mathbb{C}$ and the orientation-preserving isometry group $\mathrm{Isom}\, \mathbb{H}^2$ is the subgroup $\mathrm{PSL}(2, \mathbb{R})$ of $\mathrm{PSL}(2, \mathbb{C})$.

3. Grafting

A grafting is a cut-and-paste operation of a $\mathbb{C}\mathrm{P}^1$-structure inserting some structure along a loop, an arc or more generally a lamination, originally due to [20, 14, 18]. There are slightly different versions of grafting, but they all yield new $\mathbb{C}\mathrm{P}^1$-structures without changing the topological types of the base surfaces.

A *round circle* in $\mathbb{C}\mathrm{P}^1 = \mathbb{C} \cup \{\infty\}$ is a round circle in $\mathbb{C}$ or a straight line in $\mathbb{C}$ plus ∞. A *round disk* in $\mathbb{C}\mathrm{P}^1$ is a disk bounded by a round circle. An arc α on a $\mathbb{C}\mathrm{P}^1$-structure is *circular* if α is sent into a round circle on $\mathbb{C}\mathrm{P}^1$ by the developing map. Similarly, a loop α on a $\mathbb{C}\mathrm{P}^1$-structure C is *circular* if its lift $\tilde{\alpha}$ to the universal cover is sent into a circular arc $\mathbb{C}\mathrm{P}^1$ by the developing map.

We first define a grafting along a circular arc on a $\mathbb{C}\mathrm{P}^1$-structure. For $\theta > 0$, consider the horizontal biinfinite strip $\mathbb{R} \times [0, \theta i]$ in $\mathbb{C}$ of height θ. Then let R_θ be the $\mathbb{C}\mathrm{P}^1$-structure on the strip whose developing map is the restriction of the exponential map $\exp\colon \mathbb{C} \to \mathbb{C} \setminus \{0\}$. This $\mathbb{C}\mathrm{P}^1$-structure is called the *crescent* of angle θ or simply *θ-crescent*.

Let ℓ be a (biinfinite) circular arc properly embedded in a $\mathbb{C}\mathrm{P}^1$-surface C. Then the *grafting* of C along ℓ by θ is the insertion of this strip R_θ along ℓ (*θ-grafting*), to be precise, as follows: Notice that $C \setminus \ell$ has two boundary components isomorphic to ℓ. Then we take a union of $C \setminus \ell$ and $\mathbb{R} \times [0, \theta i]$ by an isomorphism between $\partial(C \setminus \ell)$ and $\partial(\mathbb{R} \times [0, \theta i])$ so that there is "no shearing", i.e. for each $r \in \mathbb{R}$, the vertical arc $r \times [0, \theta i]$ connects the points of the different boundary components of $C \setminus \ell$ corresponding to the same point of ℓ.

Let ℓ be a circular loop on a projective surface C. We can similarly define a grafting along ℓ by grafting the universal cover $\tilde{C}$ of C in an equivariant manner: Letting $\phi\colon \tilde{C} \to C$ be the universal covering map, $\phi^{-1}(\ell)$ is a union of disjoint circular arcs property embedded in $\tilde{C}$ which is invariant under $\pi_1(S)$.

Then, we insert a θ-crescent along each arc of $\phi^{-1}(\ell)$ as above. By quotienting out the resulting structure by $\pi_1(S)$, we obtain a new $\mathbb{C}\mathrm{P}^1$-structure homeomorphic to C, since a cylinder is inserted to C along ℓ. Indeed, the stabilizer of an arc $\tilde{\ell}$ of $\phi^{-1}(\ell)$ is an infinite cyclic group generated by an element $\gamma \in \pi_1(S)$ whose free homotopy class is ℓ, and the cyclic group $\langle\gamma\rangle$ acts on R_θ so that the quotient is the inserted cylinder (*grafting cylinder of height* θ).

Note that R_θ is foliated by horizontal lines $\mathbb{R} \times \{y\}$, $y \in [0, \theta]$. Then it has a natural transverse measure given by the difference of the second coordinates. This measured foliation descends to a measured foliation on the grafting cylinder. In addition, there is a natural projection $R_\theta \to \mathbb{R}$ to the first coordinate (*collapsing map*). This projection descends to a collapsing map of a grafting cylinder to a circle.

Let $\mathrm{Gr}_{\ell,\theta}(\mathrm{C})$ denote the resulting $\mathbb{C}\mathrm{P}^1$-structure homeomorphic to C. Notice that the holonomy along the circular loop ℓ is hyperbolic, as it has exactly two fixed points on $\mathbb{C}\mathrm{P}^1$ which are the endpoints of the developments of ℓ.

In the case that θ is an integer multiple of 2π, the holonomy C is not changed by the θ-grafting, since the developing map does not change in $\phi^{-1}(C \setminus \ell)$. In particular, the 2π-grafting along a circular loop ℓ inserts a copy of $\mathbb{C}\mathrm{P}^1$ minus a circular arc along each lift of ℓ.

In fact, a 2π-grafting is still well-defined along a more general loop. A loop ℓ on $C = (f, \rho)$ is *admissible* if $\rho(\gamma)$ is hyperbolic and an (equivalently, every) lift $\tilde{\ell}$ of ℓ embeds into $\mathbb{C}\mathrm{P}^1$ by f. Given such a loop, we can insert a copy of $\mathbb{C}\mathrm{P}^1 \setminus (f(\tilde{\ell}) \cup \mathrm{Fix}(\rho(\gamma)))$ along $\tilde{\ell}$, where $\mathrm{Fix}(\rho(\gamma))$ denotes the fixed points of $\rho(\gamma)$. Note that the quotient of $\mathbb{C}\mathrm{P}^1 \setminus \mathrm{Fix}(\rho(\gamma))$ by the infinite cyclic group generated by $\rho(\gamma)$ is a projective structure T on a torus, and the development $f(\tilde{\ell})$ covers a simple loop on T isomorphic to ℓ. By abuse of notation, we also denote the loop on T by ℓ. Then the 2π-grafting of C along ℓ is given by identifying the boundary loops of $C \setminus \ell$ and $T \setminus \ell$ by the isomorphism. Denote by $\mathrm{Gr}_\ell(\mathrm{C})$ the 2π-grafting of C along an admissible loop ℓ.

A *multiloop* is a union of (locally) finite disjoint simple closed curves. Note that if there is a multiloop M on a projective surface consisting of admissible loops, then a grafting can be done along M simultaneously.

4. The construction of Thurston's parameters

In this section, we explain the correspondence stated in Theorem A in both directions, following [17].

4.1. **The construction of $\mathbb{C}\mathrm{P}^1$-structures from measured laminations on hyperbolic surfaces.**

Let $(\tau, L) \in \mathsf{T} \times \mathsf{ML}$, where τ is a hyperbolic structure on S, and L is a measured geodesic lamination on τ. Then (τ, L) corresponds to the $\mathbb{C}\mathrm{P}^1$-structure on S obtained by grafting τ along L as follows.

Suppose first that L consists of periodic leaves. Then, for each leaf ℓ of L, letting w be its weight, we insert a grafting cylinder of height w, and obtain a projective structure $C = (f, \rho)$ on S. Let $\tilde{L}$ be the pull back of L by the universal covering map. Then there is a ρ-equivariant pleated surface $\beta\colon \mathbb{H}^2 \to \mathbb{H}^3$, obtained by bending $\mathbb{H}^2$ along $\tilde{L}$ by the angles given by the weights.

Let $\kappa\colon C \to \tau$ be the collapsing map obtained by collapsing all grafting cylinders in C in §**??**. For each point p in $\tilde{C}$, there is an open neighborhood D, called a maximal disk, such that f embeds D onto a round disk in $\mathbb{C}\mathrm{P}^1$. Then, the boundary of $f(D)$ bounds a hyperbolic plane H_p in $\mathbb{H}^3$. Denote, by $\Psi_p\colon f(D) \to H_p$, the nearest projection. Then $\beta \circ \tilde{\kappa}(p) = \Psi_p \circ f(p)$ for all $p \in \tilde{C}$, where $\tilde{\kappa}\colon \tilde{C} \to \mathbb{H}^2$ be the lift of $\kappa\colon C \to \tau$.

Suppose next that L contains an irrational sublamination. Then, pick a sequence of measured laminations L_i consisting of closed leaves, such that L_i converges to L as $i \to \infty$. Then, for each i, as above there is a $\mathbb{C}\mathrm{P}^1$-structure $C_i = Gr_{L_i}(\tau)$ and a ρ_i-equivariant pleated surface $\beta_i\colon \mathbb{H}^2 \to \mathbb{H}^3$. As L_i converges to L, then β_i converges to a pleated surface $\beta\colon \mathbb{H}^2 \to \mathbb{H}^3$ uniformly on compact sets, and therefore C_i converges to a $\mathbb{C}\mathrm{P}^1$-structure on S. (See [6].)

4.2. **The construction of measured laminations on hyperbolic surfaces from $\mathbb{C}\mathrm{P}^1$-structures.**

Let $C = (f, \rho)$ be a projective structure on S given by a developing pair. Let $\tilde{C}$ be the universal cover of C.

Identify $\mathbb{C}\mathrm{P}^1$ conformally with a unite sphere $\mathbb{S}^2$ in $\mathbb{R}^3$. Then, each round circle on $\mathbb{C}\mathrm{P}^1$ is the intersection of $\mathbb{S}^2$ with some (affine) hyperplane $\mathbb{R}^2$ in $\mathbb{R}^3$. A *(open) round disk* D in $\tilde{C}$ is an open subset of $\tilde{C}$ homeomorphic to an open disk, such that f embeds D onto an open round disk in $\mathbb{C}\mathrm{P}^1$ (we also say a maximal disk *of* $\tilde{C}$, emphasizing the ambient space for the maximality). A *maximal disk* D in $\tilde{C}$ is a round disk, such that there is no round disk in $\tilde{C}$ strictly containing D. Let

D be a maximal disk in $\tilde{C}$. Then the closure of its image, $\overline{f(D)}$, is a closed round disk in $\mathbb{C}P^1$.

We first see a basic example illustrating the pleated surface corresponding to a $\mathbb{C}P^1$-structure. (See [9].) Let U be a region of $\mathbb{C}P^1$ homeomorphic to an open disk such that $\mathbb{C}P^1 \setminus U$ contains more than one point (i.e. $U \not\cong \mathbb{C}P^1, \mathbb{C}$). Regard $\mathbb{C}P^1$ as the ideal boundary of hyperbolic three space $\mathbb{H}^3$, and consider the convex full of $\mathbb{C}P^1 \setminus U$ in $\mathbb{H}^3$. Then its boundary in $\mathbb{H}^3$ is a hyperbolic plane $\mathbb{H}^2$ bent along a measured lamination L_U [9]. This lamination corresponds to the lamination in the Thurston coordinates.

Let Ψ_U denote the orthogonal projection from U to $\partial\,\mathrm{Conv}(\mathbb{C}P^1 \setminus U)$. Then, since $\partial\,\mathrm{Conv}(\mathbb{C}P^1 \setminus U)$ is, in the intrinsic metric, a hyperbolic plane, Ψ yields a continuous map from U to $\mathbb{H}^2$. For each maximal disk D in U, let H_D be the hyperbolic plane in $\mathbb{H}^3$ bounded by its ideal boundary of D. Then H_D intersects $\partial\,\mathrm{Conv}(\mathbb{C}P^1 \setminus U)$ in either a geodesic or the closure of a complementary region of L_U in $\mathbb{H}^2$. Thus, all maximal disks in U correspond to the *strata* of $(\mathbb{H}^2, L)$, where each stratum is either the closure of a complementary region of L in $\mathbb{H}^2$, a leaf of L with atomic measure, or a leaf of L not contained in the closure of some complementary region. In particular, two distinct complementary regions R_1, R_2 of $(\mathbb{H}^2, L)$ correspond to different maximal disks D_1, D_2, and if R_1 is close enough to R_2, then D_1 intersects D_2. Accordingly, the ideal boundary circles of D_1 and D_2 bound hyperbolic planes intersecting in a geodesic. Then the transverse measure of L_U is, infinitesimally, given by the angles between such hyperbolic planes.

Moreover there is a natural measured lamination $\mathcal{L}_U$ on U which maps to L_U by Ψ_U. If a leaf ℓ has a positive atomic measure $w > 0$, then $\Psi_U^{-1}(\ell)$ is a crescent region R_w of angle w, and R_w is foliated by circular arcs ℓ' which project to ℓ. Then Ψ_U is a homeomorphism in the complement of such foliated crescents, and Ψ_U isomorphically takes $\mathcal{L}_U$ to L_U in the complement (i.e. it preserves leaves and transverse measure). The transverse measure of $\mathcal{L}$ is given by infinitesimal angles between "very close" maximal disks.

As developing maps of $\mathbb{C}P^1$-structures are, in general, not embedding, we need to find such projections somewhat more "locally" using maximal disks.

Let D be a maximal disk in the universal cover $\tilde{C}$, and let $\overline{D}$ be the closure of D in $\tilde{C}$. In other words, $\overline{D}$ is the connected component of $f^{-1}(\overline{f(D)})$ containing D. Then $\overline{f(D)} \setminus f(\overline{D})$ is a subset of the boundary circle of the round disk $f(D)$, and the points in this subset are called the *ideal points* of D. (Given a point p of the boundary circle $f(D)$, pick a path $\alpha\colon [0,1) \to f(D)$ limiting to p as the parameter goes to 1. Then p is a ideal point of D if and only if the lift of α to $\tilde{C}$ leaves every compact subset of $\tilde{C}$.)

Let $\partial_\infty D \subset \mathbb{C}\mathrm{P}^1$ denote the set of all ideal points of D. As $f|D$ is an embedding onto a round disk, we regard $\partial_\infty D$ as a subset of the boundary circle of D abstractly (not as a subset of $\mathbb{C}\mathrm{P}^1$). Then $\partial_\infty D$ is a closed subset of $\mathbb{S}^1$, since its complement is open. Identifying D with a hyperbolic disk conformally, we let $\mathrm{Core(D)} = \mathrm{Core}_{\tilde{\mathrm{C}}}(\mathrm{D})$ be the convex hull of $\partial_\infty D$.

For each point p of $\tilde{C}$, there is a round disk containing p, and moreover, as C is not $\mathbb{C}\mathrm{P}^1$ or $\mathbb{C}$, there is a maximal disk containing p. The *canonical neighborhood* U_p of C is the union of all maximal disks D_j $(j \in J)$ in $\tilde{C}$ which contain p.

In fact, $(C, \mathcal{L})$ completely determines the Thurston parameters (τ, L). Furthermore the Thurston parameters near $p \in \mathbb{C}\mathrm{P}^1$ are determined by the Thurston parameters of its canonical neighborhood, in the way similar to the region U in $\mathbb{C}\mathrm{P}^1$ homeomorphic to a disk as above. Namely Lemma 4.3 below implies that the Thurston lamination on $\tilde{C}$ near p is determined by the canonical neighborhood U_p of p, and the following Proposition states that U_p is a topological disk embedded in $\mathbb{C}\mathrm{P}^1$.

Proposition 4.1 ([17], Proposition 4.1)**.** *For every point p in $\tilde{C}$, $f\colon \tilde{S} \to \mathbb{C}\mathrm{P}^1$ embeds its canonical neighborhood U_p into $\mathbb{C}\mathrm{P}^1$. Moreover U_p is homeomorphic to an open disk.*

Proof. Set $U_p = \cup D_j$, where D_j are maximal disks in $\tilde{C}$ containing p. Let x, y be distinct points in U_p; let D_x and D_y be maximal disks containing $\{p, x\}$ and $\{p, y\}$, respectively. By the definition of maximal disks, f embeds D_i and D_j onto round disks in $\mathbb{C}\mathrm{P}^1$. Then $f(D_i) \cap f(D_j) = f(D_i \cap D_j)$ is either a crescent or a *round annulus*, i.e. a region in $\mathbb{C}\mathrm{P}^1$ bounded by disjoint round circles. If it is a round annulus, then $f|D_i \cup D_j$ must be a homeomorphism onto $\mathbb{C}\mathrm{P}^1$ and $D_i \cup D_j = \tilde{S}$, which cannot occur. Thus $f(D_i \cap D_j)$ is a crescent, and therefore f is injective on $D_x \cup D_y$. Hence $f(x) \neq f(y)$, and f embeds U_p into $\mathbb{C}\mathrm{P}^1$.

The image $f(U_p)$ is not surjective (as S is not homomorphic to a sphere). Thus we can normalize $\mathbb{C}\mathrm{P}^1 = \mathbb{C} \cup \{\infty\}$ so that $p = \infty$ and $0 \notin U_p$. Then $\mathbb{C}\mathrm{P}^1 \setminus U_p$ is the intersection of the closed disks $\mathbb{C}\mathrm{P}^1 \setminus D_j$ containing 0. Thus $\mathbb{C}\mathrm{P}^1 \setminus U_p$ is a closed convex subset containing 0, and therefore U_p is topologically an open disk. □

Then in the setting of Proposition 4.1, we have

Corollary 4.2. *When $\mathbb{C}\mathrm{P}^1 = \mathbb{C} \cup \{\infty\}$ is normalized so that $p = \{\infty\}$, the complement $\mathbb{C}\mathrm{P}^1 \setminus U_p$ is a compact convex subset K of $\mathbb{C}$.*

The canonical neighborhood U_p can be regarded as a projective structure on an open disk (Proposition 4.1), and one can consider maximal disks in U_p, which are *a priori* unrelated maximal disks in $\tilde{C}$.

Lemma 4.3. *The maximal disks of U_p bijectively correspond to the maximal disks of $\tilde{C}$ whose closure contain p by the inclusion $U_p \subset \tilde{C}$.*

Moreover, if D is a maximal disk of U_p containing p, then the ideal points of D as a maximal disk of U_p coincide with its ideal points of D as a maximal disk of $\tilde{C}$.

Proof. If D is a maximal disk in $\tilde{C}$ containing p, then clearly D is also a maximal disk in U_p by the definition of U_p. Similarly, if D is a maximal disk in $\tilde{C}$ whose boundary contains p, then there is a sequence of maximal disks D_i containing p with $D_i \to D$ as $i \to \infty$. Therefore every maximal disk D in $\tilde{C}$ whose closure contains p is a maximal disk in U_p.

We show the opposite inclusion. By Corollary 4.2, the complement $K = \mathbb{C}\mathrm{P}^1 \setminus U_p$ is a closed compact convex subset of C. If there is a (straight) line ℓ in $\mathbb{C}$ such that $\ell \cap K$ is a single point x, then, by the inclusions $\tilde{C} \supset U_p \subset \mathbb{C}\mathrm{P}^1$, x corresponds to an ideal point of a maximal ball of $\tilde{C}$ containing p. Next suppose that there is a line ℓ in $\mathbb{C}$ such that $\ell \cap K$ is a line segment. Then, letting P be the half-plane bounded by ℓ so that P and K have disjoint interiors, there is a sequence of maximal disks D_i of $\tilde{C}$ containing p such that D_i converges to P as $i \to \infty$. Thus the endpoints of the line segment correspond to ideal points of $\tilde{C}$.

Suppose that D is a maximal disk of U_p. Then $\overline{D}$ intersects K in ∂K. If ∂D intersects K in a line segment, then D is a half-plane in $\mathbb{C}$ with ∂D containing p. As the endpoints of the segment correspond to the ideal points of $\tilde{C}$, D is also a maximal disk in $\tilde{C}$.

If the closure of D does not intersect K in a line segment, then clearly D contains p. If a point on ∂K is not an interior point of any line segment of ∂K, then the point corresponds to an ideal point of $\tilde{C}$. Therefore no round disk in $\tilde{C}$ strictly contains D, and therefore D is also a maximal ball in $\tilde{C}$. Thus we have the opposite inclusion.

Finally, suppose that D is a maximal ball in U_p containing p. Then $\overline{D} \cap K$ contains no line segment, and therefore, $\overline{D} \cap K$ corresponds to the ideal points of D as a maximal ball in $\tilde{C}$. □

The following proposition yields a lamination on $\tilde{C}$ invariant under $\pi_1(S)$.

Proposition 4.4 ([17], Theorem 4.4)**.** *The cores* $\mathrm{Core(D)}$ *of the maximal disks* D *in* $\tilde{C}$ *are all disjoint and their union is* $\tilde{C}$*.*

Proof. We first show that the cores are disjoint. Let D_1 and D_2 be distinct maximal disks in $\tilde{C}$. If $D_1 \cap D_2 \neq \emptyset$, then $f(D_1)$ and $f(D_2)$ are round disks intersecting a crescent. Therefore $\mathrm{Core(D_1)}$ and $\mathrm{Core(D_2)}$ are disjoint. (Consider the circular arc in D_1 orthogonal to ∂D_1; then, indeed, this arc separates $\mathrm{Core(D_1)}$ and $\mathrm{Core(D_2)}$ in $D_1 \cup D_2$.)

Claim 4.5. *Given a convex subset* V *of* $\mathbb{C}$*, there is a unique round disk* D *in* $\mathbb{C}$ *of minimal radius containing* V*.*

Proof. Suppose, to the contrary, that there are two different round disks D_1, D_2 containing V which attain the minimal radius. Then, clearly, there is a round disk D_3 of strictly smaller radius which contains V (such that $D_3 \supset D_1 \cap D_2$ and $D_3 \subset D_1 \cup D_2$). This is a contradiction. □

Claim 4.6. *The convex hull of* $\partial D \cap \overline{V}$ *contains the center of* c *with respect to the complete hyperbolic metric on* $D(\cong \mathbb{H}^2)$ *given by the conformal identification.*

Proof. Suppose not; then the closure of V is contained in the interior of a (Euclidean) half disk of D. Then one can easily find a round disk of smaller radius containing $\overline{V}$, □

Note that, by the inversion of $\mathbb{C}\mathrm{P}^1 = \mathbb{C} \cup \{\infty\}$ about ∂D exchanges ∞ and the center of D, and it fixes ∂D pointwise. Then, by Claim 4.6, in the (conformal) hyperbolic metric on D, the convex hull of $\partial D \cap \overline{V}$ contains the center of D. Therefore, by the inversion, in the hyperbolic metric on $\mathbb{C}\mathrm{P}^1 \setminus D$, the point at ∞ is contained in the convex hull of $\partial D \cap \overline{V}$ in the interior of $\mathbb{C}\mathrm{P}^1 \setminus D$.

Using the above claims, we show that, for every $x \in \tilde{C}$, there is a maximal disk D in $\tilde{C}$ whose core contains x. Let $U_x = \cup_{j \in J} D_j$ be the canonical neighborhood of x, where D_j are the maximal disks in $\tilde{C}$ which contain x. Normalize $\mathbb{C}\mathrm{P}^1$ so that $f(x) = \infty$. Let $D_j^c = \mathbb{C}\mathrm{P}^1 \setminus f(D_j)$. Then $\mathbb{C}\mathrm{P}^1 \setminus f(U_x) = \cap_j D_j^c$. By Claim 4.6, let D be the maximal disk of U_x such that $x \in \mathrm{Core_{U_x}(D)}$. By Lemma 4.3, D is also a maximal disk of $\tilde{C}$ which contains x, and moreover the ideal points of D as a maximal of U_x coincide with those as a maximal ball of $\tilde{C}$. Then, $\mathrm{Core}_{\tilde{\mathrm{C}}}\mathrm{(D)}$ contains x. 4.4

By Proposition 4.4, $\tilde{C}$ is canonically decomposed into the cores of maximal disks in $\tilde{C}$, which yields a *stratification* of $\tilde{C}$. Note that this decomposition is invariant under $\pi_1(S)$, as the maximal balls and ideal points are preserved by the action. Moreover, for each maximal disk D in $\tilde{C}$, its Core(D) is properly embedded in $\tilde{C}$. Then the one-dimensional cores and the boundaries components of two-dimensional cores form a $\pi_1(S)$-invariant lamination $\tilde{\lambda}$ on $\tilde{C}$, which descends to a lamination λ on C.

Next we see that the angles between infinitesimally close maximal disks yield a natural transverse measure of this lamination. Given a point $x \in \tilde{C}$, let D_x be the maximal disk in $\tilde{C}$ whose core contains x. If $y \in \tilde{C}$ is sufficiently close to x, then D_y intersects D_x. Then let $\angle(D_x, D_y)$ denote the angle between the boundary circles of D_x and D_y. To be precise, it is the angle of the crescent $D_x \setminus D_y$ (or $D_y \setminus D_x$) at the vertices. Then $\angle(D_x, D_y) \to 0$ as $y \to x$.

Let x and y be distinct points of $\tilde{C}$ contained in different strata of $(\tilde{C}, \tilde{\lambda})$. Then pick a path $\alpha\colon [0,1] \to \tilde{C}$ connecting x to y such that α is transverse to $\tilde{\lambda}$. Let $\Delta : 0 = t_0 < t_1 < \cdots < t_n = 1$ be a finite division of $[0,1]$, and let $x_i = \alpha(t_i)$ for each $i = 0, \ldots, n$. Let $|\Delta| = \min_{i=0}^{n-1}(x_{i+1} - x_i)$, the smallest width of the subintervals. Then, let $\Theta(\Delta) = \Sigma_{i=1}^{n-1} \angle(D_{x_i}, D_{x_{i+1}})$ for a subdivision Δ of $[0,1]$ with sufficiently small $|\Delta|$. Pick a sequence of subdivisions Δ_i such that $|\Delta_i| \to 0$ as $i \to \infty$. Then $\lim_{i\to\infty}(\Theta(\Delta_i))$ exists and it is independent on the choice of Δ_i as $i \to \infty$ ([6, II.1]). We define the transverse measure of α to be $\lim_{i\to\infty}(\Theta(\Delta_i))$. Then $\tilde{\lambda}$ with this transverse measure yields a measured lamination $\tilde{\mathcal{L}}$ invariant under $\pi_1(S)$. Thus $\tilde{\mathcal{L}}$ descends to a measured lamination $\mathcal{L}$ on C.

By Lemma 4.3, for every $x \in \tilde{C}$, the measured lamination $\mathcal{L}$ near x is determined by the canonical neighborhood U_x of x. Let $\mathcal{L}_x$ be the measured lamination on U_x, which descends to the measured lamination on the boundary of $\mathrm{Conv}(\mathbb{C}\mathrm{P}^1 \setminus U_x)$. Then there is a neighborhood V of x in U_x such that $\mathcal{L}$ and $\mathcal{L}_x$ coincide in V by the inclusion $U_x \subset \tilde{C}$.

For each point $x \in \tilde{C}$, the boundary circle of the maximal disk D_x bounds a hyperbolic plane H_x in $\mathbb{H}^3$. Let $\Psi_x\colon f(D_x) \to H_x$ be the projection along geodesics in $\mathbb{H}^3$ orthogonal to H_x. Then H_x has a canonical normal direction pointing to D_x. By Lemma 4.3 there is a neighborhood V of x, such that $\Psi_y(y) = \Psi_x(y)$. Moreover, Ψ_x coincides with the projection onto the boundary pleated surface of $\mathrm{Conv}\, \mathbb{C}\mathrm{P}^1 \setminus U_x$. Therefore, as in the case of regions in $\mathbb{C}\mathrm{P}^1$, we have a pleated surface $\mathbb{H}^2 \to \mathbb{H}^3$ which is ρ-equivariant, as in the following paragraph.

We assume that crescents R in $\tilde{C}$ are always foliated by leaves of $\tilde{\mathcal{L}}$ sharing their endpoints at the vertices of R. We have a well-defined continuous map $\Psi\colon \tilde{C} \to \mathbb{H}^3$ defined by $\Psi(x) = \Psi_x(x)$. We shall take an appropriate quotient of $\tilde{C}$ to turn it into a hyperbolic plane. For each crescent R in $\tilde{C}$, Ψ takes each leaf in R to the geodesic in $\mathbb{H}^3$ connecting the vertices of R. Identify $x, y \in \tilde{C}$, if x, y are contained in a single crescent in $\tilde{C}$ and $\Psi_x(x) = \Psi_y(y)$; let $\tilde{\kappa}\colon \tilde{C} \to \tilde{C}/\sim$ be the quotient map by this identification, which collapses each foliated crescent region to a single leaf. Then by the equivalence relation, $\Psi\colon \tilde{C} \to \mathbb{H}^3$ induces a continuous map $\beta\colon (\tilde{C}/\sim) \to \mathbb{H}^3$ such that $\Psi_x(x) = \beta \circ \kappa$. Moreover, $\tilde{C}/\sim$ is $\mathbb{H}^2$ with respect to the path metric in $\mathbb{H}^3$ via Ψ, since, for every $x \in \tilde{C}$, Ψ coincides with the projection $U_x \to \partial\,\mathrm{Conv}(\mathbb{C}\mathrm{P}^1 \setminus U_x)$ in a neighborhood of x. Thus we have a ρ-equivariant pleated surface $\mathbb{H}^2 \to \mathbb{H}^3$.

The measured lamination $\tilde{\mathcal{L}}$ on $\tilde{C}$ descends to a measured lamination $\tilde{L}$ on $\mathbb{H}^2$ invariant under $\pi_1(S)$. By taking the quotient, we obtain a desired pair (τ, L) of a hyperbolic surface τ and a measured geodesic lamination L on τ.

Similarly, the collapsing map $\tilde{\kappa}\colon \tilde{C} \to \mathbb{H}^2$ descends to a *collapsing map* $\kappa\colon C \to \tau$. Then, for each periodic leaf ℓ of L, $\kappa^{-1}(\ell)$ is a grafting cylinder foliated by closed leaves of $\mathcal{L}$.

Finally we note that as $\beta\colon \mathbb{H}^2 \to \mathbb{H}^3$ is obtained by bending $\mathbb{H}^2$ in $\mathbb{H}^3$ along $\tilde{L}$, the pair (τ, L) corresponds to C by the correspondence in §4.1.

5. Goldman's theorem on projective structures with Fuchsian holonomy

Let C be a $\mathbb{C}\mathrm{P}^1$-structure on S with holonomy ρ, and let $(\tau, L) \in \mathsf{T} \times \mathsf{ML}$ be its Thurston parameters. Let $\psi\colon \mathbb{H}^2 \to \tau$ be the universal covering map, and $\tilde{L}$ be the measured lamination $\psi^{-1}(L)$ on $\mathbb{H}^2$. Let $\Gamma = \mathrm{Im}\rho$, and let Λ denote the limit set of $\mathrm{Im}\rho$.

Lemma 5.1. *Let $\beta\colon \mathbb{H}^2 \to \mathbb{H}^3$ be the associated pleated surface, where $\mathbb{H}^2$ is the universal cover of τ. Then, for every leaf $\tilde{\ell}$ of $\tilde{L}$, $\beta|\tilde{\ell}$ is a geodesic connecting different points of Λ.*

Proof. If $\tilde{\ell}$ is a lift of a closed leaf of L, then the assertion clearly holds.

For every closed curve α on τ, let $\tilde{\alpha}$ be a lift of α to $\mathbb{H}^2$. Since the curve $\beta|\tilde{\alpha}$ is preserved by the hyperbolic element $\rho(\alpha)$, it is a quasi-geodesic in $\mathbb{H}^3$ whose endpoints are the fixed points of $\rho(\alpha)$. Note that the endpoints are contained in Λ.

Let ℓ be a non-periodic leaf of L, and let $\tilde{\ell}$ be a lift of ℓ to $\mathbb{H}^2$. There is a sequence of simple closed geodesics ℓ_i on τ such that ℓ_i converges to ℓ in the Hausdorff topology ([6, I.4.2.14]). For each $i \in \mathbb{N}$, pick a lift $\tilde{\ell}_i$ of ℓ_i to $\mathbb{H}^2$ so that $\tilde{\ell}_i \to \ell$ uniformly on compact sets as $i \to \infty$.

Then, $\beta|\tilde{\ell}_i$ converges to $\beta|\tilde{\ell}$ uniformly on compact sets. Moreover as $\angle_{\tau_i}(\tau_i, L_i) \to 0$, $\beta_i|\tilde{\ell}_i$ is asymptotically an isometric embedding: To be precise, for large enough i, it is a bilipschitz embedding, and its bilipschitz constant converges to 1 as $i \to \infty$ [3, Proposition 4.1].

As ℓ_i are closed loops, the endpoints of $\beta|\tilde{\ell}_i$ are in Λ. Then the endpoints of $\beta|\tilde{\ell}_i$ converge to the endpoints of $\beta|\tilde{\ell}$ in $\mathbb{C}\mathrm{P}^1$. Therefore, since Λ is a closed subset of $\partial\mathbb{H}^3$, the endpoints of $\beta|\ell$ are also contained in Λ. □

We immediately have

Corollary 5.2. *For each stratum σ of $(\mathbb{H}^2, \tilde{L})$, let $D_\sigma \subset \tilde{C}$ be the maximal disk whose core corresponds to σ. Then its ideal points $\partial_\infty D_\sigma$ are contained in the limit set Λ.*

We reprove the following theorem by means of pleated surfaces.

Proposition 5.3. *(See [21, Theorem 3.7.3.]) Let C be a $\mathbb{C}\mathrm{P}^1$-structure with real holonomy $\rho\colon \pi_1(S) \to \mathrm{PSL}(2, \mathbb{R})$ and (L, τ) its Thurston parameters. Then each leaf of L is periodic, and its weight is π-multiple. If ρ is, in addition, Fuchsian, then each leaf of L is periodic and its weight is a 2π-multiple.*

Proof. We first show that L consists of periodic leaves. Suppose, to the contrary, that L contains an irrational minimal sublamination N. Then the transverse measure is continuous in a neighborhood of $|N|$ in τ (i.e. no leaf of N has an atomic measure).

Thus there are two-dimensional strata $\sigma, \sigma_1, \sigma_2, \dots$ of $\mathbb{H}^2 \setminus \tilde{L}$, such that σ_i converges to an edge of σ as $i \to \infty$. Note that, as they are two-dimensional, each $\beta(\sigma_i)$ has at least three ideal points, which lie in a round circle in $\mathbb{C}\mathrm{P}^1$. Let $H, H_1, H_2, \dots$ be the supporting oriented hyperbolic planes in $\mathbb{H}^3$ of $\sigma, \sigma_1, \dots$. Let $\angle_{\mathbb{H}^3}(H, H_i) \in [0, \pi]$ be the angle between the hyperbolic planes H and H_i with respect to their orientations, if H and H_i intersect. Then, by continuity, $\angle_{\mathbb{H}^3}(H, H_i) \to 0$ as $i \to \infty$. Thus the ideal points of σ and σ_i cannot be contained in a single round circle if i is sufficiently large. By Corollary 5.2, this cannot happen as Λ is a single round circle.

We first show that the weight of each leaf of L is a multiple of π. Let σ_1 and σ_2 be components of $\mathbb{H}^2 \setminus \tilde{L}$ adjacent along a leaf of $\tilde{L}$. Let H_1 and H_2 be the support planes of σ_1 and σ_2, respectively. Then the angle between H_1 and H_2 is the weight of ℓ. As the ideal points of σ_1 and σ_2 must lie in the round circle Σ, the angle must be a multiple of π.

Suppose, in addition, that ρ is Fuchsian. Let $\beta_0 \colon \mathbb{H}^2 (= \tilde{\tau}) \to \mathbb{H}^3$ be the ρ-equivariant embedding onto the hyperbolic plane H_Λ bounded by Λ. For each $i = 1, 2$, as each boundary component m of σ_i covers a periodic leaf of L, $\beta = \beta_0$ on m. Therefore $H_1 = H_2 = \mathrm{Conv}(\Lambda)$, and $\beta_0 = \beta$ on σ_i for each $i = 1, 2$. As the orientation of H_1 coincides with that of H_2, the weight of m must be a multiple of 2π. □

Proof of Theorem B. By Proposition, 5.3, L is a union of closed geodesics ℓ with 2π-multiple weights. For each (closed) leaf ℓ of L, let $2\pi n_\ell$ denote the weight of ℓ, where n_ℓ is a positive integer. Let $\kappa \colon C \to \tau$ be the collapsing map. Then, $\kappa^{-1}(\ell)$ is a grafting cylinder of height $2\pi n_\ell$, the structure inserted by 2π-grafting n times. Therefore, C is obtained by grafting along a multiloop corresponding to L. □

6. The path lifting property in the domain of discontinuity

Let $C = (f, \rho)$ be a $\mathbb{C}\mathrm{P}^1$-structure on S. Then, let Λ be the limit set of $\mathrm{Im}\rho$, and let $\Omega = \mathbb{C}\mathrm{P}^1 \setminus \Lambda$, the domain of discontinuity.

Proposition 6.1. *For every $x \in \Omega$, there is a neighborhood V_x in Ω such that, for every $y \in \tilde{S}$ with $f(y) \in V_x$, V_x is contained in the maximal disk whose core contains x.*

Proof. The union $\mathbb{H}^3 \cup \partial \mathbb{H}^3$ is a unit ball in the Euclidean space and the visual distance is the restriction of the Euclidean metric.

Suppose, to the contrary, that there is no such neighborhood V_x. Then there is a sequence $x_1, x_2, \dots \in f^{-1}(x)$ such that, letting $H_1, H_2, \dots$ be their corresponding hyperbolic support planes, the visual distance from H_i to x goes to zero as $i \to \infty$. Let $y_i \in \mathbb{H}^3$ be the nearest point projection of $f(x_i)$ to H_i. Then, $y_i \to x$ in the visual metric. Let σ_i be the stratum of $(\mathbb{H}^2, \tilde{L}_i)$ which contains $\tilde{\kappa}(x_i)$. Then, as the orthogonal projection of $f(x_i)$ to H_i is y_i, the visual distance between x and $\beta_i(\sigma_i)$ goes to zero as $i \to \infty$. Therefore, there is an ideal point p_i of $\beta(\sigma_i)$ which converges to x as $i \to \infty$. As Ω is open, this is a contradiction by Corollary 5.2. □

As f embeds maximal disks of $\tilde{C}$ into $\mathbb{C}\mathrm{P}^1$, we immediately have

Corollary 6.2. *For each point $x \in \Omega$, there is a neighborhood V_x of x such that, if $f(y) \in V_x$ for $y \in \tilde{S}$, then f embeds a neighborhood W_y of y in $\tilde{S}$ homomorphically onto V_x.*

Theorem C immediately follows from the corollary.

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