THE INTERSECTION OF HOLONOMY VARIETIES OF $\mathbb{C}\mathrm{P}^1$-STRUCTURES

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Abstract. Let S be a closed orientable surface of genus at least two, and let X, Y be distinct marked Riemann surface structures on S. In this paper, we construct infinite pairs of a $\mathbb{C}\mathrm{P}^1$-structure on X and a $\mathbb{C}\mathrm{P}^1$-structure Y which share holonomy $\pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$.

Contents

1. Introduction

Let Σ be a closed orientable surface of genus g at least two. A quasi-Fuchsian representation $\rho\colon \pi_1(\Sigma) \to \mathrm{PSL}_2\mathbb{C}$ is a typical discrete and faithful representation, such that the limit set is a Jordan curve Λ on $\mathbb{C}\mathrm{P}^1$.

Let S be the surface Σ with a fixed orientation, and S^* be Σ with the opposite orientation. Let $\mathcal{T}$ be a Teichmüller space of S, and let $\mathcal{T}^*$ be the Teichmüller space of S^*, S with the opposite orientation. Given a quasi-Fucsian representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$, let Λ be the limit set of $\mathrm{Im}\rho$. Then $\mathbb{C}\mathrm{P}^1 \setminus \Lambda$ is a union of disjoint topological open disks Ω^+ and Ω^-. The Bers' simultaneous uniformization theorem ([Ber60]) asserts that, for every pair of Riemann surface structures X on S and Y on S^*, there is unique quasi-Fuchsian representation

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$\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ such that X and Y are realized by $\Omega^+/\mathrm{Im}\rho$ on S and $\Omega^-/\mathrm{Im}\rho$ on S^*.

We note that the quotient surfaces $\Omega^+/\mathrm{Im}\rho$ and $\Omega^-/\mathrm{Im}\rho$ have not only Riemann surface structures but also have $\mathbb{C}\mathrm{P}^1$*-structures (or complex projective structures)* on S and S^*, which corresponds to holomorphic quadratic differentials on Riemann surfaces.

From a viewpoint of $\mathbb{C}\mathrm{P}^1$-structures, the simultaneous uniformization theorem can equivalently be stated as follows, without the notion of quasi-Fuchsian representations: Given a pair of Riemann surface structures X on S and Y in S^*, there is a unique pairs of $\mathbb{C}\mathrm{P}^1$-structure C_X on X and a $\mathbb{C}\mathrm{P}^1$-structure C_Y on Y such that

- the holonomy representation $\pi_1(\Sigma) \to \mathrm{PSL}_2\mathbb{C}$ of C_X coincides with the holonomy representation $\pi_1(\Sigma) \to \mathrm{PSL}_2\mathbb{C}$ of C_Y, and
- the developing maps $\tilde{S} \to \mathbb{C}\mathrm{P}^1$ of C_X and $\tilde{S}^* \to \mathbb{C}\mathrm{P}^1$ C_Y are injective, where $\tilde{S}$ and $\tilde{S}^*$ are the universal covers of S and S^*, respectively.

In this paper, we consider a more general realization problem of a pair of Riemann surface structures X and Y on either S or S^* by a pair of $\mathbb{C}\mathrm{P}^1$-structures C_X and C_Y sharing holonomy. In this more general setting without the restriction of the injectivity and the orientation, we show that there are infinitely many realizing pairs:

Theorem A. *Let $X, Y \in \mathcal{T} \cup \mathcal{T}^*$ with $X \neq Y$. Then, there exist exactly countablly many distinct pairs $(C_i^X, C_i^Y)_{i=1}^\infty$ of $\mathbb{C}\mathrm{P}^1$-structures C_i^X on X and C_i^Y and Y, such that the holonomy $\pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ of C_i^X coincides with the holonomy of C_i^Y for each $i = 1, 2, \ldots$.*

Note that the orientations of X and Y can be either the same or the opposite, in contrast to Bers' theorem.

Next we interpret Theorem A in the $\mathrm{PSL}_2\mathbb{C}$-character variety of Σ, a space of representations of

$$\chi := \{\pi_1(S) \to \mathrm{PSL}_2\mathbb{C}\} /\!/ \mathrm{PSL}_2\mathbb{C}.$$

There are various half-dimensional (real and complex) subvarieties of χ with geometric significance. It has been important to understand the intersection of such half-dimensional (real and complex) subvarieties in the $\mathrm{PSL}_2\mathbb{C}$-character variety (Faltins [Fal83, Theorem 12], Dumas-Wolf [DW08]).

Here we shall consider the intersection of holonomy varieties. For a Riemann surface structure X on Σ. The set $\mathcal{P}_X$ of $\mathbb{C}\mathrm{P}^1$-structures on X is identified with the affine space $QD(X) \cong \mathbb{C}^{3g-3}$ of holomoprhic quadratic differentials on X. Then, the deformation space $\mathcal{P}_X$ properly embeds into the $\mathrm{PSL}_2\mathbb{C}$-character variety of Σ by the holonomy map.

Its image is a smooth complex analytic subvariety of $\mathcal{X}$— it is called the holonomy variety of X, and we denote it by $\mathcal{X}_X$.

Theorem B. *For all distinct* $X, Y \in \mathcal{T} \cup \mathcal{T}^*$, *the intersection*

$$\mathcal{X}_X \cap \mathcal{X}_Y$$

is an infinite discrete closed subset of $\mathcal{X}$.

The infinite points of the intersection $\mathcal{X}_X \cap \mathcal{X}_Y$ bijectively correspond to the infinite points of the sequence $(C_i^X, C_i^Y)_{i=1}^\infty$ in Theorem A.

Last we relate our main theorem to the deformation space of isomonodromic pairs of $\mathbb{C}\mathrm{P}^1$-structures. Namely, consider the space $\mathcal{B}$ of (ordered) pairs of distinct $\mathbb{C}\mathrm{P}^1$-structures on Σ sharing holonomy. Then the quasi-Fuchsian space is identified with a connected component of $\mathcal{B}$ unique up to switching the ordering of paired $\mathbb{C}\mathrm{P}^1$-structures.

Let

$$\Psi\colon \mathcal{B} \to (\mathcal{T} \sqcup \mathcal{T}^*)^2 \setminus \Delta$$

be the uniformization map taking a pair (C, D) in $\mathcal{B}$ to the pair of the underlying Rieman surface structures of C and D. Then the author previously proved that the analytic mapping Ψ is a complete local branched covering map ([Bab23, Theorem A]).

Theorem C. *Each fiber of* Ψ *is an infinite discrete set.*

The Ψ-fiber over (X, Y) is exactly the infinite sequence $(C_i^X, C_i^Y)_{i=1}^\infty$ in Theorem A. The space $\mathcal{B}$ is quite mysterious. Theorem C suggests a possibily of $\mathcal{B}$ having infinitly many connected components.

Theorem A, Theorem B, and Theorem C are all equivalent, and the "infinite" property is the new discovery. The discreteness in those theorems was proven by the author ([Bab23, Theorem C]), and thus the cardinality has been known to be, at most, a countable set. In this paper, we show that this upper bound is sharp by constructing infinitely many pairs. As for the lower bound, it has only been known that the cardinality of $\mathcal{X}_X \cap \mathcal{X}_Y$ is at least two if the orientations of X and Y are opposite and the cardinality of $\mathcal{X}_X \cap \mathcal{X}_Y$ is at least one if the orientations of X and Y are the same ([Bab23, Corollary 12.7]).

In a large portion of this paper, we investigate the strong asymptotic property of Teichmüller (geodesic) rays and grafting rays, initiated by Gupta in his thesis ([Gup14, Gup15]). He namely showed that, given every conformal grafting ray in the Teichmüller space, there is a Teichmüller ray asymptotic to it, as unparametrized rays (see Definition). In his construction, typically those rays have different base points.

On the other hand, In order to prove our main theorem, we need to have such an asymptotic property for a certain family of pairs of a

Teichüller ray and a corresponding grafting ray sharing a base point, in contract, as parametrized rays. As a consequence, we have a uniform asymptotic rate for this family (Theorem 4.1), and use it for the proof of our main theorem.

1.1. **Ideal of the proof.** We outline the proof of Theorem A in the case that X, Y are Rieman surface structures on S— if the orientations of X and Y are opposite, the proof is reduced to this case. Supposing that there are already finitely many isomonodromic pairs $(C_1^X, C_1^Y), \dots, (C_n^X, C_n^Y)$ of $\mathbb{C}\mathrm{P}^1$-structures on X and Y, we construct a new isomonodromic pair as follows.

We take a "generic" Teichmüller geodesic X_t in $\mathcal{T}$ which passes very close to X and Y. Let $\rho_t \colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ be the representation uniformizing X_t, so that the marked hyperbolic surface $\mathbb{H}^2/\mathrm{Im}\rho_t$ is conformally identified with the marked Riemann surface X_t.

Take sufficiently small $t < 0$ so that ρ_t is sufficiently far from the n holonomy representations of $(C_1^X, C_1^Y), \dots, (C_n^X, C_n^Y)$. In addition, using 2π-grafting, we can construct $\mathbb{C}\mathrm{P}^1$-structures $C_{X'}, C_{Y'}$ with holonomy ρ_t whose underlying Riemann surface structure X' and Y' are very close to X and Y. To have this closeness, we utilize the uniform asymptotic properties of certain pairs of a Teichmüller ray and a corresponding grafting ray, related to the generic Teichmüller geodesic X_t (Theorem 4.1). By the compleness of Ψ, we can deform this pair $(C_{X'}, C_{Y'})$ to C_X, C_Y in $\mathcal{B}$ so that their underlying Riemann surface structures are exactly X and Y. As ρ_t is sufficiently far from the holonomy representations of the already given pairs, we can conclude that the deformed new pair (C_X, C_Y) realizing (X, Y) is different from the n pairs $(C_1^X, C_1^Y), \dots, (C_n^X, C_n^Y)$ we already have.

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3. Preliminaries

3.1. **Teichmüller rays.** (See [FM12] for instance.) The Teichmüller space $\mathcal{T}$ of S is the space of Riemann surface structures on S up to isotopy. Given two marked Riemann surfaces $X, Y \in \mathcal{T}$, let $K = K(X, Y)$ denote the infinium of the quasi-conformal dilatations K_f among all quasi-conformal mappings $X \to Y$ preserving the marking of the surface. The Teichmüller distance between X and Y on $\mathfrak{T}$ is given by

$$d(X,Y) = \frac{1}{2}\log K,$$

which gives a Finsler metric on $\mathfrak{T}$, call the Teichmüller metric.

A geodesic in $\mathfrak{T}$ in the Teichmüler metric is callled a Teichmüller geodesic. Then, given $X \in \mathfrak{T}$ and a measured foliation V on S, there is a Teichmüller geodesic X_t with $X_0 = X$ along which V shrinks. Namely, by Hubbard and Masur [HM79], there is a flat surface $E = E(X,V)$ conformal to X such that V is the vertical measured foliation. Then, we can obtain a ray of flat surfaces E_t obtained by stretching in the horizontal direction by $e^{t/2}$ and shrinking in the horizontal direction by $e^{-t/2}$. The conformal structure of E_t gives the Teichmüller geodesic at unit speed.

3.2. $\mathbb{C}$P**-structures.** (General references are [Dum09], [Kap01, Chapter 7], [Gol22, Chapter 14].) Recall that $\mathrm{PSL}_2\mathbb{C}$ is the automorphism group of $\mathbb{C}\mathrm{P}^1$. Then, a $\mathbb{C}\mathrm{P}^1$-structure on a surface is a $(\mathbb{C}\mathrm{P}^1, \mathrm{PSL}_2\mathbb{C})$-structure. Namely, an atlas of embedding open subsets covering S into $\mathbb{C}\mathrm{P}^1$ such that transition maps are given by elements in $\mathrm{PSL}_2\mathbb{C}$. Clearly, each $\mathbb{C}\mathrm{P}^1$-structure has a Riemann surface structure since transition maps preserve the complex structure.

A $\mathbb{C}\mathrm{P}^1$-structure has various perspectives including the following.

3.2.1. *Developing pairs.* A $\mathbb{C}\mathrm{P}^1$-structure can also be defined using a "global coordinate" on the universal cover $\tilde{S}$ of S. Namely, a $\mathbb{C}\mathrm{P}^1$-structure on X is a pair (f,ρ) of

- a local diffeomorphism $f\colon \tilde{S} \to \mathbb{C}\mathrm{P}^1$ (developing map) and
- a homomorphism $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ (holonomy representation),

such that f is ρ-equivariant.

3.2.2. *Schwarzian parametrization.* Next we explain an analytic viewpoint of a $\mathbb{C}\mathrm{P}^1$-structure. A $\mathbb{C}\mathrm{P}^1$-structure $C = (f,\rho)$ on S corresponds to a holomorphic quadratic differential $q = \phi\, dz^2$ on a Riemann surface. The developing map f gives a Riemann surface structure, and the Schwarzian derivative of f gives a holomorphic quadratic differential on X. Thus a space of $\mathbb{C}\mathrm{P}^1$-structures on a Riemann surface X is an affine vector space of holomorphic quadratic differentials on X.

There is a unique marked hyperbolic structure σ uniformizing X, and a hyperbolic structure is, in particular, a $\mathbb{C}\mathrm{P}^1$-structure. In this paper, we pick this hyperbolic structure to be the zero of this vector space— in other words, when we take the Schwarzian derivative of

f, the domain $\tilde{X}$ is identified with the upper half plane of $\mathbb{C}$ by the uniformization.

Let $\mathcal{P}$ be the space of all marked $\mathbb{C}\mathrm{P}^1$-structures on S. Let $\psi \colon \mathcal{P} \to \mathcal{T}$ be the projection map which takes each $\mathbb{C}\mathrm{P}^1$-structure to its complex structure.

3.2.3. *Thurston's parametriztion.* (See [KT92, KP94], also [Bab20].) The Riemann sphere $\mathbb{C}\mathrm{P}^1$ is the ideal boundary of the hyperbolic three-space, and $\mathrm{PSL}_2\mathbb{C}$ is also the orientation-preserving isometry group of $\mathbb{H}^3$.

Utilzing such relations, a $\mathbb{C}\mathrm{P}^1$-structure $C = (f, \rho)$ corresponds to a pair (σ, L) of a hyperbolic structure σ on S and a measured lamiantnion L on S. This pair is called Thurston's parameters of C.

Let $\tilde{L}$ be the $\pi_1(S)$-invariant measured lalmination on the universal cover $\mathbb{H}^2$ of σ. Then, this pair (σ, L) give an ρ-equivariant "locally convex" suface $\beta \colon \mathbb{H}^2 \to \mathbb{H}^3$, called a bending map, obtained by bending $\mathbb{H}^2$ along L by the angle given by the transversal measure of $\tilde{L}$. A bent surface is a particular type of pleated surface. The developing map $f \colon \tilde{S} \to \mathbb{C}\mathrm{P}^1$ corresponds to β in a ρ-equivariant manner by certain "locally" well-defined nearest point projections.

Let ML denote the space of all measured laminations on S. Then we have Thurston's parametriztion of the deformation space

$$\mathcal{P} \cong \mathcal{T} \times \mathrm{ML}, \tag{1}$$

by a canonical tangential homeomorphism.

For each periodic leaf ℓ of L, there is a round cylinder A_ℓ in the grafted surface $\mathrm{Gr}_L\tau$ foliated by circular close curves and its height is the weight of ℓ. If there are more than one periodic leaves, then their correspoding round cylinders are disjoint. The collapsing map $\kappa \colon \mathrm{Gr}_L\tau \to \tau$ collapses each cylinder A_ℓ to the closed geodesic ℓ on τ and the restriction of κ to the complement of the cylinders is a C^1-diffeomorphism onto the complement of closed leaves Therefore, there is a measured lamination $\mathcal{L}$ on the grafted surface $\mathrm{Gr}_L\tau$, such that

- the leaves of $\mathcal{L}$ are circular, and
- κ takes $\mathcal{L}$ to L on σ.

This lamination $\mathcal{L}$ is called Thurston's lamination.

The bending deformation of a hyperbolic surface in the three-space corresponds to a grafting deformation of a $\mathbb{C}\mathrm{P}^1$-structure. Given a pair $(\sigma, L) \in \mathcal{T} \times \mathrm{ML}$, the corresonding $\mathbb{C}\mathrm{P}^1$-structure is obtained by grafring σ along a measured lamination L. We denote this $\mathbb{C}\mathrm{P}^1$-structure by $\mathrm{Gr}_L\sigma$.

3.2.4. *Epstin surfaces.* ([Eps], see also [And98].) Let (f, ρ) be a developing pair of a projective structure C on S, where $f\colon \tilde{S} \to \ CP1$ is a fixed developing map.

Each point $x \in \mathbb{H}^3$ determines a unique spherical metric s_x on $\mathbb{C}\mathrm{P}^1$ by normalizing the Poincaé disk model of $\mathbb{H}^3$ so that x is at the center of the disk. Given a conformal metric μ on C, there is a unique mapping $\mathrm{Ep}\colon \tilde{S} \to \mathbb{H}^3$ such that, the spherical matric $s_{\mathrm{Ep}(x)}$ of $\mathbb{C}\mathrm{P}^1$ centered at $\mathrm{Ep}(x)$ coincides with the push-forward matric of μ at the tangent space $T_{f(x)}\mathbb{C}\mathrm{P}^1$. This surface is the envelope of the horospheres centered at the points $f(x)$ for $x \in \tilde{S}$, and Ep is also ρ-equivariant.

Let $C \cong (X, q)$ be the Scharzian parametrization of C. Then the quadratic differential q gives a singular Euclidean metric on C, where the singular points are the zeros of the differential. In this paper, we use the Epstein surface given by this singular Euclidean metric.

3.3. **Grafting rays.** Let σ be a hyperbolic structure on S. Let L be a measured geodesic lamination on σ.

Then, by Thurson's parametrization, we obtain a ray of $\mathbb{C}\mathrm{P}^1$-structures $\mathrm{Gr}_{tL}\sigma$ corresponding to $(\sigma, tL), t \geq 0$. This ray in $\mathcal{P}$ is called a (projective) grafting ray.

Let $\mathrm{gr}_L\sigma \in \mathcal{T}$ denote $\psi(\mathrm{Gr}_L\sigma)$, the complex structure of $\mathrm{Gr}_L\sigma$. Then, $\mathrm{gr}_t Lsigma, t \geq 0$ is called the sf conformal grafting ray from σ in the direction of L.

3.4. **Traintracks.** ([PH92]) A traintarck graph on a surface is a (localy finite) C^1-smooth graph G such that, for each vertex p of G, the edges of G with its endpoint at p are divides into two groups $e_1, \ldots, e_m$ and $f_1, \ldots, f_n$ such that

- the vectors v tangent to the deges $e_1, \ldots, e_m$ at p coincide
- the vectores u tangent to the edges $f_1, \ldots, f_m$ at p coincide
- $v = -u$ in the tangent space at p.

A marked rectangle is a rectangle such that a pair of opposite edges is marked as vertical edges and the other pair is marked as horizontal edges. A fat traintrack T is an orientable surface with boundary with singular points which is obtained by takings a union of marked rec tangles $\{R_i\}$ along horizontal edges as follows: Divide some horizontal edges into finitely many segments, pair up all horizontal edgesmens, and glue each paired horizontal edge by a diffeomorphism. Each rectangle R_i of T is called a branch.

More generally, a marked polygon is a polygon with even number $2n$ of edges, such that a set of alternating n edges are marked as vertical edges and the set of the other alternating n edges are marked as horizontal edges. A polygonal traintrack is an orientable surface with boundary with singular points obtained by gluing some marked polygons as follows: Divide each horizontal edge into finitely many segments (if necessary), pair up all horizontal segments, and glue each pair of horizontal segments by a diffeomorphism. Each polygon of the polygonal traiontrack is also called a branch

Given a lamination λ on a hyperbolic surface σ, a traintrack neighborhood τ of λ is a fat traintrack containing λ in its interior, such that, for each branch R of τ, each component of $\lambda \cap R$ is an arc connecting opposite horizontal edges of R.

Definition 3.1 (cf. [Min92]). *A traintrack neighborhood σ on a hyperbolic surface σ is ϵ-nearly straight, if all boundary geodesics have curvature less than ϵ at non-singular points and all horocyclic horizontal edges of rectangular branches have curvature less than ϵ.*

4. Grafting rays and Teichmüller rays

Recall that the Teichmüler geodesic flow is ergodic in the moduli space $\mathcal{M}$ of Riemann surfaces structures on S ([Mas82, Vee82]). Let PML denote the space of projective measured laminations on S. Let $X(t)$ be a generic Teichmüller geodesic parametrized by $t \in \mathbb{R}$, such that

- the projective vertical foliation $[V] \in$ PML and the projective horizontal foliation $[H] \in$ PML are both uniquely ergodic, and they have no saddle connections;
- its corresponding quadratic differential has only simple zeros;
- there are no vertical saddle connections;
- the projection $[X(t)]$ is dence in the unite-tangent space of the moduli space $\mathcal{M}$.

For each $t \in \mathbb{R}$, $(X(t), [V])$ conformally equvalent to a unique marked flat surface E_t of unite area with the vertical foliation $[V]$. Let V_t be the representative of $[V]$ such that V_t has length one on E_t. By the density assumption, let $0 > t_1 > t_2 > \ldots$ be the degreasing sequence diverging to $-\infty$ such that the unmarked tangent vector $[X'(t_i)]$ converges to $[X'_\infty(0)] \in T^1\mathcal{M}$ as $i \to \infty$, where $X_\infty(t) \in \mathcal{T}$ is an appropriate marked Teichmüller geodesic ray prametrized by $t \in \mathbb{R}$. Then, for each i, there is a mapping class $\nu_i \colon S \to S$ such that $\nu_i X(t_i)$ converges to $X_\infty(0) =: X_\infty$ as $i \to \infty$.

Then $\nu_i E(t_i)$ conreges to a flact surface E_∞ with unite area, and $\nu_i V_{t_i}$ converges to a vertical meausured foliation V_∞ on E_∞. Then V_∞ has lengths one on the flat surface E_∞, and it is the vertical foliation of the Teichmüller geodesic $X_\infty(t)$.

By the density assumption, we can without loss of generality, we can in addition assume

- the vertical V_∞ is uniquely ergodic and has no saddle connections;
- every singular point is three-pronged.

For each $i = 1, 2, \ldots$, let σ_i be the hyperbolic structure on S uniformizing $X(t_i)$. Let L_i be the measured geodesic lamination on σ_i representing the vertical measured foliation V_{t_i}. Let $\mathrm{gr}^u_{L_i}(\sigma_i), u \geq 0$ denote the conformal grafting ray starting from σ_i along the vertical foliation L_i.

In this section, we prove the following uniform asymptotic property of grafting rays and Teichmüller rays from $X(t_i) =: X_i$ as parametrized rays.

Theorem 4.1. *For every $\epsilon > 0$, there are constant $I_\epsilon > 0$, $d > 0$ and $s_\epsilon > 0$ such that, if $i > I_\epsilon$, then*

$$d_{\mathcal{T}}(X(t_i + s), \mathrm{gr}^{\exp(s)}_{L_i/d}(\sigma_i)) < \epsilon$$

for all $s > s_\epsilon$, where $d_{\mathcal{T}}$ denotes the Teichmüller distance.

The constant d will be explicitly given in §4.1. It is already known that, for each i, the grafting ray $\mathrm{gr}^{\exp(s)}_{kL_i}(X(t_i))$ is asymptotic to the Teichmüller ray $X(t_i + s)$ as unparametrized rays: Indeed, Gupta [Gup14, Gup15] proved that, for every grafting ray along a geodesic lamination L, there is a Teichmüller ray typically from a different basepoint which is asymptotic to it up to reparametrization.

In the case that L is maximal, the vertical foliation of the Teichmüller ray is L. Masur proved that, for an arbitrarily fixed recurrent uniquely ergodic vertical measured foliation, all Teichmüller rays with a fixed are all asymptotic [Mas80, Theorem 2]. Thus, the main contribution of Theorem 4.1 is the asymptotic property as parametrized rays and the uniformness of the asymptotic property.

Overall the strategy is Theorem 4.1 is similar to the proof of Gupta. However, as we compare the grafting ray with a Teichmüller ray from the same base point, our techniques are sometimes different and seemingly more geometric, in particular, §4.4. In particular, we do not use any Grötzsch type argument, whereas Lemma 4.24 in Gupta's paper is crucial in his paper.

4.1. **Fat traintrack structures and nearly striaght traintracks in the limit.** We first show the asymptotic property of the single Teichmüller ray $X_\infty(t)$ in the limit and its corresponding grafting ray—

the uniform asymptotic property in Theorem 4.1 is morally modeled on this asymptotic property in the limit.

Recall that V_∞ is the vertical measured foliation of the limit flat surface E_∞ of unit length. Let H_∞ be the horizontal measured foliation of E_∞ orthogonal to V_∞.

Then, let L_∞ be the measured geodesic lamination on σ_∞ obtained by straightening to a measured foliation V_∞.

Suppose that a flat surface E has a vertical measured lamination V so that the transversal measure is exactly given by the Euclidean length. Then, the Euclidean length $\mathrm{length}_E V$ of V is exactly the area of this flat surface. In particular, as $\mathrm{Area} E_\infty = 1$, $\mathrm{length}_{E_\infty} V_\infty = 1$, Then we let

$$d = \frac{\mathrm{length}_{E_\infty} V_\infty}{\mathrm{length}_{\sigma_\infty} L_\infty} = \frac{1}{\mathrm{length}_{\sigma_\infty} L_\infty},$$

where $\mathrm{length}_{\sigma_\infty} L_\infty$ denotes the hyhperbolic length of L_∞.

We first construct a sequence of fat traintracks by splitting E_∞ along vertical singular leaves. Let $N = 2(2g-2)$, the number of singular points on E_∞. Let $r_1, \dots r_N$ be vertical neighborhoods of singular points of E_∞; since V_∞ has no saddle connections, they are tripods. Then, the complement $E_\infty \setminus (r_1 \cup \dots \cup r_N)$ has a fat traintrack structure T_0, so that the branches are all Euclidean rectangles with horizontal and vertical edges—namely, we decompose $E_\infty \setminus (r_1 \cup \dots \cup r_N)$ by the horizontal line segments starting from the endpoints of $\gamma_1, \dots, \gamma_k$ and ending when the segments hit the boundary of $E_\infty \setminus (r_1 \cup \dots \cup r_N)$. Clearly T_0 is foliated by V_∞.

Let σ_∞ be the (marked) hyperbolic structure on S uniformizing X_∞. Since gL_∞ is the geodesic representative of V_∞, there is a traintrack neighborhood τ_0 of L_∞ on σ_∞, such that there is a marking preserving diffeomorphism $\sigma_\infty \to E_\infty$ which induces an isomorphism from (τ_∞, L_∞) to (T_0, V_∞) as fat-traintracks caryying .

By enlarging $r_1, \dots r_N$, we can construct a sequence $T_0, T_1, \dots$ of splitting of the traintrack T_0 so that lengths of all branches of T_j diverge to infinity as $j \to \infty$. For $j = 1, 2, \dots,$ we let $r_1(j), \dots r_N(j)$ be this increasing sequence of vertical neighborhoods $r_1, \dots r_N$ of singular points of E_∞, such that

- $r_h(j)$ is a tripod and the lengths of all smooth edges go to infinity as $j \to \infty$ for all $h = 1, \dots, N$, and
- the support $|T_j|$ of the traintrack T_j is $E_\infty \setminus (r_1(j) \cup \dots \cup r_N(j))$.

The vertical foliation V_∞ and the horizontal foliation H_0 induce horizontal and vertical foliation of each T_j. By collapsing each horizontal leaf of T_j to a point, we obtain a traintrack graph G_j. By modifying the sequence of splittings if necessary, we may, in addition, assume that G_j is trivalent.

We next construct the corresponding splitting sequence of traintrack neighborhoods of the measured lamination L_∞ on the limit hyperbolic surface σ_∞. Since L_∞ corresponds to a uniquely ergodic foliation V_∞ without saddle connections, such that the underlying lamination $|L_\infty|$ of L_∞ is maximal.

More generally, let L be a maximal geodesic lamination on a hyperbolic surface σ. Then, the complement $\sigma \setminus L$ consists of hyperbolic ideal triangles Δ, and each ideal triangle has a canonical horocyclic lamination λ_Δ (Figure 1, left): leaves are horocyclic arcs centered at the vertices of the ideal triangle, and the complement of the lamination is a triangle with the horocyclic edges. Then, the homocyclic arcs are orthogonal to the edges of the ideal triangle. Therefore, those horocyclic laminations on the complementary ideal triangles yield a horocyclic lamination λ on the hyperbolic surface σ orthogonal to L. (See [Thu].) The horocyclic lamination has a transversal measure given by the hyperbolic length in the direction orthogonal to λ.

The support $|\lambda_\Delta|$ of the horocyclic lamination is the complement of the triangle with horocyclic edges; thus the support is foliated by geodesic rays orthogonal to λ_Δ which limit to a common ideal vertex. We can extend this geodesic foliation to a singular foliation μ_Δ in Δ such that

- μ has exactly one singular leaf t, and it is a tripod connecting the center of Δ to the vertices of Δ by geodesic rays, and
- each component of $\Delta \setminus t$ is foliated by one parameter family of lines connecting a pair of adjacent vertices of Δ, and those leaves smoothly converge to the geodesic edge of Δ connecting the vertices (Figure 1, middle).

We call this singular foliation μ_Δ of Δ a nearly-striagith foliation of Δ.

The complement $\sigma \setminus L$ consists of N ideal triangles. Thus, the mostly straight foliations μ_Δ on the ideal triangles Δ yield a nearly-straight (singular) foliation μ on σ w.r.t. L, where the singular points are the center points of the ideal triangles.

Note that the horocyclic lamination λ_Δ of an ideal triangle Δ has a singular point on each edge where two horocyclic arcs centered at different vertices meet tangentially. There is a singular foliation λ'_Δ on Δ such that

- λ_Δ coincides with λ'_Δ in a small neighborhood U of the complementary triangle $\Delta \setminus |\lambda_\Delta|$ with horocyclic edges;

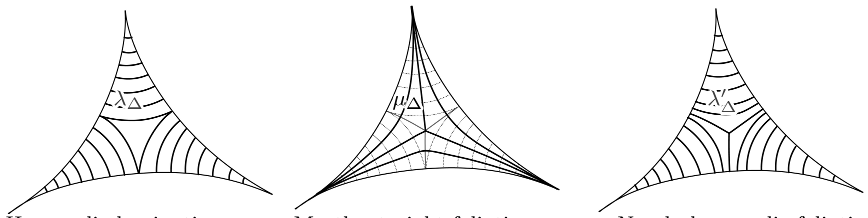

FIGURE 1. Horocylic lamination and nearly horocyclic foliation

- λ'_Δ has only one singular leaf t, and it is a tripod centered at the center of the complementary triangle;
- each connected component of $\Delta \setminus t$ is foliated by smooth parallel arcs which connect the same pair of edges of Δ and intersect the edge orthogonally (Figure 1, right).

We call this singular foliation λ'_Δ, a nearly-horocyclic foliation of the deal triangle Δ. Given a maximal geodesic lamination L on a hyperbolic surface σ, by taking a union of the nearly-horocyclic foliations on complementary ideal triangles, we obtain a nearly-horocyclic (singular) foliation λ on σ w.r.t. L. (c.f [Thu, §4].)

We pick a marking preserving collapsing map from $\kappa\colon (\sigma_\infty, L_\infty) \to (E_\infty, V_\infty)$ such that κ collapses, in each complementary triangle Δ of $(\sigma_\infty, L_\infty)$ to a "Y-shaped" graph with half-infinite edges (tripod) by collapsing each nearly-horocyclic leaves of λ'_Δ. Then $\kappa\colon \sigma_\infty \to E_\infty$ takes L_∞ to V_∞ and injective on each leaf of the maximal lamination L_∞.

Therefore, by each singular tripod leaf of V_∞ corresponds to a complementary ideal triangle of L_∞, we can construct a sequence of train track neighborhoods τ_i of L_∞ corresponding to the fat traintrack T_i, such that

- τ_j is ϵ_j-nearly straight and $\epsilon_j \searrow 0$ as $j \to \infty$;
- (T_j, V_∞) is isomorphic to (τ_j, λ_∞) by κ;
- horizontal (short) edges of branches of τ_j are horocyclic (Figure 2).

Lemma 4.2. *Let $\epsilon > 0$. Then, if $j \in \mathbb{Z}_{>0}$ is sufficiently large,*

$$(d-\epsilon)\mathrm{length}\alpha_j < \mathrm{length} a_j < (d+\epsilon)\mathrm{length}\alpha_j,$$

for all the vertical edges a_j and α_j of the branches of T_j and σ_j corresponding by the collapsing map κ.

Proof. Recall that V_∞ is uniquely ergodic and, the measured foliation V_∞ gives positive measures to arcs transversal to V_∞. Therefore, we

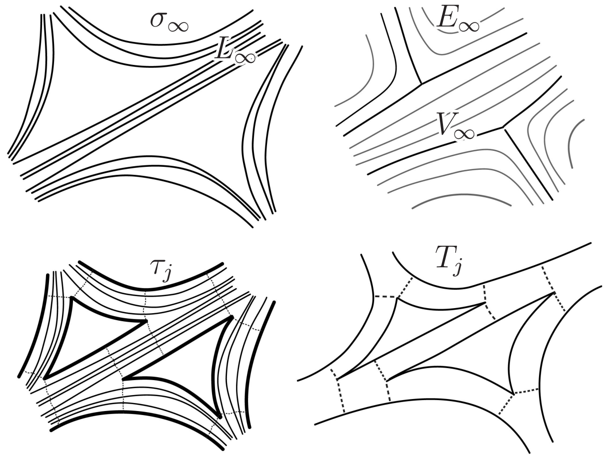

FIGURE 2. A nearly straight traintrack τ_j corresponding to a Euclidean fat traintrack T_j.

can pick sequences of weights $w_j > 0, \omega_j > 0$, such that the sequence of weighted arcs (a_j, w_j) converges to V_∞ and similarly the sequence (α_j, ω_j) converges to L_∞ as $j \to \infty$ in weak* topology.

Note that the collapsing map $\kappa\colon (\sigma_\infty, L_\infty) \to (E_\infty, V_\infty)$ isomorphically takes V_∞ to L_∞ and the vertical edge a_j to the vertical edge α_j for all $j = 1, 2, \ldots$. Therefore we can assume that $w_j = \omega_j$ for each $j = 1, 2, \ldots$.

Therefore

$$w_j \mathrm{length}(a_j) \to \mathrm{length}_{E_\infty}(V_\infty) = 1$$

as $j \to \infty$.

$$w_j \mathrm{length}(\alpha_j) \to \mathrm{length}_{\sigma_\infty}(L_\infty),$$

as $j \to \infty$.

Since

$$\frac{\mathrm{length}_{E_\infty}(V_\infty)}{\mathrm{length}_{\sigma_\infty}(L_\infty)} = d,$$

the convergences of the weighted lengths above implies

$$\frac{\mathrm{length}(a_j)}{\mathrm{length}(\alpha_j)} \to d$$

as $j \to \infty$. □

4.2. **Stretching a traintrack along a Teichmüller ray and a grafting ray.** Recall that the Teichmüller ray $X_\infty\colon [0, \infty) \to \mathcal{T}$ from X_∞ has the vertical foliation V_∞ and the horizontal foliation H_∞, and the flat surface E_∞ conformally realizes X_∞ and geometrically realizes V_∞ and H_∞.

Let $E_\infty(s)$ be the marked flat structure on S corresponding to $X_\infty(s)$ obtained by stretching E_∞ only in the horizontal direction by $\exp(s)$; then $E_\infty(s)$ realizes the vertical measured foliation $\exp(s)V_\infty$, keeping the horizontal foliation H_∞. Let $f_{\infty,s}\colon E_\infty = E_\infty(0) \to E_\infty(s)$ denote this piecewise linear stretching map in the horizontal direction. By $f_{\infty,s}$ the traintrack trauctures T_j of $E_\infty \setminus (\gamma_1(j) \cup \cdots \cup \gamma_N(j))$ descends to a traintrack structure $T_j(s)$ of $E_\infty(s) \setminus f_{\infty,s}(\gamma_1(j) \cup \cdots \cup \gamma_N(j))$.

Next we consider a corresponding grafting ray starting from the hyperbolic surface σ_∞ representing X_∞ along the geodesic representative L_∞ of V_∞. For $s > 0$, let $\mathrm{Gr}^s_{L_\infty}\sigma_\infty$ denote the projective structure on S obtained by grafting the hyperbolic surface σ_∞ along the (scaled) measured lamination sL_∞. Since L_∞ has no periodic leaves, we let $g_s\colon \sigma_\infty \to \mathrm{Gr}^s_{L_\infty}\sigma_\infty$ be the canonical grafting C^1-diffeomorphism. Then sL_∞ is geometrically realized as a circular lamination on the proejctive surface $\mathrm{Gr}^s_{L_\infty}\sigma_\infty$. Namely, the grafting map g_s takes the geodesic lamination sL_∞ to the geometric realization. (See [KT92, Bab20] for Thurston's parametrization of $\mathbb{C}\mathrm{P}^1$-structures.) Then, by $g_t\colon \sigma_\infty \to \mathrm{Gr}^s_{L_\infty}$, the nearly straight traintrack structure τ_j on σ_∞ descends to a traintrack neighborhood $\tau_j(s)$ of the circular lamination sL_∞.

For a fat traintrack, we call, by the one-skeleton, the union of the horizontal and vertical edges of the rectangular branches.

Corollary 4.3. *For every $\epsilon > 0$, there are $J_\epsilon > 0$ and $s_\epsilon > 0$ such that, if $j > J_\epsilon$ and $s > s_\epsilon$, then there is a $(d-\epsilon, d+\epsilon)$-bilipschitz "linear" isomorphism between the one-skeletons*

$$\phi^s_j\colon \tau^1_j(\exp(s)/d) \to T^1_j(s)$$

for sufficiently large $s > 0$, such that

- *ϕ^s_j is linear on each edge with respect to arc length, and*
- *ϕ^s_j extends to a marking preserving homeomorphism*

$$\mathrm{Gr}^{\exp(s)/d}_{L_\infty}\sigma_\infty \to E_\infty(s),$$

with respect to the Thurston metric on $\mathrm{Gr}^{\exp(s)/d}_{L_\infty}$ and the Euclidean metric on $E_\infty(s)$.

Proof. We first consider the bilipschitz property in the vertical direction. By Lemma 4.2, if j is sufficiently large, corresponding vertical edges of T_j and σ_j are $(d-\epsilon, d+\epsilon)$-bilipschiz. Then the grafting map $\sigma_\infty \to \mathrm{Gr}^{\exp(s)/d}_{L_\infty}\sigma_\infty$ by $\exp(s)L$ perserves the vertical length of branches of τ_j, and the horizontal stretch map $f_{\infty,s}\colon E_\infty \to E_\infty(s)$

preserves the vertical length of branches of T_j. Therefore, we can take $\phi_j^s\colon \tau_j^1(\exp(s)/d) \to T_j^1(s)$ that is $(d+\epsilon, d-\epsilon)$-bilipschitz in the vertical direction.

We next consdier the bilipschitz property in the horizontal direction. Let e_j and e'_j be corresponding horizontal edges of $T_j(s)$ and $\tau_j(d\exp(s))$, respectively. Since τ_j is ϵ_j-nearly straight traintrack with $\epsilon_j \searrow 0$ as $j \to \infty$, for every $\epsilon > 0$, there is $J_\epsilon > 0$ such that, if $j > J_\epsilon$, every horocyclic edges of branches of τ_j has length less than ϵ.

Let $e_j(s), e'_j(s)$ be the corresponding horiozntal edges of $T_j(s)$ and $\sigma_j(\exp s/d)$, respectively. Then

$$\mathrm{length}_{E_\infty(s)} e_j(s) = \exp(s) V_\infty(e_j)$$

and

$$\mathrm{length}_{\sigma_\infty^s} e'_j(s) = \exp(s) L_i(e'_j)/d + \mathrm{length}_{\sigma_\infty} e'_j,$$

where $\mathrm{length}_{\sigma_\infty^s}$ denote the length with respect to Thurston's metric on the projective surface $\mathrm{Gr}_{L_\infty}^{\exp s/d}\sigma_\infty$.

Therefore, for every $\epsilon > 0$, there is $J_\epsilon > 0$, such that, if $j > J_\epsilon$, then since $V_\infty(e_j) = L_\infty(e'_j)$ and $\mathrm{length}_{\sigma_\infty} e'_j < c$, we have

$$\left| \frac{\mathrm{length}_{E_\infty^s} e_j(s)}{\mathrm{length}_{\sigma_\infty^s} e'_j(s)} - d \right| < \epsilon,$$

for sufficiently large $s > 0$. Therefore, we can make ϕ_j^s a $(d-\epsilon, d+\epsilon)$-bilipschitz mapping also on the horizontal edges. □

4.3. **Construction of almost conformal mappings.** Recall that $X_\infty(s)$ is the Tecihmüller geodesic ray from X_∞ with the vertical measured foliation V_∞ parametrized by $s \geq 0$. Let $\mathrm{gr}_L^s(\sigma_\infty)$ be the conformal grafting ray from the hyperbolic surface σ_∞ along the measured geodesic lamination L_∞, where σ_∞ uniformizes X_∞ and L_∞ corepords to V_∞.

We prove the asymptotic property of those rays as parametrized rays without modifying their base points.

Theorem 4.4. *For every $\epsilon > 0$, there is $s_\epsilon > 0$ such that*

$$d(X_\infty(s), \mathrm{gr}_V^{\exp(s)/d}\sigma_\infty) < \epsilon.$$

We first construct a decomposition of E_∞ into rectangles and hexagons from the traintrack structure T_j of $E_\infty \setminus (\gamma_1(j) \cup \cdots \cup \gamma_N(j))$.

Given $\epsilon > 0$ and a subset A of a flat surface E with a horizontal foliation H, the ϵ-horizontal neighborhood of A is the subset of E consisting of points p can be connected to A by a segment of a leaf of H with length at most ϵ.

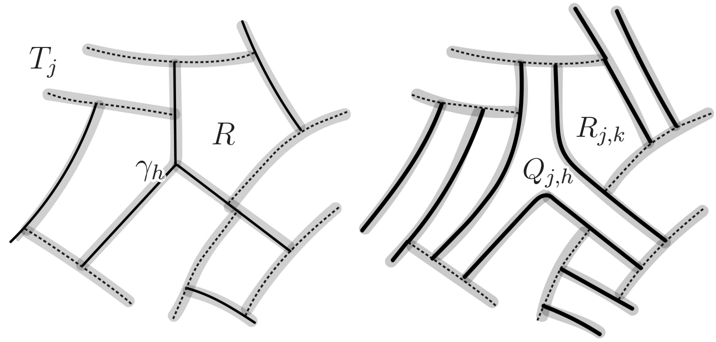

FIGURE 3. Traintrack structure T_j induces a polygonal traintrack decomposition $E_{\infty,j} = (\cup_{h=1}^{N} Q_{j,h}) \cup (\cup_{k=1}^{N'} R_{j,k})$ (right).

Let m_j be the shortest horizontal length of the rectangular branches of T_j. For each $h = 1, \dots, n$, let $Q_{j,h}$ be the horizontal $m_j/3$-neighborhood of the Y-shaped graph γ_h. Then $Q_{j,h}$ is a hexagon with horizontal and vertical edges and one singular point, and its horizontal edges have length $2m_j/3$; see Figure 3. Therefore, the definition of m_j implies that the hexagons $Q_{j,1}, Q_{j,2}, \dots, Q_{j,N}$ are pairwise disjoint. For each branch R of T_j, the Euclidean rectangle R minus the $m_j/3$-horizontal neighborhood of the vertical edges is still a rectangle. Therefore, the traintrack structure T_j of $E_\infty \setminus (\gamma_1(j) \cup \dots \cup \gamma_N(j))$ gives a rectangle decomposition of $E_\infty \setminus (Q_{j,1} \cup \dots \cup Q_{j,N}) = \cup_{k=1}^{N'} R_{j,k}$, and so that each rectangle piece $R_{j,k}$ is a branched of T_j minus the $m_j/3$-horizontal-neighborhood of its vertical edges. Thus we have a decomposition of the flat surface E_∞ into hexagons and rectangles,

$$E_{\infty,j} = (\cup_{h=1}^{N} Q_{j,h}) \cup (\cup_{k=1}^{N'} R_{j,k}).$$

By the horizontal stretch map $f_{\infty,s}\colon E_\infty \to E_\infty(s)$, this polygonal decomposition $E_{\infty,j}$ induces a corresponding polygonal decomposition of $E_\infty(s)$

$$E_{\infty,j}(s) = (\cup_{k=1}^{N} Q_{j,h}^s) \cup (\cup_{h=1}^{N'} R_{j,k}^s),$$

where $f_{\infty,s}(Q_{j,h}) = Q_{j,h}^s$ and $f_{\infty,s}(R_{j,k}) = R_{j,k}^s$.

Recall that we constructed an ϵ_j-nearly straight traintrack neighborhood τ_j of L_∞ on σ_∞, where $\epsilon_j \searrow 0$ as $j \to \infty$. This decomposition τ_j is induced by the traintrack decomposition T_j of E_∞ so that τ_j descends to T_j by the collapsing map $\kappa\colon (\sigma_\infty, L_\infty) \to (E_\infty, V_\infty)$.

Similarly, for each $j = 1, 2, \dots$, the polygonal decomposition $E_{\infty,j} = (\cup_{h=1}^{N} Q_{j,h}) \cup (\cup_{k=1}^{N'} R_{j,k})$ induces a polygonal decomposition $\sigma_{\infty,j} = (\cup_{h=1}^{N} \mathcal{Q}_{j,h}) \cup (\cup_{k=1}^{N'} \mathcal{R}_{j,k})$ such that

- $\sigma_{\infty,j} = (\cup_{k=1}^{N} \mathcal{Q}_{j,h}) \cup (\cup_{h=1}^{N'} \mathcal{R}_{j,k})$ is isomorphic to $E_{\infty,j} = (\cup_{h=1}^{N} Q_{j,h}) \cup (\cup_{k=1}^{N'} R_{j,k})$ as polygonal traintrack carrying $L_\infty \cong V_\infty$;
- this isomorphism is realized by the collapsing map κ;
- the vertical edges of the $\mathcal{Q}_{j,h}$ and $\mathcal{R}_{j,k}$ are segments of leaves of L_∞, and their horizontal edges are segments of the horocyclic foliation λ_∞.

Recall that the traintrack neighborhood τ_j of L_∞ on σ_∞ is transformed into a traintrack neighborhood $\tau_j(s)$ of the Thurston lamination sL_∞ on $\mathrm{Gr}^s_{L_\infty}\sigma_\infty$. Similarly, the polygonal decomposition $\sigma_{\infty,j}$ induces a decomposition $\sigma_{\infty,j}(s)$ after grafting:

$$\mathrm{Gr}^{\exp(s)/d}_{L_\infty}\sigma_\infty = (\cup_h \mathcal{Q}^s_{j,h}) \cup (\cup_k \mathcal{R}^s_{j,k}),$$

where $\mathcal{Q}^s_{j,h}$ is obtained by grafting $\mathcal{Q}_{j,h}$ along the restriction of $\frac{\exp s}{d} L_\infty$ to $\mathcal{Q}_{j,h}$ and $\mathcal{R}^s_{j,k}$ is obtained by grafting $\mathcal{R}_{j,k}$ along the restriction of $\frac{\exp s}{d} L_\infty$ to $\mathcal{R}_{j,k}$.

Proposition 4.5. *For every $\epsilon > 0$, there are $J_\epsilon > 0$ and $s_\epsilon > 0$ such that, if $j > J_\epsilon$ and $s > s_\epsilon$, there is a $(d-\epsilon, d+\epsilon)$-bilipschitz map between the one-skeletons of the polygonal decompositions*

$$\phi^s_j \colon (\cup_h \partial\mathcal{Q}^s_{j,h}) \cup (\cup_k \partial\mathcal{R}^s_{j,k}) \to (\cup_h \partial Q^s_{j,h}) \cup (\cup_k \partial R^s_{j,k})$$

for sufficiently large $s > 0$.

Proof. The proof is similar to that of Corollary 4.3. □

Our main of this section is to prove the following.

Proposition 4.6. *For every $\epsilon > 0$, there are $J_s > 0$ and $s_\epsilon > 0$ such that, if $j > J_s$ and $s > s_\epsilon$, then we can extend the above bilipschitz mapping between the one-skeletons*

$$\phi^s_j \colon (\cup_{h=1}^{N} \partial\mathcal{Q}^s_{j,h}) \cup (\cup_{k=1}^{N'} \partial\mathcal{R}^s_{j,k}) \to (\cup_{h=1}^{N} \partial Q^s_{j,h}) \cup (\cup_{k=1}^{N'} \partial R^s_{j,k})$$

to a $(1+\epsilon)$-quasi-conformal mapping

$$\Phi^s_j \colon \mathrm{Gr}^{\exp(s)/d}_{L_\infty}\sigma_\infty \to E_\infty(s)$$

taking the polygonal decomposition $\tau_j(s)$ to the polygonal decomposition $E_{\infty,j}(s)$.

In order to prove Proposition 4.6, we construct a desired extension on each polygonal piece in the following subsections.

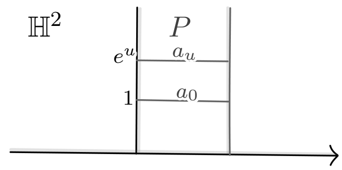

FIGURE 4. Horocylic arcs in the region P.

4.3.1. *Rectangles.* In this subsection, we extend ϕ_j^s to a quasiconformal mapping $\mathcal{R}_{j,k}^s \to R_{j,k}^s$ with small distortion for each rectangular branch $\mathcal{R}_{j,k}^s$.

Lemma 4.7. *Consider the region P in $\mathbb{H}^2$ bounded by two disjoint geodesics sharing an endpoint in the ideal boundary $\partial\mathbb{H}^2$. Then P is foliated by a one-parameter family of horocyclic arcs $\{a_u\}$ centered at the common endpoint. We can parametrize the horocyclic arc $a_u (u \in \mathbb{R})$ by between their distances, so that it corresponds to the length between the arcs. Then*

$$\frac{d}{du}\mathrm{length}_{\mathbb{H}^2}(a_u) = -\mathrm{length}_{\mathbb{H}^2} a_u$$

Proof. We first normalize the region P in the upper half plane model of $\mathbb{H}$ so that the common endpoint is at ∞. It suffices to show the derivative formula at $u = 0$, and we can further normalize the region P so that a_0 is the horizontal arc at height one; see Figure 4. Since a_u is parametrized by the vertical (hyperbolic) distance, we have

$$\frac{d}{du}\left(\frac{\epsilon}{e^u}\right) = -\epsilon e^{-u}.$$

Thus

$$\frac{d}{du}\left(\frac{\epsilon}{e^u}\right)\Big|_{u=0} = -\epsilon.$$

□

Pick a rectangular piece $\mathcal{R}_{j,k}^s$ of the polygonal decomosition $\mathrm{Gr}_{L_\infty}^{\exp s/d}\sigma_\infty = (\cup_{h=1}^N \partial \mathcal{Q}_{j,h}^s) \cup (\cup_{k=1}^{N'} \partial \mathcal{R}_{j,k}^s)$. Then $\mathcal{R}_{j,k}^s$ is folizated by the leaf segments of horocyclic foliation λ_∞ of $(\sigma_\infty, \lambda_\infty)$. Therefore, the vertical edges of $\mathcal{R}_{j,k}^s$ are geodesic segments of the same length; let $\ell (= \ell_{j,k}^s)$ demote this vertical length of $\mathcal{R}_{j,k}^s$.

Consider the branch $\mathcal{R}_{j,k}$ of the polygonal decomposition of σ_∞ which, after grafting, corresponds to $\mathcal{R}_{j,k}^s$. Then $\mathcal{R}_{j,k}$ is foliated by the horocyclic segments of the (horizontal) horocyclic lamination λ_∞, since the non-foliated parts are contained in hexagonal branches. Let $\lambda_{j,k}^s$ denote this horocyclic foliation of $\mathcal{R}_{j,k}$

The measured geodesic lamination L_∞ is orthogonal to the horocyclic lamination λ_∞. Then, the restriction of L_∞ to $\mathcal{R}_{j,k}$ extends to a (vertical) geodesic foliation $\mu = \mu_{j,k}$ in $\mathcal{R}_{j,k}$ orthogonal to the horocyclic foliation. Note that the lengths of the leaves of the foliation $\mu_{j,k}$ are the same, since there are isometries between the leaves given by the translation along the homocyclic foliation $\mu_{j,k}$.

As $\mathcal{R}^s_{j,k}$ is obtained by grafrting $\mathcal{R}_{j,k}$ along sL_∞, the horocyclic foliation $\lambda_{j,k}$ induces a horocyclic foliation $\lambda^s_{j,k}$ on $\mathcal{R}^s_{j,k}$, so that the collapsing map $\kappa_s\colon \mathrm{Gr}^s_{L_\infty}\sigma_\infty \to \sigma_\infty$ takes leaves of $\lambda^s_{j,k}$ to leaves of $\lambda_{j,k}$. Similarly, the vertical geodesic foliation $\mu_{j,k}$ induces the vertical geodesic folalition $\mu^s_{j,k}$ on $\mathcal{R}^s_{j,k}$, so taht κ_s takes $\mu^s_{j,k}$ to $\mu_{j,k}$.

Lemma 4.8. *For every $\epsilon > 0$, there is $J_\epsilon > 0$, such that, if $j > J_\epsilon$, then, for every sufficiently large $s > 0$, every rectangular branch $\mathcal{R}^s_{j,k}$ of the polygonal decomosition $\mathrm{Gr}^{\exp s/d}_{L_\infty}\sigma_\infty = (\cup_{h=1}^N \partial \mathcal{Q}^s_{j,h}) \cup (\cup_{k=1}^{N'} \partial \mathcal{R}^s_{j,k})$ is $(1-\epsilon, 1+\epsilon)$-quasiconforally equivalent to a Euclidean rectangle of the same length $\ell = \ell^s_{j,k}$ and the width $\exp(s) L(\mathcal{R}^s_{j,k})$, where $L(\mathcal{R}^s_{j,k})$ denote the transversal measure of the horizontal edge of $\mathcal{R}^s_{j,k}$ given by L.*

Proof. Let $F = F^s_{j,k}$ be the Euclidean rectangle of length $\ell^s_{j,k}$ and width $\exp(s)L(\mathcal{R}^s_{j,k})$. We construct an almost conformal mapping $\zeta^s_{j,k}\colon \mathcal{R}^s_{j,k} \to F^s_{j,k}$ preserving horizontal leaves.

Pick an horizontal (horocyclic) edge e_h of $\mathcal{R}^s_{j,k}$, and a vertical (geodesic) edge e_v of $\mathcal{R}^s_{j,k}$. Let z be a point on $\mathcal{R}^s_{j,k}$. Then z is contained in a leaf u_z of the horizontal horocyclic foliation $\lambda^s_{j,k}$, and a leaf w_z of the vertical geodesic foliation $\mu^s_{j,k}$. Let y be the length of the geodesic segment of w_z from z to e_h (along w). Let x be the length of the segment of u_z connecting z to a point in e_v. Then we define a mapping $\zeta^s_{j,k}\colon \mathcal{R}^s_{j,k} \to F^s_{j,k}$ by

$$z = (x, y) \mapsto (L(\mathcal{R}^s_{j,k}) \frac{y}{\text{length } u_z}, y),$$

so that it is linear along u_z with respect to arc length (Figure 5).

Next we show that $\zeta^s_{j,k}$ is almost conformal mapping for sufficiently large $j, s > 0$.

Each horocyclic leaf u in $R_{j,k} = R^0_{j,k}$ intersects the measured lamination L_∞ in a measure zero set. As L_∞ is a maximal lamination, we set

$$u \setminus L_\infty = \cup_{r=1}^\infty u_r,$$

where u_r are the connected segments of $u \setminus L_\infty$. Since $L \cap u$ has measure zero in u,

$$\text{length}_{\sigma_\infty} u = \Sigma_{r=1}^\infty \text{length}_{\sigma_\infty} u_r.$$

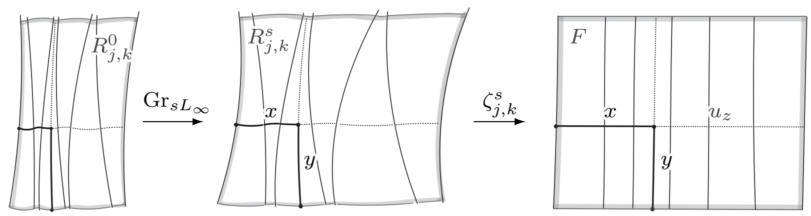

FIGURE 5. Mapping hyperbolic rectangle into a Euclidean rectangle

We parametrize the horocyclic leaves h_x of $\lambda^0_{j,k}$ with $x \in [0, \ell^s_{j,k}]$ by the length from the horizontal ledge e_h along vertical leaves of $\mu^s_{j,k}$. For every $\epsilon > 0$, there is $J_\epsilon > 0$, such that if $j > J_\epsilon$, then length $h < \epsilon$ for all horocyclic leaves u of $\lambda^s_{j,k}$. Then, by Lemma 4.7,

$$\left|\frac{d(\text{length}\, u_x)}{dx}\right| \leq \frac{d}{dx}(\sum_{r=1}^{\infty} \text{length}_{\mathbb{H}^2} u_r) = \sum_{r=1}^{\infty} \frac{d}{dx}(\text{length}_{\mathbb{H}^2} u_r) < \epsilon.$$

The grafting of σ_∞ along the measured lamination L_∞ inserts Euclidean structure along L_∞, and the length of all horocyclic leaves of $R_{j,k}$ equally increases by the constant $\exp(s)L(\mathcal{R}^s_{j,k})$ w.r.t. Thurston's metric. Clearly $\exp(s)L(\mathcal{R}_{j,k}) \to \infty$ as $s \to \infty$. Therefore, for every $\epsilon > 0$, there are $J_\epsilon > 0$ and $s_\epsilon > 0$, such that if $j > J_\epsilon$ and $s > s_\epsilon$, then the horizontal derivative $\frac{d\zeta^s_{j,k}}{dx}(z)$ is the vector $(0, t)$ with $t \in (1-\epsilon, 1+\epsilon)$ for all $z \in R^s_{j,k}$.

Since $\zeta^s_{j,k}$ preserves the height by its definition, a similar argument shows that there are $J_\epsilon > 0$ and s_ϵ, such that if $j > J_\epsilon$ and $s > s_\epsilon$, then $\frac{d\zeta}{dx}(z)$ is $(1, t)$ with $t \in (1-\epsilon, 1+\epsilon)$ for all $z \in \mathcal{R}^s_{j,k}$.

We have shown that $d\zeta^s_{j,k}$ almost preserves the orthogonal frames. Therefore, for every $\epsilon > 0$, there are $J_\epsilon > 0$ and $s_\epsilon > 0$ such that if $j > J_\epsilon$ and $s > s_\epsilon$ such that $\zeta^s_{j,k}$ is $(1+\epsilon)$-equasiconformal. □

Corollary 4.9. *For every $\epsilon > 0$, there are $J_\epsilon > 0$ and $s_\epsilon > 0$, such that if $j > J_\epsilon$ and $s > s_\epsilon$, then the edge-wise linear map ϕ^s_j on the one-skeleton extends continuously to a $(1+\epsilon)$-quasi-coformal mapping from $R^s_{j,s}$ to $\mathcal{R}^s_{j,k}$.*

Proof. Let $\xi^s_{j,k}\colon F^s_{j,k} \to R^s_{j,k}$ be the linear mapping between Euclidean rectangles which preserves horizontal and vertical edges. Then, by Corollary 4.3 and the definition of $F^s_{j,k}$, for every $\epsilon > 0$ there is $J_\epsilon > 0$ such that, if $j > J_\epsilon$, implies that the linear mapping $\xi^s_{j,k}\colon F^s_{j,k} \to R^s_{j,k}$ is a $(d-\epsilon, d+\epsilon)$-bilipschitz for sufficiently large $s > 0$. Therefore, we

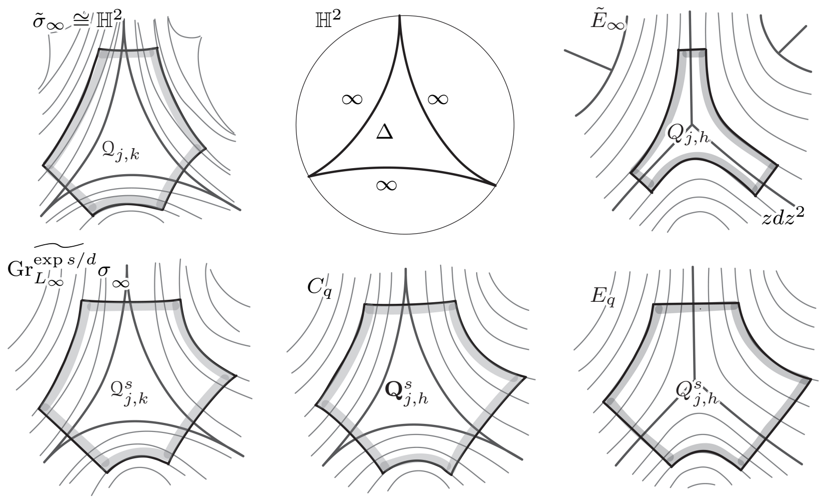

FIGURE 6.

can in addition assume that the composition $\xi^s_{j,k} \circ \zeta^s_{j,k} \colon \mathcal{R}^s_{j,k} \to R^s_{j,k}$ is a $(1+\epsilon)$-quasiconformal mapping.

The restriction of ϕ^s_j to $\partial\mathcal{R}^s_{j,k}$ is a piecewise-linear mapping which is linear on the vertical edges but not necessarily linear on the horizontal edges of $\mathcal{R}^s_{j,k}$. Since the fat traintracks correspond to trivalent graphs, a horizontal edge of $\mathcal{R}^s_{j,k}$ may be decomposed into three linear pieces for ϕ^s_j. For every $r > 0$, there are $J_r > 0$ and $s_r > 0$, such that, then if $j > J_r$ and $s > s_r$, then the vertical edge of $R^s_{j,k}$ has length at least r, and each linear segment of each horizontal edge also has length at least r. Therefore, we can easily adjust $\xi_{j,k}$ near the boundary of $F^s_{j,k}$ by a quasi-conformal mapping with small dilatation, so that the composition $\xi^s_{j,k} \circ \zeta^s_{j,k}$ is still a $(1+\epsilon)$-quasi-conformal mapping and its restriction to $\partial\mathcal{R}^s_{j,k}$ matches with ϕ^s_j. □

4.4. **Extension to hexagonal branches.** In this subsection, we construct a quasi-conformal extension of ϕ^s_j with small distorsion to each hexagonal branch $\mathcal{Q}^s_{j,h}$. First we construct a model projective structure on a hexagon interpolating between a hyperbolic hexagonal branch $\mathcal{Q}^s_{j,h}$ and its corresponding flat hexagonal branch $Q^s_{j,h}$.

Let q be the quadratic differential zdz^2 on $\mathbb{C}$. Consider the singular Euclidean metric E_q on $\mathbb{C}$ given by q. Let V_q denote the vertical measured foliation on $\mathbb{C}$ given by q. Then $\mathbb{C}$ is a union of three Euclidean half-planes with a common boundary point at 0. The vertical singular foliation V_q has a Y-shaped graph as a singular leaf.

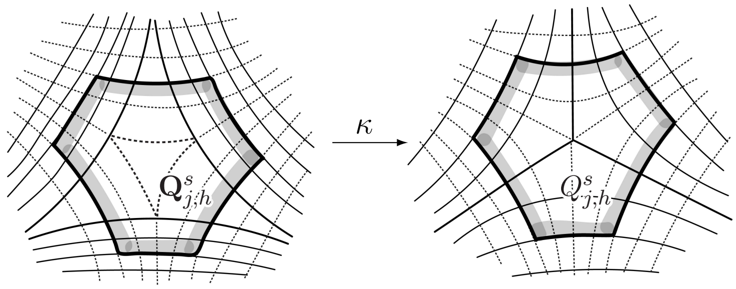

FIGURE 7. Constructing a model hexagon.

Let C_q be the $\mathbb{C}\mathrm{P}^1$-structure on $\mathbb{C}$ given by the quadratic differential q. Then Thurston's parameters of C_q are the ideal triangle Δ in $\mathbb{H}^3$ and the measured lamination L_q consisting of the boundary geodesics of Δ with weight ∞. Let $\mathcal{L}_q$ be the corresponding Thurston's lamination on $\mathbb{C}$; Then, with respect to Thurston's metric, the complementary region of $\mathcal{L}_q$ is an ideal triangle Δ', and the foliated region $|\mathcal{L}_q|$ consists of three Euclidean half-planes.

Let $\lambda_{\Delta'}$ be the horocyclic measured lamination of the ideal triangle Δ'. Then there is a collapsing map of Δ' to a Y-shaped metric graph with infinite ends, which collapses each leaf of $\lambda_{\Delta'}$ to a point and the complementary triangle to a point. Then, the collapsing map collapses each horocyclic leaf to a point and the complementary triangle to the vertex of the Y-shaped graph.

Let $C_q \to (\mathbb{C}, \frac{z}{(\sqrt{2}d)^2} dz^2)$ be the mapping which, by the collapsing map, takes the ideal triangle Δ' to the Y-shaped singular vertical leaf, such that C_q is isometric on each half plane of $C_q \setminus \Delta'$. Let $\kappa\colon C \to E_q = (\mathbb{C}, z dz^2)$ be the composition of this collapsing map with the scaling map $z \mapsto (\sqrt{2}d)z$ by $\sqrt{2}d$.

Recall that $Q^s_{j,h}$ is a hexagonal branch of the polygonal traintrack decomposition $E_{\infty,j}(s)$ of $E_\infty(s)$ associated with the fat traintrack structure $T_{\infty,j}(s)$. Then $Q^s_{j,h}$ is isometrically embedded in the singular Euclidean surface of $(\mathbb{C}, q)$, so that the horizontal foliation of $Q^s_{j,h}$ maps to the vertical foliation of $(\mathbb{C}, q)$ and the vertical foliation of $Q^s_{j,h}$ maps to the horizontal foliation of $(\mathbb{C}, q)$. By this embeding, let $\mathbf{Q}^s_{j,h}$ be $\kappa^{-1}(Q^s_{j,h})$ as in Figure 7.

We will construct a desired almost conformal mapping from the Euclidean hexagon $Q^s_{j,h}$ to the hyperbolic hexagon $\mathfrak{Q}^s_{j,h}$ through this model Euclidean hexagon $\mathbf{Q}^s_{j,h}$.

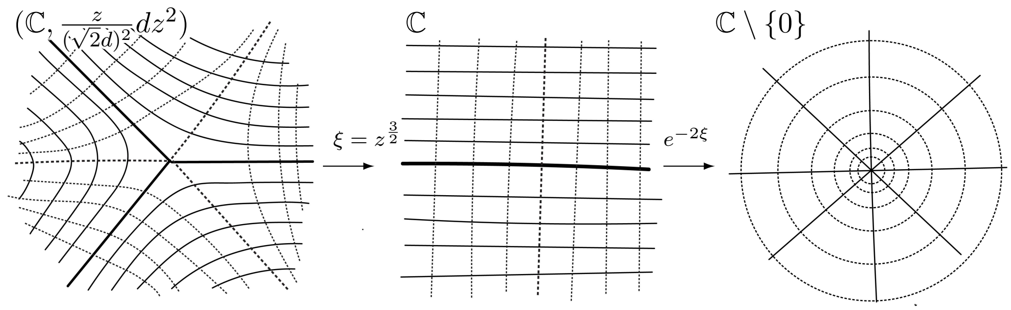

FIGURE 8. The model mapping $\exp[\sqrt{2}z^{\frac{3}{2}}]$.

4.4.1. *Almost conformal ideantification of the Euclidean hexagon $Q^s_{j,h}$ and the model projective hexagon $\mathbf{Q}^s_{j,h}$.* Let $f_q\colon \mathbb{C} \to \mathbb{CP}^1$ denote the developing map of the $\mathbb{CP}^1$-structure given by $(\mathbb{C}, \frac{z}{(\sqrt{2}d)^2}dz^2)$.

Theorem 4.10 (Corollary 4.1 in [GM21])**.** *In every anti-stokes sector, for every $m \geq 0$,*

$$(f_q(z) - \exp[\sqrt{2}z^{\frac{3}{2}}])z^m \to 0 \tag{2}$$

as $|z| \to \infty$. (Figure 8.)

Let $Z^s_{j,k}$ be the set of the boundary points of $Q^s_{j,h}$ which are vertices of the polygonal decomposition $E_\infty(s) = (\cup \partial Q^s_{j,h}) \cup (\cup \partial R^s_{j,k})$. By the construction of the polygonal decomposition, $Z^s_{j,k}$ is contained in the vertical edges of $Q^s_{j,h}$. Note that $\kappa | \partial Q^s_{j,h}$ is not a homeomorphism onto $\partial \mathbf{Q}^s_{j,k}$ as κ collapses many horizontal segments $\partial Q^s_{j,h}$, but homotopic to a homeomorphism. The restriction is a linear diffeomorphism on each vertical edge, and κ takes the measured lamination $L_\infty | Q^s_{j,h}$ to the restriction of $\mathcal{L}_q$ to $\mathbf{Q}^s_{j,k}$.

Let $\eta^s_{j,h}\colon \partial Q^s_{j,h} \to \partial \mathbf{Q}^s_{j,h}$ be the edge-wise linear homeomorphism, such that $\eta^s_{j,h}$ coincides with κ at the six verticies of $Q^s_{j,h}$. Then $\eta_{j,h}$ coincides with κ on the vertical edges.

Proposition 4.11. *For every $\epsilon > 0$, there are $J_\epsilon > 0$ and $s_\epsilon > 0$, such that, if $j > J_\epsilon$ and $s > s_\epsilon$, then there is a $(1+\epsilon)$-quasi-conformal mapping $Q^s_{j,h} \to \mathbf{Q}^s_{j,h}$ which conincides with the piecewise linear mapping $\eta^s_{j,h}$ on the boundary.*

The remaining of this subsection is the proof of Proposition 4.11. Let $\iota^s_{j,h}\colon Q^s_{j,h} \to (\mathbb{C}, q = \frac{z}{(d\sqrt{2})^2}dz^2) \cong C_q$ be the isometric embedding exchainging horizontal and vertical directions, with respect to the flat structure on C given by the differential. On the other hand, $\mathbf{Q}^s_{j,h}$ is already a subset of C_q. We show that $\iota^s_{j,h}(Q^s_{j,h})$ is in a bounded distance away from $\mathbf{Q}^s_{j,h}$ almost preserving the tangent directions.

Let $\phi^s_{j,k}\colon \partial \mathcal{Q}^s_{j,h} \to \partial \mathbf{Q}^s_{j,h}$ be the canonical homeomorphism taking vertices to corresponding vertices such that $\phi^s_{j,k}$ is edgewise linear with respect to arc length.

Theorem 4.10 immediately implies the Hausdorff distance between $\mathbf{Q}^s_{j,h}$ and $\iota^s_{j,h} Q^s_{j,h}$ are uniformly bounded.

Lemma 4.12. *There are constants $b > 0, s_\epsilon > 0, J_\epsilon > 0$, such that, if $j > J_\epsilon$ and $s > s_\epsilon$, then, for each $x \in \partial Q^s_{j,h}$,*

$$d_{Th}(\iota^s_{j,h}(x), \eta^s_{j,k}(x)) < b,$$

in the Thurston metric d_{Th} on C_q.

We now show the closeness of the tangent directions on the hexagon boundary.

Proposition 4.13. *For every $\epsilon > 0$ and $r > 0$, there are $J_\epsilon > 0$ and $s_\epsilon > 0$, such that, if $j > J_\epsilon$ and $s > s_\epsilon$, then,*

1. *for each vertical edge e of $Q^s_{j,h}$, $\iota^s_{j,h}|e$ is ϵ-almost parallel to Thurston's lamination $\mathcal{L}_q$;*
2. *for each horizontal edge e of $Q^s_{j,h}$, $\iota^s_{j,h}|e$ is ϵ-almost parallel to the horocyclic lamination $\mathcal{H}_q$ orthogonal to Thurston's lamination $\mathcal{L}_q$;*
3. *the restriction of $\iota^s_{j,h}$ to the r-neighborhood of the boundary $\partial Q^s_{j,h}$ is $(\frac{1}{d} - \epsilon, \frac{1}{d} + \epsilon)$-bilipschitz embedding onto its image in C_q w.r.t the Thurston metric.*

Proof. During the poof, we identify $Q^s_{j,h}$ and its image in E_q under $\iota^s_{j,h}$. (1) Recall that the Thurston parameters of C_q are the ideal hyperbolic triangle Δ and the geodesic lamination L_∞ consisting of the boundary geodesics of Δ with weight infinity, and $\mathcal{L}_q$ be Thurston's circular lamination on C_q. Then, the complement of $\mathcal{L}_q$ in C_q is an ideal triangle Δ' corresponding to Δ, and $C_q \setminus \Delta'$ consists of three Euclidean half-planes foliated by leaves of $\mathcal{L}_q$. Let $\kappa_q\colon C_q \to \Delta$ be the collapsing map, which collapses each complementary half-plane to its corresponding boundary geodesic of Δ, taking leaves of $\mathcal{L}_q$ diffeomorphically to the boundary geodesic.

Let ℓ be a leaf of the vertical measured foliation V_q of $(\mathbb{C}, q)$ such that ℓ contains a vertical edge e of $Q^s_{j,h}$. Let m be the boundary geodesic of the ideal triangle Δ corresponding to ℓ. By Thurston's parametrization (Δ, L_q) of C_q, the induced bending map is simply an isometric embedding of the ideal triangle into a totally geodesic plane in $\mathbb{H}^3$. By this embedding, m is isometrically identified with a geodesic in $\mathbb{H}^3$. Then, the ideal boundary $\mathbb{CP}^1$ of $\mathbb{H}^3$ minus the endpoints of m is foliated by round circles bounding disjoint hyperbolic planes orthogonal to m; let $\mathcal{C}$ denote this foliation of $\mathbb{C}\mathrm{P}^1$ minus two points by those round circles.

The leaves of Thurston lamination $\mathcal{L}_q$ corresponding to m map to circular arcs connecting the endpoints of m (Figure 10); those circular arcs are orthogonal to $\mathcal{C}$. Therefore it suffices to show that each tangent vector v along a vertical edge e (in ℓ) maps to a tangent vector ϵ-almoost orthogonal to the circle foliation $\mathcal{C}$.

Let $\mathrm{Ep}_q \colon C_q \to \mathbb{H}^3$ be the Epstein surface of C_q. For every $\epsilon > 0$, there is $R > 0$ such that, if the disctance of ℓ from the singular point, the zero, is least R, then $\mathrm{Ep}_q \ell$ is a $(1-\epsilon, 1+\epsilon)$-bilipschitz embedding and ϵ-close to the geodesic m ([Dum17, Lemma 3.4].) Recall that $f_q \colon \mathbb{C} \to \mathbb{C}\mathrm{P}^1$ denote the developing map of C_q. By the property of the Epstein surface, $df_q v$ corresponds to $d\mathrm{Ep}_q v$ by the orthogonal projection to the Epstein surface Ep_q. Therefore, if $j > 0$ and $s > 0$ are sufficiently large, then $\mathrm{Ep}_q \ell$ is tangentially very close to the geodesic m, and thus $df_q v$ is ϵ-almost orthogonal to a leaf of $\mathcal{C}$ (Figure 10).

(2) Let e be a horizontal edge of $Q^s_{j,h}$. If $j > 0$ and $s > 0$ are sufficiently large, we can pick a rectangle R_e in E_q with horizontal and vertical edges, such that R_e is sufficiently far from the zero of E_q and the vertical edges of E_q are long. Let h_s be the horizontal foliation of R_e parametrized by $s \in [0,1]$. Let v_u be the vertical foliation of R_e parametrized. by $u \in [0,1]$; then v_0 and v_1 are its vertical edges. Let ℓ_0 and ℓ_1 be the vertical leaf of V_q containing the vertical edges v_0 and v_1. Similarly to (1), let m_0, m_1 be the boundary geodesics of the ideal triangle Δ corresponding to ℓ_0 and ℓ_1.

For every $\epsilon > 0$, if $j > 0$ and $s > 0$ are sufficienlty large, then Since R_e is far from the zero of q, $\mathrm{Ep}_q v_u$ are $(\sqrt{2} - \epsilon, \sqrt{2} + \epsilon)$-bilipschitz embedding into $\mathbb{H}^3$ and $\mathrm{Ep} s_u$ has length less than ϵ ([Eps], Lemma 2.6, Lemma 3.4 in [Dum17]). Therefore, we may, in addition, assume that the long almost-geodesic curves $\mathrm{Ep}_q v_u (u \in [0,1])$ are ϵ-close to each other. Similarly to (1), let $\mathcal{C}$ be the foliation of $\mathbb{C}\mathrm{P}^1$ minus endpoints of m_0 by round circles which bound hyperbolic planes orthogonal to the geodesic m_0. Then, for every $\epsilon > 0$, if $j, s > 0$ are sufficiently large, then $f_q v_u$ are ϵ-alomost orghotonal to $\mathcal{C}$ for all $u \in [0,1]$, since $\mathrm{Ep}_q v_u$ are very close to a segment α of the geodesic m_0. Therefore $f_q h_v$ are almost parallel to $\mathcal{C}$.

The horocyclic foliation $\mathcal{H}_q$ is orthogonal to the Thurston lamination $\mathcal{L}_q$ on C_q. Since R_e is sufficiently far away from the zero of q, the boundary m_0, m_1 of the ideal triangle Δ are close to each other near α. Therefore, h_v are almost orthogonal to L_q.

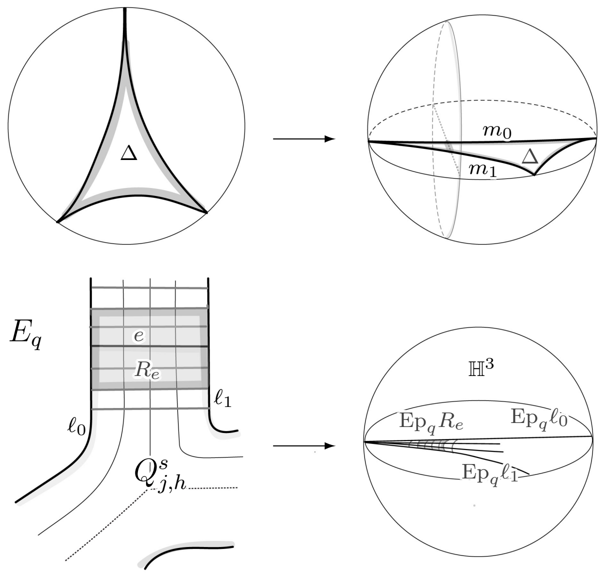

FIGURE 9.

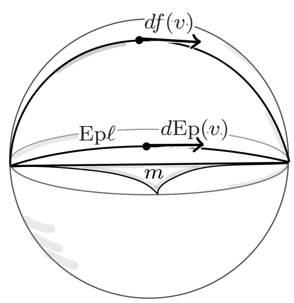

FIGURE 10.

(3) For every $R > 0$, if $j > 0$ and $s > 0$ are sufficeintly large, then the r-neighborhood of $\partial Q^s_{j,h}$ has a $\iota^s_{j,h}$-image in $(\mathbb{C}, q = \frac{z}{(d\sqrt{2})^2} dz^2)$ whose distance from 0 is at least R. Therefore, by [Bab25, Proposition 4.9], the developing map f_q on the r-neighborhood is well approximated by the exponential map. Hence, (1) and (2) imply the desired bilipschitz property.

□

Let $\iota_\phi\colon C_q \to E_q$ be the identification map given by the Schwarzian parametrization $C_q \cong (\mathbb{C}, q)$. We have seen that ι_ϕ embeds $Q^s_{j,h}$ into $\mathbb{C}$ so taht its image is bounded hausdorff distance from $\mathbf{Q}^s_{j,h}$ (Lemma 4.12) in a C^1-manner (Proposition 4.13).

$$\begin{array}{ccc} Q^s_{j,h} & \xleftarrow[a]{(d-\epsilon,d+\epsilon)} & \mathfrak{Q}^s_{j,h} \\ {\scriptstyle \frac{1}{d\sqrt{2}}-\epsilon,\frac{1}{d\sqrt{2}}+\epsilon}\Big\downarrow{\scriptstyle \iota^s_{j,h}} & & {\scriptstyle \phi^s_{j,k}}\Big\downarrow{\scriptstyle (1-\epsilon,1+\epsilon)} \\ (\mathbb{C}, zdz^2) & \xrightarrow{\sqrt{2}} & C_q \supset \mathbf{Q}^s_{j,h} \end{array}$$

Lemma 4.14. *For every $\epsilon > 0$, there are $J_\epsilon > 0$ and $s_\epsilon > 0$ such that, if $j > J_\epsilon$ and $s > s_\epsilon$, then we can modify $\iota_\phi | Q^s_{j,h}$, with respect to the (singular) Euclidean matric E_q of $(\mathbb{C}, q)$, by a $(1-\epsilon, 1+\epsilon)$-bilipschitz mapping so that*

1. *$Q^s_{j,h}$ is identified with $\mathbf{Q}^s_{j,h}$ by a $(1+\epsilon)$-quasiconformal mapping, and*
2. *the boundary of $Q^s_{j,h}$ is identified with the boundary of $\mathbf{Q}^s_{j,h}$ by the piecewise linear mapping $\eta^s_{j,h}$, with respect to arc length.*

Proof. We identify $Q^s_{j,h}$ with the image of $Q^s_{j,h}$ in E_q by $\iota^s_{j,h}$ Let H denote the (hexagonal) boundary of the hexagonal branch $Q^s_{j,h}$ in E_q. For $R > 0$, let N_R be the R-neighborhood of the hexagonal boundary H in $Q^s_{j,h}$ in the Euclidean metric E_q. Then N_R is topologically a cylinder. The outer boundary H is identified with the inner boundary of N_R by an edge-wise linear homeomorphism, which identifies a pair of parallel edges. Thus N_R has a natural product structure $H \times [0, 1]$ by linearly extending this identification of the hexagonal boundary components; for each $h \in H$, the segment $h \times [0, 1]$ is a line segment in N_R connecting a pair of identified points on the corresponding inner and outer edges. Let $b > 0$ be the Hausdorff distance bound in Lemma 4.12. Then, if $R > 2b$, then $\iota_\phi(\mathbf{Q}^s_{j,h})$ contains the inner boundary of N_R.

Let $\mathbf{N}_R$ be the region in E_q bounded by the boundary hexagon of $\iota_\phi(\mathbf{Q}^s_{j,h})$ and the inner boundary $H \times \{0\}$ of N_R (Figure 11). We shall define a natural product structure $H \times [0, 1]$ on $\mathbf{N}_R$, such that, via this product structure $\mathbf{N}_R = H \times [0, 1]$, the identification $\mathbf{N}_R = H \times [0, 1] = N_R$ agrees with the identity on the inner hexagonal boundary and $\eta^s_{j,h}$ on the outer hexagonal boundary.

If $j > 0$ and $s > 0$ are sufficiently large, The Hausdorff distance between the outer boundary hexagon of $Q^s_{j,h}$ and the boundary hexagon of $\iota_\phi \mathbf{Q}^s_{j,h}$ is less than a fixed constant b by Lemma 4.12, and then the corresponding edges are ϵ-almost parallel by Proposition 4.13 w.r.t the singular Euclidean metric E_q.

Therefore we let $f\colon N_R \to \mathbf{N}_R$ be a mapping such that

- the restriction of f to the outer boundary of N_R is equal to $\eta^s_{j,h}$;
- the restriction of f to the inner boundary of N_R is the identify map;

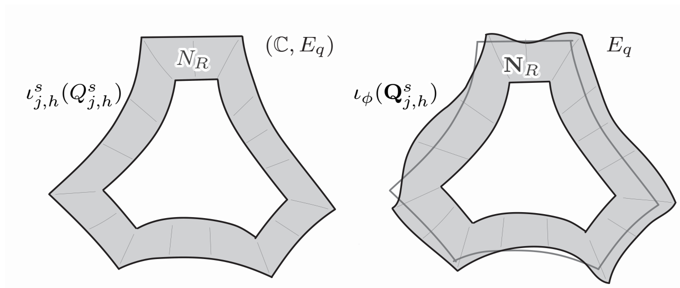

FIGURE 11. The product structures on hexagonal cylinders

- f is linear of each $\{h\} \times [0,1]$ for all $h \in H$.

The edges of the hexagons are long if $j > 0$ and $s > 0$ are large. Therefore Proposition 4.13 implies that following.

Claim 4.15. *For every $\epsilon > 0$, there are $J_\epsilon > 0$ and $s_\epsilon > 0$, such that if $j > J_\epsilon$ and $s > s_\epsilon$, then f is a $(1+\epsilon)$-quasi-conformal mapping.*

□

We complete the proof of Proposition 4.11.

4.4.2. *Hyperbolic Hexagons are almost conformal to model projective Hexagons.* For every $\epsilon > 0$, there are $J_\epsilon > 0$ and $s_\epsilon > 0$, such that, for every hexagonal branch, we constructed a piecewise linear $(d-\epsilon, d+\epsilon)$-bilipschitz mapping $\phi^s_{j,h} \colon \partial \mathcal{Q}^s_{j,h} \to \partial Q^s_{j,h}$ and a $(\frac{1}{d\sqrt{2}} - \epsilon, \frac{1}{d\sqrt{2}} + \epsilon)$-bilipschitz mapping $\eta^s_{j,h} \colon \partial Q^s_{j,h} \to \partial \mathbf{Q}^s_{j,h}$.

We shall construct an almost conformal mapping identifying $\mathcal{Q}^s_{j,h}$ and $\mathbf{Q}^s_{j,h}$ extending the picesie linear homeomorphism $\eta^s_{j,h} \circ \phi^s_{j,h} \colon \partial \mathcal{Q}^s_{j,h} \to Q^s_{j,h}$. Note that the singuar point of $\eta^s_{j,h} \circ \phi^s_{j,h}$ are the vertex point set Z of the polygonal decomposition of $\mathrm{Gr}^{\exp s/d}_{L_\infty} \sigma_\infty$. Those singular points are only on the vertical edges of $\partial \mathcal{Q}^s_{j,h}$ and the number is uniformly bounded from above by $2(2g-2)$, the number of the singular points on $E_\infty(s)$, where g is the genus of the surface.

Lemma 4.16. *For every $\epsilon > 0$, there are $J_\epsilon > 0$ and $s_\epsilon > 0$, such that $j > J_\epsilon$ and $s > s_\epsilon$, then there is a $(1+\epsilon)$-quasiconformal mapping $\xi^s_{j,h} \colon Q^s_{j,h} \to \mathbf{Q}^s_{j,h}$ preserving vertical and horizontal edges as $\eta^s_{j,h}$ does.*

Proof. In the hexagon $\mathbf{Q}^s_{j,h}$, the complement of the Thurston lamination $\mathcal{L}_q$, the hyperbolic hexagon $\mathbf{H}^s_{j,h}$ obtained by cutting by the ideal triangle along horocyclic arcs centered at the vertices. Then the complement of $\mathbf{H}^s_{j,h}$ in $\mathbf{Q}^s_{j,h}$ are three Euclidean rectangles in Thurston's metric.

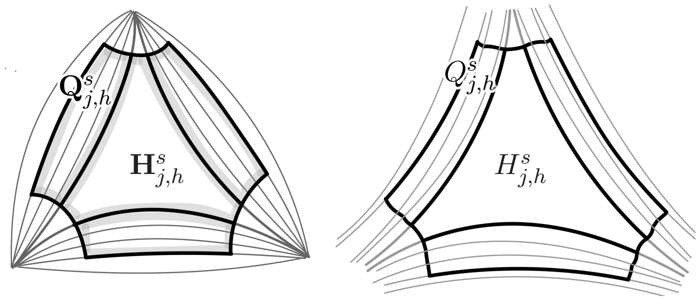

FIGURE 12. Mapping blue hexagon to blue hexagons and rectangles to rectangles.

Similarly, the hexagon $Q^s_{j,h}$ contains the hyperbolic hexagon $H^s_{j,h}$ obtained by cutting the ideal triangle along three horocyclic arcs. For every $\epsilon > 0$, there are $J_\epsilon > 0$ and $s_\epsilon > 0$, such that if $j > J_\epsilon$ and $s > s_\epsilon$, then then three components of $Q^s_{j,h} \setminus H^s_{j,h}$ are, with respect to Thurston's metric, $(1-\epsilon, 1+\epsilon)$-bilipschitz to the corresponding three complementary Euclidean rectangles of $\mathbf{Q}^s_{j,h} \setminus \mathbf{H}^s_{j,h}$ using the product structure given by the horocyclic foliation and the orthogotonal geodesic foliation as in §4.3.1. Recall that this mapping linearly preserves horizontal foliation and, in this sense, it is linear with respect to the vertical distance.

If $J_\epsilon > 0$ is sufficiently large, then the corresponding vertical edges $\mathbf{H}^s_{j,h}$ and $H^s_{j,h}$ are $(1-\epsilon, 1+\epsilon)$-bilipschitz. For every $\epsilon > 0$, there are $J_\epsilon > 0$ and $s_\epsilon > 0$, such that, if $j > J_\epsilon$ and $s > s_\epsilon$, then there is a $(1+\epsilon)$-quasiconformal mapping $H^s_{j,h} \to \mathbf{H}^s_{j,h}$, preserving vertical and horizontal edges such that it is linear on each edge of $H^s_{j,h}$ with respect to the hyperbolic length.

We have constructed $(1+\epsilon)$-quasiconformal mappings from rectangle and hexagon pieces of $Q^s_{j,h}$ to corresponding rectangle and hexagon pieces of $\mathbf{Q}^s_{j,h}$ so that they coincide along common vertical edges. Thus, by gluing those quasi-conformal mappings along vertical edges, we obtain a desired $(1+\epsilon)$-quasiconformal mapping. □

Corollary 4.17. *We can in addition assume that the* $(1+\epsilon)$*-qausiconformal mapping* $\xi^s_{j,h} \colon Q^s_{j,h} \to \mathbf{Q}^s_{j,h}$ *coinsides with* $\eta^s_{j,h} \circ \phi^s_{j,h}$ *on the boundary* $\partial Q^s_{j,h}$.

Proof. We first modify $\xi^s_{j,h}$ so that it coincides with $\eta^s_{j,h} \circ \phi^s_{j,h}$ on vertical edges of $Q^s_{j,h}$. Let v be a vertical edge of $\mathbf{Q}^s_{j,h}$. Let $\mathbf{R}$ be the corrsponding rectangular component of $\mathbf{Q}^s_{j,h} \setminus \mathbf{H}^s_{j,h}$ such that v is also a vertical edge of $\mathbf{R}$ (Figure 13).

There is a unique linear mapping from $\zeta \colon \mathbf{R} \to \mathbf{R}$ such that

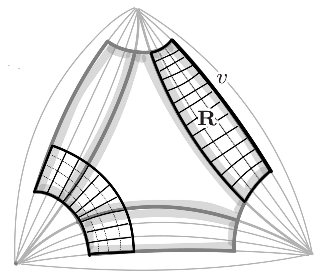

FIGURE 13. Reasions in $\mathbf{Q}^s_{j,h}$ on which $\xi^s_{j,h}$ is modified.

- the restriction of $\xi^s_{j,h}$ on v coincides with the composition of $\eta^s_{j,h} \circ \phi^s_{j,h}$ with ζ on v;
- ζ is linear on each horizontal leaf of $\mathbf{R}$;
- ζ is the identitfy map on the vertical edge of $\mathbf{R}$ opposite to v.

Then, for every $\epsilon > 0$, there are $J_\epsilon > 0$ and $s_\epsilon > 0$, such that, if $j > J_\epsilon$ and $s > s_\epsilon$, then ζ is a $(1-\epsilon, 1+\epsilon)$-bilipschitz mapping, as $\mathbf{R}$ has sufficiently long vertical and horizontal edges. Then by redefining $\xi^s_{j,h} \colon Q^s_{j,h} \to \mathbf{Q}^s_{j,h}$ by post-composing $\xi^s_{j,h}$ with $\zeta \colon \mathbf{R} \to \mathbf{R}$, $\xi^s_{j,h}$ coincides with $\eta^s_{j,h} \circ \phi^s_{j,h}$ on the vertical edge v.

By applying this to all vertical edges of $\mathbf{Q}^s_{j,h}$, we can modify $\xi^s_{j,h} \colon Q^s_{j,h} \to \mathbf{Q}^s_{j,h}$ so that $\xi^s_{j,h}$ coincides with $\eta^s_{j,h} \circ \phi^s_{j,h}$ on all three vertical edges of $Q^s_{j,h}$. Then, there are $J_\epsilon > 0$ and $s_\epsilon > 0$, such that, if $j > J_\epsilon$ and $s > s_\epsilon$, then this modified mapping $\xi^s_{j,h}$ is still $(1+\epsilon)$-quasiconformal after this modification.

Similarly, we can modify $\xi^s_{j,h}$ along appropriately large rectangular regular neighborhoods of horizontal edges, so that $\xi^s_{j,h}$ also coincides with $\eta^s_{j,h} \circ \phi^s_{j,h}$ along horizontal edges.

□

We completed the proof of Theorem 4.4.

5. UNIFORM ASYMPOTOCITY

We have proved in Theorem 4.4 that the limit of the Teichmuller ray $X_\infty \colon \mathbb{R} \to \mathcal{T}$ is asymptotic to the corresponding conformal grafting ray from the same base point $X_\infty(0)$, by directly constructing a quasi-conformal mapping between corresponding points on the rays.

Utilizing this asymptotic property, in this section, we show a uniform asymptotic property for the family of the Teichmuller rays $X_i \colon \mathbb{R} \to \mathcal{T}$ limiting to $X_\infty \colon \mathbb{R} \to \mathcal{T}$ and their corresponding grafting rays with the same base points $X_i(0)$.

Recall that the vertical measured foliation V_i on $X(t_i)$ is normalized so that V_i has length one on the corresponding flat surface E_i. For each $i = 1, 2 \ldots,$ let

$$d_i = \frac{\mathrm{length}_{E_i} V_i}{\mathrm{length}_{\sigma_i} L_i} = \frac{1}{\mathrm{length}_{\sigma_i} L_i} \in \mathbb{R}_{>0}$$

Then $d_i \to d$ as $i \to \infty$, since $[E_i]$ converges to $[E_\infty]$ and accordingly $[V_i]$ converges to $[V_\infty]$ as $i \to \infty$.

Theorem 5.1. *For every $\epsilon > 0$, there are $I_\epsilon > 0$ $s_\epsilon > 0$ such that, if $i > I_\epsilon$ and $s > s_\epsilon$, then*

$$d_{\mathfrak{T}}(R_i(s), \mathrm{gr}_{L_i}^{d_i \exp(s)} \sigma_i) < \epsilon$$

for all $s > s_\epsilon$.

5.1. **Congerecne of Euclidean polygonal structure.** We first analyze the convergence of the Euclidean surfaces. By the convergence of $\nu_i(E_i, V_i) \to (E_\infty, V_\infty)$ implies the following proposition.

Lemma 5.2. *For every $\epsilon > 0$, there are $I_\epsilon > 0, J_\epsilon > 0$ and $J_i > 0$ with $J_i \to \infty$ as $i \to \infty$, such that, if $i > I_\epsilon$ and $J_\epsilon < j < J_i$, then, there is a $(1-\epsilon, 1+\epsilon)$-bilipschitz map*

$$\nu_{i,j} \colon E_i \to E_\infty$$

homotopic to ν_i such that

- *$\nu_{i,j}$ preserves singular points;*
- *the inverse map $\nu_{i,j}^{-1}$ takes the tripods $\gamma_1(j), \ldots, \gamma_N(j)$ into tripods in singular leaves of the vertical foliation V_i;*
- *$\nu_{i,j}$ is $(1+\epsilon)$-bilipschitz both in the vertical and horizontal length.*

By the second property of $\nu_{i,j}$, we can cut E_i minus the $\nu_{i,j}^{-1}$-image of $\gamma_1(j), \ldots, \gamma_N(j)$ into Euclidean rectangles along horizontal segments from the endpoints of the tripods, we obtain a traintrack structure $T_{i,j}$ on the complement which is isomorphic to $T_{\infty,j}$ as fat-traintracks— the same construction as the traintrack structure $T_{\infty,j}$ on $E_\infty \setminus \gamma_1(j) \cup \cdots \cup \gamma_N(j)$.

In Section 4.3, we constructed, from the traintrack structure $T_{\infty,j}$, a decomposition $E_{\infty,j} = (\cup_{k=1}^{N'} R_{j,k}) \cup (\cup_{h=1}^{N} Q_{j,h})$ into rectangles $R_{j,k}$ and hexagons $Q_{j,h}$.

Similarly, let $m_{i,j} > 0$ be the shortest width of the (rectangular) branches of $T_{i,j}$. For each $h = 1, \ldots, N$, the inverse-image $\nu_{i,j}^{-1}(\gamma_h(j))$ is also a tripod embedded in a singular leaf of V_i. Let $Q_{i,j,h}$ be the hexagon which is the $(m_{i,j}/3)$-neighborhood of the tripod $\nu_{i,j}^{-1}(\gamma_h(j))$ in the horizontal direction. Then, removing, the hexagonal part $Q_{i,j,1} \cup$

$\cdots \cup Q_{i,j,N}$ from the rectangular branches of $T_{i,j}$, we obtain thiner rectangles $R_{i,j,1}, \ldots, R_{i,j,N'}$. We thus obtain a decomposition $E_{i,j}$ of E_j

$$(\cup_{k=1}^{N'} R_{i,j,k}) \cup (\cup_{h=1}^{N} Q_{i,j,h})$$

into the hexagonal $Q_{i,j,1}, \ldots, Q_{i,j,N}$ and the rectangles $R_{i,j,1}, \ldots, R_{i,j,N'}$, which have disjoint interiors.

We can isotopy $\nu_{i,j} \colon E_i \to E_\infty$ takes the decomposition $E_{i,j}$ to the decomposition $E_{\infty,j}$, keeping the properties in Lemma 5.2.

Proposition 5.3. *For every $\epsilon > 0$, there are $I_\epsilon > 0, J_\epsilon > 0$ and $J_i > 0$ with $J_i \to \infty$ as $i \to \infty$, such that if $i > I_\epsilon$ and $J_\epsilon < j < J_i$, then for all $s > 0$. There is a $(1-\epsilon, 1+\epsilon)$-bilipschitz map*

$$\nu'_{i,j} \colon E_i \to E_\infty$$

homotopic to ν_i such that

- *$\nu'_{i,j}$ preserves the singular points;*
- *$\nu'_{i,j}$ induces an isomorphism between polygonal decomposition*

$$E_{i,j} = (\cup_{k=1}^{N'} R_{i,j,k}) \cup (\cup_{h=1}^{N} Q_{i,j,h}) \to E_{\infty,j} = (\cup_{k=1}^{N'} R_{j,k}) \cup (\cup_{h=1}^{N} Q_{j,h});$$

  and
- *$\nu'_{i,j}$ is $(1+\epsilon)$-bilipschitz both in the vertical and horizontal directions.*

Similarly to $E_\infty(s)$ in §4.1, for each $s \geq 0$, we let $E_i(s)$ be the marked flat structure on S obtained by stretching E_i by $\exp(s)$ in the horizontal direction, so that $E_i(s)$ is conformally equivalent to $X_i(s)$. Similarly to $f_{\infty,s} \colon E_\infty = E_\infty(0) \to E_\infty(s)$, we let $f_{i,s} \colon E_i(0) \to E_i(s)$ denote this stretch map by $\exp(s)$ so that $f_{i,s}$ reallizes the best quasi-conformal distorsion between $E_i(0)$ and $E_i(s)$.

Then, by $f_{i,s}$, the polygonal decomposition $E_{i,j} = (\cup_k R_{i,j,k}) \cup (\cup_h Q_{i,j,h})$ descends to a polygonal decomposition of $E_i(s)$; we set

$$E_{i,j}(s) = (\cup_{k=1}^{N'} R^s_{i,j,k}) \cup (\cup_h^N Q^s_{i,j,h}).$$

Then, the $(1-\epsilon, 1+\epsilon)$-bilipschitz map

$$\nu'_{i,j} \colon E_i \to E_\infty$$

induces

$$\nu^s_{i,j} \colon E_i(s) \to E_\infty(s)$$

so that $f_{\infty,s} \circ \nu_{i,j} = \nu^s_{i,j} \circ f_{i,s}$. Since the mapping $f_{i,s}$ and $f_{\infty,s}$ both stretch E_i and E_∞ by $\exp(s)$ in the horizontal direction, $\nu^s_{i,j}$ rerains the properties of $\nu'_{i,j}$, and we obtain the following corollary.

Corollary 5.4. *Under the same assumption, for all $s > 0$, the mapping*

$$\nu_{i,j}^s : E_{i,j}(s) \to E_{\infty,j}(s)$$

is a $(1-\epsilon, 1+\epsilon)$-bilipschitz map such that

- *$\nu_{i,j}^s$ gives a polygonal isomorphism*

$$E_{i,j}(s) = (\cup_k R_{i,j,k}^s) \cup (\cup Q_{i,j,h}^s) \to E_{\infty,j}(s) = (\cup_k R_{j,k}^s) \cup (\cup Q_{j,h}^s)$$

- *this induces isomorphism is $(1+\epsilon)$-bilipschitz both in the vertical and horizontal directions.*

5.2. **Convergence of decompositions of hyperbolic surfaces.** In §5.1, we constructed a Euclidean polygonal decomposition $E_{i,j}$ of E_i which converges to the Euclidean polygonal decomposition $E_{\infty,j}$ of E_∞ as $i \to \infty$. In this subsection, we construct a corresponding polygonal decomposition of $\tau_{i,j}$ converging to the polygonal decomposition $\tau_{\infty,j}$.

By the convergence $(\sigma_i, L_i) \to (\sigma_\infty, L_\infty)$ implies the following Lemma.

Lemma 5.5. *For every $\epsilon > 0$, there are constants $I_\epsilon > 0, J_\epsilon > 0$ and a sequence $J_i > 0$ with $J_i \to \infty$ as $i \to \infty$, such that, if $i > I_\epsilon$ and $J_\epsilon < j < J_i$, then, there are*

- *a $(1-\epsilon, 1+\epsilon)$-bilipschitz map*

  $$\upsilon_{i,j} : \sigma_{i,j} \to \sigma_{\infty,j}$$

  homotopic to the diffeomorphism υ_i, and
- *an ϵ-nearly straight traintrack $\tau_{i,j}$ on σ_i combinatorially isomorphic to $\tau_{\infty,j}$,*

such that

- *$\upsilon_{i,j}$ induces a $(1-\epsilon, 1+\epsilon)$-bilipschitz isomorphism of traintrack neighborhoods*

  $$\tau_{i,j} \to \tau_{\infty,j},$$

  and
- *the L_i-weights of $\tau_{i,j}$ are $(1-\epsilon, 1+\epsilon)$-bilipschitz close to the L_∞-weights of $\tau_{\infty,j}$ (on the corresponding branches).*

Recall, from §4.3, that the polygonal decomposition $\sigma_{\infty,j} = (\cup_k \mathcal{R}_{j,k}) \cup (\cup \mathcal{Q}_{j,h})$ carrying L_∞ is constructed from the ϵ-nearly straight traintrack $\tau_{\infty,j}$ so that it realizes $E_{\infty,j} = (\cup_{k=1}^{N'} R_{j,k}) \cup (\cup_{h=1}^{N} Q_{j,h})$ carrying V_∞.

Recall that the Euclidean polygonal decomposition $E_{i,j} = (\cup_k R_{i,j,k}) \cup (\cup_h Q_{i,j,h})$ of E_i carries the vertical foliaiton V_i. Then we can similarly construct a corresponding polygonal decomposition

$$\sigma_{i,j} = (\cup_{k=1}^{N'} \mathcal{R}_{i,j,k}) \cup (\cup_{h=1}^{N} \mathcal{Q}_{i,j,h})$$

carrying L_i, where $\mathcal{R}_{i,j,k}$ and $\mathcal{Q}_{i,j,h}$ are rectangles and hexagons with horocyclic horizontal edges and with vertical edges in L_i, such that

- $\sigma_{i,j} = (\cup_{k=1}^{N'} \mathcal{R}_{i,j,k}) \cup (\cup_{h=1}^{N} \mathcal{Q}_{i,j,h})$ is combinatorially isomorphic to $E_{i,j} = (\cup_{k=1}^{N'} R_{i,j,k}) \cup (\cup_{h=1}^{N} Q_{i,j,h})$ by a marking-preserving homeomorphism σ_i to E_i;
- moreover $\sigma_{i,j} = (\cup_k \mathcal{R}_{i,j,k}) \cup (\cup \mathcal{Q}_{i,j,h})$ carries L_∞ in the same as $E_{i,j} = (\cup_{k=1}^{N'} R_{i,j,k}) \cup (\cup_{h=1}^{N} Q_{i,j,h})$ carries V_∞, respecting the identification of the geodesic measured lamination L_i and the measured foliation V_i;
- the union of the horizontal edges of $\mathcal{R}_{i,j,k}$ and $\mathcal{Q}_{i,j,h}$ is the union of (horocyclic) horizontal edges of $\tau_{i,j}$;

As the polygonal decomposition $\sigma_{i,j}$ is geometrically determined by the nearly-straight traintack neighborhood $\tau_{i,j}$, Lemma 5.5 implies the following.

Proposition 5.6. *For every $\epsilon > 0$, there are $I_\epsilon > 0, J_\epsilon > 0$ and $J_i > 0$ with $J_i \to \infty$ as $i \to \infty$, such that, if $i > I_\epsilon$ and $J_\epsilon < j < J_i$, then, there is a $(1-\epsilon, 1+\epsilon)$-bilipschitz map*

$$\upsilon'_{i,j} \colon \sigma_{i,j} \to \sigma_{\infty,j}$$

homotopic to υ_i, such that

- *$\upsilon'_{i,j}$ induces a $(1-\epsilon, 1+\epsilon)$-bilipschitz isomorphism between polygonal decompositions*

  $$\sigma_i = (\cup_k \mathcal{R}_{i,j,k}) \cup (\cup \mathcal{Q}_{i,j,h}) \to \sigma_\infty = (\cup_k \mathcal{R}_{j,k}) \cup (\cup \mathcal{Q}_{j,h}),$$

  and
- *the L_i-weights of $(\cup_k \mathcal{R}_{i,j,k}) \cup (\cup \mathcal{Q}_{i,j,h})$ are $(1-\epsilon, 1+\epsilon)$-bilipschitz close to the L_∞-weights of $(\cup_k \mathcal{R}_{j,k}) \cup (\cup \mathcal{Q}_{j,h})$ on the corresponding horizontal edges of the polygonal decompositions.*

In §4.3, we see that the grafting of σ_∞ along sL_∞ transforms the polygonal decomposition $\sigma_{\infty,j} = (\cup_k \mathcal{R}_{j,k}) \cup (\cup \mathcal{Q}_{j,h})$ to a polygonal decomposition of $\mathrm{Gr}_{sL_\infty} \sigma_\infty$

$$\mathrm{Gr}_{sL} \sigma_\infty = (\cup_k \mathcal{R}^s_{j,k}) \cup (\cup \mathcal{Q}^s_{j,h})$$

for $s \geq 0$. Similarly, by the grafting of σ_i along $sL_i \, (s \geq 0)$, the polygonal decomposition

$$\sigma_{i,j} = (\cup_k \mathcal{R}_{i,j,k}) \cup (\cup \mathcal{Q}_{i,j,h})$$

induces a polygonal decomposition

$$\mathrm{Gr}_{sL_i} \sigma_{i,j} = (\cup_k \mathcal{R}^s_{i,j,k}) \cup (\cup \mathcal{Q}^s_{i,j,h}),$$

where $\mathcal{R}^s_{i,j,k}$ is a rectangle obtained by grafting $\mathcal{R}_{i,j,k}$ along the restriction of L_i to $\mathcal{R}_{i,j,k}$ and $\mathcal{Q}^s_{i,j,h}$ is a hexagon obtained by grafting $\mathcal{Q}_{i,j,h}$ along the restriction of L_i to $\mathcal{Q}_{i,j,h}$.

Then, since the way $\sigma_{i,j}$ carries L_i geometrically converges to the way $\sigma_{\infty,j}$ carries L_∞, we Proposition 5.6 implies that the convergence of grafted decompositions.

Corollary 5.7. *Under the same assumption, for all $s > 0$, there is a $(1-\epsilon, 1+\epsilon)$-bilipschitz map*

$$\upsilon_{i,j}^s \colon \mathrm{Gr}_{sL_i}\sigma_{i,j} \to \mathrm{Gr}_{sL_\infty}\sigma_{\infty,j}$$

homotopic to υ_i, such that $\upsilon_{i,j}^s$ induces a C^1-smooth $(1-\epsilon, 1+\epsilon)$-bilipschitz isomorphism

$$\sigma_i = (\cup_k \mathcal{R}_{i,j,k}^s) \cup (\cup \mathcal{Q}_{i,j,h}^s) \to \sigma_\infty = (\cup_k \mathcal{R}_{j,k}^s) \cup (\cup \mathcal{Q}_{j,h}^s).$$

5.3. **Uniform quasi-conformal mappings.** By compositing the C^1-smooth bilipschitz mappings, we obtain a desired quasi-conformal mapping with small distortion.

Proof of Theorem 5.1. For every $\epsilon > 0$, there are $I_\epsilon > 0$, $J_\epsilon > 0$, $J_i > 0$ with $J_i \to \infty$ as $i \to \infty$ and $s_\epsilon > 0$, such that, if $i > I_\epsilon$ and $J_\epsilon < j < J_i$, and $s > s_\epsilon$, combining the quasi-conformal mappings with small distorsion

$$\nu_{i,j}^s \colon E_{i,j}(s) \to E_{\infty,j}(s)$$

in Corollary 5.4,

$$\upsilon_{i,j}^s \colon \mathrm{Gr}_{sL_i}\sigma_{i,j} \to \mathrm{Gr}_{sL_\infty}\sigma_{\infty,j}$$

in Corollary 5.7,

$$\Phi_j^s \colon \mathrm{Gr}_{L_\infty}^{d\exp(s)} \to E_\infty(s)$$

in Proposition 4.6, we obtain a desired $(1+\epsilon)$-quasiconformal mapping,

$$E_i(s)) \xrightarrow{\nu_{i,j}^s} E_\infty(s) \xrightarrow{(\Phi_j^s)^{-1}} \mathrm{Gr}_{L_\infty}^{\exp s/d}\sigma_\infty(s) \xrightarrow{(\upsilon_{i,j}^s)^{-1}} \mathrm{Gr}_{L_i}^{\exp s/d}\sigma_i(s)$$

(see Figure 14). 5.1

6. Uniform approximation of grafting rays by integral grafting

Recall that σ_i is a sequence of marked hyperbolic structures on S and $\nu_i \colon S \to S$ is a diffeomorphism such that $\nu_i\sigma_i$ converges to $\sigma_\infty \in \mathcal{T}$ as $i \to \infty$. Moreover, L_i is a maximal measured lamination on σ_i such that $\nu_i L_i$ converges to the maximal measured lamination L_∞ on σ_∞ as $i \to \infty$.

Gupta showed that every grafting ray is conformally well-approximated by a sequence of integral grafting toward infinity; see [Gup14, Lemma 6.19]. In this section, following Gupta's idea, we show that the family

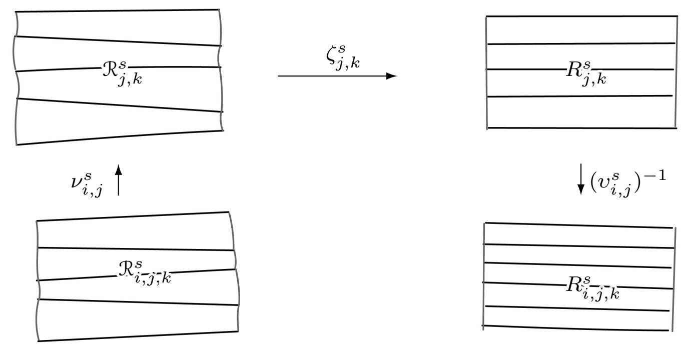

FIGURE 14. The composition $(\upsilon^s_{i,j})^{-1} \circ \zeta^s_{j,k} \circ \nu^s_{i,j}$ from $\mathcal{R}^s_{j,k}$ to $R^s_{i,j,k}$.

of graftring rays $\mathrm{gr}^s_{L_i} \sigma_i \, (s \geq 0)$ is well-approximated by the integral graftings of σ_i in a uniform manner.

Theorem 6.1. *For every $\epsilon > 0$, there are $I_\epsilon > 0$ and $s_\epsilon > 0$ such that, if $i > I_\epsilon$ and $s > s_\epsilon$, then there is a multiloop $M = M_{i,s}$ with weights multiples of 2π, such that*

$$d_{\mathcal{T}}(\mathrm{gr}^s_{L_i}(\sigma_i), \mathrm{gr}_{M_{i,s}}(\sigma_i)) < \epsilon,$$

where $d_{\mathcal{T}}$ denotes the Teichmüller distance.

The rest of this section is the proof of Theorem 6.1,

6.1. **Uniform approximation of grafting lamination rays.** By the convergence $\nu_i(\tau_i, L_i) \to (\tau_\infty, L_\infty)$, we can take a nearly-straight traintrack neighborthood of L_i convering to a nearly-straight traintrack neighborhood of L_∞.

Lemma 6.2. *For all $\epsilon > 0$, there are $I_\epsilon > 0$, $J_\epsilon > 0$, and $J_i > 0$ with $J_i \to \infty$ as $i \to \infty$, such that, if $i > I_\epsilon$ and $J_\epsilon < j < J_i$, then we can take an ϵ-nearly-straight traintrack neighborhood $\tau_{i,j} = \cup_{k=1}^{N'} R_{i,j,k}$ of L_i on σ_i and an ϵ-nearly straight traintrack neighborhood $\tau_{\infty,j} = \cup_{k=1}^{N'} R_{j,k}$, such that*

- *there is a $(1-\epsilon, 1+\epsilon)$-bilipschitz map $\nu_{i,j} \colon \sigma_i \to \sigma_\infty$ homotopic to ν_i which takes $\tau_{i,j}$ to $\tau_{\infty,j}$, and*
- *the bilispchitz constsnts of $\nu_{i,j} \colon \sigma_i \to \sigma_\infty$ both converge to one as $i \to \infty$.*

Let $\cup_{h=1}^N \Delta_{i,j,h}$ denote the complement $\sigma_i \setminus \tau_{i,j}$ where $\Delta_{i,j,h}$ are (triangular) connected components. Let $\epsilon > 0$. Then, for $i > I_\epsilon$ and

$J_\epsilon < j < J_i$, we have the decomposition

$$\sigma_{i,j} = (\cup_{k=1}^{N'} R_{i,j,k}) \cup (\cup_{h=1}^{N} \Delta_{i,j,h})$$

of σ_i into reclantular branches $R_{i,j,k}$ of $\tau_{i,j}$ and the triangular complements $\Delta_{i,j,h}$, so that $R_{i,j,k}$ and $\Delta_{i,j,h}$ have disjoint interiors.

Lemma 6.3 (see Lemma 6.14 in [Gup14])**.** *For every $i = 1, 2, \dots$ and $j = 1, 2, \dots$, there is $K_{i,j} > 0$ such that for every measured lamination L carried by $\tau_{i,j}$, there is a multiloop M carried by $\tau_{i,j}$ such that, for each branch R of τ, the difference of the weights of L and M on R is less than $K_{i,j}$.*

Recall that the traintrack structure of $\tau_{i,j}$ is identified with $\tau_{\infty,j}$ by the diffeomorphism $\nu_i \colon \sigma_i \to \sigma_\infty$, and combinatorially independent on $i = 1, 2, \dots$. Moreover there are only finitely many combinatorial types of Γ_i, we can take $K_{i,j}$ independent of the indices.

Corollary 6.4. *There is $K > 0$ such that, for every $i = 1, 2, \dots, \infty$ and $j = 1, 2, \dots$ and every measured lamination L carried by $\tau_{i,j}$, there is a multiloop M carried by $\tau_{i,j}$ such that, for each branch R of τ, the difference of the weights of L and M on R is less than K.*

Proposition 6.5. *Pick arbitrary $\epsilon > 0$ and arbitrary $J > J_\epsilon$, so that, there is $I_{\epsilon,J} > 0$ such that, if i is sufficiently large, then $J_\epsilon < J < J_i$ and the ϵ-nearly-straight traintrack neighborhood $\tau_{i,J}$ of L_i exists by Lemma 6.2.*

Then, there are $s_{\epsilon,J} > 0$ and $I_{\epsilon,J} > 0$ such that, if $i > I_{\epsilon,J}$ and $J_\epsilon < j < J_i$, and $s > s_{\epsilon,J}$, then the geodesic representative of the multiloop $M^s_{i,J}$ in Lemma 6.3 on σ_i is carried by $\tau_{i,J}$ (without isotopy).

Proof. Fix $\epsilon > 0$, and let $\tau^\epsilon_{\infty,J}$ is an ϵ-nearly straight traintrack on σ_∞ carring L_∞ from Lemma 6.2. if a neighborhood U_i of L_∞ in PML is suffiicently small, then $\tau^\epsilon_{\infty,J}$ carries all geodesic laminations on σ_∞ whose proejctive class contained in U_ϵ. For sufficiently large i, let $\tau^\epsilon_{i,J}$ be an ϵ-nearly striaght traintrack on σ_i carrying L_i, so that $\nu_i(\sigma_i, \tau_{i,J})$ converges to $(\sigma_\infty, \tau_\infty)$.

Then, as the bilispchiz constants of $\nu_{i,j}$ converge to one, if the neighborhood U_ϵ of $[L_\infty]$ in PML is sufficientely small, then there is $I_{\epsilon,J} > 0$ such that the ϵ-nearly straight traintrack $\tau^\epsilon_{i,J}$ contains all geodesic laminations whose projective classes are in U_ϵ. Let $M^s_{i,J}$ be the geodesic multiloop on σ_i so that the difference of the weights of sL_i and $M^s_{i,J}$ is less than K_J on each branch of $\tau_{i,J}$.

We can pick sufficiently large $s_{\epsilon,J} > 0$ so that if $s > s_{\epsilon,J}$ and $i > I_\epsilon$, then the projective class of $M^s_{i,J}$ is contained in U_ϵ. Thus the geodesic multiloop $M^s_{i,J}$ is carried by $\tau^\epsilon_{i,J}$ (without isotopy). □

6.2. **2π-grafting of nearly straight traintracks.** Recall that, for $J_\epsilon < J$, $\tau_{i,J}$ is the ϵ-nearly-straight traintrack neighborhood of L_i on σ_i, and

$$\sigma_{i,J} = (\cup_{k=1}^{N'} R_{i,J,k}) \cup (\cup_{h=1}^{N} \Delta_{i,J,h})$$

is the traintrack decomposition of $\tau_{i,J}$ such that horizontal edges of $R_{i,J,k}$ are contained in leaves of the horocyclice lamination λ_i of (σ_i, L_i).

Consider the projective grafting of σ_i along sL_i. Since L_i is carried by $\tau_{i,J}$, the above traintrack decomposition of σ_i induces a traintrack decomposition of $\mathrm{Gr}_{sL_i}\sigma_i$ for $s \geq 0$, and we set

$$\mathrm{Gr}_{sL_i}\sigma_i = (\cup_{k=1}^{N'} R^s_{i,J,k}) \cup (\cup_{h=1}^{N} \Delta_{i,J,h}),$$

where $R^s_{i,J,k}$ are grafting of $R_{i,J,k}$ along the restriction of sL_i to the branch $R^s_{i,J,k}$.

By Proposition 6.5, there are $I_{\epsilon,J} > 0$ and $s_{\epsilon,J}$, such that the traintrack $\tau_{i,J}$ carries the geodesic representative of $M^s_{i,J}$ on τ_i for $i > I_{\epsilon,J}$ and $s > s_{\epsilon,J}$. Then, similarly, the traintrack decomposition $\sigma_{i,J}$ induces a traintrack decomposition of the grafting of σ_i along $M^s_{i,J}$ as follows. Along each loop m of $M^s_{i,J}$, the grafting $\mathrm{Gr}_{M^s_{i,J}}$ inserts an Euclidean cylinder of width 2π times the weight along $M^s_{i,J}$ (in $\mathbb{Z}_{\geq 0}$) in Thurston metric. Then, for each branch $R_{i,J,k}$, the restriction of $M^s_{i,J}$ to $R_{i,J,k}$ is a geodesic multi-arc connecting horizontal horocyclic edges. Then, let $R^{M_s}_{i,J,k}$ denote the grafting of $R_{i,J,k}$ along the multi-arc. In Thurston metric, along each arc of the multiarc, it inserts an Euclidean rectangle of length equal to the length of the arc and width equal to 2π times the weight of the arc. Then the induced traintrack decomposition is

$$\mathrm{Gr}_{M^s_{i,J}}\sigma_i = (\cup_{k=1}^{N'} R^{M_s}_{i,J,k}) \cup (\cup_{h=1}^{N} \Delta_{i,J,h}).$$

6.3. **Model Euclidean Traintracks.** Let $F_i(sL_i)$ be the Euclidean traintrack which represents the sum of the structure inserted to the hyperbolic traintrack $\tau_{i,j}$ by Gr_{sL_i}. Namely,

- $F_i(sL_i)$ is diffeomorphic to $\tau_{i,j} = \cup_{k=1}^{N'} R_{i,j,k}$ as fat traintracks.
- the branch of $F_i(sL_i)$ corresponding to $R_{i,j,k}$ is a Euclidean rectangle of length equal to the length of $R_{i,j,k}$ and width equal to the weight of sL_i on $R_{i,j,k}$.

Similarly, let $F_i(M^s_{i,j})$ be the Euclidean traintrack representing the sum of the structure inserted to $\tau_{i,j}$ along $M^s_{i,j}$. Namely,

- $F_i(M^s_{i,j})$ is diffeomorphic to $\tau_{i,j} = \cup_{k=1}^{N'} R_{i,j,k}$ as fat traintracks, and
- if the branch of $F_i(M^s_{i,j})$ corresponding to $R_{i,j,k}$, the it is a Euclidean rectangle of length equal to the length of $R_{i,j,k}$ and width equal to the weight of $M^s_{i,j}$ on $R_{i,j,k}$.

Each branch of the grafted train track $\mathrm{Gr}_{sL_i}\tau_{i,j}$ is foliated by nearly-horocyclic foliation and nearly-straight foliation orthogonal to it. Let

$$\xi_{sL_i}\colon \mathrm{Gr}_{sL_i}\tau_{i,j} \to F_{i,j}(sL_i)$$

be the straightening mapping defined similarly to the proof of *Lemma* 4.8 using the nearly-horocyclic foliation and nearly-straight foliation orthogonal to it. Namely,

- ξ_{sL_i} takes horizontal foliation of $\mathrm{Gr}_{sL_i}\tau_{ij}$ to the horizontal foliation of $F_{i,j}(M^s_{i,j})$, and
- ξ_{sL_i} is linear on each vertical edge of $\mathrm{Gr}_{sL_i}\tau_{i,j}$ with respect to vertical distance.

Then, by the construction of $F_{i,j}(sL_i)$ we have the following.

Proposition 6.6. *For every $\epsilon > 0$, there are $I_\epsilon > 0, J_\epsilon > 0, s_\epsilon > 0$ such that, if $i > I_\epsilon, s > s_\epsilon$, then*

$$\xi_{sL_i}\colon \mathrm{Gr}_{sL_i}\tau_i \to F_{i,J_\epsilon}(sL_i)$$

is a $(1+\epsilon)$-bilipschitz homeomorphism.

Proof. Similarly to the proof of Lemma 4.8, for every $\epsilon > 0$, given sufficiently large $J_\epsilon > 0$, one can prove the derivatives of $\xi_{M^s_{i,j}}$ in both horizontal and vertical directions are ϵ-close to one. This implies the assertion. □

As $\mathrm{Gr}_{M^s_{i,j}}$ inserts to each branch $R_{i,j,k}$ of $\tau_{i,j}$ Euclidean rectangles rectangles along the geodesic arcs of $M^s_{i,j}|R_{i,j,k}$, The grafted branches $R^{M^s}_{i,j,k}$ have horizontal and vertical foliations obtained by the obvious horizontal and vertical foliations of the rectangle and the nearly-horocyclic and nearly-straight foliations of $R_{i,j,k}$. The similarly Let

$$\xi_{M^s_{i,j}}\colon \mathrm{Gr}_{M^s_{i,j}}\tau_i \to F_{i,j}(M^s_{i,j})$$

be the straightening mapping defined similarly to $\zeta^s_{j,k}$ in the proof of *Lemma* 4.8.

Proposition 6.7. *For every $\epsilon > 0$, there are $I_\epsilon > 0, J_\epsilon > 0, s_\epsilon > 0$ such that, if $i > I_\epsilon, s > s_\epsilon$, then*

$$\xi_{M^s_{i,J_\epsilon}}\colon \mathrm{Gr}_{M^s_{i,j}}\tau_i \to F_{i,J_\epsilon}(M^s_{i,J_\epsilon})$$

is a $(1-\epsilon, 1+\epsilon)$-bilipschitz mapping.

Recall that the boundary of the traintrack $\tau_{i,j}$ on σ_i is identified both with the boundary of $F_{i,j}(sL_i)$ by $\xi_{sL_i}$and the boundary of $F_{i,j}(M^s_{i,j})$ by $\xi_{M^s_{i,j}}$. Thus we have a canonical "identity" mapping $\partial\zeta^s_{i,j}$ from $\partial F_{i,j}(sV)$ to $\partial F_{i,j}(M^s_{i,j})$ by composing those idnetifications. With respect to this identification, endpoints of horizontal leaves of $F_{i,j}(sL_i)$ coincide with endpoints of horizontal leaves of $F_{i,j}(M^s_{i,j})$, since the constructions of $\mathrm{Gr}_{M^s_{i,j}}\tau_i$ and $\mathrm{Gr}_{sL_i}\tau_i$ preserve horizontal leaves.

Therefore, We can finally define a C^1-diffeomorphism $\zeta^s_{i,j}\colon F_{i,j}(sV) \to F_{i,j}(M^s_{i,j})$ so that

- $\zeta^s_{i,j}$ coincides with $\partial\zeta^s_{i,j}$ on the boundary of $F_{i,j}(sV)$, and
- $\zeta^s_{i,j}$ linear on each horizontal leaf of $F_{i,j}(sL_i)$ with respect to arc length.

Proposition 6.8. *For every $\epsilon > 0$, if $J > 0$ is sufficient large, then there are $I_\epsilon > 0, s_\epsilon > 0$ such that, if $i > \epsilon$ and $s > s_\epsilon$, then the above piecewise C^1-diffeomorphism $\zeta^s_{i,j}\colon F_{i,j}(sL_i) \to F_{i,j}(M^s_{i,j})$ is a $(1-\epsilon, 1+\epsilon)$-bilipschitz map.*

Proof. For an arbitaray branch $R_{i,j,k}$ of $\tau_{i,j}$, let R_L and R_M be its corresponding branches of $F_{i,j}(sL_i)$ and $F_{i,j}(M^s_{i,j})$, respectively. Then the width of R_L is the weight of sL_i on $R_{i,j,k}$, and the width of R_M is the weight of $M^s_{i,j}$ on $R_{i,j,k}$. As $s_\epsilon > 0$ is sufficiently large, the ratio of the width of R_L and R_M is ϵ-close to one by Corollary 6.4. Therefore, under the assumption of the assertion, $\zeta^s_{i,j}$ is $(1-\epsilon, 1+\epsilon)$-bilipschitz in the horizontal direction.

By the definition of $F_{i,j}(sL_i)$ and $F_{i,j}(M^s_{i,j})$, the lengths of the corresponding branches are the same. Since $\tau_{i,j}$ are sufficiently straight, $\zeta^s_{i,j}$ is $(1-\epsilon, 1+\epsilon)$-bilipschitz in the vertical direction as well.

Since $\zeta^s_{i,j}\colon F_{i,j}(sL_i) \to F_{i,j}(M^s_{i,j})$ is a $(1-\epsilon, 1+\epsilon)$-bilipschitz in both vertical and horizontal direction, the proposition follows. □

6.8

Corollary 6.9. *For every $\epsilon > 0$, if $J > 0$ is sufficient large, then there are $I_\epsilon > 0, s_\epsilon > 0$ such that, if $i > \epsilon$ and $s > s_\epsilon$, then the mapping $\xi^{-1}_{sM_s} \circ \zeta_{i,j,s} \circ \xi_{sL_i}$ is a $(1+\epsilon)$-quasiconformal mapping from*

$$\mathrm{Gr}_{M_s}\tau_i \to \mathrm{Gr}_{sV}\tau_i$$

which is the identity on the boundary.

Proof. By Proposition 6.8, Proposition 6.6, Proposition 6.7, under the assumotion of the corollary, the mappings $\xi^{-1}_{sM_s}$, $\zeta_{i,j,s}$ and ξ_{sL_i} are all ϵ-quasicnoformal mapping with small distorision. Therefore the assertion follows immediately.

□

We completed the proof of Theorem 6.1.

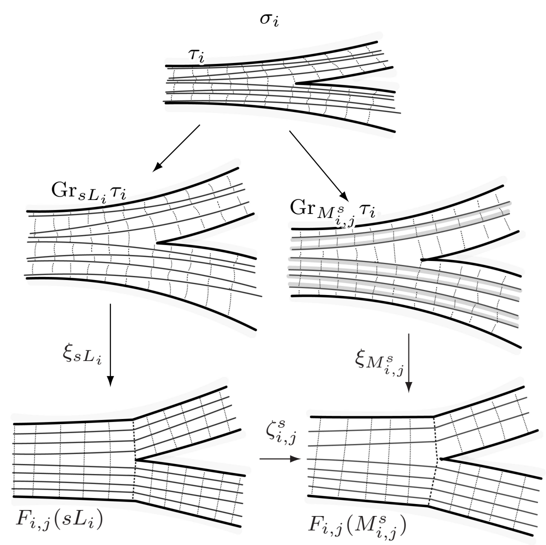

FIGURE 15.

7. Proof of the main theorem

In this section, we prove our main theorem.

Theorem 7.1. *Let X, Y be distinct Riemann surface structures in $\mathcal{T} \cup \mathcal{T}^*$. There is an infinite sequence $(C_{X,i}, C_{Y,i})_{i=1}^{\infty} \in \mathcal{B}$ of distinct pairs such that $\psi(C_{X,i}) = X$ and $\psi(C_{Y,i}) = Y$ for all $i = 1, 2, \ldots$*

We prove Theorem 7.1 by induction. Suppose that we have n pairs

$$(C_{X,1}, C_{Y,1}), \ldots, (C_{X,n}, C_{Y,n})$$

in $\Psi^{-1}(X, Y)$. Then we shall find a new pair $(C_{X,n+1}, C_{Y,n+1})$ in $\Psi^{-1}(X, Y)$. Then, for each $i = 1, \ldots, n$, there are bounded open neighborhoods U_i of $(C_{X,i}, C_{Y,i})$ in $\mathcal{B}$ and W_i of (X, Y) in a connected component of $(\mathcal{T} \cup \mathcal{T}^*)^2 \setminus \Delta$, such that the restriction of Ψ to U_i is a finite branched covering map onto W_i ([Bab23, Theorem A]).

Let W be the (open) connected component of the intersection $W_1 \cap W_2 \cap \cdots \cap W_n$ containing (X, Y). Then it suffices to show the following.

Proposition 7.2. *There is $(C, D) \in \mathcal{B}$ such that $(\psi(C), \psi(D))$ is in W and* $\mathrm{Hol}(C) = \mathrm{Hol}(D) \notin \cup_{h=1}^{n} \mathrm{Hol}(U_h)$.

Indeed, if we find such a pair (C, D), then we take a path $(X_t, Y_t), t \in [0, 1]$ in W connecting $(\psi(C), \psi(D))$ to (X, Y). Let $(C_t, D_t), t \in [0, 1]$ be the lift of (X_t, Y_t) to $\mathcal{B}$ such that

- $(C_0, D_0) = (C, D)$, and

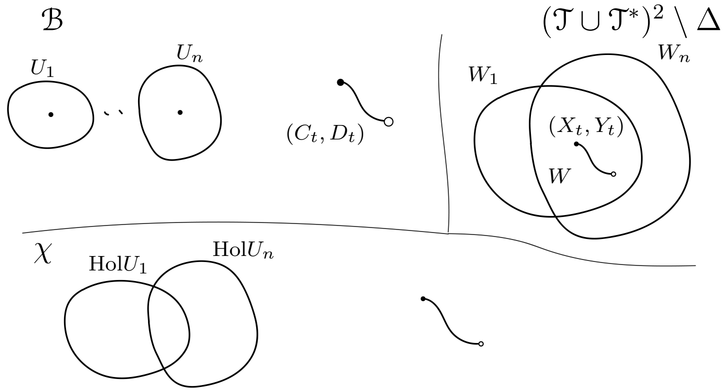

FIGURE 16. Lifting the parth (X_t, Y_t)

- $(\psi(C_1), \psi(D_1)) = (X, Y)$.

Claim 7.3. *(C_1, D_1) is different from all given n pairs $(C_{X,1}, C_{Y,1}), \ldots, (C_{X,n}, C_{Y,n})$.*

Proof. Suppose, to the contaray, that $(C_1, D_1) = (C_{X,i}, C_{Y,i})$ for some $i \in \{1, \ldots, n\}$. Then, the lifted path $(C_t, D_t), t \in [0, 1]$ is entirely contained in U_i, since $\Psi_i \colon U_i \to W_i$ is a finite branched covering map and $W (\subset W_i)$ contains the path (X_t, Y_t). Accordingly $\mathrm{Hol}(C_t) = \mathrm{Hol}(D_t), t \in [0, 1]$ is entirely contained in $\mathrm{Hol}(U_i)$. In particular, the initial holonomy $\mathrm{Hol}(C_0) = \mathrm{Hol}(D_0) = \mathrm{Hol}(C) = \mathrm{Hol}(D)$ is in $\mathrm{Hol}(U_i)$. This contradicts Proposition 7.2. Therefore, we conclude that (C_1, D_1) is a new pair in $\Psi^{-1}(X, Y)$. □

We prove Proposition 7.2 in the remaining of §7.

7.1. When the orientations of X and Y are the same.

In this subsection, supposing that the orientation of X coincides with that of Y, we prove Proposition 7.2. We, in addition, assume that $X, Y \in \mathcal{T}$, and the proof in the case $X, Y \in \mathcal{T}^*$ is essentially the same.

Then pick a sufficiently small $\epsilon > 0$ so that W contains the product of the ϵ-negihborhood of X and the ϵ-neighborhood of Y in $\mathcal{T}$ w.r.t. the Teichmüller metric.

There is a unique Teichmüller geodesic passing X and Y. By perturbing it, we obtain a "generic" Teichmüller geodesic $R \colon \mathbb{R} \to \mathcal{T}$ passing the $\epsilon/3$-neighborhood of X and the $\epsilon/3$-neighborhood of Y such that

- its corresponding quadratic differential q has only simple zeros, and
- The projection of the ray $R(-\infty, 0]$ towrad $-\infty$ is dense in the moduli space $\mathcal{M}$ of Riemann surfaces.

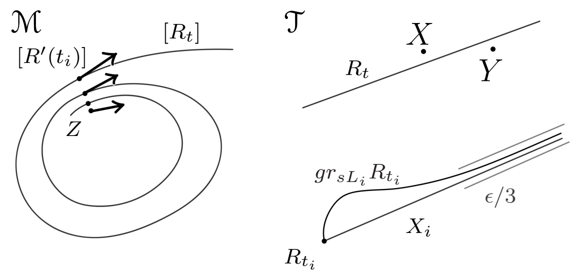

FIGURE 17.

Let V denote the vertical (singular) measured foliation of R. Since q has only simple zeros, each singular point of V has three prongs.

Let $[R(t)] \in \mathcal{M}$ denote the unmarked Riemann surface structure of $R(t)$. Let $0 > t_1 > t_2 > \dots$ be a sequence diverging to $-\infty$, such that

- its unmarked sequence $[R(t_i)]$ converges to $Z \in \mathcal{M}$ and
- the tangent vector $[R'(t_i)]$ also converges in the unite tangent space $T^1\mathcal{M}$ at Z as $i \to \infty$.

For each $i = 1, 2, \dots$, let σ_i be the marked hyperbolic structure on S corresponding to the marked Riemann surface $R(t_i)$ by the uniformization theorem. Let $L_i \in \mathrm{ML}$ denote the measured geodesic lamination on σ_i representing the vertical measured foliation V. Let $\mathrm{gr}^t_{L_i}\sigma_i \in \mathcal{T}\,(t \geq 0)$ be the conformal grafting ray from σ_i along L_i.

For each $i = 1, 2, \dots$, define $R_i\colon \mathbb{R} \to \mathcal{T}$ by $R_i(s) = R(t_i + s)$, the reprametrization of the Teichmüller geodesci R with the base point shifted backward to $R(t_i)$.

By Theorem 5.1, for every $\epsilon > 0$, there are $I_\epsilon > 0$ and $s_\epsilon > 0$ such that, if $i > I_\epsilon$, then

$$d_{\mathcal{T}}(R_i(s), \mathrm{gr}^{d_i \exp(s)}_{L_i}\sigma_i) < \epsilon/3$$

for all $s > s_\epsilon$. Since R_i passes through the $\frac{\epsilon}{3}$-neighborhoods of X and Y, if $i > I_\epsilon$, then $\mathrm{gr}^t_{V_i}\sigma_i$ passes through the $\frac{2}{3}\epsilon$-neighborhood of X and the $\frac{2}{3}\epsilon$-neighborhood of Y. Thus, $i > I_\epsilon$, there are $s^i_X, s^i_Y > s_\epsilon$, such that

$$d_{\mathcal{T}}(X, \mathrm{gr}^{d_i \exp(s^i_X)}_{L_i}\sigma_i) < 2\epsilon/3,$$

and

$$d_{\mathcal{T}}(Y, \mathrm{gr}^{d_i \exp(s^i_Y)}_{L_i}\sigma_i) < 2\epsilon/3.$$

By Theorem 6.1, there is $s_\epsilon > 0$ such that, for sufficiently large i, if $s > s_\epsilon$, there is a multi-loop M_s such that

$$d_{\mathcal{T}}(\mathrm{gr}^{d_i \exp(s)}_{L_i}(\sigma_i), \mathrm{gr}_{M_s}(\sigma_i)) < \frac{\epsilon}{3}.$$

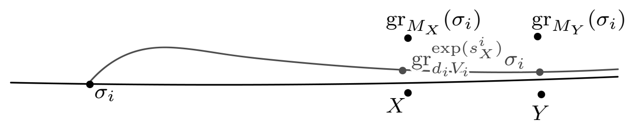

FIGURE 18. Approximating the Riemann surfaces X and Y by integral grafting

If i is sufficiently large, then $t_i < -s_\epsilon$. By this inequality, there are multiloops $M_X = M_{X,i}$ and $M_Y = M_{Y,i}$ on S with weight in $2\pi\mathbb{Z}_{>0}$ such that

$$d_{\mathcal{T}}(\mathrm{gr}_{L_i}^{d_i \exp(s^i_X)}(\sigma_i), \mathrm{gr}_{M_X}(\sigma_i)) < \frac{\epsilon}{3}$$

and

$$d_{\mathcal{T}}(\mathrm{gr}_{L_i}^{d_i \exp(s^i_Y)}(\sigma_i), \mathrm{gr}_{M_Y}(\sigma_i)) < \frac{\epsilon}{3}$$

By combining the inequalities above and the triangle inequality, we obtain

$$d_{\mathcal{T}}(X, \mathrm{gr}_{M_{X,i}}(\sigma_i)) < \epsilon$$

and

$$d_{\mathcal{T}}(Y, \mathrm{gr}_{M_{Y,i}}(\sigma_i)) < \epsilon.$$

(See Figure 18.)

The holonomy representation of the marked hyperbolic surface σ_i is a discrete and faithful representation $\rho_i\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{R}$ unique up to conjugation by $\mathrm{PSL}_2\mathbb{R}$. Since $R(t_i) = R_i(0)$ leaves every compact in $\mathcal{T}$ as $i \to \infty$, thus σ_i diverges to infinity and accordingly ρ_i diverges to infinity in the character variety $\mathcal{X}$. Thus ρ_i leaves every compact subset in the character variety as $i \to \infty$.

Then $\mathrm{Gr}_{M_X}(\sigma_i)$ is a $\mathbb{C}\mathrm{P}^1$-structure with holonomy ρ_i and its underlying Riemann surface structure is ϵ-close to X, and $\mathrm{Gr}_{M_Y}(\sigma_i)$ is a $\mathbb{C}\mathrm{P}^1$-structure with holonomy ρ_i and its underlying Riemann surface structure is ϵ-close to Y in the Teichmüller metric. Therefore, by the condition of ϵ when being picked, $(\mathrm{gr}_{M_X}(\sigma_i), \mathrm{gr}_{M_Y}(\sigma_i)) \in W$. As U_i is a bounded subset of $\mathcal{B}$, $\cup_{h=1}^n \mathrm{Hol}(U_h)$ is a bounded subset of $\mathcal{X}$. Thus, if i is sufficiently large, then $\rho_i \notin \cup_h \mathrm{Hol}(U_h)$. Therefore $(\mathrm{Gr}_{M_X}\sigma_i, \mathrm{Gr}_{M_Y}\sigma_i) \in \mathcal{B}$ has holonomy outside of $\cup_h \mathrm{Hol}(U_h)$ and the pair of their Riemann surface structures is in W, as desired.

7.2. **When the orientations of X and Y are the opposite.** We last prove Proposition 7.2, supposing that the orientations of X and Y are opposite. The proof is basically the same as in the other case (§7.1) if we appropriately reverse the orientation of the surface.

We can assume, without loss of generality, that $X \in \mathcal{T}$ and $Y \in \mathcal{T}^*$. Let Y^* be the complex conjugate of Y, so that $Y^* \in \mathcal{T}$.

Similarly to §7.1, pick $\epsilon > 0$ so that the product of the ϵ-negihborhood of X in $\mathcal{T}$ and the ϵ-neighborhood of Y is contained in

$$W = W_1 \cap W_2 \cap \cdots \cap W_n.$$

Let $R\colon \mathbb{R} \to \mathcal{T}$ be a "generic" Teichmuller ray in $\mathcal{T}$ passing the $\epsilon/3$-neighborhood of X and the $\epsilon/3$-neighborhood of the complex conjugate Y^* such that

- its corresponding quadratic differential has only simple zeros, and
- $R(t)$ is dense in the moduli space $\mathcal{M}$ of Riemann surfaces as $t \to -\infty$.

Let $t_1 > t_2 > \dots$ be a sequnce such that

- $t_i \to -\infty$ as $i \to \infty$;
- the unmarked Riemann surface $[R(t_i)]$ converges to Z in the moduli space $\mathcal{M}$ as $i \to \infty$;
- the tangent vectore $[R'(t_i)]$ converges in the unite tangent vector of $\mathcal{M}$ at Z as $i \to \infty$.

For each $i = 1, 2, \dots$, let σ_i be the marked hyperbolic structure on S uniformizaing $R(t_i)$. Let $\rho_i\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{R}$ be the discrete faithful representation corresponding to the hyperbolic surface σ_i.

As $X, Y^* \in \mathcal{T}$, by Section 7.1, for sufficiently large i,

- $\rho_i \notin \cup_{i=1}^n \mathrm{Hol}(U_i)$,
- there are a multiloop M_X and M_Y on σ_i such that
$$d(X, \mathrm{gr}_{M_X}\sigma_i) < \epsilon, d(Y^*, \mathrm{gr}_{M_{Y^*}}\sigma_i) < \epsilon,$$
- $(\mathrm{Gr}_{M_X}(\sigma_i), \mathrm{Gr}_{M_{Y^*}}(\sigma_i)) \in \mathcal{B}$.

Clearly, the conjugate σ^* of the hyperbolic structure σ is a hyperbolic structure on S^* with holonomy ρ_i. Let M_Y denote the multiloop on Y corresponding to M_{Y^*} on Y^* by the complex conjugation, so that M_Y and M_{Y^*} represent the same loop on the unoriented surface Σ. Therefore $d(Y^*, \mathrm{gr}_{M_{Y^*}}\sigma_i) < \epsilon$ implies $d(Y, \mathrm{gr}_{M_Y}\sigma_i^*) < \epsilon$. Hence $(\mathrm{gr}_{M_X}\sigma_i, \mathrm{gr}_{M_Y}\sigma_i^*) \in W$. Therefore, if i is sufficiently large, the projective grafting pair $(\mathrm{Gr}_{M_X}\sigma_i, \mathrm{Gr}_{M_Y}\sigma_i^*)$ in $\mathcal{B}$ has holonomy outside $\cup_{i=1}^n \mathrm{Hol}(U_i)$, and the pair of their Riemann surface structures is in W, as desired.

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