# $2\pi$-GRAFTING AND COMPLEX PROJECTIVE STRUCTURES, I

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ABSTRACT. Let $S$ be a closed oriented surface of genus at least two. Gallo, Kapovich, and Marden [13] asked if $2\pi$-grafting produces all projective structures on $S$ with arbitrarily fixed holonomy (Grafting Conjecture). In this paper, we show that the conjecture holds true "locally" in the space $\mathcal{GL}$ of geodesic laminations on $S$ via a natural projection of projective structures on $S$ into $\mathcal{GL}$ in Thurston coordinates. In the sequel paper ([1]), using this local solution, we prove the conjecture for generic holonomy.

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## 1. INTRODUCTION

Let $F$ be a connected and oriented surface. A *(complex) projective structure* on $F$ is a $(\hat{\mathbb{C}}, \mathrm{PSL}(2,\mathbb{C}))$-structure, where $\hat{\mathbb{C}}$ is the Riemann sphere. Equivalently, a projective structure on $F$ is a pair $(f, \rho)$ of

- an immersion $f\colon \tilde{F} \to \hat{\mathbb{C}}$ (*developing map)*, where $\tilde{F}$ is the universal cover of $F$, and
- a homomorphism $\rho\colon \pi_1(F) \to \mathrm{PSL}(2,\mathbb{C})$ (*holonomy representation*)

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such that $f$ is $\rho$-equivariant, i.e. $f \cdot \gamma = \rho(\gamma) \cdot f$ for all $\gamma \in \pi_1(F)$; see for example [32, §3.4]. The pair $(f, \rho)$ is defined up to an element $\alpha$ of $\mathrm{PSL}(2,\mathbb{C})$, i.e. $(f,\rho) \sim (\alpha f,\, \alpha\, \rho\, \alpha^{-1})$. (For general background about projective structures, see [10, 24].) Throughout this paper let $S$ be a closed oriented surface of genus $g > 1$.

We aim to characterize the set $\mathcal{P}_\rho$ of all projective structures with fixed holonomy $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$. This basic question is discussed in [18, p 274] [23, §7.1, Problem 2] [13, Problem 12.1.1.] [10, §1]; see also [14, §1.10]. This aims for understanding of the geometry behind general representations $\pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$, which are not necessarily discrete.

Let $\mathcal{P}$ be the space of all (marked) projective structures on $S$, and let $\chi$ be the $\mathrm{PSL}(2,\mathbb{C})$-character variety of $S$, i.e. the space of homomorphisms $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$, roughly, up to conjugation by an element of $\mathrm{PSL}(2,\mathbb{C})$; see [24]. Then there is an obvious forgetful map $\mathrm{Hol}\colon \mathcal{P} \to \chi$, called the *holonomy map*. Clearly $\mathcal{P}_\rho$ is a fiber of Hol. In addition $\mathcal{P}$ is diffeomorphic to $\mathbb{R}^{2(6g-6)}$ and moreover it enjoys a natural complex structure (see [10]). Then Hol is a local biholomorphism ([17, 18, 11]), and thus $\mathcal{P}_\rho$ is a discrete subset of $\mathcal{P}$.

There is a surgery operation of a projective structure, called *(2π-)grafting*, that produces a different projective structure, preserving its holonomy representation (§3.2): It inserts a cylinder along an appropriate essential loop (*admissible loop*) on a projective surface. Given $n \in \mathbb{Z}_{>0}$, we can graft a projective surface $n$ times along the same admissible loop; we denote it by assigning weight $2\pi n$ to the loop.

If there are disjoint admissible loops on a projective surface, we can simultaneously graft along all loops and obtain a new projective structure with the same holonomy. Similarly we use a multiloop with $2\pi$-multiple weights (*weighted multiloop*) to specify a general grafting along a multiloop.

For some special discrete representations $\pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$, graftings are known to produce all projective structures in $\mathcal{P}_\rho$ ([15, 20, 3]; see also §1.1). Then, more generally, Gallo, Kapovich and Marden [13, Problem 12.1.2] asked the following question.

**Question 1.1** (Grafting Conjecture)**.** *Given two projective structures sharing (arbitrary) holonomy $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$, is there a sequence of graftings and ungraftings that transforms one to the other?*

Holonomy representations are quite general. In fact, a homomorphism $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ is the holonomy representation of some projective structure (on $S$) if and only if $\rho$ satisfies:

(i) $\mathrm{Im}(\rho)$ is a nonelementary subgroup of $\mathrm{PSL}(2,\mathbb{C})$ and

(ii) $\rho$ lifts to $\tilde{\rho}\colon \pi_1(S) \to \mathrm{SL}(2, \mathbb{C})$

([13]). Recall that the character variety $\chi$ consists of two connected components ([16]), one of which consists of the representations with the lifting property in (ii). Thus Hol is almost onto this component. In particular, holonomy representations are *not* necessarily discrete or faithful, and many holonomy representations have dense images in $\mathrm{PSL}(2, \mathbb{C})$ (c.f. [28, Lemma 2.1]). Moreover if $\rho\colon \pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$ satisfies (i) and (ii) then $\mathcal{P}_\rho$ contains infinitely many distinct projective structures, which can be constructed by grafting (implicitly in [13]; see also [2]).

1.1. **Projective structures with fuchsian holonomy.** We recall the characterization of $\mathcal{P}_\rho$ when $\rho\colon \pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$ is a discrete and faithful representation into $\mathrm{PSL}(2, \mathbb{R})$, called a *fuchsian (holonomy) representation.* Then $\mathrm{Im}(\rho) =: \Gamma$ is called a *fuchsian group*, and its domain of discontinuity is a union of two disjoint round disks in $\hat{\mathbb{C}}$. Then, by quotienting out the domain by $\Gamma$, we obtain two distinct projective structures with fuchsian holonomy $\rho$ (*uniformizable projective structures*), which have different orientations. Let $C_0$ denote the one of our fixed orientation. Then $C_0$ is isomorphic to the hyperbolic surface $\mathbb{H}^2/\,\mathrm{Im}(\rho)$ as projective surfaces and every essential loop on $C_0$ is admissible.

**Theorem 1.2** (Goldman [15]; also [22])**.** *If $C \in \mathcal{P}_\rho$, then $C$ is obtained by grafting $C_0$ along a weighted multiloop $M$,*

$$C = \mathrm{Gr}_M(C_0).$$

In Theorem 1.2, $M$ is unique up to an isotopy, and the same assertion holds moreover for *quasifuchsian representations* (although the proof is easily reduced to a fuchsian case by a quasiconformal map). Let $\mathcal{ML}$ be the space of measured laminations on $S$. Then $\mathcal{P}_\rho$ is naturally identified with the discrete subset $\mathcal{ML}_{\mathbb{N}}$ of $\mathcal{ML}$ that consists of weighted multiloops.

Let $C$ and $C'$ be the projective structures sharing the fuchsian holonomy $\rho$. Then $C = Gr_M(C_0)$ and $C' = \mathrm{Gr}_{M'}(C_0)$ for unique weighted multiloops $M$ and $M'$ on $C_0$ by Theorem 1.2. Then it follows from Ito's work ([19, Theorem 1.3]) that:

**Theorem 1.3.** *$C$ and $C'$ can be transformed to a common projective structure in $\mathcal{P}_\rho$ by grafting $C$ along $M'$ and $C'$ along $M$,*

$$\mathrm{Gr}_{M'}(C) = \mathrm{Gr}_M(C')$$

*(see also* [6]*).*

1.2. **Thurston coordinates.** (More details in §3.3.) In a geometric manner, Thurston gave a natural homeomorphism

$$\mathcal{P} \to \mathcal{T} \times \mathcal{ML},$$

where $\mathcal{T}$ is the space of marked hyperbolic structures $S$ (Teichmüller space). Thus, given $C \in \mathcal{P}$ , we denote *its Thurston coordinates* by $C \cong (\tau, L)$ with $\tau \in \mathcal{T}$ and $L \in \mathcal{ML}$.

For example, suppose that a projective structure $C \cong (\tau, L)$ has fuchsian holonomy $\rho\colon \pi_1(S) \to \mathrm{PSL}(2, \mathbb{R})$. Then $\tau$ is $\mathbb{H}^2/\operatorname{Im}(\rho)$, and $L$ is the weighted multiloop $M$ given by Theorem 1.2, so that $C = \mathrm{Gr}_L(\tau)$; see [15].

Without restricting holonomy, let $C \cong (\tau, L) \in \mathcal{P}$. Suppose that there is a weighted multiloop $M$ on $\tau$ such that each loop of $M$ does not intersect $L$ transversally (for example, $M$ is supported on some closed leaves of $L$). Note that, by the transversality, the addition $M+L$ is a well-defined measured lamination. Then there is a corresponding circular admissible (weighted) multiloop $\mathcal{M}$ on $C$ that is equal to $M$ in $\mathcal{ML}$, such that $\mathrm{Gr}_{\mathcal{M}}(C) \cong (\tau, L + M)$. Then we may simply write $\mathrm{Gr}_M(C)$ to denote $\mathrm{Gr}_{\mathcal{M}}(C)$, abusing notation.

Given a projective structure $C \cong (\tau, L)$ in Thurston coordinates, there is a natural marking preserving map $\kappa\colon C \to \tau$, called the collapsing map, which is a diffeomorphism except on the inverseimage of the closed leaves of $L$. Nonetheless, there is a natural measured lamination $\mathcal{L}$ on $\tau$ such that leaves of $\mathcal{L}$ are circular and $\mathcal{L}$ descends to $L$ by $\kappa$. Thus $\mathcal{L}$ is a natural representative of $L$ on $C$. (See §3.3.)

1.3. **Traintracks and measured laminations.** (See §6.3 for details.) A *(fat) traintrack* $T$ on a surface is a subsurface that is a union of rectangles (*branches*) with disjoint interiors glued along vertical edges in a certain manner. We say that a traintrack *carries* a measured lamination if $T$ contains a measured lamination in a natural manner without "backtracks". Then the transversal measure assigns each branch of $T$ a non-negative real number (*weight*).

When a single traintrack $T$ carries two measured laminations $L$ and $M$, the *sum* $L+M$ is defined to be a measured lamination carried by $T$ that is given by adding the weights of $L$ and $M$ branch-wise. Similarly, when appropriate, we obtain the *difference* $L-M$ that is a measured lamination carried by $T$ represented by the differences on the weights of $L$ and $M$.

1.3.1. *Existence of admissible traintracks.* Given an admissible loop $\ell$ on a projective surface $C$, an isotopy of $\ell$ on $C$ does not necessarily keep $\ell$ admissible. On the other hand, given a loop $\ell$ on $C$ whose holonomy is loxodromic, in general it is hard to tell if $\ell$ can be isotoped an admissible loop. Thus, we introduce *admissible traintracks* in order to specify admissible loops (Definition 6.1), still allowing a "uniform amount" of isotopies. If a traintrack $\mathcal{T}$ on $C$ is admissible, then it is foliated by circular arcs parallel to vertical edges (§6.1). Indeed, if a loop $\ell$ is carried by $\mathcal{T}$ and it is transversal to this *circular foliation*, then $\ell$ is admissible (Lemma 7.2). Note that we do *not* need to isotope $\ell$ to make it admissible, and in addition $\ell$ stays admissible under an isotopy through such transversal loops carried by $\mathcal{T}$. Moreover, such an isotopy preserves the resulting projective structure $\mathrm{Gr}_\ell(C)$. In fact

**Corollary 7.5.** Given $C \cong (\tau, L) \in \mathcal{P}$ and a geodesic lamination $\nu$ on $\tau$ containing the underlying lamination $|L|$, there is an admissible traintrack on $C$ fully carrying $\nu$.

Suppose that there is an admissible traintrack $\mathcal{T}$ on $C \in \mathcal{P}$, and let $\ell$ be a loop carried by $\mathcal{T}$ transversal to the circular foliation so that $\ell$ is admissible. Then the grafting of $C$ along $\ell$ restricts to the grafting of $\mathcal{T}$ along $\ell$. Then $\mathrm{Gr}_\ell(\mathcal{T})$ is naturally an admissible traintrack on $\mathrm{Gr}_\ell(C)$. In this paper, in oder to compare different projective structures sharing holonomy, we construct admissible traintracks on them that are related by grafting.

In [2], given arbitrary $C \cong (\tau, L) \in \mathcal{P}$, the author constructed an admissible loop $\ell$ on $C$, so that $\ell$ is a good approximation of a minimal sublamination of $L$ in the Chabauty topology on the space $\mathcal{GL}$ of geodesic laminations (using the "Closing Lemma" in [7, I.4.2.15]). Corollary 7.5 a provides more general way of constructing admissible loops. In particular, if a loop on $\tau$ is close to $L$ in $\mathcal{ML}$ or if it intersects leaves of $L$ only at uniformly small angles (see Definition 1.5), then there is an admissible loop on $C$ in the same isotopy class.

1.4. **Local characterization of $\mathcal{P}_\rho$ in $\mathcal{PML}$.** Let $\mathcal{PML}$ be the space of projective measured laminations on $S$. Note that $\mathcal{PML}$ is homeomorphic to the sphere of dimension $6g-7$. We show that, if two projective structures in $\mathcal{P}_\rho$ are close in $\mathcal{PML}$ in Thurston coordinates, then they are related by a single grafting along a weighted multiloop:

**Theorem A.** (see Theorem 8.7.)

*Let $C \cong (\tau, L)$ be a projective structure on $S$ with (arbitrary) holonomy $\rho$. Then for every $\epsilon > 0$, there is a neighborhood $U$ of the projective class $[L]$ in $\mathcal{PML}$ such that if another projective structure $C' \cong (\tau', L')$ with holonomy $\rho$ satisfies $[L'] \in U$, then we have either*

(i) $[L] = [L']$, *and* $L - L'$ *is a weighted multiloop* $M'$ *such that* $C = Gr_{M'}(C')$, *or*

(ii) *there are an admissible traintrack* $\mathcal{T}$ *on* $C$ *carrying both* $L$ *and* $L'$ *and a weighted multiloop* $M$ *carried by* $\mathcal{T}$, *such that* $M$ *is* $\epsilon$*-close to* $L' - L$ *on* $\mathcal{T}$ *(§6.3) and*

$$Gr_M(C) = C'.$$

**Remark 1.4.** *In* (ii)*, by "*$\epsilon$*-close", we mean, roughly, that* $M$ *is a good approximation of* $L - L'$ *for sufficiently small* $\epsilon > 0$ *(see §6.3).*

*Case* (i) *may happen only when* $L$ *and* $L'$ *are both multiloops. Since generic measured laminations are not multiloops, for generic* $C \in \mathcal{P}$ *, only* (ii) *occurs.*

*In the case of* (i)*, the weight of* $L$ *is larger than that of* $L'$ *on every branch; whereas, in the case of* (ii)*, the weight is smaller on every branch. This dichotomy is due to the discreteness of* $\mathcal{P}_\rho$ *in* $\mathcal{P}$ *and the smallness of* $U$*.*

Let $\rho\colon \pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$ be a fuchsian representation. Let $C$ be $\mathbb{H}^2/\operatorname{Im}\rho =: \tau$, the uniformizable structure with holonomy $\rho$ as in §1.1. Then $C \cong (\tau, \emptyset)$, where $\emptyset$ is the empty lamination. Then, for every $C' \cong (\tau, M)$ with holonomy $\rho$, we have $C' = \mathrm{Gr}_M(C)$ by Theorem 1.2. Thus theorem A (i) holds true with $U = \mathcal{PML}$ and $M = M - \emptyset$. Hence Theorem A generalizes Theorem 1.2.

### 1.5. Local characterization of $\mathcal{P}_\rho$ in $\mathcal{GL}$.

Let $\mathcal{GL}$ be the set of geodesic laminations on $S$. Naturally $\mathcal{ML}$ projects to $\mathcal{GL}$ by forgetting transversal measures. Theorem B below gives a local characterization $\mathcal{P}_\rho$ in $\mathcal{GL}$ analogous to Theorem A. Note that geodesic laminations are more essential to pleated surfaces than measured laminations are. Indeed, Theorem B is essentially stronger than Theorem A, and in particular it generalizes not only Theorem 1.2 but also Theorem 1.3.

**Definition 1.5.** *If* $\ell$ *and* $\ell'$ *are simple geodesics on a hyperbolic surface* $\tau$ *intersecting at a point* $p$*, we let* $\angle_p(\ell, \ell')$ *denote the angle, taking a value in* $[0, \pi/2]$*, between* $\ell$ *and* $\ell'$ *at* $p$*. Let* $\lambda$ *and* $\lambda'$ *be (possibly measured) geodesic laminations on* $\tau$*. Then the* <u>*angle*</u> *between* $\lambda$ *and* $\lambda'$ *is*

$$sup_p \angle_p(\ell_p, \ell'_p) \ \in [0, \pi/2],$$

*taken over all points* $p$ *in the intersection of* $\lambda$ *and* $\lambda'$*, where* $\ell_p$ *and* $\ell'_p$ *are the leaves of* $\lambda$ *and* $\lambda'$*, respectively, intersecting at* $p$*. We denote this angle by* $\angle_\tau(\lambda, \lambda')$ *or simply* $\angle(\lambda, \lambda')$*.*

In Definition 1.5, if $\lambda$ or $\lambda'$ is a geodesic lamination on a different hyperbolic surface homeomorphic to $\tau$, then we always take its geodesic representative on $\tau$ in order to measure the angle $\angle_\tau(\lambda, \lambda')$.

Let $C \cong (\tau, L)$ be a projective structure on $S$ with holonomy $\rho\colon \pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$. Then $L$ determines whether $C$ is obtained by grafting another projective structure in an obvious way as described in §1.2. If $L$ contains a closed leaf $\ell$ and its weight $w(\ell)$ is equal to or more than $2\pi$, then change the weight of $\ell$ by subtracting a $2\pi$-multiple so that $0 \leq w(\ell) < 2\pi$. By applying this weight reduction to all closed leaves of $L$, we obtain a measured lamination $L_0$ such that every closed leaf of $L_0$ has weight less than $2\pi$. Let $M = L - L_0$, so that $M$ is a weighted multiloop.

Throughout this paper, let $C_0$ denote the projective structure given by $(\tau, L_0)$ in Thurston coordinates, with $L_0$ as above. Then $C = \mathrm{Gr}_M(C_0)$, and the holonomy of $C_0$ is also $\rho$. Note that, since generic $L \in \mathcal{ML}$ contains no closed loops, for generic $C \in \mathcal{P}$ in Thurston coordinates, we have $C = C_0$ and $M = \emptyset$.

Then analogues of Theorem 1.2 and Theorem 1.3 hold for projective structures in $\mathcal{P}_\rho$ whose geodesic laminations are close to $L$ in terms of the angle defined above:

**Theorem B.** *(See Theorem 8.1.) For every $\epsilon > 0$ and every projective structure $C \cong (\tau, L)$ on $S$ with holonomy $\rho$, there exists $\delta > 0$, such that, if another projective structure $C' \cong (\tau', L')$ with holonomy $\rho$ satisfies $\angle_\tau(L, L') < \delta$, then there are admissible traintracks on $\mathcal{T}_0$, $\mathcal{T}$, and $\mathcal{T}'$ on $C_0$, $C$, $C'$, respectively, that are isotopic on $S$ and carry both $L$ and $L'$ (thus also $L_0$), so that*

(i) *$C'$ is obtained by grafting $C_0$ along a weighted multiloop $M'$ carried by $\mathcal{T}_0$, such that $M'$ is $\epsilon$-close to the measured lamination given by $L' - L_0$ on $\mathcal{T}_0$, and*

(ii) *if weighted multiloops $\hat{M}$ and $\hat{M}'$ are carried by $\mathcal{T}$ and $\mathcal{T}'$, respectively, and $\hat{M} + M = \hat{M}' + M'$ on the traintracks, then we have*
$$\mathrm{Gr}_{\hat{M}}(C) = \mathrm{Gr}_{\hat{M}'}(C').$$

(See Figure 1.) In (ii), there are infinitely many choices for $\hat{M}$ and $\hat{M}'$ satisfying the equality; in particular we can let $\hat{M} = M'$ and $\hat{M}' = M$.

Moreover, given a compact subset $K$ in the moduli space of (unmarked) hyperbolic structures on $S$, there is $\delta > 0$ such that Theorem B holds for all projective structures $C \cong (\tau, L)$ on $S$ with its unmarked $\tau$ in $K$; see Theorem 8.4. In addition Theorem B (i) implies that, if $L$

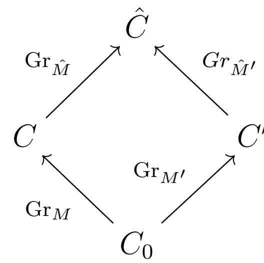


FIGURE 1.

contains no closed leaves of weight at least $2\pi$, then $C'$ is obtained by grafting $C$ along a multiloop.

In the case that $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ is fuchsian, Theorem B (i) corresponds to Theorem 1.2: Let $C \cong (\tau, \emptyset)$ denote the unique uniformizable structure in $\mathcal{P}_\rho$. Then indeed $\angle_\tau(\emptyset, L') = 0$ for every $L' \in \mathcal{ML}$. For generic $C' \in \mathcal{P}_\rho$, we have $C_0 = C$, and $C'$ is obtained by grafting $C$ along the weighted multiloop $L' - \emptyset$. Moreover Theorem B (ii) implies Theorem 1.3 (see Theorem 8.6).

1.6. **Pleated surfaces.** Consider (abstract) pleated surfaces equivariant via a fixed representation $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$. We show some continuity of $\rho$-equivariant pleated surfaces in terms of their pleating laminations. (In comparison, [4, 25] yield continuity of the pleated surfaces bounding the convex cores of hyperbolic three-manifolds, when associated discrete representations vary.)

The correspondence between a projective structure $C = (f, \rho)$ and its Thurston coordinates $(\tau, L)$ is given via a pleated surface $\beta\colon \mathbb{H}^2 \to \mathbb{H}^3$ (§3.1, §3.3). In particular $\beta$ is equivariant under the holonomy representation of $C$, and it "realizes" $(\tau, |L|)$, where $|L|$ is the underlying geodesic lamination of $L$. A pair of $\tau \in \mathcal{T}$ and $\lambda \in \mathcal{GL}$ is *realized* by a pleated surface $\beta\colon \mathbb{H}^2 \to \mathbb{H}^3$, if $\beta$ bend $\mathbb{H}^2$ (in $\mathbb{H}^3$) exactly along the *total lift* $\tilde{\lambda}$ of $\lambda$ to $\mathbb{H}^2$ (§2) and it is totally geodesic elsewhere (this is slightly stronger than a usual notion of a realization); see §3.3.1.

Then, in fact, the assumptions in Theorem A and B can be interpreted in term of pleated surfaces, by the following theorem.

**Theorem C** (See Theorem 5.2 for the precise statement.) *Let* $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ *be a homomorphism. Suppose that there is a $\rho$-equivariant pleated surface* $\beta\colon \mathbb{H}^2 \to \mathbb{H}^3$ *realizing* $(\tau, \lambda) \in \mathcal{T} \times \mathcal{GL}$.

*For every $\epsilon > 0$, there exists $\delta > 0$ such that, if there is another $\rho$-equivariant pleated surface* $\beta'\colon \mathbb{H}^2 \to \mathbb{H}^3$ *realizing* $(\sigma, \nu) \in \mathcal{T} \times \mathcal{GL}$

*with $\angle_\tau(\lambda, \nu) < \delta$, then $\sigma$ is $\epsilon$-close to $\tau$ in $\mathfrak{T}$ and $\beta'$ and $\beta$ are $\epsilon$-close.*

If we apply Theorem C to pleated surfaces associated with projective structures, Theorem A and B may seem natural. For example, in Theorem B, if $\delta > 0$ is sufficiently small, then $\tau'$ must be very close to $\tau$ in $\mathfrak{T}$ by Theorem C. Thus the differences of the projective structures $C, C_0, C'$ are captured, mostly, by the differences of the measured laminations in Thurston coordinates.

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1.7. **Outline of the proofs.** *Theorem C.* (§5.) The closeness of $\tau$ and $\tau'$ and of $\beta$ and $\beta'$ is given by constructing a marking-preserving homeomorphism $\phi\colon \tau \to \sigma$ that is almost an isometry (more precisely, it is a rough isometry with small distortion). We here outline the construction of $\phi$. First we define a homeomorphism $\psi$ from the geodesic representative $\nu_\tau$ of $\nu$ on $\tau$ onto the geodesic lamination $\nu$ on $\sigma$ so that it induces bilipschitz maps of small distortion between corresponding leaves: If $\ell$ and $\ell'$ are corresponding leaves of the total lifts $\tilde{\nu}_\tau$ and $\tilde{\nu}$ to $\mathbb{H}^2$, then $\beta'|\ell'$ is a geodesic in $\mathbb{H}^3$ and, since $\angle_\tau(\lambda, \nu)$ is sufficiently small, $\beta|\ell$ is a bilipschitz embedding of small distortion (by Proposition 4). Then $\beta(\ell)$ and $\beta'(\ell')$ are Hausdorff-close and they share their endpoints on $\hat{\mathbb{C}}$. The nearest point projection of $\beta(\ell)$ onto the geodesic $\beta'(\ell')$ yields a desired bilipschitz map $\ell \to \ell'$. By applying this to all corresponding leaves of $\tilde{\nu}_\tau$ and $\tilde{\nu}$, we obtain $\psi\colon \nu_\tau \to \nu$.

Extend $\nu_\tau$ and $\nu$ to maximal laminations on $\tau$ and $\sigma$ that are isomorphic (as topological laminations), so that they divides $\tau$ and $\sigma$ into ideal triangles. Then we extend $\psi\colon \nu_\tau \to \nu$ to $\phi\colon \tau \to \sigma$, so that $\psi$ is a quasi-isometry of small distortion between all corresponding complementary ideal triangles. It turns out that $\psi$ almost preserves horocycle laminations of the triangulation, and therefore $\psi$ has almost no "shearing" between nearby ideal triangles. Thus $\psi\colon \tau \to \sigma$ is almost an isometry.

*Theorem A.*(The proof of Theorem B is similar) If $\angle_\tau(L, L') > 0$ is sufficiently small, then we can apply theorem C to the pleated surfaces corresponding to $C$ and $C'$; then we can naturally identify $\tau$ and $\tau'$ by an almost isometric homeomorphism preserving the marking.

There is a nearly-straight traintrack $T$ on $\tau$ carrying $L$ (Lemma 7.10). By the almost isometry between $\tau$ and $\tau'$, we can regard $T$ also as a nearly-straight traintrack on $\tau'$ carrying $L'$. Then $T$ yields corresponding traintracks $\mathcal{T}$ on $C$ and $\mathcal{T}'$ on $C'$ that descend to $T$ via the collapsing maps $C \to \tau$ and $C' \to \tau'$. Moreover $\mathcal{T}$ and $\mathcal{T}'$ decompose $C$ and $C'$, isomorphic, into subsurfaces in a compatible manner (§7). In particular, $C \setminus \mathcal{T}$ and $C' \setminus \mathcal{T}'$ are isomorphic (as projective surfaces). In addition, if $\mathcal{B}$ and $\mathcal{B}'$ are corresponding branches of $\mathcal{T}$ and $\mathcal{T}'$, respectively, then they are related by a grafting along a multiarc. Then the multiloop for each grafting in Theorem A is obtained as the union of such multiarcs (see §8, c.f. [3]).

Moreover the number of the arc times $2\pi$ is approximately the difference of the weights of $L$ and $L'$ on the branch of $T$ corresponding to $\mathcal{B}$ and $\mathcal{B}'$. Accordingly $M$ is approximately the difference of $L$ and $L'$ on the traintrack $T$.

## 2. Conventions and notation

- Given a projective structure $C \cong (\tau, L)$ on $S$ in Thurston coordinates, we let $C_0$ denote the "reduced" structure $(\tau, L_0)$ constructed in §1.5 .
- By a *component*, we mean a *connected* component.
- For a geodesic metric space $X$ and points $x, y \in X$, we denote the geodesic segment connecting $x$ to $y$ by $[x, y]$. Then $\mathrm{length}[x, y]$ denotes the length of $[x, y]$.
- Let $X$ be a manifold, and let $Y$ be a subset of $X$. Given a covering map $\phi\colon \tilde{X} \to X$, the *total lift* of $Y$ is the inverse image $\phi^{-1}(Y)$.
- We say two submanifolds intersect (at a point) $\epsilon$-*nearly orthogonally* for $\epsilon > 0$, if the intersection angle is $\epsilon$-close to $\pi/2$.
- By a *loop*, we mean a simple closed curve.
- By a *marking homeomorphism*, we mean a homeomorphism that represents a given marking on a geometric structure.

## 3. Preliminaries

3.1. **Geodesic laminations and pleated surfaces.** (See [7, 9] for details) A *geodesic lamination* $\lambda$ on a hyperbolic surface $\tau$ is a set of disjoint simple geodesics whose union is a closed subset of $\tau$. The simple geodesics of $\lambda$ are called *leaves.* Let $|\lambda|$ denote the closed subset. Occasionally, $\lambda$ may refer to the closed subset $|\lambda|$, when it is clear from the context. A geodesic lamination $\lambda$ is *minimal* if there is no nonempty sublamination of $\lambda$.

A *measured (geodesic) lamination* $L$ on $\tau$ is a pair $(\lambda, \mu)$ of a geodesic lamination $\lambda$ and a transversal measure $\mu$ of $\lambda$. If $L$ is non-empty, by identifying $\mu$ with its scalar multiples by positive real numbers, we obtain a projective measured lamination $[L]$ of $L$.

Given a geodesic lamination $\lambda$ on a hyperbolic surface $\tau$, a *stratum* is a leaf of $\lambda$ or the closure of a (connected) component of $\tau \setminus |\lambda|$. A *pleated surface* $\beta\colon \mathbb{H}^2 \to \mathbb{H}^3$ *realizing* a geodesic lamination $\lambda$ on $\mathbb{H}^2$ is a continuous map such that $\beta$ preserves the lengths of paths on $\mathbb{H}^2$ and it isometrically embeds each stratum of $(\mathbb{H}^2, \lambda)$ into a (totally geodesic) hyperbolic plane in $\mathbb{H}^3$. More generally, if $\lambda$ is a geodesic lamination on a hyperbolic surface $\tau$, then a pleated surface $\mathbb{H}^2 \to \mathbb{H}^3$ *realizes* $(\tau, \lambda)$ if it realizes the total lift of $\lambda$ to $\mathbb{H}^2$. If $\beta$ realizes $(\tau, \lambda)$, then, unless otherwise stated, we in addition assume that there is no proper sublamination of $\lambda$ that $\beta$ realizes, so that $\beta$ "exactly" realizes $(\tau, \lambda)$.

3.2. **Grafting.** (see [15, 24].) Let $C = (f, \rho)$ be a projective structure on $S$. A loop $\ell$ on $C$ is called *admissible* if

(i) $\rho(\ell) \in \mathrm{PSL}(2, \mathbb{C})$ is loxodromic, and
(ii) $f$ embeds $\tilde{\ell}$ into $\hat{\mathbb{C}}$, where $\tilde{\ell}$ is a lift of $\ell$ to the universal cover of $S$.

If $\ell$ is admissible, the loxodromic element $\rho(\ell)$ generates an infinite cyclic group $Z$ in $\mathrm{PSL}(2, \mathbb{C})$. Then its limit set $\Lambda(Z)$ is the union of the attracting and repelling fixed points of $\rho(\ell)$ (on $\hat{\mathbb{C}}$), and $Z$ acts on its complement $\hat{\mathbb{C}} \setminus \Lambda(Z)$ freely and properly discontinuously. Thus the quotient $(\hat{\mathbb{C}} \setminus \Lambda(Z))/Z$ is a projective torus $T_\ell$ (*Hopf torus*). Then, by (ii), $\ell$ is isomorphically embedded in $T_\ell$. Since $\ell$ is also a loop on $C$, there is a canonical way to combine the projective surfaces $C$ and $T_\ell$ by cutting and pasting along $\ell$ as follows. We see that $T_\ell \setminus \ell$ is a cylinder and $C \setminus \ell$ is a surface with two boundary components. Thus we obtain a new projective structure on $S$ by pairing up the boundary components of $T_\ell \setminus \ell$ and $C \setminus \ell$ in an alternating manner and isomorphically identifying them. This surgery operation is called ($2\pi$-)*grafting* of $C$ along $\ell$, and we denote the new projective structure by $\mathrm{Gr}_\ell(C)$. It turns out that $\rho$ is holonomy representation $\mathrm{Gr}_\ell(C)$.

3.3. **Thurston coordinates.** (see [21, 26] and also [10, 30, 2].)

We here explain more about the parametrization

$$\mathcal{P} \cong \mathcal{T} \times \mathcal{ML}. \tag{1}$$

discussed in §1.2.

For example, suppose that a projective structure $C \in \mathcal{P}$ is isomorphic, as a projective surface, to an ideal boundary component of a hyperbolic three-manifold. Then the Thurston coordinates of $C$ are the structure on the corresponding boundary component of the convex core of the three-manifold: a hyperbolic surface bent along a measured lamination.

3.3.1. *Bending maps.* Let $(\tau, L) \in \mathcal{T} \times \mathcal{ML}$, and regard the measured lamination $L$ as a geodesic measured lamination on the hyperbolic surface $\tau$. Set $L = (\lambda, \mu)$, where $\lambda \in \mathcal{GL}(S)$ and $\mu$ is a transversal measure supported on $\lambda$. Let $\tilde{L} = (\tilde{\lambda}, \tilde{\mu}) \in \mathcal{ML}(\mathbb{H}^2)$ be the total lift of $L$ to $\mathbb{H}^2$. Then there is a corresponding pleated surface $\mathbb{H}^2 \to \mathbb{H}^3$ obtained by bending a hyperbolic plane inside $\mathbb{H}^3$ along $\tilde{\lambda}$ by the angles given by $\tilde{\mu}$. This map is called the *bending map* $\beta\colon \mathbb{H}^2 \to \mathbb{H}^3$ *induced by* $(\tau, L)$. Then $\beta$ is unique up to a postcomposition with an element of $\mathrm{PSL}(2, \mathbb{C})$. If $C = (f, \rho) \in \mathcal{P}$ corresponds to $(\tau, L) \in \mathcal{T} \times \mathcal{ML}$ by (1), then we say that $\beta$ is the bending map *associated* with $C$. Since the $\pi_1(S)$-action on $\mathbb{H}^2$ preserves $\tilde{L}$, there is a homomorphism $\pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$ under which $\beta$ is equivariant. Then this homomorphism is unique up to a conjugation by an element of $\mathrm{PSL}(2, \mathbb{C})$, and it indeed is the holonomy representation $\rho$ of $C$.

On the other hand, given a measured lamination $L$ on $\mathbb{H}^2$, this pair $(\mathbb{H}^2, L)$ gives a projective structure on an open disk.

3.3.2. *Maximal balls and collapsing maps.* (See [26, §4],[21, §1.1].)

Let $C \in \mathcal{P}$. Let $\tilde{C}$ be the universal cover of $C = (f, \rho)$. An open topological ball $B$ in $\tilde{C}$ is called a *maximal ball* if the developing map $f\colon \tilde{C} \to \hat{\mathbb{C}}$ embeds $B$ onto a round open ball in $\hat{\mathbb{C}}$ and there is no such a ball in $\tilde{C}$ properly containing $B$. Let $B$ be a maximal ball, and let $H$ be the hyperplane in $\mathbb{H}^3$ bounded by the round circle $\partial f(B)$. Then, recalling $\hat{\mathbb{C}}$ is naturally the ideal boundary of $\mathbb{H}^3$, let $\Phi\colon f(B) \to H$ be the canonical conformal map obtained from the nearest point projection onto $H$. Let $\partial_\infty B$ be $\partial f(B) \setminus f(cl(B))$, where "*cl*" denotes the closure on $\tilde{C}$.

Suppose that $(\tau, L) \in \mathcal{T} \times \mathcal{ML}$ corresponds to $C = (f, \rho)$ in (1). Let $\beta\colon \mathbb{H}^2 \to \mathbb{H}^3$ be the bending map induced by $(\tau, L)$. Then the maximal $B$ corresponds to a *stratum* $X$ of $\tilde{L}$, which is either a leaf of $\tilde{L}$ or the closure of a component of $\mathbb{H}^2 \setminus |\tilde{L}|$. Indeed $\beta$ isometrically embeds $X$ into the hyperbolic plane bounded by $\partial_\infty f(B)$.

Clearly $\Phi \circ f$ embeds $B$ onto $H \subset \mathbb{H}^3$. The *core* of the maximal ball $B$, denoted by $\mathrm{Core}(B)$, is the convex hull of $\partial_\infty B$ with $B$ conformally identified with $\mathbb{H}^2$. Thus we have a unique embedding $\kappa_B$ of $\mathrm{Core}(B)$ onto $X \subset \mathbb{H}^2$ so that $\beta \circ \kappa_B = \Phi \circ f$ on $\mathrm{Core}(B)$. Therefore, for each $x \in \mathrm{Core}(B)$, the hyperplane $H$ is called a *hyperbolic support plane* of

$\beta$ at $x$. It turns out that, for different maximal balls $B$ in $\tilde{C}$, their cores Core($B$) are disjoint. Moreover $\tilde{C}$ decomposes into these cores. Therefore we can define a continuous map $\tilde{\kappa}\colon \tilde{C} \to \mathbb{H}^2$ by $\tilde{\kappa} = \kappa_B$ on Core($B$) for all maximal balls $B$. Then $\tilde{\kappa}$ commutes with the action of $\pi_1(S)$, and thus it descends to the *collapsing map* $\kappa\colon C \to \tau$, which respects the markings by homeomorphisms from $S$. (See [26, §8]), [21, §2.3]

3.3.3. *Canonial lamination on projective surfaces.* Each boundary component of Core($B$) is a biinfinite line properly embedded in $\tilde{C}$. Then, by taking the union of $\partial\,\mathrm{Core}(B)$ over all maximal balls $B$ in $\tilde{C}$, we obtain a (topological) lamination $\tilde{\nu}$ on $\tilde{C}$. Then $\tilde{\kappa}$ embeds each leaf of $\tilde{\nu}$ onto a leaf of $\tilde{\lambda}$. Since $\pi_1(S)$ preserves the decomposition of $\tilde{C}$ into the cores, $\tilde{\nu}$ descends to a lamination $\nu$ on $C$, and $\kappa$ embeds each leaf of $\mu$ onto a leaf of $\lambda$.

Moreover $\nu$ is equipped with a natural transversal measure $\omega$ so that $\mathcal{L} := (\nu, \omega)$ descends to $L$ by $\kappa$. Then $\mathcal{L}$ is called the *canonical measured lamination* on $C$. If $\alpha$ is a curve transversal to $\omega$ and it is of infinitesimal length, its transversal measure $\omega(\alpha)$ is the angle between the hyperbolic support planes in $\mathbb{H}^3$ corresponding the leaves of $\nu$ containing the endpoints of $\alpha$. The transversal measure $\omega$ is infinitesimally given by the angles between hyperplanes supporting of $\beta$.

Let $M$ be the union of the closed leaves $\ell_1, \ell_2, \ldots, \ell_n$ of $L$. In particular $\kappa$ is a $C^1$-diffeomorphism in the complement of $\kappa^{-1}(M)$, and there $\omega$ is exactly the pullback of $\nu$ by $\kappa$. We describe $\mathcal{L}$ on $\kappa^{-1}(M)$ below.

For each $i = 1, 2, \ldots, n$, $\kappa^{-1}(\ell_i)$ is a compact cylinder embedded in $C$. This cylinder is foliated by closed leaves $m$ of $\nu$ that are diffeomorphic to $\ell$ by $\kappa$. The total transversal measure of $\kappa^{-1}(\ell_i)$ is the weight $w_\mu(\ell_i)$ of $\ell_i$ given by $\mu$.

In addition, for every $s \in \ell$, $\kappa^{-1}(s)$ is a circular arc connecting the boundary circles of $\kappa^{-1}(\ell_i)$ and it is orthogonal to each closed leaf $\nu$ in $\kappa^{-1}(\ell_i)$.

3.3.4. *Thurston metric on projective structures.* Every projective surface $C$ on $S$ has a natural Hyperbolic/Euclidean type metric associated with $\kappa\colon (C, \mathcal{L}) \to (\tau, L)$.

The cylinder $\kappa^{-1}(\ell_i)$ has a natural Euclidean metric, and it is isometric to a product of a circle of length $\mathrm{length}_\tau(\ell_i)$ and the interval $[0, w_\mu(\ell_i)]$. The Riemannian metric respects the conformal structure of $C$ on $\kappa^{-1}(\ell_i)$. For each closed leaf $\ell$ of $\nu$ in $\kappa^{-1}(\ell_i)$, $\kappa|\ell$ is an isometry onto $\ell$. For each $s \in \ell$, the metric on the circular arc $\kappa^{-1}(s)$ is given by the transversal measure $\omega$. This Euclidean metric is the restriction of the Thurston metric on $C$ to $\kappa^{-1}(\ell_i)$.

On the other hand, the restriction of $\kappa\colon C \to \tau$ to $C \setminus \kappa^{-1}(M)$ is a $C^1$-diffeomorphism onto $\tau \setminus M$. Thus $C \setminus \kappa^{-1}(M)$ has the hyperbolic metric obtained by pulling back the hyperbolic metric of $\tau$ via $\kappa$.

On each stratum $R$ of $(C, \mathcal{L})$, the Thurston metric is the restriction of the Euclidean or Hyperbolic metric defined above. In this paper, it suffices to use the Thurston metric on each stratum. (If $L$ is a union of disjoint weighted loops, the Thurston metric on $C \cong (\tau, L)$ is the piecewise Euclidean/hyperbolic metric that is the sum of the Euclidean metric on the cylinders and the hyperbolic metric in the complement. For general $L$, we can take a sequence of weighted loops $\ell_i$ converging to $L$ as $i \to \infty$. Then the Thurston metric on $C \cong (\tau, L)$ is the limit of the Thurston metrics on the projective surfaces given by $(\tau, \ell_i)$.)

### 3.4. **Equivariant homotopies.**

**Lemma 3.1.** *Let $\rho\colon \pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$ be a homomorphism. Suppose that there are two continuous maps $\beta\colon \tilde{S} \to \mathbb{H}^3$ and $\beta'\colon \tilde{S} \to \mathbb{H}^3$ that are $\rho$-equivariant. Then $\beta$ and $\beta'$ are $\rho$-equivariantly homotopic, i.e. homotopic through $\rho$-equivariant maps $\tilde{S} \to \mathbb{H}^3$.*

*Proof.* We proceed in steps. *Step 1.* We first construct an equivariant homotopy for each loop $l$ on $S$. Let $\tilde{l}$ be a lift of $l$ to the universal cover $\tilde{S}$ of $S$. Then $\beta_1|\tilde{l}$ and $\beta_2|\tilde{l}$ are equivariant under the restriction of $\rho$ to $\langle l \rangle$, the infinite cyclic subgroup of $\pi_1(S)$ generated by $l \in \pi_1(S)$. Note that $\rho(l)$ may be of any type of hyperbolic isometry, i.e. parabolic, elliptic, or loxodromic. Then, in each case, we can easily construct a homotopy between $\beta_1|\tilde{l}$ and $\beta_2|\tilde{l}$ that is equivariant under $\rho|\langle l \rangle$.

*Step 2.* Next let $P$ be a pair of pants embedded in $S$. Let $l_1, l_2, l_3$ be the boundary loops of $P$. Let $\tilde{P}$ be a lift of $P$ to $\tilde{S}$. Then we show that there is a homotopy between $\beta_1|\tilde{P}$ and $\beta_2|\tilde{P}$ equivariant under $\rho|\pi_1(P)$. For each $j = 1, 2, 3$, pick a lift $\tilde{l}_j$ of $l_j$ to $\tilde{P}$. Then, by Step 1, we have a homotopy connecting $\beta_1|\tilde{l}_j$ and $\beta_2|\tilde{l}_j$ equivariant under $\rho|\pi_1(l_j)$. By equivariantly extending those homotopies, we have a homotopy $\Phi_{\partial \tilde{P}}\colon \partial\tilde{P} \times [0, 1] \to \mathbb{H}^3$ between $\beta_1|\partial\tilde{P}$ and $\beta_2|\partial\tilde{P}$ that is $\rho|\pi_1(P)$-equivariant. Pick disjoint arcs $a_1, a_2, a_3$ properly embedded in $P$ that decompose $P$ into two hexagons. Then we can easily extend the homotopy $\Phi_{\partial\tilde{P}}$ to a homotopy between the lifts of arcs $a_i$ (for $i = 1, 2, 3$) to $\tilde{P}$ so that the extension is still equivariant under $\rho|\pi_1(P)$. Since $a_i$'s decompose $P$ into simply connected surfaces, we can further extend the homotopy to the $\rho|\pi_1(P)$-equivariant homotopy between $\beta_1|\tilde{P}$ to $\beta_2|\tilde{P}$.

*Step 3.* Pick a *maximal* multiloop $M$ on $S$, which decomposes $S$ into pairs of pants $P_k$ $(k = 1, 2, \ldots, 2(g-1))$. Let $\tilde{M}$ denote the total lift of $M$ to $\tilde{S}$. Then we can obtain a $\rho$-equivariant homotopy $\Phi_{\tilde{M}}$ between $\beta_1|\tilde{M}$ and $\beta_2|\tilde{M}$ similarly to the way we obtained the homotopy $\Phi_{\partial\tilde{P}}$ in Step 2. For each $k \in \{1, 2, \ldots, 2(g-1)\}$, let $\tilde{P}_k$ be a lift of $P_k$ to $\tilde{S}$. Then $\Phi_{\tilde{M}}$ induces a homotopy $\Phi_{\partial\tilde{P}_k}$ between $\beta_1|\partial\tilde{P}_k$ and $\beta_2|\partial\tilde{P}_k$ that is equivariant under $\rho|\pi_1(P_k)$. Similarly to Step 2, we can extend this induced homotopy to a homotopy $\Phi_{\tilde{P}_k}$ between $\beta_1|\tilde{P}_k$ and $\beta_2|\tilde{P}_k$ that is equivariant under $\rho|\pi_1(P_k)$. By $\rho$-equivalently extending the homotopies $\Phi_{\tilde{P}_k}$ $(k = 1, 2, \ldots, 2(g-1))$, we obtain a $\rho$-equivariant homotopy between $\beta_1$ and $\beta_2$. □

### 3.5. **Isomorphisms of projective structures via developing maps.**

**Definition 3.2.** *Let $F$ be a surface and $\rho\colon \pi_1(F) \to \mathrm{PSL}(2, \mathbb{C})$ be a homomorphism. Let $C_1 = (f_1, \rho)$ and $C_2 = (f_2, \rho)$ be projective structures on $F$ sharing holonomy $\rho$, where $\tilde{F}$ is the universal cover of $F$ and $f_1, f_2\colon \tilde{F} \to \hat{\mathbb{C}}$ are their developing maps. Then $C_1$ and $C_2$ are isomorphic (as projective structures) via $f_1$ and $f_2$, if there is a homeomorphism $\phi\colon F \to F$ homotopic to the identify map, such that, letting $\tilde{\phi}\colon \tilde{F} \to \tilde{F}$ be the lift of $\phi$, we have $f_1 = f_2 \circ \tilde{\phi}\colon \tilde{C}_1 \to \hat{\mathbb{C}}$. We also say that the isomorphism $\phi$ is compatible with $f_1$ and $f_2$.*

## 4. Bilipschitz curves on pleated surfaces

Let $L$ be a measured geodesic lamination on $\mathbb{H}^2$ with $\mathrm{Area}_{\mathbb{H}^2}(|L|) = 0$. Let $\beta_L\colon \mathbb{H}^2 \to \mathbb{H}^3$ be the bending map induced by $L$. In this section, we prove

**Proposition 4.1.** *For every $\epsilon > 0$, there is $\delta > 0$ such that, if $l$ is a geodesic on $\mathbb{H}^2$ with $\angle_{\mathbb{H}^2}(l, L) < \delta$, then,*

(i) *$\beta_L$ is a $(1+\epsilon)$-bilipschitz embedding $l \to \mathbb{H}^3$,*

*and, letting $m$ be the geodesic in $\mathbb{H}^3$ connecting the endpoints of the quasigeodesic $\beta_L|l$,*

(ii) *for each point $x \in l$, $\beta_L(x)$ is $\epsilon$-close to $m$, and, if $\beta_L$ is differentiable at $x$, then the tangent vector of $\beta_L|l$ at $x$ is $\underline{\epsilon\text{-parallel}}$ to $m$,*

*that is, the tangent vector of $\beta_L|l$ in $\mathbb{H}^3$ at $x$ is $\epsilon$-nearly orthogonal to the(totally geodesic) hyperbolic plane orthogonal to $m$ and containing $\beta(x)$.*

**Remark 4.2.** *Similar statements are in* [8, 12, 2]. *However the condition on $\angle_{\mathbb{H}^3}(l, L)$ is new.*

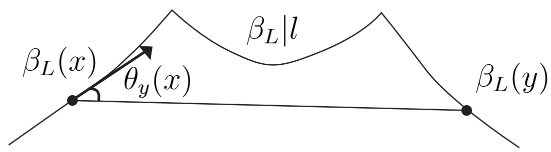


FIGURE 2.

Let $\Phi_m\colon \mathbb{H}^3 \to m$ be the nearest point projection. Then

**Corollary 4.3.** *(iii) $\Phi_m \circ \beta_L|\, l$ is a $(1+\epsilon)$-bilipschitz map $l \to m$.*

*Proof of corollary.* For each point $y \in m$, $\Phi_m^{-1}(y)$ is the hyperbolic lane in $\mathbb{H}^3$ orthogonal to $m$. Then $\mathbb{H}^3$ is foliated by the hyperplanes. Since $\mathrm{Area}_{\mathbb{H}^2}(\lambda) = 0$, $\beta|l$ is differentiable almost everywhere. By Proposition 4.1 (ii), the curve $\beta_L|l$ stays in a small neighborhood of $m$ and $\epsilon$-orthogonally intersects the hyperplanes of $\mathbb{H}^3$ at almost every point of $l$. If $\delta > 0$ is sufficiently small, at almost every point on $l$, the ratio of the lengths of the tangent vector along $\beta_L|l$ and of its $\Phi_m$-image is bounded by $(1+\epsilon)$. □

We first prove an analogue of Proposition 4.1 for geodesic segments of bounded lengths:

**Proposition 4.4.** *For every (large) $K > 0$ and (small) $\epsilon > 0$, there exists a $\delta > 0$ such that:*

(i) *If $L$ is a measured geodesic lamination on $\mathbb{H}^2$, and $l\colon \mathbb{R} \to \mathbb{H}^2$ is a parametrized geodesic at unit speed such that $\angle(l, L) < \delta$, then, if points $x, y$ on $l\,(\cong \mathbb{R})$ satisfies $0 < y - x < K$, then we have $(1-\epsilon)\cdot \mathrm{dist}_{\mathbb{H}^2}(x,y) < \mathrm{dist}_{\mathbb{H}^3}(\beta_L(x), \beta_L(y))$.*

(ii) *If $\beta|l$ is differentiable at $x \in \mathbb{R}$, for all $y \in \ell$ with $0 < y - x < K$, then $\theta_y(x) < \epsilon$, where $\theta_y(x) \in [0, \pi]$ is the angle between the geodesic segment from $\beta_L(x)$ to $\beta_L(y)$ and the tangent vector of $\beta_L|l$ at $\beta(x)$; see Figure 2.*

*Proof.* First consider a right hyperbolic triangle $\triangle ABC$ in $\mathbb{H}^2$ (with geodesic edges) with $\angle C = \pi/2$, where $A, B, C$ are its vertices. Then it is easy to prove

**Lemma 4.5.** *For every $K > 0$ and $\epsilon' > 0$, there is $\delta > 0$ such that, if $\angle B < \delta$ and $\mathrm{dist}(A,B) < K$, then*

(i) *$(1-\epsilon')\cdot \mathrm{dist}(A,B) < \mathrm{dist}(B,C) - \mathrm{dist}(C,A)$, and*

(ii) *$\angle A'BC < \epsilon'$ for every $A' \in \mathbb{H}^2$ with $\mathrm{dist}(C,A') < \mathrm{dist}(C,A)$.*

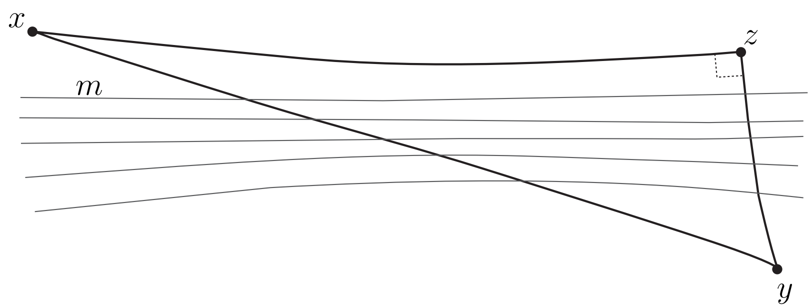


FIGURE 3.

Let $K > 0$ and $\epsilon > 0$. Let $\epsilon' = \epsilon/2$. Then let $\delta > 0$ be the number obtained by applying Lemma 4.5 to $K$ and $\epsilon'$. Then we can in addition assume that $\delta < \epsilon/2$.

Let $x$ and $y$ be distinct points on $l$ with $0 < y - x < K$. Let $I$ be the minimal sublamination of $L$ containing the leaves that intersect $[x, y]$. We can assume that $[x, y]$ intersects at least one leaf of $L$ transversally, since otherwise Proposition 4.4 clearly holds. Let $m$ denote the leaf of $I$ closest to $x$. Then, there is a unique point $z \in \mathbb{H}^2$ such that $\triangle xyz$ is a hyperbolic triangle with $\angle z = \pi/2$ and such that $[x, z] \subset \eta(m)$, where $\eta\colon \mathbb{H}^2 \to \mathbb{H}^2$ is the translation along $l$ taking the point $l \cap m$ to $x$. (See Figure 3.)

Then $[x, z]$ is disjoint from $I$ if $x$ is in the complement of $I$. Then, since $\angle(l, L) < \delta$, in particular $\angle yxz < \delta$. Let $\beta_I\colon \mathbb{H}^2 \to \mathbb{H}^3$ be the bending map induced by $I$. Then $\beta_I$ isometrically embeds $[x, z]$ into $\mathbb{H}^3$. Therefore $\mathrm{dist}_{\mathbb{H}^3}(\beta_I(x), \beta_I(z)) = \mathrm{dist}_{\mathbb{H}^2}(x, z)$. Since bending maps are 1-lipschitz, $\mathrm{dist}(\beta_I(z), \beta_I(y)) \leq \mathrm{dist}(z, y)$. By the triangle inequality, we have

$$\begin{aligned} \mathrm{dist}(\beta_I(x), \beta_I(y)) &\geq \mathrm{dist}(\beta_I(x), \beta_I(z)) - \mathrm{dist}(\beta_I(z), \beta_I(y)) \\ &\geq \mathrm{dist}(x, z) - \mathrm{dist}(z, y). \end{aligned}$$

Then, by Lemma 4.5 (i), we have

$$\mathrm{dist}(\beta_I(x), \beta_I(y)) > (1 - \epsilon') \cdot \mathrm{dist}(x, y).$$

Since $\beta_I = \beta_L$ on $[x, y]$, $\mathrm{dist}(\beta_L(x), \beta_L(y)) > (1 - \epsilon') \cdot \mathrm{dist}(x, y)$; thus we have shown (i).

By Lemma 4.5 (ii) applied to $\triangle A'BC = \triangle \beta_I(y)\beta_I(x)\beta_I(z)$, we have $\angle \beta_I(y)\beta_I(x)\beta_I(z) < \epsilon'$. By the triangle inequality on the sphere in $\mathbb{H}^3$ centered at $\beta_I(x)$ of infinitesimal radius,

$$\theta_y(x) \leq \angle yxz + \angle \beta_I(y)\beta_I(x)\beta_I(z) < \delta + \epsilon' < \epsilon.$$

Thus we have proved (ii). $\boxed{4.4}$

*Proof of Proposition 4.1.* (i) We first show for every $\epsilon' > 0$, there is $\delta > 0$ such that, if a geodesic lamination $L$ on $\mathbb{H}^2$ and a geodesic $l\colon \mathbb{R} \to \mathbb{H}^2$ satisfy $\angle_{\mathbb{H}^2}(l, L) < \delta$, then $\theta_x(y) < \epsilon'$ for all distinct points $x, y$ on $l$ with $x < y$ such that $\beta_L|l$ is differentiable at $y$. Pick $K > 0$ and $\epsilon'' > 0$ with $\epsilon'' < \epsilon'/2$. Then we can assume that $[x, y]$ is not contained in a leaf of $L$ and, by Proposition 4.4 (ii), that $\mathrm{dist}(x, y) > K$. Let $\delta' = \delta'(K, \epsilon'') > 0$ be the number obtained by applying Lemma 4.4 to $K$ and $\epsilon''$. Then divide the geodesic segment $[x, y]$ into subsegments $[p_0, p_1], [p_1, p_2], \ldots, [p_{n-1}, p_n]$, where $x = p_0 < p_1 < \ldots < p_n = y$, so that

- $K/2 < p_{i+1} - p_i < K$ for $i = 0, 1, \ldots, n-1$.
- $p_1, p_2, \ldots, p_{n-1}$ are in the complement of $|L|$ (since $\mathrm{Area}_{\mathbb{H}^2}(|L|) = 0$).

Let $\beta$ be the bending map $\beta_L$. Then the union of the geodesic segments $[\beta(p_i), \beta(p_{i+1})]$ in $\mathbb{H}^3$ over $i = 0, \ldots, n-1$ is a piecewise-geodesic curve in $\mathbb{H}^3$ connecting $\beta(x)$ to $\beta(y)$. If $\angle(l, L) < \delta'$, then, by Lemma 4.4 (ii), we have $\theta_{p_{n-1}}(y) < \epsilon''$ and $\pi - \angle_{\mathbb{H}^3}(\beta(p_{i-1}), \beta(p_i), \beta(p_{i+1})) < 2\epsilon''$ for all $i = 1, 2, \ldots, n-1$. By Lemma 4.4 (i), $\mathrm{dist}(\beta(p_i), \beta(p_{i+1})) > (1 - \epsilon'') \cdot (K/2)$ for $i = 0, 1, \ldots, n-1$. Then, if $\epsilon'' > 0$ is sufficiently small, since the exterior angles of the piecewise-geodesic curve are sufficiently small relative to the lengths of the segments, we have $\angle\beta(x)\beta(y)\beta(p_{n-1}) < \epsilon'/2$ (see [8, §I.4.2]; also [12, 2]). Then, by the triangle inequality,

$$0 < \theta_x(y) \leq \angle\beta(p_{n-1})\beta(y)\beta(x) + \theta_{p_{n-1}}(y) < \epsilon'/2 + \epsilon''.$$

Hence $0 < \theta_x(y) < \epsilon'$. We have

$$\frac{d\ \mathrm{dist}(\beta(x), \beta(y))}{dy} = \cos(\theta_x(y))$$

(see [8, §I.4.2]; also [12, 2]). Then, for every $\epsilon > 0$, by taking a smaller $\epsilon' > 0$ if necessary, we have $\frac{1}{1+\epsilon} < \cos(\theta_x(y)) \leq 1$ for all different $x, y$ on $l$ such that $\beta|l$ is differentiable at $y$. Since $\beta|l$ is differentiable at almost all points of $l$, $\beta|l$ is a $(1+\epsilon)$-bilipschitz embedding.

(ii) For every $\epsilon > 0$, pick $\epsilon' > 0$ with $2\epsilon' < \epsilon$. Then we have shown, in proving (i), that there exists $\delta > 0$, such that, if $\angle_{\mathbb{H}^2}(l, L) < \delta$, then $\theta_x(y) < \epsilon'$ for all different $x, y \in l$ such that $\beta|l$ is differentiable at $y$. Since $\beta|l$ is bilipschitz, it takes the endpoints $\pm\infty$ of the geodesic $l\colon \mathbb{R} \to \mathbb{H}^2$ to the distinct points $\beta(-\infty), \beta(\infty)$ of the ideal boundary of $\mathbb{H}^3$. Thus taking the limits as $x$ goes to the end points of $l$, we have $\theta_{-\infty}(y), \theta_\infty(y) \leq \epsilon'$. Thus, we have $\beta(-\infty)\beta(y)\beta(\infty) > \pi - 2\epsilon'$. Let $m$ be the geodesic in $\mathbb{H}^3$ connecting $\beta(-\infty), \beta(\infty)$ so that $m$ is a bounded distance away from $\beta|l$. It is well-known that the area of a triangle in $\mathbb{H}^2$ is equal to $\pi$ minus the sum of the angles of its vertices. Thus the area of the geodesic triangle $\triangle\beta(-\infty)\beta(y)\beta(\infty)$ is less than $2\epsilon'$. Thus, if necessary by taking smaller $\epsilon' > 0$, we can assume that $\mathrm{dist}_{\mathbb{H}^3}(\beta(y), m) < \epsilon$. Recalling that $\Phi_m\colon \mathbb{H}^3 \to m$ is the nearest point projection, $\triangle\beta(-\infty)\beta(y)(\Phi_m \circ \beta)(y)$ has area less than $\epsilon'$. Applying the same formula to this ideal triangle, we have $\angle_{\mathbb{H}^3}\beta(-\infty)\beta(y)(\Phi_m \circ \beta)(y) < \pi/2 - \epsilon'$. Since $\theta_{-\infty}(y) < \epsilon'$ and $2\epsilon' < \epsilon$, by the triangle inequality, we see that the tangent vector of $\beta|l$ at $y$ is $\epsilon$-parallel to $m$. $\boxed{4.1}$

## 5. Local stability of bending maps in $\mathcal{GL}$

### 5.1. Bending maps with a fixed bending lamination.

**Definition 5.1.** *Let $X, Y$ be metric space with distance functions $d_X, d_Y$. For every $\epsilon > 0$, a map $\phi\colon X \to Y$ is an $\epsilon$-rough isometric embedding, if $d_X(p,q) - \epsilon < d_Y(\phi(p), \phi(q)) < d_X(p,q) + \epsilon$ for all $p, q \in X$. It is an $\epsilon$-rough isometry if, in addition, the $\epsilon$-neighborhood of $\mathrm{Im}(\phi)$ is $Y$.*

**Theorem 5.2.** *Let $(\tau, \lambda) \in \mathcal{T} \times \mathcal{GL}$ and $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ be a homomorphism. Suppose that there is a $\rho$-equivariant pleated surface $\beta\colon \mathbb{H}^2 \to \mathbb{H}^3$ realizing $(\tau, \lambda)$. Then, for every $\epsilon > 0$, there is a $\delta > 0$, such that, if there is a pair $(\sigma, \nu) \in \mathcal{T} \times \mathcal{GL}$ and a $\rho$-equivariant pleated surface $\beta'\colon \mathbb{H}^2 \to \mathbb{H}^3$ realizing $(\sigma, \nu)$ and $\angle_\tau(\lambda, \nu) < \delta$, then $\beta'$ and $\beta$ are $\epsilon$-close in the following sense: There is a marking-preserving homeomorphism $\psi\colon \tau \to \sigma$ such that $\psi$ is an $\epsilon$-rough isometry and, letting $\tilde\psi\colon \mathbb{H}^2 \to \mathbb{H}^2$ be its lift, the maps $\beta$ and $\beta' \circ \tilde\psi$ are $\epsilon$-close in the $C^0$-topology and moreover in the $C^1$-topology in the complement of the $\epsilon$-neighborhood of $|\tilde\lambda| \cup |\tilde\nu|$ in the universal cover $\tilde\tau = \mathbb{H}^2$, where $\tilde\lambda$ and $\tilde\nu$ are the total lifts of $\lambda$ and (the geodesic representative of) $\nu$ on $\tau$ to $\tilde\tau$.*

The rest of §5.1 is the proof of Theorem 5.2. Suppose that there is a sequence of $\rho$-equivariant pleated surfaces $\beta_i\colon \mathbb{H}^2 \to \mathbb{H}^3$ realizing some $(\sigma_i, \nu_i) \in \mathcal{T} \times \mathcal{GL}$. Let $\nu_{i,\tau}$ denote the geodesic representative of $\nu_i$ on $\tau$. Assuming $\angle_\tau(\nu_{i,\tau}, \lambda) \to 0$ as $i \to \infty$, we will construct a homeomorphism $\psi_i\colon \tau \to \sigma_i$ such that, for every $\epsilon > 0$, if $i$ is sufficiently large, then $\beta$ and $\beta_i \circ \tilde\psi_i\colon \tilde\tau \to \mathbb{H}^3$ are $\epsilon$-close in the $C^0$-topology and in the $C^1$-topology in the complement of the $\epsilon$-neighborhood of $\nu_{i,\tau}$.

*Outline of the construction.* We first construct $\psi_i\colon \nu_{i,\tau} \to \nu_i$ such that $\psi_i$ is $(1+\epsilon)$-bilipschitz on every leaf of $\nu_{i,\tau}$ for sufficiently large

$i$ and, as desired, $\beta' \circ \tilde{\psi}_i \to \beta$ in the $C^0$-topology. This bilipschitz property is given by Proposition 4.1. Then we continuously extend it to $\psi_i \colon \tau \to \sigma_i$ so that $\psi_i$ is $(1+\epsilon)$-bilipschitz on each stratum of $(\tau, \nu_{i,\tau})$ and $\beta' \circ \tilde{\psi}_i \to \beta$ as desired. In particular, given a compact subset $K$ of a stratum of $(\tau, \nu_{i,\tau})$, $\psi_i$ is an $\epsilon$-rough isometry on $K$ for sufficiently large $i$ (Lemma 5.5). We take a"sufficiently thick part" of the stratum to be the compact subset $K$. Finally, in order to show that $\psi_i$ is an $(1+\epsilon)$-rough isometry, we show that $\psi_i$ is an $\epsilon$-rough isometry along arcs transversal to the lamination $\lambda$ (Lemma 5.7).

Since $\mathcal{GL}$ is compact with the Chabauty topology, we can assume that $\nu_{i,\tau}$ converges to some $\nu_\infty \in \mathcal{GL}(\tau)$ as $i \to \infty$. Then $\angle_\tau(\nu_\infty, \lambda) = 0$. We moreover have

**Proposition 5.3.** *$\lambda$ is a sublamination of $\nu_\infty$.*

*Proof.* Since $\angle_\tau(\nu_\infty, \lambda) = 0$, the union $\nu_\infty \cup \lambda$ is a geodesic lamination on $\tau$. Suppose that $\lambda$ is *not* a sublamination of $\nu_\infty$. Then there is a leaf of $\lambda$ *not* contained in $\nu_\infty$. Below each tilde symbol " $\sim$ " denotes either the universal cover of a surface, e.g. $\tilde{\tau} \cong \mathbb{H}^2$, or the total lift of a geodesic lamination to the universal cover, e.g. $\tilde{\lambda}$. Then there are distinct components $R$ and $R'$ of $\tilde{\tau} \cong \mathbb{H}^2$ minus the total lift of $\nu_\infty \cup \lambda$, such that

- a leaf of $\tilde{\lambda}$ separates $R$ and $R'$,
- yet $R$ and $R'$ are contained in a single component $P$ of $\tilde{\tau} \setminus \tilde{\nu}_\infty$, and
- either
  - $R$ and $R'$ share a boundary geodesic and $\beta$ bends $\mathbb{H}^2$ along the geodesic by the angle $\pi$, or
  - $\beta(R)$ and $\beta(R')$ are contained in distinct copies of $\mathbb{H}^2$ in $\mathbb{H}^3$.

Since $\nu_{i,\tau} \to \nu_\infty$, for every $i \in \mathbb{N}$, we can pick a component $P_i$ of $\tilde{\tau} \setminus \tilde{\nu}_{i,\tau}$ such that $P_i$ converges to $P$ uniformly on compacts as $i \to \infty$. Then $P_i \cap R \to R$ and $P_i \cap R' \to R'$ as $i \to \infty$. Then, let $Q_i$ be the component of $\tilde{\sigma}_i \setminus \tilde{\nu}_i$ that corresponds to $P_i$ so that a marking-preserving homeomorphism $\sigma_i \to \tau$ induces a homeomorphism $Q_i \to P_i$. Then $\beta_i \colon \mathbb{H}^2 \to \mathbb{H}^3$ isometrically embeds $Q_i$ in a copy $H_i$ of $\mathbb{H}^2$. By Lemma 3.1, $\beta$ and $\beta_i$ are $\rho$-equivariantly homotopic, and thus $\beta | P_i$ and $\beta_i | Q_i$ are a bounded distance apart pointwise via the homeomorphism $Q_i \to P_i$.

**Claim 5.4.** *For every $\epsilon > 0$, if $i$ is sufficiently large, then $\beta | P_i$ and $\beta_i | Q_i$ are $\epsilon$-close in the $C^0$-topology via $Q_i \to P_i$.*

*Proof.* For every $\epsilon > 0$, if $i$ is sufficiently large, then $\angle_{\tilde{\tau}}(\ell, \lambda) < \epsilon$ for each boundary geodesic $\ell$ of $P_i$. Therefore, by Proposition 4.1, if $i$ is sufficiently large, $\beta_i|\partial Q_i$ is $\epsilon$-close to $\beta|\partial P_i$. Thus we can in addition assume that $\beta_i|\ell$ is an $(1+\epsilon, \epsilon)$-quasiisometric embedding for every geodesic or geodesic segment $\ell$ in $P_i$ not transversal to $\lambda$. This implies the claim. □

By this claim, $\beta|P_i$ become more and more totally geodesic as $i \to \infty$. Since $P_i \cap R \to R$ and $P_i \cap R' \to R$, the hyperbolic plane containing $\beta(R)$ must coincide with the hyperbolic plane containing $\beta(R')$, and moreover $\beta(R)$ and $\beta(R')$ must be disjoint. This is a contradiction to the third hypothesis of $R$ and $R'$ above. 5.3

For each $i$, we enlarge the geodesic lamination $\nu_i$ to a *maximal* lamination, which decomposes $\sigma_i$ into ideal triangulations. By taking a subsequence if necessary, we can assume that the maximal lamination $\nu_i$ converges to a maximal lamination containing $\nu$. Thus accordingly we denote the limit by $\nu$. Similarly let $\nu_{i,\tau}$ be the geodesic lamination on $\tau$ representing $\nu_i$. Then we still have $\angle_\tau(\nu_{i,\tau}, \lambda) \to 0$ as $i \to \infty$ (by Proposition 5.3). Although $\nu_i$ is not a "minimal" lamination realizing $\beta_i$, it will not affect our arguments.

We construct a homeomorphism $\psi_i \colon \tau \to \sigma_i$ for all sufficiently large $i$. First, since $\angle_\tau(\nu_i, \lambda) \to 0$ as $i \to \infty$, for every $\epsilon > 0$, if $i$ is sufficiently large, by Lemma 3.1, and Corollary 4.3, there is a bijection $\psi_i \colon \nu_{i,\tau} \to \nu_i$ that is a $(1+\epsilon)$-bilipschitz map on each leaf of $\nu_{i,\tau}$ (to be precise $\psi_i \colon |\nu_{i,\tau}| \to |\nu_i|$). If there is a sequence of leaves $\ell_j$ of $\nu_{i,\tau}$ converging to a leaf $\ell_\infty$ of $\nu_{i,\tau}$, then $\beta_i|\ell_j$ converges to $\beta_i|\ell_\infty$ uniformly on compacts as $j \to \infty$. Then, since if $i$ is sufficiently large, $\beta_i|\ell_j$ are $(1+\epsilon)$-bilipschitz for all $j$, the endpoints of $\beta_i|\ell_j$ converge to the endpoints to $\beta_i|\ell_\infty$ on $\hat{\mathbb{C}}$ as $j \to \infty$. Since $\psi_i$ is obtained from Corollary 4.3, we see that the entire map $\psi_i \colon \nu_{i,\tau} \to \nu_i$ is a homeomorphism with the topology induced from $\tilde{\tau}$ and $\tilde{\sigma}_i$.

Given $\epsilon > 0$ and a connected component $\Delta$ of $\tau \setminus \nu_{i,\tau}$, let $\Delta_\epsilon$ be the *$\epsilon$-thick part* of $\Delta$, that is, the union of disks of radius $\epsilon$ embedded in $\Delta$. Then $p \in \partial\Delta \cap \partial\Delta_\epsilon$ if and only if $\Delta$ contains a disk of radius $\epsilon$ tangent to $\partial\Delta$ at $p$. Since $\Delta$ is an ideal triangle, $\partial\Delta \cap \partial\Delta_\epsilon$ is compact, and if $\epsilon > 0$ is sufficiently small, then it is a union of three long (but finite) segments of the edges of $\Delta$.

For every $\zeta > 0$, if $\epsilon > 0$ is sufficiently small, every $(1+\epsilon)$-bilipschitz curve in $\mathbb{H}^3$, in particular $\beta_i|\ell_j$ above, is contained in the $\zeta$-neighborhood of the geodesic in $\mathbb{H}^3$ connecting its endpoints on $\hat{\mathbb{C}}$. Thus, since $\partial\Delta \cap \partial\Delta_\epsilon$ is bounded, we have

**Lemma 5.5.** *For every $\epsilon > 0$, if $i \in \mathbb{N}$ is sufficiently large, then, for every component $\Delta$ of $\tau \setminus \nu_{i,\tau}$, the map $\psi_i$ restricts to an $\epsilon$-rough isometric embedding on $\partial\Delta \cap \partial\Delta_\epsilon$ into the corresponding component $\Delta'$ of $\sigma_i \setminus \nu_i$ with respect to the path metrics on the ideal triangles $\Delta$ and $\Delta'$.*

Next we extend $\psi_i \colon \nu_{i,\tau} \to \nu_i$ to $\tau \to \sigma_i$ by extending $\psi_i$ to the interior each component $\Delta$ of $\tau \setminus \nu_{i,\tau}$ in a natural way. The ideal triangle $\Delta$ contains a unique inscribed circle, which is tangent to each edge of $\Delta$ at a single point. Then, by connecting those tangency points, we obtain a hyperbolic triangle inscribed in $\Delta$. Each component of $\Delta$ minus the inscribed triangle is a hyperbolic triangle $\check{\Delta}$ with a single ideal vertex $v$. Then $\check{\Delta}$ has two edges of infinite length sharing $v$, and a point of one edge corresponds to a point on the other edge so that a horocycle centered at $v$ passes through both points. By connecting all pairs of such corresponding points by geodesic segments, we obtain a foliation of $\check{\Delta}$ by geodesic segments. Then continuously extend $\psi_i$, which is so far defined on $\partial\Delta$, to $\check{\Delta}$ so that $\psi_i$ linearly takes each such geodesic segment connecting points on $\partial\Delta$ to geodesic segments connecting $\psi_i$-images of the points on $\partial\Delta'$ (see Figure 4).

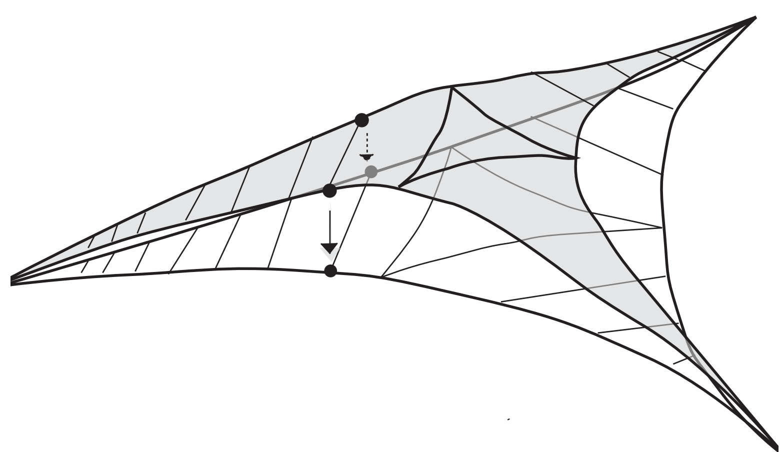

FIGURE 4.

If $\epsilon > 0$ is sufficiently small, then the inscribed triangle in $\Delta$ is contained in the $\epsilon$-thick part $\Delta_\epsilon$ for all components $\Delta$ of $\tau \setminus \nu_{i,\tau}$. Thus, by Lemma 5.5, we can further extend $\psi_i$ to the inscribed triangle of $\Delta$ so that $\psi_i$ restricts to an $\epsilon$-rough isometric embedding on $\Delta_\epsilon$ into $\Delta'$.

For every $\epsilon > 0$, if $i$ is sufficiently large, then $\beta|\tilde{\nu}_{i,\tau}$ and $\beta_i|\tilde{\nu}_i$ are $\epsilon$-close pointwise via $\tilde{\psi}_i \colon \tilde{\tau} \to \tilde{\sigma}_i$ since $\psi_i|\nu_{i,\tau}$ is defined using Corollary

4.3. In addition, for every $\zeta > 0$, if $\epsilon > 0$ is sufficiently small, then $\tilde{\tau}\,(\cong \mathbb{H}^2)$ is covered by the $\zeta$-neighborhoods of the $\epsilon$-thick parts of the ideal triangles of $\tilde{\tau} \setminus \tilde{\nu}_{i,\tau}$. For every $\epsilon > 0$, if $i$ is sufficiently large, then $\beta$ and $\beta_i \circ \tilde{\psi}_i$ are also $\epsilon$-close in the $C^0$-topology via $\tilde{\psi}$.

For very $\epsilon > 0$, if $i$ is sufficiently large, then the $\epsilon$-neighborhood of $\nu$ contains $\nu_{i,\tau}$ in $\tau$. In the complement of the $\epsilon$-neighborhood of $\tilde{\nu}$, $\beta$ and $\beta_i \circ \tilde{\psi}_i$ are totally geodesic. We can in addition assume that $\beta$ and $\beta_i \circ \tilde{\psi}_i$ are $\epsilon$-close, moreover, in the $C^1$-topology, for sufficiently large $i$.

Note that this $C^1$-convergence is weaker than that in Theorem 5.2, since we have enlarged each $\nu_i$ to a maximal lamination. The $\epsilon$-neighborhood of the extended lamination $\nu_{i,\tau}$ may be bigger than the $\epsilon$-neighborhood $N_{i,\epsilon}$ of the original lamination $\nu_{i,\tau}$. However, since $\beta$ and $\beta_i \circ \tilde{\psi}_i$ are totally geodesic in the complement of $N_{i,\epsilon}$, it is easy to make it $C^1$-convergence there for sufficiently large $i$ by a small perturbation.

Thus, it only remains to show:

**Proposition 5.6.** *For every $\epsilon > 0$, if $i \in \mathbb{N}$ is sufficiently large, then $\psi_i \colon \tau \to \sigma_i$ is an $\epsilon$-rough isometry.*

*Proof.* Let $x$ be a point of $|\tilde{\nu}|$, and let $\ell$ be the leaf of $\tilde{\nu}$ containing $x$. Consider a (totally geodesic) hyperbolic plane $H$ of $\mathbb{H}^3$ transversally intersecting the geodesic $\beta(\ell)$ at $\beta(x)$. Then, by the transversality, there is a neighborhood $a$ of $x$ in $\beta^{-1}(H)$ homeomorphic to an arc, which we call an *orthogonal arc* through $x$

If a sequence of leaves of $\tilde{\nu}$ converges to $\ell$, then accordingly their $\beta$-images are geodesics in $\mathbb{H}^3$ converging to $\beta(\ell)$ uniformly on compacts. Thus, for every $\epsilon > 0$, if $a$ is sufficiently short then, for a stratum $R$ of $(\mathbb{H}^2, \tilde{\nu})$ which intersects $a$, the angle $\angle_{\mathbb{H}^3}(\beta(R), H)$ is $\epsilon$-close to $\angle(H, \beta(\ell))$.

Since $\lambda$ has measure zero in $\tau$ and the pleated surface $\beta\colon \mathbb{H}^2 \to \mathbb{H}^3$ preserves the length of paths, the length of $a$ is equal to the total length of the arcs $a \setminus |\tilde{\nu}|$.

In particular, $a$ intersects $\ell$ only at $x$ and it is "transversal" in the sense that there is a $\delta > 0$ such that, if $s$ is a geodesic segment in $\mathbb{H}^2$ with its endpoints on different components of $a \setminus x$, then $s$ intersects $\ell$ transversally at an angle of more than $\delta$.

We have shown that $\beta_i \circ \tilde{\psi}_i$ converges to $\beta$ as $i \to \infty$ uniformly everywhere in the $C^0$-topology and pointwise almost everywhere in the $C^1$-topology. By this convergence, if $i$ is sufficiently large and $\tilde{\psi}_i(a)$ intersects a stratum $R_i$ of $(\mathbb{H}^2, \tilde{\nu}_i)$, then $\beta_i(R_i)$ is transversal to $H$ and $\angle(\beta_i(R_i), H) > \delta$ for some fixed $\delta > 0$.

Therefore, for sufficiently large $i \in \mathbb{N}$, there is an arc $b_i$ embedded in $\beta_i^{-1}(H) \subset \mathbb{H}^2$ such that $\beta_i|b_i$ converges to $\beta|a$ in the $C^0$-topology uniformly. Note that, since $\nu_i$ on $\sigma_i$ and $\lambda$ on $\tau$ have measure zero, $\beta_i|b_i$ and $\beta|a$ are almost everywhere smooth. Thus the uniform convergence $\beta_i|b_i \to \beta|a$ is moreover in the $C^1$-topology almost everywhere. Similarly the length of $b_i$ is the sum of the lengths of the segments of $b_i \setminus |\tilde{\nu}_i|$.

**Lemma 5.7.** length$(b_i)$ *converges to* length$(a)$ *as* $i \to \infty$.

*Proof.* Since $\beta_i|b_i$ converges to $\beta|a$ and pleated surfaces $\mathbb{H}^2 \to \mathbb{H}^3$ preserve length, for every $\epsilon > 0$, we have length$(a) < \epsilon +$ length$(b_i)$ for sufficiently large $i$. Thus it suffices to show the opposite length$(b_i) < \epsilon +$ length$(a)$ for sufficiently large $i$.

Recall that $\psi_i \colon \tau \to \sigma_i$ is a marking-preserving homeomorphism taking $\nu_{i,\tau} \to \nu_i$. Since the endpoints of $a$ are in the complement of $\nu$, for sufficiently large $i$, the components of $\mathbb{H}^2 \setminus \tilde{\nu}_{i,\tau}$ intersecting $a$ bijectively correspond to the components of $\mathbb{H}^2 \setminus \tilde{\nu}_i$ intersecting $b_i$. Therefore, there is a homeomorphism $\eta_i \colon a \to b_i$ such that, if $\Delta$ and $\Delta_i$ are corresponding complementary ideal triangles of $\tilde{\nu}_{i,\tau}$ and $\tilde{\nu}_i$, respectively, then $\eta_i$ takes the arc $a \cap \Delta$ to the arc $b_i \cap \Delta_i$ homeomorphically.

Let $\hat{a}_i$ be the union of arcs of $a \setminus |\tilde{\nu}_{i,\tau}|$ intersecting the $\epsilon$-thick part of $\mathbb{H}^2 \setminus \tilde{\nu}_{i,\tau}$. Then $\hat{a}_i$ is union of finitely many disjoint arcs. Let $\check{a}_i = a \setminus \hat{a}_i$. For every $\epsilon > 0$, if $i$ is sufficiently large, then $\psi_i$ is an $\epsilon$-rough isometry in the $\epsilon$-thick part of $\tau \setminus \nu_{i,\tau}$. Therefore, if $i$ is sufficiently large, $\eta_i$ changes the total length of $\hat{a}_i$ by at most $\epsilon$.

For all $i$, we have Area$(\tau) =$ Area$(\sigma_i)$. For all $\epsilon > 0$ and $\delta > 0$, if $i$ is sufficiently large, $\psi_i$ changes the total area of the $\epsilon$-thick part $\tau \setminus \nu_{i,\tau}$ by at most $\delta$.

There is a $\theta > 0$ such that, for sufficiently large $i$,

- If $\Delta$ is a component of $\mathbb{H}^2 \setminus \tilde{\nu}$ and $a$ intersects a boundary geodesic $\ell$ of $\Delta$, then the angle between $a \cap \Delta$ and $\ell$ is at least $\theta$.
- if $\Delta_i$ is a component of $\mathbb{H}^2 \setminus \tilde{\nu}_i$ and a boundary geodesic $\ell_i$ of $\Delta_i$ intersects $b_i$, then the angle between $b_i \cap \Delta_i$ and $\ell_i$ is at least $\theta$.
- If $\Delta_{i,\tau}$ is a component of $\mathbb{H}^2 \setminus \tilde{\nu}_{i,\tau}$ and $a$ intersects a boundary geodesic $\ell$ of $\Delta_{i,\tau}$, then the angle between $a \cap \Delta_{i,\tau}$ and $\ell$ is at least $\theta$.

For every $\zeta > 0$, if $i$ is sufficiently large, then the $\psi_i$ takes the $\epsilon$-thick part of $\tau \setminus \nu_{i,\tau}$ into the $(\epsilon - \zeta)$-thick part of $\sigma_i \setminus \nu_i$ and the $\epsilon$-thin part of $\tau \setminus \nu_{i,\tau}$ maps into $(\epsilon + \zeta)$-thick part of $\sigma \setminus \nu_i$. Therefore, for every $\delta > 0$, if $\epsilon > 0$ is sufficiently small and $i \in \mathbb{Z}_{>0}$ is sufficiently large, then the length of $\check{a}_i$ is $\delta$-close to the length of $\eta_i(\check{a}_i)$, since otherwise $\psi_i$ must increase the total area of the $\epsilon$-thin part of $\tau \setminus \nu$ some definite amount, such that $\mathrm{Area}(\sigma_i) > \mathrm{Area}(\tau)$; this is a contraction. Therefore for every $\epsilon > 0$, if $i$ is large enough, $\mathrm{length}(b_i) < \mathrm{length}(a) + \epsilon$. $\square$

**Lemma 5.8.** *For every $p \in \tilde{\tau}$ and $\epsilon > 0$, there is a neighborhood $U$ of $p$ in $\tilde{\tau}$ such that, if $i \in \mathbb{N}$ is sufficiently large, then $\tilde{\psi}_i(U) \subset \tilde{\sigma}_i$ has diameter less than $\epsilon$.*

*Proof.* First suppose that $p$ is in $\tilde{\tau} \setminus \tilde{\nu}$. Let $\Delta$ is the component of $\tilde{\tau} \setminus \tilde{\nu}$ containing $p$. Then take a sufficiently small closed ball centered at $p$ so that it is contained in $\Delta$. Let $U$ be the interior of the closed ball, which is an open ball centered at $p$. Then for sufficiently large $i$, $U$ is contained also in a component $\Delta_i$ of $\tilde{\sigma}_i \setminus \tilde{\nu}_{i,\tau}$. There is a $\delta > 0$ such that $U$ is contained in the $\delta$-thick part of $\Delta_i$. Thus, for every $\epsilon > 0$, if $i$ large enough, $\psi_i$ is an $\epsilon$-rough isometry near $p$. Thus if $U$ is a sufficiently small neighborhood of $p$, then $\tilde{\psi}_i(U)$ has diameter less than $\epsilon$.

Next suppose that $p$ is on a leaf $\ell$ of $\tilde{\nu}$. Then we construct a small "rectangular" neighborhood bounded by geodesic segments disjoint from $\tilde{\nu}$ and curves, as above, mapping into hyperbolic planes orthogonal to $\beta(\ell)$ by $\beta$. For $\delta > 0$, let $x_1$ and $x_2$ be the points on $\ell$ that have distance $\delta$ from $p$, so that $p$ bisects the geodesic segment $[x_1, x_2]$. Let $H_1$ and $H_2$ be the hyperbolic planes in $\mathbb{H}^3$ that are orthogonal to $\beta(\ell)$ at $\beta(x_1)$ and $\beta(x_2)$, respectively. Given $\delta > 0$, let $a_i$ be an orthogonal curve on $\tilde{\tau}$ passing through $x_i$ for each $i = 1, 2$ such that

- $\mathrm{length}_{\tilde{\tau}}(a_i) < \delta$,
- $\beta(a_i)$ is contained in $H_i$.
- the corresponding endpoints of $a_1$ an $a_2$ are in the same component of $\tilde{\tau} \setminus \tilde{\nu}$.

By the third condition, the corresponding endpoints of $a_1$ and $a_2$ are in the complements of $\tilde{\tau} \setminus \tilde{\nu}$. Thus let $b_1$ and $b_2$ be geodesic segments in $\tilde{\tau} \setminus \tilde{\nu}$ that connect the corresponding endpoints of $a_1$ and $a_2$.

Then $\mathrm{length}(b_1) \to 0$ and $\mathrm{length}(b_2) \to 0$ as $\delta \to 0$. Then let $U_\delta$ be the rectangular neighborhood of $p$ bounded by $a_1, a_2, b_1, b_2$, so that $p \in U_\delta$.

We claim that, for every $\epsilon > 0$, the diameter of $\psi_i(U_\delta)$ is less than $\epsilon$, if $\delta \to 0$ is sufficiently small and $i$ is sufficiently large. Since $\psi_i$ are homeomorphisms, it is suffices to show that the $\psi_i$-images of the edges $a_1, a_2, b_1, b_2$ have length less than $\epsilon$.

Since $\mathrm{length}(a_1), \mathrm{length}(a_2) < \delta$, by Lemma 5.7, if $\delta > 0$ is sufficiently small and $i$ is sufficiently large, then $\psi_i(a_1)$ and $\psi_i(a_2)$ have length less than $\epsilon$.

The geodesic segments $b_1$ and $b_2$ are disjoint from $\tilde{\nu}_{i,\tau}$ for sufficiently large $i$. Therefore, for every $\epsilon > 0$, the restrictions of $\psi_i$ to $b_1, b_2$ are $\epsilon$-rough isometric embeddings for sufficiently large $i$. Hence, if $\delta > 0$ is sufficiently small and $i$ is sufficiently large, then $\psi_i(b_1)$ and $\psi_i(b_2)$ have length less than $\epsilon$. □

**Proposition 5.9.** *For every $p, q \in \tilde{\tau}(\cong \mathbb{H}^2)$ and $\epsilon > 0$, if $i \in \mathbb{N}$ is sufficiently large, then*

$$-\epsilon < \mathrm{length}_{\tilde{\tau}}[p,q] - \mathrm{length}_{\tilde{\sigma}}[\psi_i(p), \psi_i(q)] < \epsilon.$$

*Proof.* First Suppose that $p$ and $q$ are in the interior of a single stratum $\Delta$ of $(\tilde{\sigma}, \tilde{\nu})$. Then the assertion holds true since, given $\epsilon > 0$, $\tilde{\psi}_i$ is a $(1+\epsilon, \epsilon)$-quasiisometric embedding on $\Delta$ for sufficiently large $i$ .

Second suppose that $p$ and $q$ are contained in a single leaf $\ell$ of $\tilde{\nu}$. For every $\epsilon > 0$, if $i$ is sufficiently large, then, since $\angle_\tau(\lambda, \nu_{i,\tau}) \to 0$, we can pick orthogonal arcs $a_i$ from $p$ and $a_i'$ from $q$ on $\tilde{\tau}$ such that $\mathrm{length}(a_i), \mathrm{length}(a_i') < \epsilon$ and, letting $r_i$ and $s_i$ be the other endpoints of $a_i$ and $a_i'$, such that $r_i$ and $s_i$ are in a single stratum of $(\tilde{\tau}_i, \tilde{\nu}_{i,\tau})$, using Lemma 5.7. Let $b_i$ and $b_i'$ be the arcs on $\tilde{\sigma}_i$ that correspond to $a_i$ and $a_i'$, as discussed just before Lemma 5.7, so that $\beta_i|b_i$ and $\beta_i|b_i'$ are $\epsilon$-close to $\beta|a_i$ and $\beta|a_i'$, respectively, in hyperbolic planes orthogonal to $\beta(\ell)$. Then we can in addition assume that $\mathrm{length}(b_i), \mathrm{length}(b_i') < \epsilon$. Let $c_i = [\psi_i(r_i), \psi_i(s_i)]$, which is contained in a stratum of $(\tilde{\sigma}_i, \tilde{\nu}_i)$. Therefore, if $i$ is sufficiently large, then $|\mathrm{length}(c_i) - \mathrm{length}[r_i, s_i]| < \epsilon$. Hence, for every $\epsilon > 0$, since we can assume that the lengths of $a_i, a_i', b_i, b_i'$ are less than $\epsilon$ for sufficiently large $i$, we have $-\epsilon < \mathrm{length}[p,q] - \mathrm{length}[\psi_i(p), \psi_i(q)] < \epsilon$.

Last suppose that $p, q$ are in different strata, so that $[p,q]$ transversally intersects $\tilde{\nu}$. Since $\beta_i \circ \tilde{\psi}_i$ converges to $\beta$ as $i \to \infty$ and $\beta, \beta_i$ preserve length of curves, thus $\mathrm{length}[p,q] < \epsilon + \mathrm{length}[\psi_i(p), \psi_i(q)]$ for sufficiently large $i$.

For each $x \in [p,q] \cap \tilde{\nu}$, let $\ell_x$ be the leaf of $\tilde{\nu}$ containing $x$. Then, as discussed above, there is an orthogonal $a_x$ passing through $x$ so that $\beta(a_x)$ is contained in the hyperbolic plane, $\mathbb{H}^3$, orthogonal to the geodesic $\beta(\ell_x)$ at $\beta(x)$. We can in addition assume that the endpoints of $a_x$ are in the complement of $\tilde{\lambda}$.

Next, using orthogonal curves, we pick a curve approximating the geodesic segment $[p,q]$ that intersects $\nu$ almost "orthogonally". Namely, take finitely many points $x_1, \ldots, x_n$ on $[p,q] \cap \tilde{\nu}$, and pick orthogonal curves $a_1, \ldots, a_n$ so that an endpoint of $a_k$ and an endpoint of $a_{k+1}$ are in the interior of a single stratum of $(\tilde{\tau}, \tilde{\nu})$ for each $k$. Then let $c_k$ be the geodesic segments connecting the endpoints. Taking an union of such $a_j$ and $c_k$, we can construct a curve $\alpha$ connecting $p$ to $q$, such that $\alpha \cap \nu \subset \cup_{j=1}^n a_j$ and that $\alpha \setminus \cup_{j=1}^n a_j$ is a union of disjoint geodesic segments. For every $\epsilon > 0$, taking large $n$ so that the orthogonal curves $a_j$ are sufficiently short, we can in addition assume that

- $\alpha$ is $\epsilon$-close to $[p, q]$ in the Hausdorff metric,
- $\Sigma_j \operatorname{length}(a_j) < \epsilon$, and
- $-\epsilon < \operatorname{length}[p, q] - \Sigma_i \operatorname{length}(c_k) < \epsilon$.

(For the last assertion, consider the nearest point projection of $\alpha$ to $[p, q]$). Thus, for every $\epsilon > 0$, there is such an approximating curve $\alpha$ with $-\epsilon < \operatorname{length}(\alpha) - \operatorname{length}[p, q] < \epsilon$. Therefore, in order to show $\operatorname{length}[p, q] + \epsilon > \operatorname{length}[\tilde{\psi}_i(p), \tilde{\psi}_i(q)]$ for sufficiently large $i$, it suffices to find a curve $\alpha_i$ on $\tilde{\sigma}_i$ connecting $\tilde{\psi}_i(p)$ to $\tilde{\psi}_i(q)$ such that $-\epsilon < \operatorname{length}(\alpha_i) - \operatorname{length}(\alpha) < \epsilon$.

For each orthogonal segment $a_j$, as defined for Lemma 5.7, there is a corresponding curve $b_{i,j}$ on $(\mathbb{H}^2, \tilde{\nu}_i)$, such that $\beta | a_j$ and $\beta_i | b_{i,j}$ are contained in a single hyperbolic plane and $\beta_i | a_{i,j}$ converges to $\beta | a_j$ as $i \to \infty$. Then, by Lemma 5.7, $\operatorname{length}(b_{i,j}) \to \operatorname{length}(a_j)$ as $i \to \infty$. Thus $\Sigma_j \operatorname{length}(b_{i,j}) < \epsilon$ for sufficiently large $i$. Then, for each $j$, we can connect the endpoints of $a_{i,j}$ and $b_{i,j+1}$ by geodesic segments $c_{i,j}$ in the complement of $\tilde{\nu}_i$, to obtain a curve $\alpha_i$ connecting $\tilde{\psi}_i(p)$ and $\tilde{\psi}_i(q)$. Then, for every $\epsilon > 0$, if $i$ is sufficiently large, then the endpoints of $c_{i,j}$ are $\epsilon$-close to the $\tilde{\psi}_i$-image of the endpoints of $c_j$ for all $j$. Since $\tilde{\nu}_{i,\tau} \to \tilde{\nu}$, the segment $c_j$ is disjoint from $\tilde{\nu}_{i,\tau}$ for sufficiently large $i$. Therefore $\operatorname{length}(c_{i,j})$ is $\epsilon$-close to $\operatorname{length}(c_j)$ for all $j$. Since $\epsilon > 0$ is arbitrary, $-\epsilon < \operatorname{length}(\alpha) - \operatorname{length}(\alpha_i) < \epsilon$ for sufficiently large $i$. □

Proposition 5.6 immediately follows from Proposition 5.9 and Lemma 5.8. [5.6]

## 6. Rectangular projective structures

### 6.1. Projective structures on rectangle supported on cylinders.

Let $c$ be a round circle on $\hat{\mathbb{C}}$. A geodesic $g$ in $\mathbb{H}^3$ is an *axis* of $c$ on $\hat{\mathbb{C}}$ if $g$ is orthogonal to the (totally geodesic) hyperbolic plane in $\mathbb{H}^3$ bounded by $c$.

Let $\mathcal{A}$ be a *round cylinder* in $\hat{\mathbb{C}}$, that is, $\mathcal{A}$ is bounded by disjoint round circles $c_{-1}$ and $c_1$. Then the *axis* of $\mathcal{A}$ is the unique geodesic in $\mathbb{H}^3$ that is orthogonal to both hyperbolic planes bounded by $c_{-1}$ and $c_1$. Then, there is a unique foliation $\mathcal{F}_\mathcal{A}$ of $\mathcal{A}$ given by the continuous family of round circles, $\{c_t\}_{t \in [-1,1]}$, sharing the axis $g$. We call it the *circular foliation* on $\mathcal{F}_\mathcal{A}$. Then each round circle $c_t$ has a smooth metric

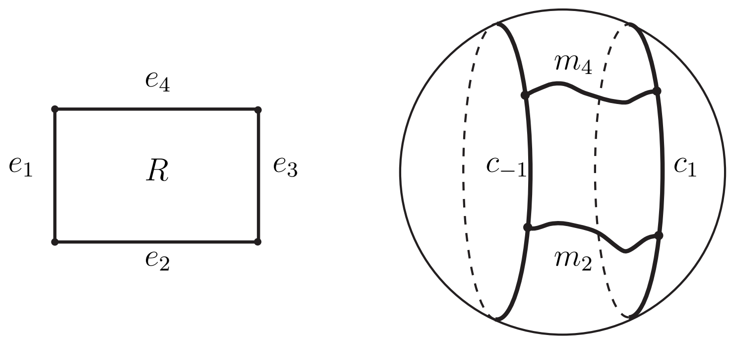


Figure 5.

invariant under by elliptic isometries of $\mathbb{H}^3$ fixing $g$. It is unique up to scaling, and thus we normalize it so that the length of $c_t$ is $2\pi$ for all $t \in [-1, 1]$ (*canonical metric*).

**Definition 6.1.** *Let $C = (f, \rho)$ be a projective structure on a simply connected surface $F$. (In particular $\rho$ is trivial.) Let $e$ be a simple curve on $C$. Then we say that $e$ is supported on the round cylinder $\mathcal{A}$ if $f$ embeds $e$ properly into $\mathcal{A}$ so that $e$ transversally intersects all leaves $c_t$ of $\mathcal{A}$.*

Let $R$ be a rectangle, and let $e_1, e_2, e_3, e_4$ denote the edges of $R$, cyclically indexed along $\partial R\,(\cong \mathbb{S}^1)$; Figure 5. A projective structure $C = (f, \rho)$ on $R$ is *supported* on the round cylinder $\mathcal{A}$ if

(i) $f$ immerses $e_1$ and $e_3$ into $c_{-1}$ and $c_1$, respectively, and
(ii) $e_2$ and $e_4$ are supported on $\mathcal{A}$.

Then we say that $C$ is *supported* on the round cylinder $\mathcal{A}$ and *bounded* by the arcs $f|e_2$ and $f|e_4$ supported on $\mathcal{A}$.

Then, if $C$ is supported on $\mathcal{A}$, we can pull back, via $f$, the circular foliation $\mathcal{F}_\mathcal{A}$ on $\mathcal{A}$ to a circular foliation $\mathcal{F}_C$ on the rectangle $C$. Each leaf of $\mathcal{F}_C$ immerses into a closed leaf of $\mathcal{F}_\mathcal{A}$, and thus it has a metric obtained by pulling back the canonical metric of the closed leaf. Then we say that the *height* of $C$ is $\epsilon$-close to $W$ for some $W > 0$, if every leaf of $\mathcal{F}_C$ has length $\epsilon$-close to $W$.

6.2. **Grafting a rectangle supported on a cylinder.** (Compare [3, §3.5].) Let $C$ be a projective structure on the rectangle $R$ supported on the round cylinder $\mathcal{A}$ as above. Let $m$ be a simple arc on $C$ supported on $\mathcal{A}$. Then $m$ is an arc properly embedded in $\mathcal{A}$. Then, similarly to grafting a projective surface along a loop (§3.2)), we can combine two projective structures $C$ and $\mathcal{A}$, by cutting and pasting along $m$, and obtain a new projective structure on $R$ supported on $\mathcal{A}$. Namely, we can pair up the boundary arcs of $C \setminus m$ and the boundary arcs of $\mathcal{A} \setminus m$ and isomorphically identify them to create a new projective structure on $R$ supported on $\mathcal{A}$. We call this operation the *grafting* of $C$ along $m$ and denote this resulting projective structure by $\mathrm{Gr}_m(C)$. We call $m$ an *admissible* arc on $C$. If there is a multiarc $M$ on $C$ consisting of arcs supported on $\mathcal{A}$ (*admissible multiarc*), then we can graft $C$ along all arcs of $M$ simultaneously and obtain a new projective structure on $R$ supported on $\mathcal{A}$. We accordingly denote it by $\mathrm{Gr}_M(C)$.

**Lemma 6.2.** *Let $C_1$ and $C_2$ be projective structures on a rectangle $R$. Suppose that they are supported on the same round cylinder and bounded by the same pair of arcs supported on the cylinder. Then, we have either $C_1 = \mathrm{Gr}_M(C_2)$ or $C_2 = \mathrm{Gr}_M(C_1)$ for some admissible multiarc $M$. Furthermore, the multiarc $M$ is unique up to an isotopy of $M$ on $R$ through admissible multiarcs.*

*Moreover the number of arcs of $M$ times $2\pi$ is equal to the length difference of the corresponding vertical edges of $C_1$ and $C_2$.*

*Proof.* Let $\mathcal{A}$ be the round cylinder supporting $C_1$ and $C_2$. Let $f_1\colon R \to \mathcal{A}$ and $f_2\colon R \to \mathcal{A}$ be the developing maps of $C_1$ and $C_2$, respectively. Let $\tilde{\mathcal{A}}$ be the universal cover of $\mathcal{A}$ and $\Psi\colon \tilde{\mathcal{A}} \to \mathcal{A}$ be the universal covering map. Let $m_2$ and $m_4$ be the simple arcs properly embedded in $\mathcal{A}$ that bound both $C_1$ and $C_2$ so that $m_2 = f_1(e_2) = f_2(e_2)$ and $m_4 = f_1(e_4) = f_2(e_4)$. Pick a lift $\tilde{m}_4$ of $m_4$ to $\tilde{\mathcal{A}}$. Then, for each $k = 1, 2$, $f_k\colon R \to \mathcal{A}$ uniquely lifts to $\tilde{f}_k\colon R \to \tilde{\mathcal{A}}$ so that $f_k = \Psi \circ \tilde{f}_k$ and $\tilde{f}_k$ embeds $e_4$ onto $\tilde{m}_4$. Clearly $\tilde{f}_k$ is an embedding (although $f_k$ may *not* be). We see that $\tilde{f}_k(e_2)$ is a lift of $m_2$ to $\tilde{\mathcal{A}}$. Since projective structures have fixed orientation, $\tilde{f}_1(e_2)$ and $\tilde{f}_2(e_2)$ are in the same component of $\tilde{\mathcal{A}} \setminus \tilde{m}_4$. If $\tilde{f}_1(e_2) = \tilde{f}_2(e_2)$, then clearly $C_1 = C_2$. If $\tilde{f}_1(e_2) \neq \tilde{f}_2(e_2)$, then, without loss of generality, we can assume that $Im(\tilde{f}_2)$ is a proper subset of $\mathrm{Im}(\tilde{f}_1)$, if necessary, by exchanging $C_1$ and $C_2$. Thus we can naturally regard $\mathrm{Im}(\tilde{f}_1) \setminus \mathrm{Im}(\tilde{f}_2)$ as a projective structure on a rectangle supported on $\mathcal{A}$, where its developing map is the restriction of $\Psi$ to $\mathrm{Im}(\tilde{f}_1) \setminus \mathrm{Im}(\tilde{f}_2)$. Then its supporting arc are both $m_4$. Let $d$ be the generic degree of the developing map of $\mathrm{Im}(\tilde{f}_1) \setminus \mathrm{Im}(\tilde{f}_2)$ to $\mathcal{A}$ (i.e. the degree over a point in $\mathcal{A} \setminus m_4$). Note that

$$2\pi d = \text{length}\, f_1(e_1) - \text{length}\, f_2(e_1) = \text{length}\, f_1(e_3) - \text{length}\, f_2(e_3).$$

Note that the grafting along an admissible arc on $C_2$ increases the length of its vertical edges by $2\pi$. Thus, if $M$ is the union of $n$ disjoint admissible arcs on $C_2$, then the length of vertical edges increases by $2\pi n$ Therefore $Gr_M(C_2) = C_1$ if and only if $n = d$. □

6.3. **Fat traintracks.** Given a rectangle, pick a pair of opposite edges and call them *horizontal edges* and the other edges *vertical*, to distinguish them.

**Definition 6.3** ([24, 29])**.** *Let $F$ be a topological surface. A (fat) traintrack $T$ on $F$ is a collection of rectangles $R_j\,(j \in J)$ embedded in $F$, called branches, such that*

- *$\{R_j\}_{j \in J}$ is locally finite,*
- *branches can intersect only along their vertical edges, and*
- *if $e$ is a vertical edge then either*
  - *$e$ is (homeomorphically) identified with another vertical edge of a rectangle,*
  - *$e$ is a union of two other vertical edges, which share an endpoint, of some rectangles, or*
  - *$e$ is identified with a segment of another vertical edge containing an endpoint.*

*Then let $|T| \subset S$ denote the union of the rectangles $R_i$.*

Let $T = \{R_j\}_{j \in J}$ denote a traintrack on $F$, where $R_j$ are its branches. The vertical edges of the branches $R_j$ decompose $T$ into the branches. The boundary of $|T|$ is the union of the horizontal edges. If a point of $\partial|T|$ is the common end point of the second possibility for $e$, then it is called a *switch point*.

Let $\lambda$ be a lamination on $F$. Then the traintrack $T = \{R_j\}_{j \in J}$ *carries* $\lambda$, if

- the interior of $|T|$ contains $\lambda$ and
- each leaf $\ell$ of $\lambda$ is transversal to the vertical edges of $T$ and each component of $\ell \cap R_j$ is an arc connecting the vertical edges of $R_j$ for each $j \in J$,

If, in addition, $R_j \cap \lambda \neq \emptyset$ for all $j \in J$, then we say $T$ *fully carries* $\lambda$. Suppose that $L = (\lambda, \mu)$ is a measured lamination carried by $T$. The *weight* of $L$ on a branch $R_j$ is the transversal measure $\mu$ of a vertical edge of $R_j$; we denote it by $\mu(R_j)$. The weights of branches satisfy some simple equations, called *switch conditions*.

Suppose that $T$ carries two measured laminations $L_1$ and $L_2$. Then the weights $L_1$ and $L_2$ are nonnegative real numbers on each branch of $T$. Thus there is a unique measured lamination $L_1 + L_2$ carried by

$T$ such that, the weight of $L_1 + L_2$ on $R_j$ is the sum of the weights of $L_1$ and $L_2$ on $R_j$ for each $j$. Suppose that the weight of $L_1$ is at least the weight of $L_2$ on each branch of $T$. Then similarly there is a unique measured lamination $L_1 - L_2$ carried by $T$ such that the weight of $L_1 - L_2$ on $R_j$ is the weight of $L_1$ minus the weight of $L_2$ on $R_j$ for each $j$.

Given $\epsilon > 0$, we say $L_1$ is *$\epsilon$-close* to $L_2$ on $T$, if the weight of $L_1$ is $\epsilon$-close to that of $L_2$ on each branch of $T$. We say that $L_1$ is a *good approximation* of $L_2$ on $T$, if $L_1$ is $\epsilon$-close to $L_2$ for a sufficiently small $\epsilon > 0$.

We remark that, if a traintrack $T = \{R_j\}_j$ has weights on its branches satisfying the switch conditions, it corresponds to a unique measured lamination. Indeed there is a measured foliation $\mathcal{F}$ of $|T|$ such that $\mathcal{F}$ foliates each branch $R_i$ by arcs connecting its vertical edges and the transversal measure of $R_i$ given by $\mathcal{F}$ realizes the weight of $R_i$. Then by "straightening" leaves of $F$ fixing a hyperbolic metric on $F$, we obtain a measured (geodesic) lamination.

Moreover the above addition and subtraction respect the piecewise linear structure on the space of measured laminations. In particular, the set of all possible weight-systems on a traintrack is a piecewise linear cone in a vector space.

Let $\tau$ be a hyperbolic structure on $F$. Then the traintrack $T$ is *smooth* if all branches are smooth (i.e. the edges of its branches are smooth) and $\partial|T|$ is smooth except at the switch points.

**Definition 6.4.** *For $\epsilon > 0$, a smooth traintrack $T = \{R_j\}_{j \in J}$ on $\tau$ is called $\epsilon$-nearly straight if each branch $R_j$ is $(1+\epsilon)$-bilipschitz to some Euclidean rectangle and, at each switch point, the angle of $\partial|T|$ is less than $\epsilon$.*

*For $\epsilon, K > 0$, $T$ is $(\epsilon, K)$-nearly straight if $T$ is $\epsilon$-nearly straight and, for each branch $R_i$ of $T$, $K$ is less than the length of the horizontal edge of such a Euclidean rectangle corresponding to $R_i$ (which we call the length of $R_i$).*

Such a nearly straight (non-fat) traintrack is introduced in [31, ch. 8]; see also [5, 27]]. If $\epsilon > 0$ is sufficiently small, each branch of an $(\epsilon, K)$-nearly straight traintrack is hausdorff close to an almost straight curve.

## 7. Decomposition of projective structures by traintracks

**Definition 7.1.** *Let $R$ be a branch of a traintrack $T$ in a projective structure $C = (f, \rho)$ on $S$. (Recall that a branch is a rectangle.) Then* *the branch $R$ is supported on a round cylinder $A$ on $\hat{\mathbb{C}}$ if $A$ supports the restriction of $C$ to $R$ so that horizontal edges of $R$ are supported on $A$.*

**Lemma 7.2.** *Let $T = \{R_k\}$ be a traintrack on a projective surface $(S, C)$ such that each branch $R_k$ is supported on a round cylinder (admissible traintrack). Note that the circular foliations on $R_k$ yield a circular foliation on $|T|$. Then, if a loop $\ell$ is carried by $T$ and transversal to the circular foliation on $|T|$, then $\ell$ is admissible.*

*Proof.* Let $\tilde{\ell}$ be a lift of $\ell$ to $\tilde{S}$. Let $\tilde{T}$ be the lift of $T$ of $\tilde{S}$. Let $\tilde{R}_{k\in\mathbb{Z}}$ denote the branches of $T$ intersecting $\tilde{\ell}$, so that $\tilde{R}_k$ and $\tilde{R}_{k+1}$ are adjacent. Let $C = (f, \rho)$, where $f$ is the developing map and $\rho$ is the holonomy. Then $f$ injects $\ell \cap \tilde{R}_k$ for each $k$. The supports of $\tilde{R}_k$ have disjoint interiors. Thus $f$ embeds $\tilde{\ell}$ into $\hat{\mathbb{C}}$. Since this embedding extends the endpoint of $\tilde{\ell}$, taking them to distinct points, $\rho(\ell)$ is loxodromic. □

The following proposition will yield the traintracks on projective surfaces in Theorem A and Theorem B.

**Proposition 7.3.** *Let $C_i \cong (\tau_i, L_i)$, $i \in \mathbb{Z}_{>0}$, be a sequence of projective structures on $S$ with fixed holonomy $\rho$, and let $f_i$ be the developing map of $C_i$. Let $\mathcal{L}_i$ be the canonical lamination on $C_i$, which descends to $L_i$ by the collapsing map $\kappa_i \colon C_i \to \tau_i$.*

*Suppose that $\tau_i$ converges to $\tau_\infty$ in $\mathfrak{T}$ as $i \to \infty$, and there is a geodesic lamination $\lambda_\infty$ (on $\tau_\infty$ ), such that, for every $\epsilon > 0$, the $\epsilon$-neighborhood of $|\lambda_\infty|$ contains $|L_i|$ for sufficiently large $i$.*

*Then there are a traintrack $\mathcal{T} = \{\mathcal{R}_k\}_{k=1}^n$ on $S$ (which depends only on $\rho, \tau_\infty, |\lambda_\infty|, \epsilon$) and a homeomorphism $\phi_i \colon S \to C_i$ for every $i$, such that for every $\epsilon > 0$ if $i, j$ are sufficiently large, then*

(I) *$\phi_i(\mathcal{T})$ carries $\mathcal{L}_i$, and $\phi_i(\mathcal{T})$ descends, by $\kappa_i$, to an $(\epsilon, K)$-nearly straight traintrack $T_i$ on $\tau_i$ carrying both $L_i$ and $\lambda_\infty$, where $K > 0$ is an arbitrarily fixed constant that is less than one third of a shortest closed leaf on $\lambda_\infty$ (if $\lambda_\infty$ contains no closed leaves, then $K > 0$ is arbitrarily).*

(II) *$f_i$ and $f_j$ induce an isomorphism from $C_i \setminus \phi_i(\mathcal{T})$ to $C_j \setminus \phi_j(\mathcal{T})$ as projective surfaces; thus we can assume that $\phi_i \circ \phi_j^{-1} \colon C_j \to C_i$ induces this isomorphism.*

(III) (i) *For each branch $\tilde{\mathcal{R}}_k$ of $\tilde{\mathcal{T}}$, there exists a round cylinder $\tilde{\mathcal{A}}_k$ on $\hat{\mathbb{C}}$ that supports its corresponding rectangle $\tilde{\phi}_i(\tilde{\mathcal{R}}_k)$ in $\tilde{C}_i$ for every sufficiently large $i$, where $\tilde{\phi}_i \colon \tilde{S} \to \tilde{C}_i$ is the lift of $\phi_i$.*

(ii) *Moreover, if a is a vertical edge of $\tilde{\phi}_i(\tilde{\mathcal{R}}_k)$ then the length of a (§6.1) is $\epsilon$-close to the transversal measure of $\phi_i(\mathcal{R}_k)$ with respect to $\mathcal{L}$ (§6.3), where $\mathcal{R}_k$ is the branch of $\mathcal{T}$ that lifts to $\tilde{\mathcal{R}}_k$.*

**Remark 7.4.** *To be precise, by "descends" in (I), we mean that $\kappa_i$ takes $\phi_i(\mathcal{T})$ to $T_i$ up to an $\epsilon$-small perturbation of vertical edges as given in Theorem 7.12. Yet $\kappa_i$ takes $|\phi_i(\mathcal{T})| \subset C$ exactly onto $|T_i| \subset \tau_i$.*

In particular, by taking all $C_i$ to be a fixed projective structure, we obtain

**Corollary 7.5.** *Let $C \cong (\tau, L)$ be a projective structure on $S$, and let $\lambda$ be a geodesic lamination $\lambda$ on $\tau$ containing $|L|$. Let $\kappa \colon C \to \tau$ be its collapsing map. Let $\Lambda$ be the lamination on $C$ that descends to $\lambda$ by $\kappa$. Then for every $\epsilon > 0$, there is an admissible traintrack $\mathfrak{T}$ on $C$ carrying $\Lambda$, so that it descends to an $\epsilon$-nearly straight track on $\tau$ by $\kappa$ up to an $\epsilon$-small perturbation of vertical edges.*

The rest of §7 is the proof of Proposition 7.3.

*Outline of the proof of Proposition 7.3.* Construct a nearly straight traintrack $T_\infty$ on $\tau_\infty$ carrying $\lambda_\infty$ (Lemma 7.10). Indeed $T_\infty$ yields all other traintracks in the proposition. There is a $\rho$-equivariant pleated surface realizing $(\tau_\infty, \lambda_\infty)$, and it is the limit of the $\rho$-equivariant pleated surface for $C_i$ (see Lemma 7.6). By this convergence, for sufficiently large $i$, there is a corresponding nearly straight traintrack $T_i$ on $\tau_i$. The traintrack $\mathcal{T}_i$ in the proposition is obtained by pulling back $T_i$ by the collapsing map $\kappa_i \colon C_i \to \tau_i$ and perturbing of vertical edges a little bit ( see Proposition 7.11. The estimate of the lengths of vertical edges is given in §7.3.

**Lemma 7.6.** *Let $C_i \cong (\tau_i, L_i)$ be a sequence of projective structures on $S$ with fixed holonomy $\rho$, such that $\tau_i$ converges to $\tau_\infty$. For each $i$, let $\beta_i \colon \mathbb{H}^2 \to \mathbb{H}^3$ be the $\rho$-equivariant pleated surface corresponding to $C_i$.*

*Suppose that there is a geodesic lamination $\lambda_\infty$ on $\tau_\infty$ such that, given any $\epsilon > 0$, the $\epsilon$-neighborhood of $|\lambda_\infty|$ contains (the geodesic representative of) $|L_i|$ for all sufficiently large $i$. Then there is a $\rho$-equivariant pleated surface $\beta_\infty$ realizing the pair $(\tau_\infty, \lambda_\infty)$, and $\beta_i$ converges to $\beta_\infty$. This convergence is uniform in the $C^0$-topology on $S$ and uniform on compacts in the $C^1$-topology in the complement of $|\lambda_\infty|$.*

**Remark 7.7.** *In this lemma, there may be a sublamination of $\lambda_\infty$ that realizes $\beta$ as well.*

*Proof.* First we show that $\lambda_\infty$ is realizable by a $\rho$-equivariant pleated surface. The assumption on $\lambda_\infty$ and $L_i$ implies that $\angle_{\tau_\infty}(\lambda_\infty, L_i) \to 0$ as

$i \to \infty$. Thus, for every $\epsilon > 0$, if $i$ is sufficiently large, then $\beta_i$ restricts to a $(1+\epsilon)$-bilipschitz map on every leaf of $\tilde\lambda_\infty$ on the universal cover of $\tau_i$ (Proposition 4.4).

Since $\epsilon > 0$ is arbitrary, there a unique $\rho$-equivariant map $\beta_\infty$ taking leaves of $\tilde\lambda_\infty$ to geodesics in $\mathbb{H}^3$, so that $\beta_i$ converges to $\beta_\infty$ on $\tilde\lambda_\infty$ uniformly as $i \to \infty$. Let $\tilde\lambda_i$ be the total lift of $\lambda_i$ to $\mathbb{H}^2$.

**Claim 7.8.** *For every $\epsilon > 0$, if $i$ is sufficiently large, then the restriction of $\beta_i$ to each component $R$ of $\mathbb{H}^2 \setminus |\tilde\lambda_\infty|$ is a $(1+\epsilon, \epsilon)$-quasiisometric embedding.*

*Proof.* The proof is similar to arguments in [2, §7]. For every $\delta > 0$, if $i$ is sufficiently large, then, for each component $R$ of $\mathbb{H}^2 \setminus |\tilde\lambda_\infty|$, the restriction $\beta_i|R$ is totally geodesic away from the $\delta$-neighborhood of $\partial R$. Moreover, if $\delta > 0$ is sufficiently small and $i \in \mathbb{Z}_{>0}$ is sufficiently large, then we can in addition assume that, for every geodesic segment $s$ in the $\delta$-neighborhood of $\partial R$ in $R$, either $\mathrm{length}_{\mathbb{H}^2}(s) < \epsilon$ or $\angle(s, \tilde L_i)$ is quite small, so that $\beta_i|s$ is a $(1+\epsilon, \epsilon)$-quasiisometric embedding.

Then the claim immediately follows. □

The $\epsilon > 0$ is arbitrary in Claim 7.8, and therefore $\beta_\infty$, which is defined on $\partial R$, must extends to $R$ so that $\beta_\infty|R$ is a totally geodesic isometric embedding. Thus $\beta_i$ uniformly converges to $\beta_\infty$ realizing $(\tau_\infty, \lambda_\infty)$ in the $C^0$-topology. Since $\beta_i$ and $\beta_\infty$ are totally geodesic in the complements of $\lambda_i$ and $\lambda_\infty$, respectively, the convergence is, in the $C^\infty$-topology, uniform on compacts in the complement of $\tilde\lambda_\infty$. [7.6]

**Remark 7.9.** *The bending map $\beta_i$ has a natural normal vector field at all points $x$ away from the lifts of closed leaves of $L_i$: Namely, it is the direction of the ray from $\beta(x)$ to the $f_i$-image of the point on $\tilde S$ corresponding to $x$. Then the limit of this normal vector field yields a normal vector field on $\beta_i$ away from $|\tilde\lambda_\infty|$.*

### 7.1. **Construction of Traintracks.**

**Lemma 7.10.** *Let $|\nu|$ be a geodesic lamination on a hyperbolic surface $\sigma$ homeomorphic to $S$. Then there exists a $K > 0$, such that, for every $\epsilon > 0$, there exists an $(\epsilon, K)$-nearly straight traintrack $T = \{R_j\}_j$ on $\sigma$ fully carrying $\nu$, such that if a vertical edge of $T$ intersects a leaf of $\nu$, then the angle is $\epsilon$-close to $\pi/2$.*

*If $\nu$ contains a closed leaf, then we can take $K > 0$ to be any number less than one third of the length of the shortest closed leaf of $\nu$ and, otherwise, we can take $K$ to be any positive number.*

Such a traintrack can be obtained by taking a $\delta$-neighborhood of $|\nu|$ with sufficiently small $\delta > 0$ and splitting it so that each branch has a certain amount of length; the details are left to the reader.

By Lemma 7.6, we have a $\rho$-equivariant pleated surface $\beta_\infty$ realizing $(\tau_\infty, \lambda_\infty)$. Similarly to Thurston coordinates on projective structures on $S$, the pair $(\tau_\infty, \lambda_\infty)$ defines a projective structure $C_\infty$ on $S \setminus |\lambda_\infty|$. Indeed, since $\tilde\beta_\infty$ is a locally totally geodesic embedding away from the total lift $\tilde\lambda_\infty$ of $\lambda_\infty$, it induces a $\rho$-equivariant developing map $f_\infty$ from $\tilde S$ minus $|\tilde\lambda_\infty|$, so that $f_\infty(x)$ projects orthogonally to the image of $\beta_\infty(x)$ for all $x \in \tilde S \setminus \tilde\lambda_\infty$. In particular there is a natural embedding $\kappa_\infty$ of $C_\infty$ onto $\tau_\infty \setminus |\lambda_\infty|$.

For every $\epsilon > 0$, let $T_\infty(= T_{\infty,\epsilon})$ be an $(\epsilon, K)$-nearly straight traintrack on $\tau_\infty$ given by Lemma 7.10 carrying $\lambda_\infty$. Then let $\mathcal{T}_\infty = \kappa^{-1}(|T_\infty|)$ be the subset of $C_\infty$, such that the closure of $C_\infty \setminus \mathcal{T}_\infty$ is a compact subset of $C_\infty$, on which $C_\infty$ deformation retracts.

Suppose that there is a nearly straight traintrack $T_i$ on $\tau_i$ carrying $L_i$. Then, since vertical edges of $T_i$ are transversal to $L_i$, thus $\kappa_i^{-1}(T_i) =: \mathcal{T}_i$ is a traintrack on $C_i$ carrying the canonical lamination $\mathcal{L}_i$. Note that $\mathcal{T}_i$ and $T_i$ are the same traintrack as topological traintracks on $S$ and this identification is given by $\kappa_i$. The measured laminations $\mathcal{L}_i$ and $L_i$ represent the same element on $\mathcal{ML}(S)$. Then if $\mathcal{R}$ and $R$ are corresponding branches of $\mathcal{T}_i$ and $T_i$, then the weight of $R$ given by $L$ is equal to the weight of $\mathcal{R}$ given by $\mathcal{L}_i$. In this sense, $(\mathcal{T}_i, \mathcal{L}_i)$ is *isomorphic* to $(T_i, L_i)$ (as weighted traintracks).

Then we show (I) and (II):

**Proposition 7.11.** *Let $\epsilon > 0$. Then, for sufficiently large $i \in \mathbb{N}$, there exists a traintrack $T_i$ on $\tau_i$ isotopic to $T_\infty$ on $\tau_\infty$ as a topological traintrack on $S$, such that*

(i) *$T_i$ is $(\epsilon, K)$-nearly straight,*
(ii) *$T_i$ carries $L_i$,*
(iii) *there is a marking-preserving $\epsilon$-rough isometry from $\tau_\infty$ to $\tau_i$ that takes $|T_\infty|$ to $|T_i|$, and*
(iv) *$C_\infty \setminus |\mathcal{T}_\infty|$ is isomorphic to $C_i \setminus |\mathcal{T}_i|$ (as projective surfaces) via their developing maps, where $\mathcal{T}_i$ the traintrack on $C_i$ that descends to $T_i$ via the collapsing map $\kappa_i\colon C_i \to \tau_i$.*

*Proof.* For each component $P$ of $\tau_\infty \setminus |T_\infty|$, let $P'$ be the component of $\tau_\infty \setminus |\lambda_\infty|$ containing $P$. For each $i$, let $\psi_i\colon \tau_\infty \to \tau_i$ be a marking-preserving $\delta_i$-rough isometry, obtained by Theorem 5.2, with its distortion $\delta_i$ limiting to 0 as $i \to \infty$. Then since a small neighborhood of $|\lambda_\infty|$ contains $|\lambda_i|$ for sufficiently large $i$, there is a corresponding component $P_i'$ of $\tau_i \setminus |\lambda_i|$ such that $\phi_i(P_i')$ contains $P$. Let $\tilde P'$ be a lift of $P'$ to $\mathbb{H}^2$. Then $\beta_\infty$ takes $\tilde{P}'$ isometrically into a (totally geodesic) hyperbolic plane in $\mathbb{H}^3$. The ideal boundary of this hyperplane cuts $\hat{\mathbb{C}}$ into two round open balls. Then, one of those round balls is in the normal direction of $\beta_\infty|\tilde{P}'$ (see Remark 7.9). Let $\tilde{Q}'$ be the region in this round ball that conformally projects onto $\beta_\infty(\tilde{P}')$ via the orthogonal projection to the hyperplane. Since $P$ is a subset of $P'$, let $\tilde{Q}$ be the regain in $\tilde{Q}'$ conformal to $\beta_\infty(P)$ via the projection.

Similarly, $\beta_i$ isometrically embeds $\tilde{P}'_i$ into $\mathbb{H}^2$ in $\mathbb{H}^3$, and we let $\tilde{Q}'_i$ be the region in $\hat{\mathbb{C}}$ conformal to $\tilde{P}'_i$ via the orthogonal projection to this hyperplane. Then, for sufficiently large $i$, $\tilde{Q}'_i$ contains $\tilde{Q}$ since $\psi_i(\tilde{P}'_i)$ contains $\tilde{P}$ and $\beta_i$ is sufficiently close to $\beta_\infty$. Then via the conformal isomorphisms $\tilde{Q}' \cong \tilde{P}'$ and $\tilde{Q}_i \cong \tilde{P}_i$, we have an embedding $\tilde{\eta}_i \colon \tilde{P} \to \tilde{P}'_i$. Then $\beta_i \circ \tilde{\eta}_i$ smoothly converges to $\beta_\infty$ as $i \to \infty$. Thus, for every $\epsilon > 0$, if $i$ is sufficiently large, $\tilde{\eta}_i$ and $\tilde{\psi}_i$ are $\epsilon$-close on $\tilde{P}$ (in the $C^\infty$ topology).

Moreover, since $\beta_i \to \beta_\infty$, if $i$ is sufficiently large, then different complementary components of $|\tilde{T}_\infty|$ have disjoint images in $\tilde{\tau}_i \setminus |\tilde{\lambda}_i|$. Thus we have a conformal embedding $\tilde{\eta}_i \colon \tilde{\tau}_\infty \setminus |\tilde{T}_\infty| \to \tilde{\tau}_i \setminus |\tilde{\lambda}_i|$ that commutes with the action of $\pi_1(S)$. Then it descends to an embedding $\eta_i \colon \tau_\infty \setminus |T_\infty| \to \tau_i \setminus |\lambda_i|$. Moreover, for every $\epsilon > 0$, if $i$ is sufficiently large, $\eta_i$ and $\psi_i$ are $\epsilon$-close on $\tau_\infty \setminus |T_\infty|$. Since $\psi_i$ converges to an isometry as $i \to \infty$, therefore $\tau_i \setminus \operatorname{Im} \eta_i$ enjoys a traintrack structure $T_i$ carrying $L_i$ (proving (ii)) that satisfies (i) and (iii) for sufficiently large $i$. There is a unique isomorphic embedding of $C \setminus |\mathcal{T}_\infty|$ into $C_i \setminus |\mathcal{L}_i|$ compatible with their developing maps, such that it descends to $\eta_i$ via the collapsing maps of $C$ and $C_i$. Hence $T_i$ also satisfies (iv). □

7.2. **Proof of Proposition 7.3 (III) - (i).** By the arguments above, we can assume that $T_\infty$ is $(\epsilon, K)$-nearly straight with the fixed constant $K > 0$ and sufficiently small $\epsilon > 0$.

Let $b$ be a switch point of $T_\infty$. Then, since $b$ is in the complement of $|\lambda_\infty|$, the pleated surface $\beta_\infty \colon \mathbb{H}^2 \to \mathbb{H}^3$ is smooth at each lift $\tilde{b}$ of $b$ to $\mathbb{H}^2$. Thus there is a unique point $\tilde{\mathsf{b}}$ on $\hat{\mathbb{C}}$ that orthogonally projects to $\beta_\infty(\tilde{b})$ on $\beta_\infty$ from its normal direction (see Remark 7.9).

Pick a round circle $\mathsf{c}(\tilde{b})$ on $\hat{\mathbb{C}}$ containing $\tilde{\mathsf{b}}$ so that the hyperplane bounded by $\mathsf{c}(\tilde{b})$ contains $\beta_\infty(\tilde{b})$ and so that it is "*nearly orthogonal* to the traintrack of $\tilde{T}_\infty$ near $\tilde{b}$": Namely, given any $\zeta > 0$, if $T_\infty$ is sufficiently straight (i.e. $\epsilon > 0$ is sufficiently small), then, for each leaf $\ell$ of $\lambda_\infty$ passing through the vertical edge of $\tilde{T}_\infty$ containing $b$, the geodesic $\beta_\infty(\ell)$ is $\zeta$-nearly orthogonal to the hyperplane. For different lifts $\tilde{b}$ of $b$, we $\rho$-equivariantly take such round circles $\mathsf{c}(\tilde{b})$. (Note that the upper bound of the number of branches points on $T_\infty$ depends only on the topological surface $S$.)

Then, since each branch of $\tilde{T}_\infty$ has length at least $K$ and we can pick sufficiently small $\zeta > 0$, for each branch $\tilde{R}_\infty$ of $\tilde{T}_\infty$, letting $\tilde{b}_1$ and $\tilde{b}_2$ be the switch points on the different vertical edges of $\tilde{R}_\infty$, their corresponding round circles $\mathsf{c}(\tilde{b}_1)$ and $\mathsf{c}(\tilde{b}_2)$ are disjoint (in $\hat{\mathbb{C}}$). Let $\mathcal{A}(\tilde{R}_\infty)$ denote the round cylinder bounded by $\mathsf{c}(\tilde{b}_1)$ and $\mathsf{c}(\tilde{b}_2)$. Then the convex hull, in $\mathbb{H}^3$, of $\mathcal{A}(\tilde{R}_\infty)$ contains most of $\beta_\infty(\tilde{R}_\infty)$.

Consider the two copies of $\mathbb{H}^2$ in $\mathbb{H}^3$ bounded by $\mathsf{c}(\tilde{b}_1)$ and $\mathsf{c}(\tilde{b}_2)$. Then, for every $\zeta > 0$, if $\epsilon > 0$ is sufficiently small, the distance between these hyperplanes is at least $K - \zeta$ for all branches $\tilde{R}_\infty$ of $\tilde{\mathcal{T}}_\infty$. Therefore the modulus of $\mathcal{A}(\tilde{R}_\infty)$ is at least $(K - \zeta)/2\pi$.

For all sufficiently large $i$, set $T_i = \{R_{i,j}\}$ to be the $(\epsilon, K)$-nearly straight traintrack on $\tau_i$ obtained by Proposition 7.11. Note that, for different $i$, the traintracks $T_i$ are isomorphic as smooth traintracks and those isomorphisms, restrict, for each $j$, to a diffeomorphism between corresponding branches $R_{i,j}$. Accordingly, set $\mathcal{T}_i = \{\mathcal{R}_{i,j}\}_j$ to be the corresponding traintrack on $C_i$, so that $\kappa_i$ maps $\mathcal{R}_{i,j}$ to $R_{i,j}$ for each $j$. In addition there is a branch $R_{\infty,j}$ of $T_\infty$ corresponding to $\mathcal{R}_{i,j}$ and $R_{i,j}$.

**Proposition 7.12.** *For every $\zeta > 0$, if $\epsilon > 0$ is sufficiently small, then, for $i$ sufficiently large so that $T_i$ is $(\epsilon, K)$-nearly straight, we can isotope $\mathcal{T}_i$ on $C_i$ by a $\zeta$-small isotopy of the vertical edges of $\mathcal{T}_i$, so that, if $R$, $\mathcal{R}$ and $R_\infty$ are corresponding branches of $\tilde{T}_i$, $\tilde{\mathcal{T}}_i$ and $\tilde{T}_\infty$, respectively, then*

- (i) *$(\mathcal{T}_i, \mathcal{L}_i)$ remains isomorphic to $(T_i, L_i)$ as weighted traintracks,*
- (ii) *each branch $\mathcal{R}$ of $\tilde{\mathcal{T}}_i$ is supported on the round cylinder $\mathcal{A}(R_\infty)$, and the modulus of $\mathcal{A}(R_\infty)$ is at least $(K - \zeta)/2\pi$.*
- (iii) *both horizontal edges of $\mathcal{R}$ intersect each leaf of the circular foliation of $\mathcal{A}(R_\infty)$ at angles in $(\pi/2 - \zeta, \pi/2 + \zeta)$, and*
- (iv) *the isotopy moves each vertical edge of $\mathcal{T}_i$ at most $\zeta$ in the Hausdorff distance with respect to the Thurston metric on $C_i$.*

*Proof.* In this proof, we can assume that $i$ is sufficiently large. Let $R$ and $\mathcal{R}$ be corresponding branches of $\tilde{T}_i$ and $\tilde{\mathcal{T}}_i$, respectively. Then let $P$ be a stratum of $(\mathbb{H}^2, \tilde{\lambda}_i)$ that intersects $R$. Let $\mathcal{P} = \tilde{\kappa}_i^{-1}(P)$, where $\tilde{\kappa}_i \colon \tilde{C}_i \to \mathbb{H}^2$ is the lift of $\kappa_i \colon C_i \to \tau_i$.

The horizontal edges of $R$ are contained in different 2-dimensional strata of $(\mathbb{H}^2, \tilde{\lambda}_i)$. Then those strata bound a region in $\mathbb{H}^2$ containing all other strata intersecting $R$.

First suppose that $P$ is one of those other strata, so that $P$ intersects no horizontal edge of $R$. Let $\mathcal{P}_{\mathcal{A}(R_\infty)}$ be $\mathcal{P} \cap f_i^{-1}(\mathcal{A}(R_\infty))$. Then we show that

- $\mathcal{P}_{\mathcal{A}(R_\infty)}$ is a rectangle if $\dim \mathcal{P} = 2$ or an arc supported on $\mathcal{A}(R_\infty)$ if $\dim \mathcal{P} = 1$, and
- $\mathcal{P}_{\mathcal{A}(R_\infty)}$ is $\zeta$-close to $\mathcal{R} \cap \mathcal{P}$ with the Thurston metric on $C$.

Suppose, in addition, that $P$ is a leaf of $\tilde{\lambda}_i$. If $\epsilon > 0$ is sufficiently small, then the geodesic $\beta_i(P)$ is $\zeta$-close to the axis of $\mathcal{A}(R_\infty)$ since $\beta_i \to \beta_\infty$ as $i \to \infty$. Thus, we can assume that, for each leaf $c$ of the circular foliation of $\mathcal{A}(R_\infty)$, if we let $H_c \subset \mathbb{H}^3$ be the hyperbolic plane bounded by $c$, then $\beta_i(P)$ intersects $H_c$ in a single point at an angle $\zeta$-close to $\pi/2$. If $P$ has no atomic measure, $f_i(\mathcal{P})$ is a circular arc on $\hat{\mathbb{C}}$. Then, if $\epsilon > 0$ is sufficiently small, $f_i(\mathcal{P})$ intersects each leaf of the foliation of $\mathcal{A}(\mathcal{R}_\infty)$ at an angle $\zeta$-close to $\pi/2$. In particular $f_i(\mathcal{P}) \cap \mathcal{A}(R_\infty)$ is a connected circular curve supported on $\mathcal{A}(R_\infty)$. Thus, if $\epsilon > 0$ is sufficiently small, $\beta_i(P)$ is sufficiently close to the axis of $\mathcal{A}(R_\infty)$, and therefore $f_i^{-1}(\mathcal{A}(R_\infty)) \cap \mathcal{P}$ is $\zeta$-close to $\mathcal{R} \cap \mathcal{P}$.

If $P$ has positive atomic measure, then $\mathcal{P}$ is foliated by leaves of $\tilde{\mathcal{L}}_i$. Then if $\ell$ is a leaf of this foliation, then $f_i(\ell) \cap \mathcal{A}(R_\infty)$ is supported on $\mathcal{A}(R_\infty)$ as above. In addition, its $f_i^{-1}$-image is $\zeta$-close to $\ell \cap \mathcal{R}$, and there is a small isotopy between them in $\ell$. Since $\mathcal{P} \cap f_i^{-1}(\mathcal{A}(R_\infty))$ is the union of such arcs, similarly it is a rectangle supported on $\mathcal{A}(R_\infty)$ and $\zeta$-close to $\mathcal{R} \cap \mathcal{P}$.

Suppose that $P$ is a complementary region of $\tilde{L}_i$. Then accordingly $\mathcal{P}$ is a complementary region of $\tilde{\mathcal{L}}_i$. If $\epsilon > 0$ is small enough, $R \cap P$ is a very thin rectangle bounded by the vertical edges of $R$ and two boundary geodesics of $P$ intersecting them. Then, regardless of the choice of $P$, those boundary geodesics are $\zeta$-close in $R$ and their $\beta_i$-images are geodesics in $\mathbb{H}^3$ that are $\zeta$-close to the axis of $\mathcal{A}(R_\infty)$ in the convex hull $\mathrm{Conv}(\mathcal{A}(R_\infty))$. On the other hand, the other boundary geodesics of $\beta_i(P)$ are far away from $\mathrm{Conv}\,\mathcal{A}(R_\infty)$. Then $f_i(\mathcal{P}) \cap \mathcal{A}(R_\infty)$ is a rectangle supported on $\mathcal{A}(R_\infty)$. Hence, if $\epsilon > 0$ is sufficiently small, since $f_i$ embeds $\mathcal{P}$ into $\hat{\mathbb{C}}$, then $\mathcal{P} \cap f_i^{-1}(\mathcal{A}(R_\infty))$ is a rectangle supported on $\mathcal{A}(R_\infty)$ and $\zeta$-close to $\mathcal{P} \cap \mathcal{R}$.

Next suppose that $P$ contains a horizontal edge of $R$. Then $P \cap R$ is a thin rectangle bounded by the vertical edges of $R$, a leaf $\ell$ of $\tilde{L}_i$, and a smooth boundary segment $m$ of $|\tilde{T}_i|$. Then, if $\epsilon > 0$ is sufficiently small, $m$ is an *almost geodesic* (i.e. a bilipschitz curve with very small distortion), and thus $\beta_i(\ell)$ and $\beta_i(m)$ are $\zeta$-close to the axis of $\mathcal{A}(R_\infty)$ in $\mathrm{Conv}\,\mathcal{A}(R_\infty)$, independent of the choice of $P$. Let $\mathtt{l}$ be the boundary leaf of $\mathcal{P}$ corresponding to $\ell$ and $\mathtt{m}$ be the smooth boundary segment of $|\tilde{\mathcal{T}}_i|$ corresponding to $m$. Then $f_i$ embeds $\mathcal{P}$ into $\hat{\mathbb{C}}$. Thus $f_i(\mathtt{l})$ is a circular arc and $f_i(\mathtt{m})$ is an "almost" circular arc on $\hat{\mathbb{C}}$, and they intersect $\mathcal{A}(R_\infty)$ almost orthogonally. Therefore, if $\epsilon > 0$ is sufficiently small, each leaf of the foliation $\mathcal{A}(R_\infty)$ intersects $f_i(\mathtt{l})$ and $f_i(\mathtt{m})$ in single points $\zeta$-orthogonally. Thus there is a unique thin rectangular component of $f_i(\mathcal{P} \cap \tilde{\mathcal{T}}_i) \cap \mathcal{A}(R_\infty)$ bounded by $f_i(\mathtt{l})$, $f_i(\mathtt{m})$ and the boundary circular loops of $\mathcal{A}(R_\infty)$. Let $\mathcal{P}_{\mathcal{A}(R_\infty)}$ be the subset of $\mathcal{P}$ that diffeomorphically maps onto the component by $f_i$. Then similarly $\mathcal{P}_{\mathcal{A}(R_\infty)}$ is $\zeta$-close to $\mathcal{R} \cap \mathcal{P}$.

We have shown that, if $P$ is a stratum of $(\mathbb{H}^2, \tilde{L}_i)$ intersecting $R$, then $\mathcal{P}_{\mathcal{A}(R_\infty)}$ is either an arc or a rectangle supported on $\mathcal{A}(R_\infty)$. Thus $\mathcal{P}_{\mathcal{A}(R_\infty)}$ is diffeomorphic to the product of a point and an interval or of two intervals, where the second factor is in the direction orthogonal to the circular leaves of $\mathcal{A}(R_\infty)$. Let $\mathcal{R}'$ be the union of all $\mathcal{P}_{\mathcal{A}(R_\infty)}$ over all strata $P$ of $(\mathbb{H}^2, \tilde{L}_i)$ intersecting $R$. Then since the developing map is a local homeomorphism, by continuity, the product structures of $\mathcal{P}_{\mathcal{A}(R_\infty)}$ match up. Therefore $\mathcal{R}'$ is a (smooth) rectangle supported on $\mathcal{A}(R_\infty)$. In addition, since each $\mathcal{P}_{\mathcal{A}(R_\infty)}$ is $\zeta$-close to $\mathcal{R} \cap \mathcal{P}$, thus $\mathcal{R}'$ is $\zeta$-close to $\mathcal{R}$ if $\epsilon > 0$ is sufficiently small.

One can construct a desired small isotopy of the vertical edges of $\mathcal{R}$. Construct small isotopies all $\mathcal{R} \cap \mathcal{P}$ and $\mathcal{P}_{\mathcal{A}(R_\infty)}$ in $\mathcal{P}$ for all strata $P$ of $(\mathbb{H}^2, \tilde{L}_i)$ intersecting $R$ so that they match up. □

7.3. **Estimates of admissible multiarcs by transversal measure.** Recall that $T_\infty$ is an $(\epsilon, K)$-nearly straight traintrack carrying $\lambda_\infty$ and $T_i$ is, for $i$ sufficiently large, an $(\epsilon, K)$-nearly straight traintrack carrying $L_i$. Set $L_i = (\lambda_i, \mu_i)$ for each $i \in \mathbb{N}$ with a geodesic lamination $\lambda_i$ and its transversal measure $\mu_i$. We prove Proposition 7.3, III - ii:

**Proposition 7.13.** *For every $\delta > 0$, if $\epsilon > 0$ is sufficiently small and $i \in \mathbb{N}$ is sufficiently large, then for all corresponding branches $R$ and $\mathcal{R}$ of $\tilde{T}_i$ and $\tilde{\mathcal{T}}_i$, respectively, we have $|\operatorname{length}_{\mathcal{A}(R)}(a) - \tilde{\mu}_i(R)| < \delta$ for each vertical edge $a$ of $\mathcal{R}$.*

*Idea of proof.* Let $R_\infty$ be the branch of $\tilde{T}_\infty$ corresponding to $R$ and $R_i$. Then there are hyperbolic planes in $\mathbb{H}^3$ almost orthogonal to the $\beta_\infty$-image of $R_\infty$. The collapsing map $\tilde{\kappa}_i\colon \tilde{C}_i \to \tilde{\tau}_i$ on $R_\infty$ corresponds to the nearest point projections in $\mathbb{H}^3$ to the hyperbolic planes supporting $\beta_\infty$ on $R_\infty$. Thus this proposition is proven by carefully relating $\operatorname{length}_{\mathcal{A}(R)}(a)$ and $\tilde{\mu}_i(R)$ in a hyperbolic plane almost orthogonal to $\beta_\infty | R_\infty$.

**Remark 7.14.** *Let $\mathcal{L}_i$ be the canonical measured lamination on $C_i$, which descends to $L_i$. Then the transversal measure of $R$ given by $\tilde{L}_i$ is equal to that of $\mathcal{R}$ given by $\tilde{\mathcal{L}}_i$. Recall that $\mathcal{R}$ is foliated by the circular arcs parallel to its vertical edges (§6.1). Then Proposition 7.13 holds for the length of each leaf of the foliation, since the leaves have almost the same length.*

*Proof of Proposition 7.13.* We have the convergence $\beta_i \to \beta_\infty$ (Lemma 7.6). Recall that $T_\infty$ is $\epsilon$-nearly straight with a sufficiently small $\epsilon > 0$. Throughout this proof, $\epsilon > 0$ is a sufficiently small number, which depends on $\delta > 0$ but not on the choices of sufficiently large $i$ and the corresponding branches $R$ and $\mathcal{R}$.

For an arbitrary branch $R$ of $\tilde{T}_i$, let $I = I(R)$ be the minimal sublamination of $\tilde{L}$ containing the leaves of $\tilde{L}_i$ intersecting $R$ (c.f. [2]). Set $I = (\lambda_I, \mu_I)$, where $\lambda_I \in \mathcal{GL}(\mathbb{H}^2)$ and $\mu_I = \tilde{\mu}_i | \lambda_I$. Accordingly, let $\beta_I \colon \mathbb{H}^2 \to \mathbb{H}^3$ denote the bending map induced by $I$. Then the total transversal measure of $I$ is $\tilde{\mu}_i(R)$. In particular, for every geodesic segment $s$ on $\mathbb{H}^2$ transversal to $I$, we have $\mu_I(s) \le \tilde{\mu}_i(R)$. Therefore $\beta_I \colon \mathbb{H}^2 \to \mathbb{H}^3$ continuously extends to $\partial_\infty \mathbb{H}^2 \cong \mathbb{S}^1 \to \partial_\infty \mathbb{H}^3$.

Each complementary component of $|\lambda_I|$ is bounded by at most two leaves of $\lambda_i$. Thus the geodesic lamination $\lambda_I$ extends to a geodesic foliation $F_I$ on $\mathbb{H}^2$. Furthermore, since $T_i$ is a $(1+\epsilon, K)$-nearly straight traintrack carrying $\lambda_i$, for every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then each vertical edge of $R$ is $\delta$-nearly orthogonal to every leaf of $F_I$ unless they are disjoint.

Let $C_I$ denote the projective structure on the open disk $\mathbb{D}^2$ associated with the measured lamination $I$ on $\mathbb{H}^2$. Since $R$ is connected and $I$ is a sublamination of $\tilde{L}_i$, then $C_I$, as a projective surface, isomorphically embeds into $\tilde{C}_i$ in a canonical way (see [2, §3.8.1]). In particular, since $I = \tilde{L}_i$ on $R$, then $\mathcal{R}$ canonically embeds into $C_I$. Let $f_I \colon \mathbb{D}^2 \to \hat{\mathbb{C}}$ denote the developing map of $C_I$, and let $\mathcal{F}_I$ denote the canonical foliation on $C_I$ corresponding to $F_I$, so that the collapsing map $\kappa_I \colon C_I \to \mathbb{H}^2$ takes each leaf of $\mathcal{F}_I$ to a leaf of $F_I$ diffeomorphically. Then the dual tree of $F_I$ is homeomorphic to an open interval. With the Thurston metric on $C$, the collapsing map $\kappa_I$ continuously extends to a homeomorphism between ideal boundaries $\partial_\infty C_I$ and $\partial_\infty \mathbb{H}^2$ ($\cong \mathbb{S}^1$).

There are exactly two points on $\partial_\infty C_I$ that are not endpoints of leaves of $\mathcal{F}_I$; they divide $\partial_\infty C_I$ ($\cong \mathbb{S}^1$) into two open intervals. Pick one of the intervals, and let $\Phi$ be the projection of $C_I$ onto the interval along leaves of $\mathcal{F}_I$. With respect to the Thurston metric, $C_I$ is divided into Euclidean and hyperbolic regions. Then each connected component of the Euclidean region is foliated by leaves of $\mathcal{F}_I$ sharing endpoints on $\partial_\infty C_I$, and $\Phi$ takes the component to the end point in the chosen interval.

Since each vertical edge $a$ of $\mathcal{R}$ is transversal to the foliation $\mathcal{F}_I$, $\Phi(a)$ is an arc $b$ in the open interval in $\partial C_I$. Recalling that $\beta_I\colon \mathbb{H}^2 \to \mathbb{H}^3$ extends to $\partial_\infty \mathbb{H}^2 \to \partial_\infty \mathbb{H}^3$ and $\kappa_I\colon C_I \to \mathbb{H}^2$ to $\partial_\infty C_I \to \partial_\infty \mathbb{H}^2$, we have $f_I = \beta_I \circ \kappa_I$ on $\partial_\infty C_I$. Since $\kappa_I\colon \partial_\infty C_I \to \partial_\infty \mathbb{H}^2$ is a homeomorphism, we can identify $b$ with its image in $\partial \mathbb{H}^2$. Thus let $\beta_b\colon b \to \hat{\mathbb{C}}$ denote the (continuous) path obtained by restricting $\beta_I$ to $b$.

The transversal measure $\mu_I$ of $I$ is defined for arcs in $\mathbb{H}^2$ transversal to $I$. Since the total measure of $\mu_I$ is finite, $\mu_I$ continuously extends to arcs in $\partial \mathbb{H}^2$. Then $\beta_I | \partial_\infty \mathbb{H}^2$ is determined by $\mu_I | \partial_\infty \mathbb{H}^2$. In particular, $\beta_b$ can be regarded as a bending map of $b \subset \partial_\infty \mathbb{H}^2$ by the measure $\mu_I$ on $b$.

Let $g \subset \mathbb{H}^3$ be the axis of the round cylinder $\mathcal{A}(R)$. For each vertical edge $a$ of $\mathcal{R}$, there is a map to its corresponding boundary circle $h$ of $\mathcal{A}(R)$. Identifying $\hat{\mathbb{C}}$ with $\mathbb{S}^2$ conformally, we normalize $\hat{\mathbb{C}}$ by an element of $\mathrm{PSL}(2, \mathbb{C})$ so that $h$ is the equator. Let $\mathrm{Conv}(A)$ be the convex hull of $\mathcal{A}(R)$ in $\mathbb{H}^3$. Then, for every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then, for all branches $R$ of $\tilde{T}_i$ and for all leaves $l$ of $\tilde{L}_i$ intersecting $R$, since the geodesic $\beta(l)$ is nearly orthogonal to the boundary hyperplanes of $\mathrm{Conv}(\mathcal{A}(R))$, the geodesic segments $\beta_I(l \cap R)$ and $g \cap \mathrm{Conv}(\mathcal{A}(R))$ are $\delta$-close in the Hausdorff metric. Thus, for any (small) $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then $\mathrm{Im}(\beta_b)$ is contained in a $\delta$-neighborhood of an endpoint $O$ of $g$ on $\hat{\mathbb{C}}$ in the spherical metric. In particular, $\mathrm{Im}(\beta_b)$ is contained in a round disk $D$ on $\hat{\mathbb{C}}$ bounded by $h$.

By the definition of $\Phi\colon C_I \to \partial C_I$, for each $x \in a$, $\Phi(x)$ is an endpoint of the leaf $\ell$ of $\mathcal{F}_I$ containing $x$. Consider the ray from $x$ to $\Phi(x)$ contained in $\ell$. Then $f_I$ homeomorphically takes this ray onto a circular arc in $\hat{\mathbb{C}}$ that connects the point $f_I(x)$ to the point $f_I(\Phi(x)) = \beta_b(\Phi(x))$. Let $r_x\colon [0,1] \to D$ denote this circular arc with $r_x(0) = f_I(\Phi(x))$ and $r_x(1) = f_I(x)$. Then, since the geodesic $\beta_I \circ \kappa_I(\ell)$ is nearly orthogonal to the hyperplane $\mathrm{Conv}(h)$, for every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then $r_x$ and $h$ are $\delta$-nearly orthogonal (at the point $f_I(x)$) and $\gamma_x(0)$ is $\delta$-close to the center $O$.

There is a unique maximal ball in $C_I$ whose core contains $x$. By the definition of a maximal ball, its $f_I$-image is a round open ball $R_x$ in $\hat{\mathbb{C}}$. The boundary circle of $R_x$ bounds a hyperbolic plane $H_x$ in $\mathbb{H}^3$. Then $r_x$ orthogonally intersects $\partial R_x$ at the endpoint $r_x(0)$. For all $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then, since $\beta_I \circ \kappa_I(\ell)$ is nearly orthogonal to $\mathrm{Conv}(h)$, the curvature of $r_x\colon [0,1] \to D \subset \mathbb{S}^2$ is less than $\delta$ (in the induced spherical metric on $D$). Let $\mathtt{r}_x\colon [0,1] \to D$ be the spherical geodesic segment in $D$ connecting the endpoints of $r_x$. Then, since $r_x(0)$ is sufficiently close to the center $O$ of $D$ and the curvature of $r_x(0)$ is sufficiently small, for every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then, (in particular) for all $x \in a$, the "almost" geodesic segment $r_x$ is $\delta$-close to the geodesic segment $\mathtt{r}_x$ in the hausdorff metric in $D$.

**Lemma 7.15.** *For every $x$ on the arc $a$, there exists a small neighborhood $U_x$ of $x$ in $a$, such that if $y \in U_x$, then $\mathtt{r}_x$ and $\mathtt{r}_y$ are disjoint, except possibly at their endpoints close to $O$ (i.e. possibly $\mathtt{r}_x(0) = \mathtt{r}_y(0)$).*

*Proof.* Since $f_I|a$ is an immersion into $h$, if $y \in a$ is sufficiently close to $x$, then $\mathtt{r}_x(1) = f_I(x)$ is different from $\mathtt{r}_y(1) = f_I(y)$. If $\mathtt{r}_x(0) = \mathtt{r}_y(0)$, since $\mathtt{r}_x$ and $\mathtt{r}_y$ are geodesic segments in the hemisphere $D$, they are disjoint except at $\mathtt{r}_x(0) = \mathtt{r}_y(0)$.

Next assume that $\mathtt{r}_x(0) \neq \mathtt{r}_y(0)$ for $y \in a$ sufficiently close to $x$. The foliation $\mathcal{F}_I$ on $C_I$ carries a canonical transversal measure that descends to $I$ on $\mathbb{H}^2$ and it has no atomic measure. Then, by continuity, if $y$ is sufficiently close to $x$, the transversal measure of the segment in $a$ connecting $x$ to $y$ is sufficiently small. Therefore $r_x$ and $r_y$ are disjoint. Let $\ell$ be the spherical geodesic in $D$ through $r_x(0)$ and $r_y(0)$ with its endpoints on $h$. Then $r_x$ and $r_y$ are contained in a component $P$ of $D \setminus \ell$, so that the endpoints of $r_x$ and $r_y$ on the boundary $P$. Since $P$ is convex and $r_x$ and $r_y$ are disjoint, by the uniqueness of geodesics, $\mathtt{r}_x$ and $\mathtt{r}_y$ must be disjoint. □

For each $x \in a$, let $\gamma_x\colon [0,1] \to D$ denote the geodesic segment on $D \subset \mathbb{S}^2$ connecting the center $O$ to $r_x(1)$. Then $\gamma_x$ intersects $h = \partial D$ orthogonally at $\gamma_x(1)$. Let $\alpha = f_I|a$. Parametrize $a$ so that $\alpha\colon a \to h \subset \mathbb{S}^2$ is an isometric immersion, and identify $a$ with the closed interval $[0, A]$, where $A$ denotes the length of $\alpha$. Then, as $x \in [0, A]$ increases, the circular arc $\gamma_x\colon [0,1] \to D$ ($x \in [0,1]$) changes by the continuous rotation of $D$ about $O = \gamma_x(0)$ monotonically. Thus define $\gamma\colon a \times [0,1] \to D$ by $\gamma(x,t) = \gamma_x(t)$. Then $\gamma(x,1) = \alpha(x)$ for all $x \in a$. Then $\gamma|a \times (0,1]$ is an immersion. (The parametrized surface $\gamma$ is a *fan* where the vertex of the fan is $O$ and the angle of the fan is $A$.) Let $E$ denote the domain $a \times [0,1]$ of $\gamma$ with the metric obtained by pulling back the spherical metric on $D$ via $\gamma$. Then we have

$$\mathrm{Area}(E) = \mathrm{Area}_{\mathbb{S}^2}(D) \cdot (A/2\pi) = A. \tag{2}$$

Similarly define $\mathtt{r}\colon a \times [0,1] \to D$ by $\mathtt{r}(x,t) = \mathtt{r}_x(t)$. Then, by Lemma 7.15, the restriction of $\mathtt{r}$ to $a \times (0,1]$ is an immersion. Clearly $\mathtt{r}|a \times \{0\}$ is $\beta_b \circ \Phi\colon a \to D$, and $\mathtt{r}|a \times \{1\}$ is $\alpha\colon a \to \partial D$. Let $F$ be the rectangular domain $a \times [0,1]$ of $\mathtt{r}$ with the pull-back metric of the spherical metric on $D$. Then the boundary edges of the rectangle $F$ correspond to the four curves $\alpha$, $\beta_b$, $\mathtt{r}_0$, and $\mathtt{r}_1$. Applying the Gauss-Bonnet theorem to $F$ with respect to the spherical metric, we have

$$\text{Area}(F) + \int_{\partial F} k ds + \Sigma\theta_p = 2\pi \cdot 1,$$

where $k$ is the curvature at smooth points of $\partial F$, and $\theta_p$ are the exterior angles at non-smooth points $p$ of $\partial F$, which include (infinitesimal) bending angles of $\beta_b$ corresponding to $\mu_I$. Then, since $\beta_b \colon b \to D$ be obtained by bending $b$ with respect to $\mu_I|b$, the third term $\Sigma\theta_p$ is $-\tilde{\mu}_i(R) + 2\pi$.

Next consider the second term

$$\int_{\partial F} k\, ds = \int_{\beta_b} k\, ds + \int_{\mathtt{r}_0} k\, ds + \int_{\mathtt{r}_1} k\, ds + \int_{\alpha} k\, ds.$$

Since $\mathtt{r}_0$ and $\mathtt{r}_1$ are geodesic segments, $\int_{\mathtt{r}_0} k\, ds = 0$ and $\int_{\mathtt{r}_1} k\, ds = 0$. Since $\alpha$ is a segment of the geodesic loop $\partial D$ on $\mathbb{S}^2$, we have $\int_\alpha kds = 0$. For every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then the curvature at every smooth point $x$ of $\alpha$ is less than $\delta$, since $\beta_b(x)$ is sufficiently close to $O$ and $H_x$ is almost orthogonal to the hyperplane, in $\mathbb{H}^3$, bounded by $\partial D$. Since we can assume that the length of $\beta_b$ is sufficiently small, we have $\int_{\partial F} k\, ds < \delta$. Thus

**Lemma 7.16.** *For every $\delta > 0$, if $i \in \mathbb{N}$ is sufficiently large and $\epsilon > 0$ is sufficiently small, then,*

$$-\delta < \text{Area}(F) - \tilde{\mu}_i(R) < \delta,$$

*for all branches $R$ of $\tilde{T}_i$.*

By (2) and Lemma 7.16, to prove Proposition 7.13, it suffices to show

**Proposition 7.17.** *For every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then, for every pair of corresponding branches $R$ and $\mathcal{R}$ of $\tilde{T}_\infty$ and $\tilde{\mathcal{T}}_\infty$, respectively, and every vertical edge $a$ of $\mathcal{R}$, we have*

$$-\delta < \text{Area}(F) - \text{Area}(E) < \delta,$$

*where $F$ and $E$ are defined as above.*

*Proof.* Fix sufficiently small $\delta > 0$. For each $y \in b$, let $g_y \colon [0,1] \to D$ be the geodesic segment from $O = g_y(0)$ to $\beta_b(y) = g_y(1)$. If $\epsilon > 0$ is sufficiently small, then $\text{Im}(\beta_b)$ is contained in the $\delta$-neighborhood of $O$. Thus $\text{length}_{\mathbb{S}^2}(g_y) < \delta$ for all $y \in b$. Define $g \colon b \times [0,1] \to D$ by $g(y,t) = g_y(t)$ for $y \in b$ and $t \in [0,1]$. Then $g$ is smooth almost everywhere since so is $\beta_b$. Let $G = b \times [0,1]$ equipped with the 2-form obtained by pulling back the spherical Riemannian metric of $D$ via $g$. In particular, this form induces the arc length of $\beta_b \colon b \to D$ when restricted to $b \times \{1\} = b$.

Then we see that $0 \leq \mathrm{Area}(G) = \int_{y \in b} \frac{1}{2} \mathrm{length}(g_y) dy$. Therefore, if $\epsilon > 0$ is sufficiently small, then $\mathrm{Area}(G) < \delta$, since $\mathrm{length}(\beta_b)$ and $\mathrm{length}(g_y)$ for all $y \in b$ are sufficiently small.

Let $\Delta_0$ be the (geodesic) triangle in $D$ bounded by $\mathtt{r}_0, \gamma_0, g_0$, and $\Delta_A$ the triangle bounded by $\mathtt{r}_A, \gamma_A, g_A$. Then, if $\epsilon > 0$ is sufficiently small, then $g_0$ and $g_A$ are sufficiently short so that $\mathrm{Area}_{\mathbb{S}^2}(\Delta_0), \mathrm{Area}_{\mathbb{S}^2}(\Delta_A) < \delta$. Thus, it suffices to show that, for sufficiently small $\epsilon > 0$, we have

(3) $|\mathrm{Area}(F) - \mathrm{Area}(E)| < \mathrm{Area}(G) + \mathrm{Area}(\Delta_0) + \mathrm{Area}(\Delta_A)$.

In order to prove (3), we decompose the interval $[0, A]$ so that it accordingly decomposes $F$, $E$, $G$ into subsets, and we show similar inequalities for the corresponding subsets. Let $X$ be the set of points $x$ in $[0, A]$ such that $\gamma_x$ and $\mathtt{r}_x$ are parallel, so that either $\gamma_x \supset \mathtt{r}_x$ (Type I) or $\gamma_x \subset \mathtt{r}_x$ (Type II) holds. Then accordingly either $\gamma_x = \mathtt{r}_x \cup g_x$ or $\gamma_x \cup g_x = \mathtt{r}_x$.

The supporting lamination $|I|$ has measure zero in $\mathbb{H}^2$, since it is obtained from the measured lamination on a closed hyperbolic surface, which has measure zero. On a subinterval $J$ of $[0, A]$ where $\beta_b$ is smooth, $J \cap X$ is a finite set. Thus $X$ has measure zero in $[0, 1]$. Therefore, letting $F_X, E_X, G_X$ be the respective subsets of $F, E, G$ corresponding to $X \times [0, 1]$ in $I \times [0, 1]$, we have

$$\mathrm{Area}(F_X) = \mathrm{Area}(E_X) = \mathrm{Area}(G_X) = 0.$$

The definition of $X$ implies that $X$ is a closed subset of $[0, A]$. Then the complement of $X$ is the union of, at most, countably many disjoint intervals $Y_k\,(k \in K)$. Then $Y_k$ have open endpoints except at 0 and $A$. For each $k \in K$, let $0 \leq y_k < z_k \leq 1$, be the endpoints of $Y_k$. In addition let $F_k$ and $E_k$ be the subsurfaces of $F$ and $E$, respectively, corresponding to $Y_k \times [0, 1]$. Let $G_k$ be the subsurface of $G$ corresponding to $[\Phi(y_k), \Phi(z_k)] \times [0, 1]$.

Then it suffices to show that, if $Y_k$ does not contain an endpoint of $[0, A]$,

$$|\mathrm{Area}(F_k) - \mathrm{Area}(E_k)| \leq \mathrm{Area}(G_k),$$

and, if $Y_k$ contains an endpoint $p$ of $[0, A]$,

$$|\mathrm{Area}(F_k) - \mathrm{Area}(E_k)| \leq \mathrm{Area}(G_k) + \mathrm{Area}(\Delta_p).$$

(i) First we suppose that the interval $Y_k$ is an open interval $(y_k, z_k)$. At least one of the endpoints $y_k$ and $z_k$ must be of Type I , since $\beta_b$ is obtained by bending $b$. Therefore the endpoints of $Y_k$ are either: (i - i) both of Type I (Figure 6) or (i - ii) of the different types, Type I and II (Figure 7).

Suppose the case of (i - i). Then $\mathtt{r}(F_k)$ is disjoint from $O$. Then $E_k$ is naturally the union of $F_k$ and $G_k$ such that $F_k$ and $G_k$ have disjoint interiors in $E_k$. In particular $\text{Area}(E_i) = \text{Area}(F_i) + \text{Area}(G_i)$.

Suppose the case of (i - ii). Then there is a point $t \in (y_k, z_k)$ such that $g_t$ is "tangent" to $\beta_b$ at $\Phi(t)$ so that the restrictions of $g$ to $[\Phi(y_k), \Phi(t)] \times [0,1]$ and $[\Phi(t), \Phi(z_k)] \times [0,1]$ are immersions of the opposite orientations (Figure 7). Let $G'_i$ be the component of $G_i \setminus \{\Phi(t)\} \times [0,1]$ that contains $\{\Phi(y_k)\} \times [0,1]$ if $y_k$ is of Type I and $\{\Phi(z_k)\} \times [0,1]$ if $z_k$ is of Type I. Then $E_k$ is the union of $F_k$ and $G'_k$. In particular $\text{Area}(E_k) < \text{Area}(F_k) + \text{Area}(G'_k) < \text{Area}(F_k) + \text{Area}(G_k)$.

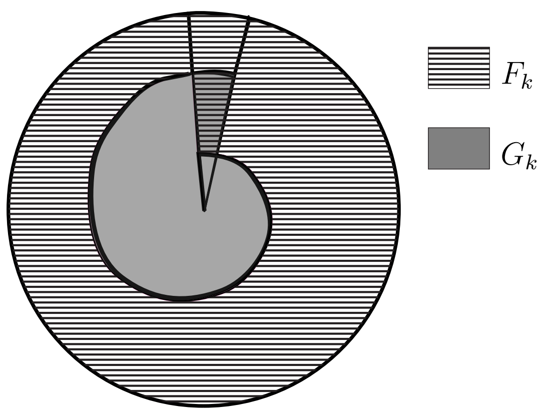


FIGURE 6. $(y_k, z_k)$ with Type I ends

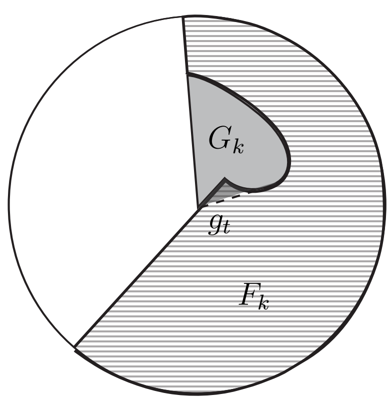


FIGURE 7. $(y_k, z_k)$ with Type I and II ends

(ii) Next suppose that exactly one of the endpoints of $Y_k$ is closed. Consider the case of $Y_k = (y_k, z_k]$ so that $z_k = A$. (The case of $[y_k, z_k)$

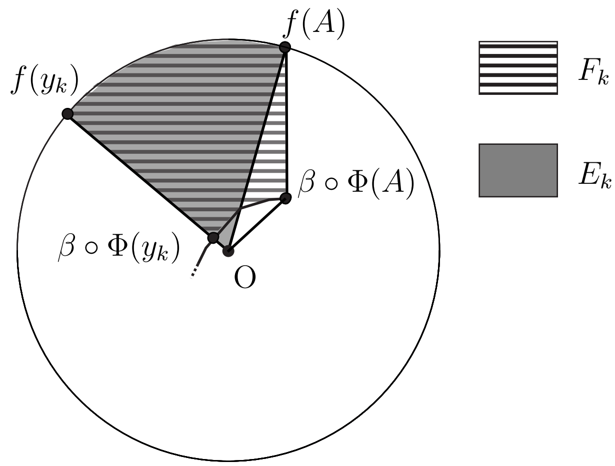


FIGURE 8. $(y_k, z_k]$ with Type I at $y_k$, Case I

is similar.) First suppose, in addition, that $y_k$ is of Type I. Then if $\epsilon > 0$ is sufficiently small, then either $\mathtt{r}_{y+\epsilon}$ is disjoint from all $\gamma_t$ for $y_k \leq t < y_k + \epsilon$ (Case I, Figure 8) or $\gamma_{y+\epsilon}$ is disjoint from $\mathtt{r}_t$ for $y_k \leq t < y_k + \epsilon$ (Case II, Figure 9).

In Case I, we naturally have

$$E_k \cup \Delta_A = G_k \cup F_k,$$

so that $E_k$ and $\Delta_A$ have disjoint interiors and $G_k$ and $F_k$ have disjoint interiors. Then $E_k \setminus F_k \subset G_k$ and $F_k \setminus E_k \subset \Delta_A$. Therefore

$$|\operatorname{Area}(E_k) - \operatorname{Area}(F_k)| < \operatorname{Area}(G_k) + \operatorname{Area}(\Delta_A).$$

In Case II , we naturally have

$$E_k = F_k \cup G_k \cup \Delta_A,$$

so that $F_k, G_k, \Delta_A$ have disjoint interiors. Then

$$0 < \operatorname{Area}(E_k) - \operatorname{Area}(F_k) = \operatorname{Area}(G_k) + \operatorname{Area}(\Delta_A).$$

Next suppose instead that $y_k$ is of Type II (Figure 10). Then $(E_k \setminus F_k) \sqcup (F_k \setminus E_k)$ is naturally embedded in $G_k \sqcup \Delta_k$. Thus

$$|\operatorname{Area}(E_k) - \operatorname{Area}(F_n)| < \operatorname{Area}(G_k) + \operatorname{Area}(\Delta_k).$$

Lastly suppose that $Y_k$ is the closed interval. Then $Y_k$ must be the entire interval $[0, A]$. Then a similar argument proves (3). $\boxed{7.17}$

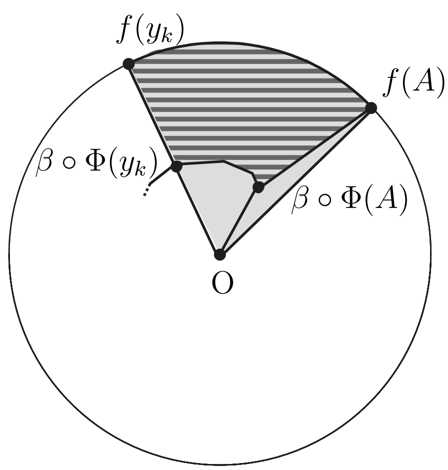


FIGURE 9. $(y_k, z_k]$ with Type I at $y_k$, Case II

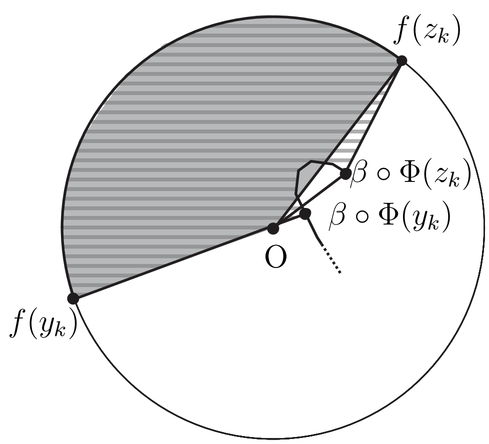


FIGURE 10. Type II at $y_k$

## 8. CHARACTERIZATION OF $\mathcal{P}_\rho$ VIA THURSTON COORDINATES

8.1. **Local characterization of $\mathcal{P}_\rho$ in $\mathcal{GL}$.** Let $C \cong (\tau, L)$ be a projective structure on $S$ with holonomy $\rho\colon \pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$, and let $\kappa\colon C \to \tau$ be the collapsing map. Let $\mathcal{L}$ be the canonical lamination on $C$, which descends to $L$ by $\kappa$. Let $C_0 \cong (\tau, L_0)$ be the corresponding projective structure with holonomy $\rho$ as in §1.5, so that $C = \mathrm{Gr}_M(C_0)$, where $M$ is the maximal weighted multiloop "contained in $L$" so that $M = L - L_0$. Similarly let $\kappa_0 \colon C_0 \to \tau$ be its collapsing map and $\mathcal{L}_0$ be the canonical lamination on $C_0$. Then $C$ and $C_0$ correspond to the same $\rho$-equivariant pleated surface $\mathbb{H}^2 \to \mathbb{H}^3$. Given another projective structure $C' \cong (\tau', L') \in \mathcal{P}_\rho$, let $\kappa' \colon C' \to \tau'$ be its collapsing map and $\mathcal{L}'$ be the canonical lamination on $C'$. Consider a shortest closed geodesic loop on $\tau$, and let $K > 0$ be one-fourth of its length (or any positive number less than one-third of it).

**Theorem 8.1.** *For every $\epsilon > 0$, there is a $\delta > 0$, which depends only on $C$ and $\epsilon$, such that, if $C' \cong (\tau', L')$ satisfies $\angle_\tau(L, L') < \delta$, then there are a traintrack $\mathcal{T} = \{\mathcal{R}_j\}$ on $S$ and marking homeomorphisms*

$$\phi\colon S \to C,\ \phi_0\colon S \to C_0,\ \phi'\colon S \to C'$$

*taking $\mathcal{T}$ to admissible traintracks (on $C, C_0, C'$) such that:*

(I)
- *$\phi(\mathcal{T})$ carries $\mathcal{L}$ on $C$, and it descends, by $\kappa$, to an $(\epsilon, K)$-nearly straight traintrack $T$ on $\tau$ carrying $L$.*
- *$\phi_0(\mathcal{T})$ carries $\mathcal{L}_0$ on $C_0$, and it descends, by $\kappa_0$, also to $T$ on $\tau$ (carrying $L_0$).*
- *$\phi'(\mathcal{T})$ carries $\mathcal{L}'$ on $C'$, and it descends, by $\kappa'$, to an $(\epsilon, K)$-nearly straight traintrack $T'$ on $\tau'$ carrying $L'$.*

*By identifying $\mathcal{T}$ and its images by the homeomorphisms $\phi, \phi_0, \phi'$, we have:*

(II) *$C'$ is obtained by grafting $C_0$ along a weighted multiloop $M'$ carried by $\mathcal{T}$, and moreover $M'$ is $\epsilon$-close $\mathcal{L}' - \mathcal{L}_0$ on $\mathcal{T}$.*

(III) *We can graft $C$ and $C'$ along some weighted multiloops carried by $\mathcal{T}$, respectively, to a common projective structure. Indeed, if there are weighted multiloops $\hat{M}$ and $\hat{M}'$ carried by $\mathcal{T}$ such that $\hat{M} + M = \hat{M}' + M'$ on $\mathcal{T}$, then*

$$\mathrm{Gr}_{\hat{M}}(C) = \mathrm{Gr}_{\hat{M}'}(C').$$

**Remark 8.2.** *In (II) and (III), we make the multiloops transversal to the circular foliations of $\mathcal{T}$'s on $C, C_0, C'$, so that they are admissible (by Lemma 7.2).*

*In (I), the correspondences between the traintracks on the projective surfaces and the hyperbolic surfaces are up to small perturbations of the vertical edges (as in Proposition 7.12).*

*The traintracks $T, T'$ are obtained by Proposition 7.11. Thus we can in addition assume that there is an $\epsilon$-rough isometry that takes $T$ to $T'$ that preserves the marking.*

*In Theorem B, $\mathcal{L}' - \mathcal{L}_0$ is replaced by $L' - L_0$ as they coincide as topological measured laminations.*

The following proposition immediately yields Theorem 8.1 (I), and in addition it will be promoted to Theorem 8.1 (II) and (III).

**Proposition 8.3.** *For every $\epsilon > 0$, there is a $\delta > 0$ such that, if $C' \cong (\tau', L')$ is a projective structure with holonomy $\rho$ and $\angle_\tau(L, L') < \delta$, then there are a traintrack $\mathcal{T} = \{\mathcal{R}_j\}_{j=1}^n$ on $S$ and marking homeomorphisms*

$$\phi\colon S \to C,\ \phi_0\colon S \to C_0,\ \phi'\colon S \to C'$$

*taking $\mathcal{T}$ to admissible traintracks (on $C, C_0, C'$), such that:*

(I)
- *$\phi(\mathcal{T})$ descends, by $\kappa$, to an $(\epsilon, K)$-nearly straight traintrack on $\tau$ carrying $\mathcal{L}$.*
- *$\phi_0(\mathcal{T})$ descends, by $\kappa_0$, to the same $(\epsilon, K)$-nearly straight traintrack on $\tau$ (which carries $\mathcal{L}_0$).*
- *$\phi'(\mathcal{T})$ descends, by $\kappa'$, to an $(\epsilon, K)$-nearly straight traintrack on $\tau'$ carrying $\mathcal{L}'$.*

(II) *The developing maps of $C, C_0, C'$ induce isomorphisms between $C \setminus \phi(\mathcal{T}), C_0 \setminus \phi_0(\mathcal{T}), C' \setminus \phi'(\mathcal{T})$ as projective surfaces (see Definition 3.2).*

*Thus we can assume that $\phi, \phi_0, \phi'$ induce those isomorphisms, and*

(III)
- i. *For every branch $\mathcal{R}_j$ of $\mathcal{T}$ and its lift $\tilde{\mathcal{R}}_j$ to $\tilde{S}$, the corresponding branches $\tilde{\phi}'(\tilde{\mathcal{R}}_j)$, $\tilde{\phi}(\tilde{\mathcal{R}}_j)$, $\tilde{\phi}_0(\tilde{\mathcal{R}}_j)$ are supported on a common round cylinder on $\hat{\mathbb{C}}$, where $\tilde{\phi}\colon \tilde{S} \to \tilde{C}$ and $\tilde{\phi}'\colon \tilde{S} \to \tilde{C}'$, $\tilde{\phi}_0\colon \tilde{S} \to \tilde{C}_0$ are the lifts of $\phi, \phi', \phi_0$.*
- ii. *By identifying $\mathcal{T}$ and its images under $\phi, \phi_0, \phi'$, then $\phi'(\mathcal{R}_j) \subset C'$ is obtained by grafting $\phi_0(\mathcal{R}_j) \subset C_0$ along a multiarc $M'_j$ that is $\epsilon$-close to $\mathcal{L}' - \mathcal{L}_0$ on $\mathcal{R}_j$, and $\phi(\mathcal{R}_j) \subset C$ is obtained by grafting $\phi_0(\mathcal{R}_j) \subset C_0$ along the multiarc $M_j$ corresponding exactly to $\mathcal{L} - \mathcal{L}_0$ on $\mathcal{R}_j$.*

*Proof.* Let $C'_i \cong (\tau'_i, L'_i)$ be a sequence in $\mathcal{P}_\rho$ such that $\angle_\tau(L, L'_i) \to 0$. Then it suffices to show the proposition for $C'_i$ with sufficiently large $i$. By Theorem 5.2, $\tau'_i \to \tau$ as $i \to \infty$.

Without loss of generality, we can assume that both $C$ and $C_0$ appear in the sequence $\{C'_i\}$ infinitely many times. Since $\mathcal{GL}(S)$ is compact, we can in addition assume that $|L'_i|$ converges to a geodesic lamination $\lambda_\infty$ in the Hausdorff topology as $i \to \infty$, by taking a subsequence if necessary. For every $\epsilon > 0$, by applying Proposition 7.3 to $\rho, \tau, \lambda_\infty$ and Lemma 6.2, we obtain the proposition. Note that, since $L_0$ has no leaves of weight at least $2\pi$, for every $\epsilon > 0$, if $\delta > 0$ is sufficiently small, the weight of $\mathcal{L}'$ is more than the weight of $\mathcal{L}_0$ minus $\epsilon$ on each branch of $\mathcal{T}$. $\square$

*Proof of Theorem 8.1.* For $\epsilon > 0$, let $\delta > 0$ be the constant obtained by applying Proposition 8.3. Then, for every $C' = (f', \rho) \in \mathcal{P}_\rho$ with $\angle_\tau(L, L') < \delta$, Proposition 8.3 yields a topological traintrack $\mathcal{T} = \{\mathcal{R}_j\}_{j=1}^n$ on $S$ and marking-preserving homeomorphisms $\phi\colon S \to C$, $\phi_0\colon S \to C_0$ and $\phi'\colon S \to C'$. Thus we have (I) by Proposition 8.3 I. In particular $\kappa$ and $\kappa_0$ take $\phi(\mathcal{T})$ and $\phi_0(\mathcal{T})$, respectively, to the same $(\epsilon, K)$-nearly straight traintrack $T$ on $\tau$ carrying both $L$ and (the geodesic representative of) $L'$ on $\tau$. Recall that $L_0$ has no closed leaf of weight at least $2\pi$.

First we prove (II). We have a natural decomposition of $S$ by the traintrack $\mathcal{T}$:

$$S = (S \setminus |\mathcal{T}|) \cup \mathcal{T} = (S \setminus |\mathcal{T}|) \cup (\cup_{j=1}^n \mathcal{R}_j).$$

Then, this decomposition of $S$ descends to decompositions of $C$ and $C'$ via the homeomorphisms $\phi$ and $\phi'$, respectively:

$$C' = (C' \setminus \phi'(\mathcal{T})) \cup \phi'(\mathcal{T}) = (C' \setminus \phi'(\mathcal{T})) \cup (\cup_j \phi'(\mathcal{R}_j))$$

$$C_0 = (C_0 \setminus \phi_0(\mathcal{T})) \cup \phi_0(\mathcal{T}) = (C_0 \setminus \phi_0(\mathcal{T})) \cup (\cup_j \phi_0(\mathcal{R}_j)) \ .$$

Then by Proposition 8.3 II, $\phi' \circ \phi^{-1}$ yields an isomorphism from $(C_0 \setminus \phi(\mathcal{T}))$ to $(C' \setminus \phi'(\mathcal{T}))$ compatible with the developing maps $f$ and $f'$.

In addition, by Proposition 8.3 III - i, for each $j = 1, 2, \ldots, n$, the corresponding branches $\phi'(\mathcal{R}_j)$ and $\phi_0(\mathcal{R}_j)$ are supported on a common round cylinder. Since $L_0$ has no closed leaf of weight at least $2\pi$, by Proposition 8.3 III- ii, we have $\phi'(\mathcal{R}_j) = \mathrm{Gr}_{M'_j}(\phi_0(\mathcal{R}_j))$ for some multiarc $M'_j$ that is $\epsilon$-close to $\mu'(\phi'(\mathcal{R}_j)) \setminus \mu_0(\phi_0(\mathcal{R}_j))$, where $\mu'$ and $\mu_0$ are the transversal measures of $\mathcal{L}'$ and $\mathcal{L}_0$, respectively. Let $\kappa_0\colon C_0 \to \tau$ and $\kappa'\colon C' \to \tau'$ be the collapsing maps. Then, by Proposition 8.3 I, $\kappa_0$ takes the traintrack $\phi_0(\mathcal{T})$ on $C_0$ to an $(\epsilon, K)$-nearly straight traintrack on $\tau$ carrying $L_0$, and $\kappa'$ takes the traintrack $\phi'(\mathcal{T})$ on $C'$ to an $(\epsilon, K)$-nearly straight traintrack on $\tau'$ carrying $L'$.

Since $\phi'(\mathcal{T})$ carries $\mathcal{L}'$, the $n$-tuple $\{\mu'(\phi'(\mathcal{R}_j))\}_{j=1}^n$ satisfies the switch conditions of the traintrack $\phi'(\mathcal{T}) \cong \mathcal{T}$. Similarly, since $\phi_0(\mathcal{T})$ carries $\mathcal{L}$, the $n$-tuple $\{\mu(\phi_i(\mathcal{R}_j))\}_{j=1}^n$ satisfies the switch conditions of the traintrack $\phi(\mathcal{T}) \cong \mathcal{T}$. Thus the $n$-tuple of their differences, $\{\mu'(\kappa' \circ \phi'(\mathcal{R}_j)) - \mu(\kappa_0 \circ \phi(\mathcal{R}_j))\}_{j=1}^n$, satisfies the switch conditions as well. Therefore, the $n$-tuple of the numbers of the arcs of $M'_j$ $(j = 1, 2, \ldots, n)$ also satisfies the switch conditions. Thus, after isotoping $M'_j$ on $\phi(\mathcal{R}_j)$ through admissible multiarcs so that their endpoints match up on the vertical edges, the union $\cup_j M'_j =: M'$ is a multiloop carried by the traintrack $\phi(\mathcal{T})$. Note that the isomorphisms $\phi'(\mathcal{R}_j) = \mathrm{Gr}_{M'_j}(\phi_0(\mathcal{R}_j))$,

$j = 1, \ldots, n$, as projective surfaces remain true under such an isotopy (Lemma 6.2). Since $\phi_0(\mathcal{T})$ carries $\mathcal{L}$ and $\phi'(\mathcal{T})$ carries $\mathcal{L}'$, we can regard $\mathcal{L}' - \mathcal{L}$ as a measured lamination on $S$ carried by $\mathcal{T}$. Since $\mu'(\phi'(\mathcal{R}_j)) - \mu_0(\phi_0(\mathcal{R}_j))$, therefore $\mathcal{L}' - \mathcal{L}$ is $\epsilon$-close to $M'$ on $\mathcal{T}$.

Next we compare the traintracks $\phi_0(\mathcal{T}) \subset C_0$ and $\phi'(\mathcal{T}) \subset C'$ as projective structures on $\mathcal{T}$ (compare with [3]). Let $\mathcal{R}_i$ and $\mathcal{R}_j$ be branches of $\mathcal{T}$ that are adjacent along a vertical edge $e$. Then let $m_i$ and $m_j$ be arcs of $M'_i$ and $M'_j$, respectively, that share an endpoint on $e$, so that $m_i \cup m_j$ is a simple arc on $\mathcal{R}_i \cup \mathcal{R}_j$, which is obtained by naturally gluing $\mathcal{R}_i$ and $\mathcal{R}_j$ along $e$. Since $\mathcal{R}_j$ and $\mathcal{R}_i$ are supported on a round cylinder, the projective structure inserted by the grafting of $\mathcal{R}_i \cup \mathcal{R}_j$ along $m_i \cup m_j$ is exactly the union of projective structures inserted by the graftings of $\mathcal{R}_i$ along $m_j$ and of $\mathcal{R}_j$ along $m_j$. Since this holds for all adjacent arcs, we have

$$\phi'(\mathcal{T}) = \cup_j \phi'(\mathcal{R}_j) = \cup_j \operatorname{Gr}_{M'_j}(\phi_0(\mathcal{R}_j)) = \operatorname{Gr}_{M'}(\phi_0(\mathcal{T})).$$

Hence

$$\begin{aligned} C' &= (C' \setminus \phi'(\mathcal{T})) \cup \phi'(\mathcal{T}) \\ &= (C_0 \setminus \phi_0(\mathcal{T})) \cup (\operatorname{Gr}_{M'}(\phi_0(\mathcal{T}))) = \operatorname{Gr}_{M'}(C_0). \end{aligned}$$

Next we prove (III), that is, $\operatorname{Gr}_{\hat{M}}(C) = \operatorname{Gr}_{\hat{M}'}(C')$. Since $C = \operatorname{Gr}_M(C_0)$, we have

$$C = (C \setminus \phi(\mathcal{T})) \cup \phi(\mathcal{T}) = (C_0 \setminus \phi_0(\mathcal{T})) \cup \operatorname{Gr}_M(\phi_0(\mathcal{T})) = \operatorname{Gr}_M(C_0).$$

and

$$C \setminus \phi(\mathcal{T}) = C_0 \setminus \phi_0(\mathcal{T}).$$

The traintrack $\phi_0(\mathcal{T}) = \{\phi_0(\mathcal{R}_j)\}(\cong \mathcal{T})$ carries $\mathcal{L}_0$. Thus the grafting of $C_0$ along $M$ naturally decomposes into grafting of all branches:

$$\cup_j \phi(\mathcal{R}_j) = \phi(\mathcal{T}) = \operatorname{Gr}_M(\phi_0(\mathcal{T})) = \cup_j \operatorname{Gr}_{M|\mathcal{R}_j}(\phi_0(\mathcal{R}_j)).$$

In particular $\phi(\mathcal{R}_j) = \operatorname{Gr}_{M|\mathcal{R}_j}(\phi_0(\mathcal{R}_j))$.

Since $\hat{M}$ is also carried by $\phi(\mathcal{T}) \cong \mathcal{T}$ and each branch $\phi(\mathcal{R}_j)$ of $\phi(\mathcal{T})$on $C$ is supported on a round cylinder, $\hat{M}$ is admissible on $C$ (Lemma 7.2) Then $\operatorname{Gr}_{\hat{M}}(C)$ is well defined, and $\operatorname{Gr}_{\hat{M}}(C) = \operatorname{Gr}_{\hat{M}} \circ \operatorname{Gr}_M(C_0)$.

Recall that the homeomorphism $\phi\colon S \to C$ represents the marking of $C$. Then there is a marking homeomorphism $\hat{\phi}\colon S \to \operatorname{Gr}_{\hat{M}}(C)$ so that $\hat{\phi} \circ \phi^{-1}$ induces an isomorphism from $C \setminus \phi(|\mathcal{T}|)$ to $\operatorname{Gr}_{\hat{M}}(C) \setminus \hat{\phi}(|\mathcal{T}|)$ and $\operatorname{Gr}_{\hat{M}_j} \phi(\mathcal{R}_j) = \hat{\phi}(\mathcal{R}_j)$ for all $j = 1, \ldots, n$, where $\hat{M}_j = \hat{M}|\phi(\mathcal{R}_j)$. Note that $(M + \hat{M})|\mathcal{R}_j$ is the multiarc on $\mathcal{R}_j$ such that the number of its arcs is the sum of the number of the arcs of $M|\phi_0(\mathcal{R}_j)$ and $\hat{M}|\phi(\mathcal{R}_j)$. Then $\hat{\phi}(\mathcal{R}_j)$ is obtained by grafting $\phi_0(\mathcal{R}_j)$ along $(M+\hat{M})|\mathcal{R}_j$, and

$$\mathrm{Gr}_{\hat{M}}(C) = \cup_j \mathrm{Gr}_{(M+\hat{M})|\mathcal{R}_j}\, \phi_0(\mathcal{R}_j) \cup (C_0 \setminus \phi_0(\mathcal{T})).$$

Similarly, since $\mathrm{Gr}_{\hat{M}'}(C') = \mathrm{Gr}_{\hat{M}'} \circ \mathrm{Gr}_{M'}(C_0)$, the traintrack $\mathcal{T}$ yields a decomposition of $\mathrm{Gr}_{\hat{M}'}(C')$.

$$\mathrm{Gr}_{\hat{M}'}(C') = \cup_j \mathrm{Gr}_{(M'+\hat{M}')|\mathcal{R}_j}\, \phi_0(\mathcal{R}_j) \cup (C_0 \setminus \phi_0(\mathcal{T})).$$

Since $M+\hat{M} = M'+\hat{M}'$ on $\mathcal{T}$, therefore $\mathrm{Gr}_{\hat{M}}(C) = \mathrm{Gr}_{\hat{M}'}(C')$. 8.1

**Theorem 8.4.** *For every $\epsilon > 0$ and every compact set $X$ in the moduli space of $S$, there is $\delta > 0$, such that the assertions of Theorem 8.1 hold true for every projective structure $C \cong (\tau, L)$ on $S$ with unmarked $\tau$ in $X$.*

*Proof.* We can observe that, for the proof of Theorem 8.1, the assumption $\angle_\tau(L, L') < \delta$ is only used to guarantee that, there is an $(\epsilon, K)$-nearly straight traintrack $T$ on $\tau$ carrying both $L$ and $L'$, where $K = K_\tau$ is one-fourth of the length of the shortest closed geodesic loop on $\tau$. (All traintracks in Theorem 8.1 are constructed from $T$.)

Since $X$ is compact, now let $K$ be the infimum of $K_\tau$ over all $\tau \in \mathfrak{T}$ that project to $X$. Then it suffices to show that, for every $\epsilon > 0$, there is a $\delta > 0$, such that, if $C \cong (\tau, L)$ is a projective structure on $S$ with holonomy $\rho$ and with unmarked $\tau$ in $X$ and $C' \cong (\tau', L')$ is another projective structure with the same holonomy $\rho$ with $\angle_\tau(L, L') < \delta$, then there is an $(\epsilon, K)$-nearly straight traintrack $T$ on $\tau$ carrying both $L$ and $L'$. This claim is equivalent to

**Claim 8.5.** *If there are two sequences, $C_i \cong (\tau_i, L_i)$ and $C'_i \cong (\tau'_i, L'_i)$, of projective structures on $S$ with all unmarked $\tau_i$ in $X$, such that $\mathrm{Hol}(C_i) = \mathrm{Hol}(C'_i)$ and $\angle_{\tau_i}(L, L_i) < \delta$ for all $i$, then there is an $(\epsilon, K)$-nearly straight traintrack $T_i$ on $\tau_i$ carrying both $L_i$ and $L'_i$ for sufficiently large $i$.*

For each $i$, without loss of generality, we can change the markings of $C_i$ and $C'_i$ simultaneously by a single homeomorphism of $S$. Thus, by the compactness of $\mathcal{GL}$ and $X$, we can in addition assume that $\tau_i$ converges to $\tau \in \mathfrak{T}$ and that $|L_i|$ and $|L'_i|$ converge to $\lambda$ and $\lambda'$, respectively, in $\mathcal{GL}$. Then since $\angle_{\tau_i}(L_i, L'_i) < \delta$, we have $\angle_\tau(\lambda, \lambda') \leq \delta$. Thus if $\delta > 0$ is sufficiently small, for every $\epsilon > 0$, there is an $(\epsilon, K)$-nearly straight traintrack $T$ on $\tau$ carrying both geodesic laminations $\lambda$ and $\lambda'$. By the convergence of $\tau_i$, $|L_i|$, $|L_i|$, for sufficiently large $i$, there is an $(\epsilon, K)$-nearly straight traintrack $T_i$ on $\tau_i$ that carries both $L_i$ and $L'_i$. □

8.2. **Alternative proof of Ito's Theorem.** Suppose that $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ is a fuchsian representation. Let $\tau$ be the marked hyperbolic structure $\mathbb{H}^2/\operatorname{Im}(\rho)$ corresponding to $\rho$. Then, As discussed in §1, given arbitrary $C, C' \in \mathcal{P}_\rho$, we can express them in Thurston coordinates as $C \cong (\tau, M)$ and $C' \cong (\tau, M')$ with unique multiloops $M$ and $M'$. Then, Theorem 1.3 also follows from Theorem 8.4.

**Theorem 8.6.** *$C$ and $C'$ can be transformed to a common projective structure by grafting $C$ along $M'$ and $C'$ along $M$,*

$$\mathrm{Gr}_{M'}(C) = \mathrm{Gr}_M(C').$$

*Proof.* recall that $\mathcal{ML}_\mathbb{N}$ denotes the set of weighted multiloops. Since $\rho$ is fuchsian, $\mathcal{P}_\rho$ is canonically identified with $\mathcal{ML}_\mathbb{N}$ by Thurston coordinates (see Theorem 1.2). Thus, for a different fuchsian representation $\eta$, there $\mathcal{P}_\rho$ and $\mathcal{P}_\eta$ are naturally identified via $\mathcal{ML}_\mathbb{N}$. In fact, there is a quasiconformal map $\Theta = \Theta_\eta\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that conjugates $\rho$ to $\eta$ and realizes the identification $\mathcal{P}_\rho \cong \mathcal{P}_\eta$ by postcomposing the developing maps of structures in $\mathcal{P}_\rho$ with $\Theta$. Then, if $M$ is an admissible multiloop on $C \in \mathcal{P}_\rho$, then $\Theta$ takes $M$ to an admissible loop $\Theta(M)$ on $\Theta(C)$. Then we have

$$\Theta(\mathrm{Gr}_M(C)) = \mathrm{Gr}_{\Theta(M)}(\Theta(C)).$$

Thus it suffices to show

$$\mathrm{Gr}_{\Theta_\eta(M')}(\Theta_\eta(C)) = \mathrm{Gr}_{\Theta_\eta(M)}(\Theta_\eta(C'))$$

for some fuchsian representation $\eta$.

Let $D_M$ be the (simultaneous) Dehn twist of $S$ along all loops of $M$. Then $D_M^k(\tau)$ denote the $k$-iterates of $D_M^k$ on $\tau$ for $k \in \mathbb{Z}_{>0}$. Then $\angle_{D_M^k(\tau)}(M, M') \to 0$ as $k \to \infty$. Note that $D_M$ acts trivially on the moduli space, and in particular it preserves unmarked $\tau$. Let $\eta_k\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ be the fuchsian representation realizing $D_M^k(\tau)$. Then, by Theorem 8.4, if $k$ is sufficiently large, then

$$\mathrm{Gr}_{\Theta_{\eta_k}(M')}(\Theta_{\eta_k}(C)) = \mathrm{Gr}_{\Theta_{\eta_k}(M)}(\Theta_{\eta_k}(C')).$$

□

8.3. **Local characterization of $\mathcal{P}_\rho$ in $\mathcal{PML}$.**

**Theorem 8.7.** *Let $C \cong (\tau, L)$ be a projective structure on $S$ with (arbitrary) holonomy $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$. For every $\epsilon > 0$, there is a neighborhood $U$ of the projective class $[L]$ in $\mathcal{PML}$, such that, if another projective structure $C' \cong (\tau', L')$ with holonomy $\rho$ satisfies*

*$[L'] \in U$, then there are a traintrack $\mathcal{T}$ on $S$ and marking homeomorphisms $\phi\colon S \to C$ and $\phi'\colon S \to C'$ such that, by identifying $\mathcal{T}$ and its images under those homeomorphisms:*

- *$\mathcal{T}$ is an admissible traintrack on both $C$ and $C'$,*
- *the traintrack $\mathcal{T}$ carries $\mathcal{L}$ on $C$, and the collapsing map $\kappa\colon C \to \tau$ descends $\mathcal{T}$ to an $(\epsilon, K)$-nearly straight traintrack on $\tau$,*
- *the traintrack $\mathcal{T}$ carries $\mathcal{L}'$ on $C'$, and the collapsing map $\kappa'\colon C \to \tau'$ descends $\mathcal{T}$ to an $(\epsilon, K)$-nearly straight traintrack on $\tau'$,*

*and indeed we have either*

(i) *$Gr_M(C) = C'$ for a weighted multiloop $M$ on $C$ carried by $\mathcal{T}$ that is $\epsilon$-close to $\mathcal{L}' - \mathcal{L}$ calculated on $\mathcal{T}$ or*
(ii) *$C = Gr_{M'}(C')$ for a weighted multiloop $M' = \mathcal{L} - \mathcal{L}'$ on $\mathcal{T}$.*

**Remark 8.8.** *Similarly to Theorem 8.1, to be precise, an $\epsilon$-small perturbation of the vertical edges is needed to make the image of traintracks under the collapsing maps $(\epsilon, K)$-nearly straight (as in Proposition 7.12).*

*Proof.* If $L = \emptyset$ or $L' = \emptyset$, then $C$ or $C'$ is accordingly a hyperbolic structure. In particular $\rho$ is fuchsian.

Then, by Theorem 1.2, if $L = \emptyset$, then (i) holds and if $L' = \emptyset$ then (ii) holds. Thus we can suppose that $L, L' \neq \emptyset$. Since the holonomy map $\mathrm{Hol}\colon \mathcal{P} \to \chi$ is a local homeomorphism, $\mathcal{P}_\rho$ is a discrete subset of $\mathcal{P}$. For a neighborhood $U$ of $[L]$ in $\mathcal{PML}$, let $\mathcal{P}(\rho, U)$ be the set of projective structures with holonomy $\rho$ such that, in Thurston coordinates, their projective measured laminations are in $U$. By Theorem 5.2, for every neighborhood $V$ of $\tau$ in $\mathcal{T}$, if $U$ is sufficiently small, then, for $C' \cong (\tau', L')$ in $\mathcal{P}(\rho, U)$, we have $\tau' \in V$. Thus if two projective structures in $\mathcal{P}(\rho, U)$ share a measured lamination in Thurston coordinates, then they must coincide.

For every $\epsilon > 0$, if $U$ is sufficiently small, then, for every $C' \cong (\tau', L')$ in $\mathcal{P}(\rho, U)$ with $[L'] \in U$, then $\angle_\tau(L, L') < \epsilon$. Thus, as in Theorem 8.1, we can decompose $C$ and $C'$ by a traintrack $\mathcal{T} = \{\mathcal{R}_j\}_j$ on $S$ given by Proposition 8.3.

As in the proof of Theorem 8.1. Let $\phi\colon S \to C$ and $\phi'\colon S \to C'$ be the marking homeomorphisms obtained by Proposition 8.3, so that $\phi(\mathcal{T})$ and $\phi'(\mathcal{T})$ are corresponding admissible traintracks on $C$ and $C'$, respectively. Let $T$ be the $(\epsilon, K)$-nearly straight traintrack on $\tau$ carrying $L$ and $L'$ such that $\phi(\mathcal{T})$ descends by $\kappa$. Let $T'$ be the $(\epsilon, K)$-nearly straight traintrack on $\tau'$ such that $\phi'(\mathcal{T})$ maps to by $\kappa'$.

Then, for every $\epsilon > 0$, if $U$ is sufficiently small then, there is a constant $c > 0$ such that, the weight ratios $\frac{\mu'(\phi'(\mathcal{R}_j))}{\mu(\phi(\mathcal{R}_j))}$ are $\epsilon$-close to $c$.

Since $\mathcal{P}_\rho$ is a discrete subset of $\mathcal{P}$, unless $C = C'$, we can in addition assume that either (Case One) $c > 1$, or (Case Two) $0 < c < 1$ and the ratios are exactly $c$ for all $j$.

In Case One, $\mu'(\phi'(\mathcal{R}_j)) - \mu(\phi(\mathcal{R}_j))$ is $\epsilon$-close to a positive multiple of $2\pi$ for each $j$. Thus, similarly to the proof of Theorem 8.1 (II), we can show that $C' = Gr_M(C)$ and $M$ is $\epsilon$-close to $\mathcal{L}' - \mathcal{L}$ calculated on $\mathcal{T}$.

In Case Two, we have $c[L] = [L']$ with $0 < c < 1$. Since the ratio $c$ is independent on $j$, we see that $L$ and $L'$ must be multiloops: Otherwise, letting $F$ be a subsurface of $S$ such that $F \cap L$ is a minimal irrational lamination, the holonomy of $C$ must be different from that of $C'$ on $\pi_1(F)$. Then $L - L'$ must be a weighted multiloop, whose weights are $2\pi$-multiples, since otherwise the holonomy $C$ must be different from that of $C'$. Therefore, letting $M'$ be a multiloop $\mathcal{L} - \mathcal{L}'$ on $\phi'(\mathcal{T})$, we have $C = \mathrm{Gr}_{M'}(C')$. □

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