2π-GRAFTING AND COMPLEX PROJECTIVE STRUCTURES WITH GENERIC HOLONOMY

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Abstract. Let S be an oriented closed surface of genus at least two. We show that, given a generic representation $\rho\colon \pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$ in the character variety, (2π-)grafting produces all projective structures on S with holonomy ρ.

Contents

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1. Introduction

Let S be a closed oriented surface of genus at least two throughout this paper. A (complex) projective structure on S is a $(\hat{\mathbb{C}}, \mathrm{PSL}(2,\mathbb{C}))$-structure (see §2.1). It induces a (holonomy) representation $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ unique up to conjugation by an element of $\mathrm{PSL}(2,\mathbb{C})$.

Let $\mathcal{P}_\rho$ be the set of all marked projective structures on S with fixed holonomy $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$. It is a basic question to understand $\mathcal{P}_\rho$, in order to understand geometry behind the homomorphism ρ, which is not necessarily discrete. This question goes back to a foundational paper of Heijal [Hej75, p2, (B)], and it also appeared in various articles ([Hub81, p 274], [Kap95, §7.1], [GKM00, §12.1], [Dum09, §1]; see also [Gol80, §1.10]).

A *(2π-)graft* is a surgery operation that transforms a projective structure in $\mathcal{P}_\rho$ to another in $\mathcal{P}_\rho$ (§2.2). In the preceding paper ([Bab15]), the author showed that projective structures in $\mathcal{P}_\rho$ are related by grafting if they are "close" in the space of geodesic laminations $\mathcal{GL}$ in Thurston coordinates. In this paper, we aim to relate, without the "closeness" assumption, all projective structures in $\mathcal{P}_\rho$ by grafting.

Let $\mathcal{P}$ be the set of all marked projective structures on S. Then $\mathcal{P}$ is diffeomorphic to $\mathbb{C}^{6g-6}$. Let χ be the $\mathrm{PSL}(2,\mathbb{C})$-character variety of S, that is, the set of all representations $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$, roughly, up to conjugation (§2.7). Then χ is a complex affine algebraic variety, and it consists of exactly two connected components ([Gol88]). Let χ_0 be the *canonical component* of χ consisting of representations that lift to $\pi_1(S) \to \mathrm{SL}(2,\mathbb{C})$. Let

$$\mathrm{Hol}\colon \mathcal{P} \to \chi$$

be the *holonomy map*, which takes each projective structure to its holonomy representation. Then the image of Hol is contained in χ_0, and moreover Hol is almost onto χ_0 ([GKM00]). For example, there are many holonomy representations whose images are dense in $\mathrm{PSL}(2,\mathbb{C})$.

Noting $\mathcal{P}_\rho = \mathrm{Hol}^{-1}(\rho)$, we are interested in understanding fibers of Hol. The holonomy map Hol is a local homeomorphism [Hej75] (moreover a local biholomorphism [Hub81, Ear81]); however is not a covering map onto its image. Thus each fiber $\mathrm{Hol}^{-1}(\rho)$ is a discrete subset of $\mathcal{P}$, but $\mathcal{P}_\rho$ may possibly be quite different depending on $\rho \in \chi_0$.

A graft of a projective surface inserts a projective cylinder along an appropriate loop, called an admissible loop, on the surface. Then an ungraft is the opposite of a graft, which removes such a projective cylinder; thus it also preserves holonomy. Then

Question 1.1 ([GKM00]; Grafting Conjecture)**.** *Given two projective structures with holonomy* $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$*, is there a composition of grafts and ungrafts that transforms one to the other?*

A basic known case is when ρ is a discrete and faithful representation onto a quasifuchsian group. Then $\mathcal{P}_\rho$ contains a unique *uniformizable* projective structure (i.e. its developing map is an embedding into $\hat{\mathbb{C}}$); Then every projective structure in $\mathcal{P}_\rho$ is moreover obtained by grafting the uniformizable structure along a *multiloop*, a union of disjoint essential simple closed curves [Gol87]. On the other hand, if $\rho \in \chi_0$ is a generic representation outside the quasifuchsian space, then ρ has a dense image in $\mathrm{PSL}(2,\mathbb{C})$; in particular, there is no uniformizable structure with holonomy ρ. Thus, for general holonomy, Question 1.1 seems an appropriate analogy of the quasifuchsian case.

In this paper we answer Question 1.1 in the affirmative for generic representations in χ_0, namely, of the following type: An element $\alpha \in \mathrm{PSL}(2,\mathbb{C})$ is *loxodromic* if its trace $\mathrm{Tr}(\alpha) \in \mathbb{C}$, which is well-defined up to a sign, is not contained in $[-2,2] \subset \mathbb{R}$. A representation $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ is called *purely loxodromic* if $\rho(\gamma)$ is loxodromic for all $\gamma \in \pi_1(S)$. Then almost all elements of χ_0 are purely loxodromic (Proposition 2.7).

Theorem 1.2. *Let* $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ *be a purely loxodromic representation in* χ_0*. Then, given any* $C_\sharp, C_\flat$ *in* $\mathcal{P}_\rho$*, there is a composition of grafts and ungrafts that transforms* $C_\sharp$ *to* $C_\flat$*. Namely there is a finite composition of grafts* Gr_{ℓ_i} *along loops* ℓ_i *starting from* $C_\sharp$*,*

$$C_\sharp = C_0 \xrightarrow{Gr_{\ell_1}} C_1 \xrightarrow{Gr_{\ell_2}} C_2 \to \ldots \xrightarrow{Gr_{\ell_n}} C_n, \tag{1}$$

such that the last projective structure C_n *is a graft of* $C_\flat$ *along a multiloop* M*,*

$$C_\flat \xrightarrow{Gr_M} C_n. \tag{2}$$

Here a "graft along a multiloop M" means simultaneous grafts along all loops of M.

In the case where ρ is a quasifuchsian representation, [Ito07, Theorem 3] implies Theorem 1.2 even in a stronger form: Namely the sequence (1) can be replace by a single graft along a multiloop. (See also [CDF14, Bab15].)

Although Theorem 1.2 answers Question 1.1 in a generic setting, the question in full generality remains open. Nonetheless many techniques in this paper, including Theorem 1.3 and Theorem 1.4 below, applies to arbitrary representations in $\mathrm{Im}\,\mathrm{Hol}$.

In our proof of Theorem 1.2, we utilize Thurston coordinates on $\mathcal{P}$, which are given by (not necessarily 2π-)grafting of hyperbolic surfaces. Namely there is a natural homeomorphism

$$\mathcal{P} \cong \mathcal{T} \times \mathcal{ML},$$

where $\mathcal{T}$ is the space of marked hyperbolic structures on S and $\mathcal{ML}$ the space of measured laminations on S (see §3). Note that $\mathcal{T}$ is diffeomorphic to $\mathbb{R}^{6g-6}$ and $\mathcal{ML}$ is PL diffeomorphic to $\mathbb{R}^{6g-6}$. We denote Thurston coordinates of a projective structure C using " $\cong$ " as $C \cong (\tau, L) \in \mathcal{T} \times \mathcal{ML}$.

Let $\mathcal{GL}$ be the space of geodesic laminations on S. Then we obtain an obvious projection $\mathcal{ML} \to \mathcal{GL}$, forgetting transversal measures. In the preceding paper [Bab15], the author shows that any $C \in \mathcal{P}_\rho$ is related, by grafting, to all projective structures in $\mathcal{P}_\rho$ that are, in Thurston coordinates, "close" to C in $\mathcal{GL}$ by the projection map (see Theorem 2.8). This local relation yields the graft (2). Thus our main work in this paper is to construct the sequence (1) so that C_n is "close" to $C_\flat$ in $\mathcal{GL}$.

In order to have a control on geodesic laminations of projective structures, we observe an asymptotic, in Thurston coordinates, of projective structures given by the iteration of grafts along a fixed loop. In particular

Theorem 1.3. *Let $C \cong (\tau, L)$ be a projective structure on S in Thurston coordinates, where $(\tau, L) \in \mathcal{T} \times \mathcal{ML}$. Let ℓ be an admissible loop on C. For $i \in \mathbb{Z}_{>0}$, let $C_i \cong (\tau_i, L_i)$ be the projective structure obtained by i-times grafting C along ℓ (i.e. $2\pi i$-graft). Then τ_i converges in $\mathcal{T}$, and L_i converges to a (heavy) measured lamination L_∞ as $i \to \infty$ such that ℓ is a unique leaf of L_∞ of weight infinity. (See Theorem 7.1.)*

In this paper, some closed leaves of laminations may have weight infinity if stated, as in Theorem 1.3 (see §2.3).

In the special case that C is a hyperbolic surface (i.e. the developing map of C is an embedding onto a round disk), Theorem 1.3 is clear. Namely, $\tau_i = \tau$ for all i, and L is equal to ℓ with weight $2\pi i$ ([Gol87]). Thus Theorem 1.3 asserts that, asymptotically, C_i behaves similarly to the iteration of grafts of a hyperbolic surface. (In contrast, the conformal structure of C_i diverges but converges to a point in the Thurston boundary of $\mathcal{T}$ along every grafting ray starting from a hyperbolic structure [CDR12, DK12, Hen11, Gup14].)

By projectivizing transversal measures, $\mathcal{ML}$ minus the empty lamination projects onto the space of projective measured laminations, $\mathcal{PML}$ ($\cong \mathbb{S}^{6g-7}$). In the appendix, we prove

Theorem 1.4. *Given arbitrary $\rho \in \operatorname{Im}\operatorname{Hol}$, in Thurston coordinates, $\mathcal{P}_\rho$ projects onto a dense subset of $\mathcal{PML}$, unless $\mathcal{P}_\rho$ is empty. (see Theorem 12.2.)*

With Theorem 1.4, it seems quite natural to use Thurston coordinates in order to answer Question 1.1. A similar density is well-known for geodesic laminations realized by homotopic pleated surfaces in a fixed hyperbolic three-manifold (see [CEG87]).

Theorem 1.4 is obtained by carefully observing the construction of a projective structure with given holonomy in [GKM00] and applying Theorem 1.3.

1.1. **Outline of the proof of Theorem 1.2.** *Part 1.* Each projective structure on the surface S with holonomy ρ corresponds to a ρ-equivariant pleated surface $\mathbb{H}^2 \to \mathbb{H}^3$ (§3). Thus, given two projective structures with the same purely loxodromic holonomy, we first consider their ρ-equivariant pleated surfaces $\beta_\sharp$ and $\beta_\flat$. In §4, we construct a ordered family of ρ-equivariant pleated surfaces, so that $\beta_\sharp$ is transformed to $\beta_\flat$ through a sequence pleated surfaces in this family, by composition of certain type of simple changes (up to very small perturbations).

In §5, given a ρ-equivariant pleated surface β in the family and a projective structure C with holonomy ρ, if the pleating lamination of the pleated surface of C is sufficiently close to the pleating lamination of β, then every loop ℓ close to the pleating lamination of the next pleated surface is admissible (Proposition 5.5). Note we can graft C along the admissible loop ℓ as many as we want. In Section 6, using such graftings, we prove Theorem 1.2, modulo the result in Part 2 regarding the limit of iterated grafting.

Part 2. Let ℓ be an admissible loop on a projective structure C on S with (arbitrary) holonomy ρ. We consider the n-times grafting of C along an admissible loop ℓ and characterize its limit, as $n \to \infty$, in Thurston coordinates (§3).

The as a projective structure $\operatorname{Gr}_\ell^n(C)$ converges, in a certain sense, to a projective structure $\mathcal{C}_\infty$ on $S \setminus \ell$. In §8, we show that the Thurston coordinates of $\mathcal{C}_\infty$ are a hyperbolic structure σ_∞ on $S \setminus \ell$ and a measured lamination N_∞ on it.

In §9, we identify the boundary components of σ_∞ naturally so that it corresponds to a ρ-equivariant pleated surface. Then we have a hyperbolic structure τ_∞ on S and a measured lamination L_∞ on τ_∞, which is the expected limit of Thurston coordinates (τ_i, L_i) of C_i as $i \to \infty$. Here the measured lamination L_∞ is a bit more generalized than its usual notion: ℓ is a leaf of L_∞ with weight infinity.

Given a point on a projective surface C, A *canonical neighborhood* (§3.3) of p is a nice neighborhood homeomorphic to an open disk: which is embedded in $\hat{\mathbb{C}}$ and yet large enough to capture the Thurston coordinates of C near the point. By embedding ℓ isomorphically to each C_i appropriately, so that we have the inclusions $C_1 \setminus \ell \subset C_2 \setminus \ell \subset C_3 \setminus \ell \ldots$ and $C_i \setminus \ell$ converges to $\mathcal{C}_\infty$. In §10, given a point $p \in \mathcal{C}_\infty$, we show the convergence of the canonical neighborhoods of p in C_i when $i \to \infty$.

In §11, given a converging sequence of projective structures on a open disk which are embedded in $\hat{\mathbb{C}}$, we show the convergence of the sequence in Thurston coordinates.

In §12 we prove the convergence of $(\tau_i, L_i) \to (\tau_\infty, L_\infty)$, combining the results above.

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2. Preliminaries

[Kap01] is a general background reference. For hyperbolic geometry in particular, see [CEG87, EM87]. See also the preceding paper [Bab15].

2.1. **Projective structures.** (c.f. [Thu97].) Let F be an oriented connected surface, and let $\tilde{F}$ be the universal cover of F. Let $\hat{\mathbb{C}}$ denote the Riemann sphere $\mathbb{C} \cup \{\infty\}$. A *projective structure* C on F is a $(\hat{\mathbb{C}}, \mathrm{PSL}(2,\mathbb{C}))$-structure, i.e. it is an atlas modeled on $\hat{\mathbb{C}}$ with transition maps in $\mathrm{PSL}(2,\mathbb{C})$. (In particular C is a refinement of a complex structure.) In this paper all projective structures C are marked by a homeomorphism $F \to C$. Then, equivalently, a projective structure on F is a pair (f, ρ), where $f\colon \tilde{F} \to \hat{\mathbb{C}}$ is an immersion and $\rho\colon \pi_1(F) \to \mathrm{PSL}(2,\mathbb{C})$ is a homomorphism such that f is ρ-equivariant. The immersion f is called the *developing map*, which we denote by $dev(C)$, and ρ the *holonomy representation* of C. The equivalence of projective structures on F is given by the isotopies of F and $(f, \rho) \sim (\gamma \circ f, \gamma\rho\gamma^{-1})$ for all $\gamma \in \mathrm{PSL}(2,\mathbb{C})$.

2.2. **Grafting.** ([Gol87].) Let $C = (f, \rho)$ be a projective structure on F. A loop ℓ on C is *admissible* if $\rho(\ell)$ is loxodromic and f embeds $\tilde{\ell}$ into $\hat{\mathbb{C}}$, where $\tilde{\ell}$ is a lift of ℓ to $\tilde{F}$. Then $\rho(\ell)$ fixes exactly two points on $\hat{\mathbb{C}}$, and $\rho(\ell)$ generates an infinite cyclic group in $\mathrm{PSL}(2, \mathbb{C})$. Its domain of discontinuity is $\hat{\mathbb{C}}$ minus the two points, and its quotient by the cyclic group is a two-dimensional torus T_ℓ has a projective structure. Then ℓ is naturally embedded in T_ℓ. Therefore we can naturally combine two projective surfaces C and T_ℓ by cutting and pasting along ℓ, so that it results a new projective structure $\mathrm{Gr}_\ell(C)$ on F. (Namely we identify boundary components $C \setminus \ell$ and $T_\ell \setminus \ell$ by the identification of ℓ on C and on T_ℓ in an alternating manner.) Then it turns out that ρ is also the holonomy of $\mathrm{Gr}_\ell(C)$.

2.3. **Measured laminations.** Let F be a surface (possibly with boundary), and let τ be a hyperbolic surface homeomorphic to F (with geodesic boundary). A *geodesic lamination* λ is a set of disjoint geodesics whose union is a closed subset of τ; we denote this closed subset by $|\lambda|$. Those geodesics are call *leaves* of the lamination. A geodesic lamination λ is *maximal* if its complement is a union of disjoint ideal triangles. A *stratum* of λ is either a leaf of λ or the closure of a complementary region of λ.

Let $\mathcal{A}(\lambda)$ be the set of all smooth simple arcs α on τ, containing their endpoints, such that α is transversal to λ at its interior points and not tangent to λ at its endpoints. Let $\mathring{\mathcal{A}}(\lambda)$ be the set of all smooth simple arcs α on τ such that α is transversal to λ and the end points of α are not in $|\lambda|$. Then $\mathring{\mathcal{A}}(\lambda)$ is a dense subset of $\mathcal{A}(\lambda)$ in the Hausdorff topology.

A *transversal measure* on λ is a function $\mu\colon \mathring{\mathcal{A}}(\lambda) \to \mathbb{R}_{\geq 0}$ such that

- $\mu(\alpha) > 0$ if and only if $\alpha \in \mathring{\mathcal{A}}(\lambda)$ intersects λ,
- if $\alpha \in \mathring{\mathcal{A}}(\lambda)$ is a composition of two arcs $\alpha_1, \alpha_2 \in \mathring{\mathcal{A}}(\lambda)$, then $\mu(\alpha) = \mu(\alpha_1) + \mu(\alpha_2)$, and
- $\mu(\alpha)$ is invariant under any isotopy of α through arcs in $\mathring{\mathcal{A}}(\lambda)$.

For $p, q \in \tau$, if there is a unique shortest geodesic segment connecting p to q, then let $\mu(p, q)$ denote the transversal measure of the segment. The *measured lamination* L is a pair (λ, μ) of a geodesic lamination λ and the transversal measure μ supported on λ. Let $\mathcal{TM}(\lambda)$ denote the set of all transversal measures supported on λ. Let $\mathcal{ML}(F)$ be the set of all measured laminations on (F, τ). Note that we do not need to specify τ, since, for different hyperbolic structures on F, the corresponding spaces $\mathcal{ML}(F)$ are naturally isomorphic (see [Bon97, §1]). Suppose that λ contains a closed leaf ℓ. Then ℓ carries an atomic measure (*weight*), which is a positive real number.

Notation 2.1. $[a, b]$ *denotes the geodesic segment connecting* a *and* b.

Let $L = (\lambda, \mu)$ be a measured lamination on τ. Then, for $\alpha \in \mathcal{A}(\lambda) \setminus \mathring{\mathcal{A}}(\lambda)$, we can naturally define its transversal measure $\mu(\alpha)$ to be a closed interval in $\mathbb{R}_{\geq 0}$ as follows. Let $(a_i) \subset \mathring{\mathcal{A}}(\lambda)$ be a sequence converging to α with $a_i \subset \alpha$. Similarly let $(b_i) \subset \mathring{\mathcal{A}}(\lambda)$ be a sequence converging to α with $\alpha \subset b_i$. Then, the transversal measure $\mu(\alpha)$ is the closed interval

$$[\lim_{i \to \infty} \mu(a_i), \lim_{i \to \infty} \mu(b_i)].$$

Note that the width of the interval is the sum of the atomic measures on leaves through the endpoints of α:

In this paper, if stated, we allow closed leaves ℓ of λ to have weight infinity (*heavy leaves*), i.e. if $\alpha \in \mathring{\mathcal{A}}(\lambda)$ transversally intersects ℓ, then $\mu(\alpha) = \infty$. A measured lamination is *heavy*, if it has a leaf with weight infinity.

2.4. **Pleated surfaces.** A continuous map $\beta \colon \mathbb{H}^2 \to \mathbb{H}^3$ is a pleated plane if there exists a geodesic lamination λ on $\mathbb{H}^2$ such that

- for each stratum P of $(\mathbb{H}^2, \lambda)$, the map β isometrically embeds P into a (totally geodesic) copy of $\mathbb{H}^2$ in $\mathbb{H}^3$, and
- β preserves the length of (rectifiable) paths.

Then we say that the geodesic lamination λ is *realized* by the pleated surface β. In this paper, we in addition assume that the realizing lamination is minimal, i.e. there is no proper sublimation of λ satisfying the two conditions above.

Definition 2.2 (Total lift)**.** *Let* $Y \to X$ *be a covering map, and let* Z *be a subset of* X*. Then the total lift of* Z *to* Y *is the inverse image of* Z *by the map.*

Then, suppose, in addition, that λ is the total lift of a geodesic lamination ν on a complete hyperbolic surface τ. Let F be the underlying topological surface of τ. Then the $\pi_1(F)$-action on $\mathbb{H}^2$ preserves λ. Let $\rho \colon \pi_1(F) \to \mathrm{PSL}(2, \mathbb{C})$ be a homomorphism. Then the pleated surface $\beta \colon \mathbb{H}^2 \to \mathbb{H}^3$ is *ρ-equivariant* if $\beta \circ \gamma = \rho(\gamma) \circ \beta$ for all $\gamma \in \pi_1(F)$. Then we say that the pair (τ, ν) is *realized* by the ρ-equivariant pleated surface β.

Definition 2.3. *Let* $\psi \colon X \to Y$ *be a map between metric spaces* (X, d_X) *and* (Y, d_Y)*. Then, for* $\epsilon > 0$*, the map* ψ *is an* <u>*ϵ-rough isometric embedding*</u> *if*

$$d_Y(\psi(a), \psi(b)) - \epsilon < d_X(a, b) < d_Y(\psi(a), \psi(b)) + \epsilon,$$

for all $a, b \in X$*. Then* ψ *is an* ϵ*-rough isometry if, in addition,* Y *is the* ϵ*-neighborhood of the image of* ψ*.*

Two geodesic laminations are, in a sense, "close" if they possibly intersect at angles very close to zero (see §2.8). Then, in the preceding paper, we proved that a please surfaces change a little when realizing laminations change a little. Namely

Theorem 2.4 ([Bab15], Theorem C)**.** *Suppose that there are a representation* $\rho\colon \pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$ *and a* ρ*-equivariant pleated surface* $\beta_0\colon \mathbb{H}^2 \to \mathbb{H}^3$ *realizing* $(\sigma_0, \nu_0) \in \mathcal{T} \times \mathcal{GL}$*. Then, for every* $\epsilon > 0$*, there exists* $\delta > 0$*, such that if there is another* ρ*-equivariant pleated surface* $\beta\colon \mathbb{H}^2 \to \mathbb{H}^3$ *realizing* $(\sigma, \nu) \in \mathcal{T} \times \mathcal{GL}$ *with* $\angle_{\sigma_0}(\nu_0, \nu) < \delta$*, then* β_0 *and* β *are* ϵ*-close: Namely there is a marking-preserving* ϵ*-rough isometry* $\psi\colon \sigma_0 \to \sigma$ *such that, letting* $\tilde{\psi}\colon \mathbb{H}^2 \to \mathbb{H}^2$ *is the lift of* ψ*,* β_0 *and* $\beta \circ \tilde{\psi}\colon \mathbb{H}^2 \to \mathbb{H}^3$ *are* ϵ*-close in the* C^0*-topology on* $\mathbb{H}^2$ *and, moreover, in the* C^∞*-topology in the complement of the total lift to the* ϵ*-neighborhood of* $|\nu| \cup |\nu_0|$ *in* σ*.*

2.5. **Traintracks.** (See [Kap01]. Also [PH92]) Given a rectangle R, pick a pair of opposite edges as *horizontal edges* and the other pair *vertical edges*. A *(fat) traintrack* T is a collection $\{R_i\}_i$ of rectangles, called *branches*, embedded in a surface F so that R_i are disjoint except overlaps of their vertical edges in a particular manner: Each vertical edge e may contain at most finitely many points that are, on F, identified with some vertices of the rectangles R_i, and those points divide e into finitely many (sub)edges; After applying this decomposition to all vertical edges, all vertical edges are uniquely divided into pairs that are homeomorphically identified on F. The points dividing (original) vertical edges are called *branch points* of T. Let $|T| \subset F$ denote the union of the branches R_i over all i. Then the boundary of $|T|$ is the union of the horizontal edges of R_i, and it contains the branch points of T. In this paper, we assume that traintracks are at most trivalent, i.e. for all i, each vertical edge of R_i is a union of, at most, two other vertical edges.

A lamination λ on F is *carried* by a traintrack T if

- $|\lambda|$ is in the interior of $|T|$,
- leaves of λ are transversal to the vertical edges of the branches of T, and
- if R is a branch of T, then $R \cap \lambda$ is a lamination on R consisting of arcs property embedded in R connecting the vertical edges of R;

then we say that T is a *traintrack neighborhood* of λ.

In addition, suppose that the surface F is a hyperbolic surface and that the branches R_i are smooth rectangles. Then the boundary of $|T|$ is the disjoint union of piecewise-smooth curves, and its non-smooth points are endpoints of vertical edges. In particular, the branch points are non-smooth points.

For $\epsilon > 0$, a (smooth) traintrack $T = \{R_i\}$ on F is ϵ-*nearly straight*, if each rectangle R_i is smoothly $(1+\epsilon)$-bilipschitz to a Euclidean rectangle and at each branch point, the angle of the boundary curve of $|T|$ is ϵ-close to 0. For $K > 0$, a traintrack $T = \{R_i\}$ is (ϵ, K)-*nearly straight*, if in addition, a horizontal edge of each branch has length at least K. Note that for fixed $K > 0$, if $\epsilon > 0$ is sufficiently small, every (ϵ, K)-nearly straight traintrack is hausdorff close to a geodesic lamination.

(See also [Bab15].) A *round circle* is, by identifying $\hat{\mathbb{C}}$ with the unite sphere in $\mathbb{R}^3$, a circle which is the intersection of $\hat{\mathbb{C}}$ with a hyperplane in $\mathbb{R}^3$. A *round cylinder* in the Riemann sphere $\hat{\mathbb{C}}$ is a cylinder bounded by disjoint round circles. The *axis* of a round cylinder A in $\hat{\mathbb{C}}$ is the geodesic in $\mathbb{H}^3$ orthogonal to both hyperbolic planes bounded by the boundary circles of A. Then A admits a canonical *circular foliation* by one parameter family of round circles bounding disjoint hyperbolic planes orthogonal to the axis of A.

Let C be a projective structure on a surface F. Suppose that $T = \{R_i\}$ is a smooth traintrack on C. Then C induces a projective structure on each rectangle R_i. Then $dev(R_i)$ is an immersion of R_i to $\hat{\mathbb{C}}$, which is defined up to a postcomposition with an element of $\mathrm{PSL}(2, \mathbb{C})$. Then a branch R_i of T is a *supported* on a round cylinder A on $\hat{\mathbb{C}}$ if

- $dev(R_i)$ maps into A,
- different vertical edges of R_i immerse into different boundary circles of A, and
- the horizontal edges of R_i immerses transversally to the circular foliation of A.

A curve on a projective surface is *circular* if its lift (to the universal cover) immerses into a round circle in $\hat{\mathbb{C}}$ by the developing map. Then, the circular foliation of A induces a foliation on R_i by *circular arcs* connecting the horizontal edges. Suppose that each branch R_i of T is supposed on a round cylinder. Then the circular foliations on the branches R_i yield a foliation of $|T|$ by circular arcs. Note that, if a loop ℓ is carried by T, then we can isotope ℓ through loops carried by T so that ℓ is transversal to the circular foliation on T. Then

Lemma 2.5. *Let T be a traintrack on a projective surface C such that the branches on T are supported on round cylinders on $\hat{\mathbb{C}}$. Then, if a loop ℓ is carried by T and ℓ is transversal to the circular foliation of T, then ℓ is admissible. (Lemma 7.2 in* [Bab15]*.)*

2.6. **Pants Graph.** ([HT80, Bro03]) Recall that S is a closed oriented surface of genus $g \geq 2$. Then a *maximal multiloop* M on S is a multiloop such that $S \setminus M$ is a union of disjoint pairs of pants. Then M consists of exactly $3(g-1)$ non-parallel loops.

An *elementary move* transforms a maximal multiloop M to a different maximal multiloop by removing a loop ℓ of M and adding another loop m disjoint from the multiloop $M \setminus \ell$ such that m intersects ℓ minimally. Namely m intersects ℓ in either one or two points. Then there is a unique connected component F of S minus $M \setminus \ell$ such that F contains ℓ. Then either

- F is a one-holed torus, and m intersects ℓ in a single point, or
- F is a four-holed sphere, and m intersects ℓ in two points.

The *pants graph* $\mathcal{PG}$ of S is a one-dimensional simplicial complex whose vertices bijectively correspond to (the isotopy classes of) the maximal multiloops on S and the edges to the elementary moves connecting different maximal multiloops. Then it turns out that $\mathcal{PG}$ is connected ([HT80]).

2.7. **Purely loxodromic representations.**

Lemma 2.6. *Let $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ be a purely loxodromic representation (see §1). Then, if $\gamma, \eta \in \pi_1(S)$ are non-commuting elements, the axes of the loxodromics $\rho(\gamma), \rho(\eta)$ share no endpoint.*

Proof. Since γ and η do not commute, $\gamma, \eta \neq id$ and $[\gamma, \eta] \neq id$. Suppose that, to the contrary, the axes of $\rho(\gamma)$ and $\rho(\eta)$ share an endpoint. Then we can show that their commuter $[\rho(\gamma), \rho(\eta)]$ is parabolic element, by computing its trace. This is a contradiction since ρ is purely loxodromic. □

(See [Kap01, §4.3] for example.) Recall that S is a closed oriented surface of genus at least two. Let $\{\gamma_1, \cdots, \gamma_m\}$ be a generating set of $\pi_1(S)$. Then, by the adjoint representation, $\mathrm{PSL}(2,\mathbb{C})$ embeds into $\mathrm{GL}(3,\mathbb{C})$ as a complex affine group. Since $\mathrm{GL}(3,\mathbb{C}) \subset \mathbb{C}^9$, representations $\rho\colon \pi_1(S) \to \mathrm{GL}(3,\mathbb{C})$ injectively correspond to tuples $\{\rho(\gamma_1), \ldots, \rho(\gamma_m)\}$ in $\mathrm{PSL}(2,\mathbb{C})^m \subset \mathbb{C}^{9m}$. Thus we can regard the space $\mathcal{R}$ of representations $\rho\colon \pi_1(S) \to \mathrm{GL}(3,\mathbb{C})$ as an affine algebraic variety. This variety is called the $\mathrm{PSL}(2,\mathbb{C})$ *representation variety* of S.

Then $\mathrm{PSL}(2,\mathbb{C})$ acts on $\mathcal{R}$ by conjugation, and its orbits give the equivalent classes of representations. By quotienting out $\mathcal{R}$ by a slightly stronger equivalent relation, we obtain the $\mathrm{PSL}(2,\mathbb{C})$-*character variety* χ of S (see [BZ98, HP04] about the $\mathrm{PSL}(2,\mathbb{C})$-character varieties and the quotient). It turns out that two representations $\rho_1, \rho_2 \colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ are equivalent if and only if $tr^2(\rho_1(\gamma)) = tr^2(\rho_2(\gamma))$ for all $\gamma \in \pi_1(S)$ (see Theorem [HP04, Theorem 1.3]).

Then χ has exactly two connected components ([Gol88]). Let χ_0 be the component consisting of representations $\pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ that lift to $\pi_1(S) \to \mathrm{SL}(2,\mathbb{C})$. Then χ_0 contains the quasifuchsian space.

For every $\gamma \in \pi_1(S)$, let $\mathrm{Tr}^2_\gamma \colon \chi \to \mathbb{C}$ denote the square trace function of γ given by $\rho \to \mathrm{Tr}^2(\rho(\gamma))$. Then Tr^2_γ is a regular function (see ([BZ98]). Note that the singular part of the χ has complex codimension at least one. In addition the image of the holonomy map $\mathrm{Hol}\colon \mathcal{P} \to \chi$ is contained in the smooth part of χ_0.

Lemma 2.7. *Almost all elements of χ_0 are purely loxodromic.*

Proof. Since χ_0 contains the quasifuchsian space, if $\gamma \in \pi_1(S) \setminus \{id\}$, then Tr^2_γ is nonconstant on χ_0. If $\mathrm{Tr}^2_\gamma(\rho) = 4$ if $\rho(\gamma)$ is parabolic and $\mathrm{Tr}^2_\gamma(\rho) \in [0,4)$ if $\rho(\gamma)$ is elliptic. Since $[0,4] \subset \mathbb{R}$ has measure zero in $\mathbb{C}$ and Tr^2_γ is regular, almost every element of χ_0 takes γ to a loxodromic element. Since $\pi_1(S)$ contains only countably many elements, if ρ is a generic representation in χ_0, then $\rho(\gamma)$ is loxodromic for all $\gamma \in \pi_1(S)$. □

2.8. **Local characterization of projective structures in $\mathcal{GL}(S)$.** Let τ be a hyperbolic surface homeomorphic to S. If two geodesics ℓ and m on τ intersect at a point p, then let $\angle_p(\ell, m)$ denote the angle between ℓ and m at p that takes a value in $[0, \pi/2]$. Let λ and ν are geodesic laminations on τ. Then the *angle* between λ and ν is

$$\sup \angle_p(\ell_p, m_p),$$

where the supremum runs over all points $p \in |\lambda| \cap |\nu|$ and ℓ_p and m_p are the leaves of λ and ν, respectively, intersecting at p. If λ and ν are laminations on the topological surface S or different hyperbolic surfaces homeomorphic to S, then $\angle_\tau(\lambda, \nu)$ is given by taking their geodesic representatives on τ.

Let ν be a geodesic lamination on τ, and let (λ_i) be a sequence of geodesic laminations on τ. Suppose that $\angle_\tau(\lambda_i, \nu) \to 0$ as $i \to \infty$. Note that this convergence is independent on the choice of the hyperbolic structure $\tau \in \mathcal{T}$. Since, in this paper, we typically require such an angle to be sufficiently small, we may denote $\angle_\tau(\lambda_i, \nu)$ simply by $\angle(\lambda_i, \nu)$ without specifying τ. If σ is a subsurface of τ, then let $\angle_\sigma(\lambda, \nu)$ be $\sup \angle_p(\ell, m)$ over all leaves $\ell \in \lambda$ and $m \in \nu$ intersecting at points p contained in σ.

Given measured geodesic laminations M and L on (S, τ), their angle $\angle_\tau(M, L)$ is the angle of the under lying geodesic laminations $|M|$ and $|L|$.

Theorem 2.8 ([Bab15], Theorem B). *Let $C \cong (\tau, L)$ be a projective structure on S with holonomy $\rho\colon \pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$. Then there is $\delta > 0$ such that, if another projective structure $C' \cong (\tau', L')$ with holonomy ρ satisfies $\angle_\tau(L, L') < \delta$, then we can graft C and C' along multiloops to a common projective structure. That is, there are admissible multiloops M on C and M' on C' such that*

$$\mathrm{Gr}_M(C) \cong \mathrm{Gr}_{M'}(C').$$

3. Thurston's grafting coordinates on $\mathcal{P}$

([KT92, KP94] are general references of this section.) The space $\mathcal{P}$ of all (marked) projective structures on S is naturally homeomorphic to the product of the Teichmüller space $\mathcal{T}$ of S and the space of measured laminations $\mathcal{ML}$ on S:

$$\mathcal{P} \cong \mathcal{T} \times \mathcal{ML} \tag{3}$$

Let $C = (f, \rho) \in \mathcal{P}$, and let $(\tau, L) \in \mathcal{T} \times \mathcal{ML}$ be its Thurston coordinates, i.e. the corresponding pair via (3). We briefly describe the correspondence between $\mathcal{P}$ and $\mathcal{T} \times \mathcal{ML}$. First there is a measured lamination $\mathcal{L} = (\nu, \omega)$ on C, called the *(canonical) circular lamination*, and a marking-preserving continuous map

$$\kappa\colon C \to \tau,$$

called *collapsing map*, such that κ descends $\mathcal{L}$ to L. The leaves of $\mathcal{L}$ are circular and they have no atomic measure, in comparison to L. The pair (τ, L) corresponds to a pleated surface $\beta\colon \mathbb{H}^2 \to \mathbb{H}^3$ equivariant with respect to ρ, constructed as follows. Let $\tilde{L}$ be the total lift of L under the covering map $\mathbb{H}^2 \to \tau$. Then $\tilde{L}$ is the $\pi_1(S)$-invariant measured lamination on $\mathbb{H}^2$. Thus, intuitively speaking by bending $\mathbb{H}^2$ along $\tilde{L}$ by the angle given by the transversal measure of L, (τ, L) yields a pleated surface $\beta\colon \mathbb{H}^2 \to \mathbb{H}^3$. (To be precise, there is a sequences (L_i) of measured laminations with finitely many leaves that converges to $\tilde{L}$ uniformly on compacts in $\mathbb{H}^2$. Then the pleated surface β is given by the limit of pleated surfaces $\beta_i\colon \mathbb{H}^2 \to \mathbb{H}^3$ corresponding to L_i; see [EM87, 3.11.6]).

Note that $f\colon \tilde{C} \to \hat{\mathbb{C}}$ and $\beta\colon \mathbb{H}^2 \to \mathbb{H}^3$ are both ρ-equivariant and $\hat{\mathbb{C}}$ is the ideal boundary of $\mathbb{H}^3$. In fact, for every $x \in \tilde{C}$, its image under $f\colon \tilde{C} \to \hat{\mathbb{C}}$ maps to its image under $\beta \circ \kappa\colon \tilde{C} \to \mathbb{H}^3$ by a certain nearest point projection given by the *maximal ball* in $\tilde{C}$ associated with x (see §3.1 for the precise correspondence).

The collapsing map κ takes each stratum of $(C, \mathcal{L})$ diffeomorphically onto a stratum of (τ, L). If ℓ is a closed leaves of L, which carries positive atomic measure, then $\kappa^{-1}(\ell)$ is a cylinder $\mathcal{A}_\ell$ foliated by closed leaves of $\mathcal{L}$. Conversely κ takes each closed leaf of $\mathcal{L}$ on $\mathcal{A}_\ell$ onto ℓ diffeomorphically. Let $\mathcal{A}$ be the union of the disjoint cylinders $\mathcal{A}_\ell$ over all closed leaves ℓ of L. Then, on the other hand, the strata of $(C, \mathcal{L})$ not in $\mathcal{A}$ bijectively correspond, via κ, to the strata of (τ, L) that are not the closed leaves of L. For different closed leaves ℓ of L, their corresponding $\mathcal{A}_\ell$ are disjoint on C.

Recall that $\kappa\colon C \to \tau$ preserves its marking (and thus κ is homotopic to a homeomorphism). Then, given a cover of C, its Thurston coordinates are given by the corresponding cover of (τ, L). In particular, the universal cover of C is a projective structure on an open disk, and its Thurston coordinates are $\mathbb{H}^2$ and the total lift of L to $\mathbb{H}^2$.

More generally, we say that a projective structure $C = (f, \rho)$ on a connected orientable surface F *has Thurston coordinates* (X, L), where the universal cover $\tilde{X}$ of X is a convex subset of $\mathbb{H}^2$ bounded by geodesics and a measured lamination L on X, if the maximal balls in the universal cover $\tilde{C}$ yields a $\pi_1(C)$-invariant a stratification of C and it descends, by the construction in §3.1, to a ρ-equivariant pleated surface from $\tilde{X} \to \mathbb{H}^3$ given by (X, L).

Indeed, a projective structure on an open disk has Thurston coordinates unless it is isomorphic to $\mathbb{C}$ as a projective surface (see §3.2). However its first coordinate is not necessarily the entire hyperbolic space. Let X be a convex subset of $\mathbb{H}^2$ bounded by disjoint geodesics. Note that X can be the entire hyperbolic plane or a single (biinfinite) geodesic. In addition, we suppose that each boundary geodesic of X is either a subset of X or its complement $\mathbb{H}^2 \setminus X$. Let $L = (\lambda, \mu)$ be a measured lamination on X. Then L induces a pleated surface $\beta\colon X \to \mathbb{H}^3$ by bending X, inside $\mathbb{H}^3$, which is unique up to a post-composition with an element of $\mathrm{PSL}(2, \mathbb{C})$.

Let (X, L) be the Thurston coordinates of a projective structure C of a open disk. Then, if X has a boundary geodesic ℓ, the transversal measure of L must be infinite near ℓ. More precisely,

- if ℓ is a subset of X, then ℓ has weight infinity, and
- if ℓ is a subset of $\mathbb{H}^2 \setminus X$, then the transversal measure of L is infinite "near ℓ", i.e. if an arc α on X is transversal to L and it has an open end point at ℓ, then $\mu(\alpha)$ is infinite.

3.1. **Maximal balls.** Let C be a projective structure on an open disk. Let $f\colon C \to \hat{\mathbb{C}}$ be $dev(C)$. Conformally identifying $\hat{\mathbb{C}}$ with $\mathbb{S}^2$, we fix a spherical metric on $\hat{\mathbb{C}}$, which is unique up to an element of $\mathrm{PSL}(2, \mathbb{C})$. Pullback this metric to C by f and obtain an incomplete spherical metric on C. The metric completion of C minus C is called the *ideal boundary* of C and denoted by $\partial_\infty C$. Note that this completion $C \cup \partial_\infty C$ is (topologically) independent of the choice of the spherical metric on $\hat{\mathbb{C}}$.

A *maximal ball* in C is a topological open ball B such that

- B is round, i.e. f embeds B onto a round open ball in $\hat{\mathbb{C}}$, and
- B is maximal, i.e. there is no round open ball in C strictly containing B.

The *ideal boundary* $\partial_\infty B$ of a maximal ball B in C is the intersection of $\partial_\infty C$ and ∂B in $C \cup \partial_\infty C$. Then, by identifying B with $\mathbb{H}^2$ conformally, the ideal boundary of B is a subset of the ideal boundary of $\mathbb{H}^2$ (which is $\mathbb{S}^1$). The *core* $\mathrm{Core}(B)$ of a maximal ball B in C is the convex hull of the ideal boundary of B in $\mathbb{H}^2$. It turns out that $\mathrm{Core}(B)$ is a stratum of $(C, \mathcal{L})$. In particular, for different maximal balls B in C, their corresponding cores are disjoint [KP94, Proposition 4.3]. Moreover, taking the cores of all maximal balls B, we obtain the stratification of $(C, \mathcal{L})$. In other words, for every point $p \in C$, there is a unique maximal ball B in C such that $p \in \mathrm{Core}(B)$ [KP94, Proposition 4.4]. Then we say that B is the maximal ball *centered* at p.

Let H be the hyperbolic plane in $\mathbb{H}^3$ bounded by the boundary circle of B. Then the nearest point projection from $\mathbb{H}^3$ to H extend to p. Let $\beta\colon X \to \mathbb{H}^3$ be the pleated surface for C, and $\kappa\colon C \to X$. Then $\beta \circ \tilde{\kappa}(p)$ is the projection of p to H, where $\tilde{\kappa}\colon \tilde{C} \to \tilde{X}$ is the lift (§8). [KP94]

3.1.1. *Thurston metric.* ([KP94, Tan97]) Every projective structure C on a surface, unless its universal cover is $\mathbb{C}$, admits a canonical C^1-smooth Riemannian metric, called *Thurston metric*. It is given by a $\pi_1(S)$-invariant Riemannian metric on its universal $\tilde{C}$ defined as follows. Let $x \in \tilde{C}$. For every maximal ball B in $\tilde{C}$ containing x, by conformally identifying B with $\mathbb{H}^2$, it defines a Riemannian metric tensor at x. Taking the infimum of the metric tensors over all maximal ball B containing x, we obtain Thurston metric at x.

Let $C \cong (\tau, L)$ be the Thurston coordinates. The Thurston metric is isometric to the hyperbolic metric on τ by κ on each stratum of $(C, \mathcal{L})$. For each closed leaf ℓ of L, the Thurston metric on the cylinder $\mathcal{A}_\ell$ is Euclidean, so that each leaf of $\mathcal{L}$ in $\mathcal{A}_\ell$ is a closed geodesic whose length is $\mathrm{lenth}_\tau(\ell)$ and the height of the cylinder is twice as much as the weight of ℓ given by L. We call $\mathcal{A}$ the *Euclidean region* of C. The Thurston metric changes continuously in the deformation space $\mathcal{P}$ of projective structures on S.

3.2. **Existence of Thurston coordinates on disks.** [KP94, Theorem 11.6] implies

Theorem 3.1. *Let C be a projective structure on a simply connected surface not isomorphic to $C \neq \mathbb{C}, \hat{\mathbb{C}}$ as a projective surface. Then C admits unique Thurston coordinates (X, L) such that*

- *X is a closed convex subset of $\mathbb{H}^2$ bounded by geodesics and each boundary geodesic of X is either contained in X (*closed boundary*) or in $\mathbb{H}^2 \setminus X$ (*open boundary*), and*

- *L is a measured lamination on X, and if a geodesic boundary of X is contained in X, then it is a leaf of L with weight ∞ (and no leaves in the interior of X have weight infinity).*

Remark 3.2. *The boundary leaf with weight infinity corresponds to a complex affine half infinite cylinder in the projective surface, and this cylinder is foliated by round circles which descend to the boundary leaf.*

Corollary 3.3. *Let F be a connected surface (possibly with open boundary). Let C be a projective structure on F with holonomy $\rho\colon \pi_1(F) \to \mathrm{PSL}(2,\mathbb{C})$. Suppose that $\operatorname{Im}\rho$ is non-elementary. Then C has Thurston coordinates*

$$C \cong (\tau, L),$$

where τ is a convex hyperbolic surface possibly with geodesic boundary such that the interior of τ is homeomorphic to F and L is a measured geodesic lamination on τ. In addition each boundary component of τ is either open or closed; therefore the closed boundary components of τ are the only leaves of L with weight infinity.

Remark 3.4. *If $\operatorname{Im}\rho$ is elementary but the limit set of $\operatorname{Im}\rho$ has cardinality two, then C still has Thurston coordinates (τ, L). However τ may be a single close geodesic with weight infinity, and the interior of τ is not homeomorphic to F. Nonetheless the regular neighborhood of τ is still homeomorphic to F.*

Proof of Corollary 3.3. Since the limit set of $\operatorname{Im}(\rho)$ has cardinality more than one, the universal cover of C can not be isomorphic to $\mathbb{C}$ or $\hat{\mathbb{C}}$ (as a projective structure). Thus applying Theorem 3.1, we obtain the Thurston coordinates $(\tilde{\tau}, \tilde{L})$ of the universal cover of C so that, if exists, the boundary geodesics of $\tilde{\tau}$ are the only leaves of L with weight infinity. Then $\pi_1(F)$ acts on $\tilde{\tau}$. Then $\tilde{\tau}$ is the convex hull of the limit set of $\rho(\pi_1(F))$. Thus, since $\operatorname{Im}(\rho)$ is non-elementary, $\tilde{\tau}$ has interior whose closure is $\tilde{\tau}$. Since $\pi_1(F)$ preserves $(\tilde{\tau}, \tilde{L})$, it descends to the Thurston coordinates (τ, L) of C. Then the interior of τ is homeomorphic to F. 3.3

3.3. **Canonical neighborhoods.** Suppose that C is a projective structure on an open disk with $C \neq \mathbb{C}$. Then let (X, L) denote its Thurston coordinates, where X is a convex subset of $\mathbb{H}^2$ bounded by geodesics and L is a (possibly heavy) measured lamination on X (Proposition 3.1). Let $\beta\colon X \to \mathbb{H}^3$ be the corresponding pleated surface and $\kappa\colon C \to X$ be the collapsing map. Let $\mathcal{L}$ be the measured lamination on C that descends to L by κ.

Let p be a point on C, and let $B(p)$ be the maximal ball in C centered at p. Let $U(p)$ denote the union of all maximal balls in C containing p. Then $U(p)$ is an open neighborhood of p, and it is called the *canonical neighborhood* of p in C. It turns out that $U(p)$ is homeomorphic to an open disk and $dev(C)$ embeds $U(p)$ into $\hat{\mathbb{C}}$ (see [KP94] [KT92]). Note, since $C \neq \mathbb{C}$, thus $U(p) \neq \mathbb{C}$. The ideal boundary $\partial_\infty C$ intersects the closure of $U(p)$ in the completion $C \cup \partial_\infty C$, and points in the intersection are called *ideal points* of $U(p)$.

Recall that C decomposes into strata by $\mathcal{L}$, which are leaves of $\mathcal{L}$ and closures of the complementary regions of $C \setminus |\mathcal{L}|$. Then we can take a quotient T of C by collapsing each stratum to a point. Let $\Psi\colon C \to T$ be the quotient map. Then, for two strata P, Q of $(C, \mathcal{L})$, the distance between $\Psi(P)$ and $\Psi(Q)$ on T is the infimum of the measures, given by $\mathcal{L}$, over all transversal arcs connecting P and Q in C. Then, as C is a disk, it is well-known that the quotient T is a metric $\mathbb{R}$-tree (see, for example, [Kap01, §11.12]).

Let $W(p)$ be the union of $\mathrm{Core}(B(x))$ for all $x \in C$ with $p \in B(x)$. Then $W(p)$ is the open neighborhood of p bounded by the leaves ℓ of $\mathcal{L}$ such that their corresponding maximal balls B_ℓ satisfy $\partial B_\ell \ni p$ (Figure 1). If B_1 and B_2 are different maximal balls in C, then B_1 intersects exactly one connected component of $C \setminus B_2$; this implies $W(p)$ is connected.

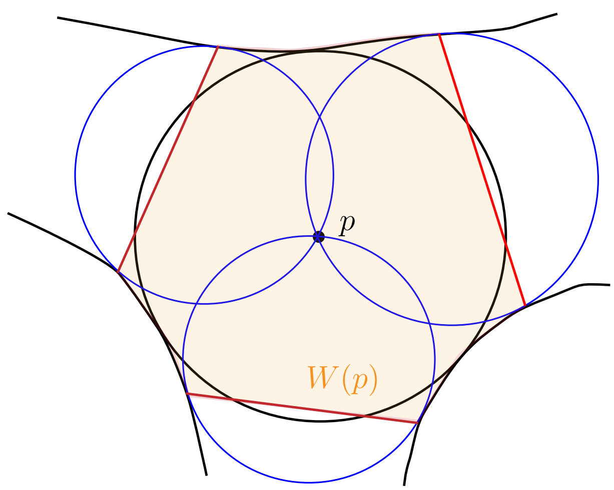


Figure 1.

$U(P) \setminus W(p)$ are disjoint half disks, which are cores of the Thurston coordinates of $U(P)$. However, typically those half disks are not strata of Thurston coordinates of C via the inclusion $U(P) \subset C$.

Lemma 3.5. *For $p \in C$, if a neighborhood V_p of $\Psi(p)$ in T is contained in $\Psi(W(p))$, then the ideal boundary $\partial_\infty B(p)$ is the boundary circle*

$\partial B(p)$ minus the union of maximal balls of C whose cores map into V_p by Ψ.

Proof. If $x \in W(p)$, then $B(x)$ is contained in $U(p)$. In particular, if the neighborhood V_p of p is contained in $\Psi(W_p)$, for all $x \in C$ with $\Psi(x) \in V_p$, $B(x)$ is contained in $U(p)$. By the definition of $U(p)$, the maximal ball $B(x)$ contains p.

The ideal boundary $\partial_\infty B(p)$ is naturally embedded in the boundary of $U(p)$ in $\hat{\mathbb{C}}$. Therefore, since $B(x)$ is in the interior of $U(p)$, the ideal boundary $\partial_\infty B(p)$ is contained in $\partial B(p) \setminus \cup_x B(x)$ over all $x \in C$ with $\Psi(x) \in V_p$.

To show the opposite inclusion, let s be the connected component of $\partial B(p) \setminus \partial_\infty B(p)$; see Figure 2. Then s is a circular arc on $\hat{\mathbb{C}}$ with open ends. Then there is a unique leaf ℓ of $\mathcal{L}$ connecting the endpoints of s. Namely ℓ is a boundary leaf of $\operatorname{Core} B(p)$.

Consider the connected component of $C \setminus \operatorname{Core} B(x)$ bounded by ℓ. Pick a sequence of points x_i in the component limiting to an interior point of ℓ. Then $B(x_i)$ converges to $B(p)$ as $i \to \infty$, and $B(x_i) \cap s$ converges to s. Since Ψ takes $\operatorname{Core} B(x_i)$ to a point in V_p for sufficiently large i, we have $s \subset \cup_x B(x)$ over $x \in C$ with $\Psi(x) \in V_p$. □

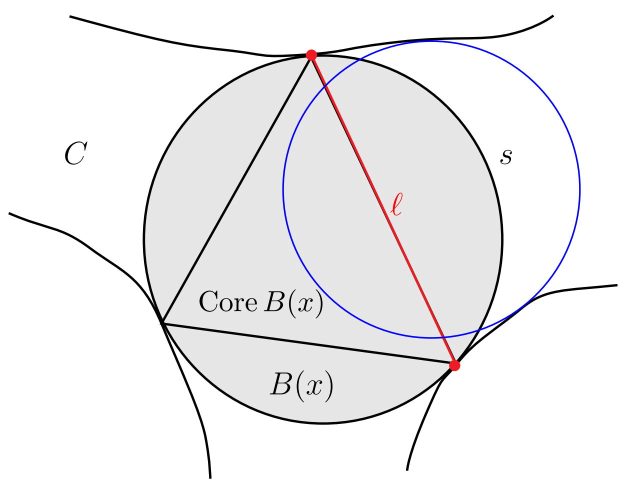


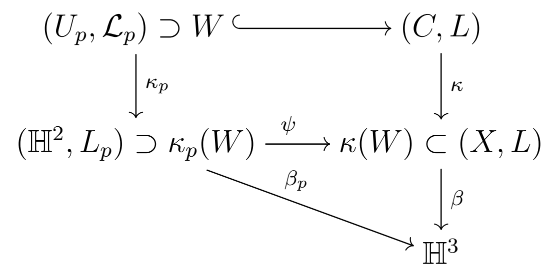
FIGURE 2.

Let $(\mathbb{H}^2, L_p)$ be the Thurston coordinates of $U(p)$. As $U(p)$ is embedded in $\hat{\mathbb{C}}$, its Thurston coordinates correspond to the boundary of the convex hull of $\hat{\mathbb{C}} \setminus U(p)$. In particular the first coordinate is the entire hyperbolic plane since $U(p)$ is embedded in $\hat{\mathbb{C}}$.

Let $\beta_p \colon \mathbb{H}^2 \to \mathbb{H}^3$ be the pleated surface corresponding to $(\mathbb{H}^2, L_p)$. Let $\mathcal{L}_p$ be the circular measured lamination on $U(p)$ that descends to L_p by the collapsing map $\kappa_p \colon U(p) \to \mathbb{H}^2$.

Note that $W(p) \subset U(p) \subset C$. Then we next show that the Thurston coordinates of $U(p)$ on $W(0)$ coincide with that of C (which typically fails on $W(p) \setminus U(p)$). Namely

Proposition 3.6.
- *In $W(p)$, $\mathcal{L}_p$ is isomorphic to $\mathcal{L}$, and thus the Thurston metric on $W(p)$ is isometric to that on C.*
- *There exists a natural isometry $\psi\colon \kappa_p(W) \to \kappa(W)$ such that $\psi \circ \kappa_p = \kappa$ on W and $\beta \circ \psi = \beta_p$ on $\kappa_p(W)$.*



Proof. Recall that $U(p) = \cup_x B(x)$ where x runs over all points C with $p \in B(x)$. Since $U(p) \sup C$, such a maximal ball $B(x)$ in C is also maximal in $U(p)$. Since $W(p)$ is connected and it contains no boundary leaves (Figure 1), $\Psi(W(p))$ is an open connected subset of T. Therefore, by Lemma 3.5, if $q \in W(p)$, the ideal boundary of the maximal ball $B(q)$ in C is equal to that in $U(p)$. Since the maximal balls and their ideal boundary determine the circular laminations, $\mathcal{L}$ is isomorphic to $\mathcal{L}_p$ on $W(p)$ by the inclusion $U_p \subset C$. The second assertion similarly holds.

3.6

Part 1. Grafting Conjecture for purely loxodromic holonomy

4. Sequence of pleated surfaces

Fix an arbitrary representation $\rho\colon \pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$ that is purely loxodromic. Let $C_\sharp \cong (\tau_\sharp, L_\sharp)$ and $C_\flat \cong (\tau_\flat, L_\flat)$ be projective structures with holonomy ρ. Then they correspond to ρ-equivariant pleated surfaces realizing $(\tau_\sharp, L_\sharp)$ and $(\tau_\flat, L_\flat)$. In this section, we construct an infinite family of pleated surfaces, in a coarse sense, "connecting" those pleated surfaces corresponding to $C_\sharp$ and $C_\flat$.

Pleated surfaces are invented by William Thurston for the study of three-dimensional hyperbolic manifolds, and in particular he used a sequence of homotopy equivalent pleated surfaces in order to understand geometry of the convex hull of the manifolds (see [Thu81, Chapter 9]), and it was been widely used (for example see [Bro03] [Min99]).

The family of pleated surfaces in this paper is motivated by the study hyperbolic 3-manifolds, but on the other hand, the homomorphisms $\rho\colon \pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$ of our interest here are not necessarily discrete or faithful. In order to adapt the theory of pleated surfaces for hyperbolic three-manifolds, we carefully use the assumption of ρ being purely loxodromic.

Lemma 4.1. *Let L be a measured (geodesic) lamination on a hyperbolic surface τ. For every $\epsilon > 0$, there is a neighborhood U of $[L]$ in $\mathcal{PML}(S)$, such that if $L' \in U$ then $\angle_\tau(L, L') < \epsilon$.*

Proof. Suppose, to the contrary, that there is a sequence of measured laminations L_i converging L as $i \to \infty$, but $\limsup_{i\to\infty} \angle_\tau(L, L_i) > 0$. The space of geodesic laminations on τ is compact. Thus, up to a subsequence, the underlying geodesic laminations $|L_i|$ converge, as $i \to \infty$, to a geodesic lamination λ_∞ which contains a leaf transversally intersecting a leaf of L. This contradicts to the assumption $L_i \to L$. □

Theorem 4.2 ([FLP79])**.** *In the space of measured laminations $\mathcal{ML}(S)$, a weighted loop is dense.*

By this theorem, we can pick maximal multiloops $M_\sharp$ and $M_\flat$ on $C_\sharp$ and $C_\flat$, respectively, so that

- $\angle_{\tau_\sharp}(M_\sharp, L_\sharp)$ and $\angle_{\tau_\flat}(M_\flat, L_\flat)$ are sufficiently small, and
- sufficiently small neighborhoods of $M_\sharp$ and $M_\flat$ contain $L_\sharp$ on $\tau_\sharp$ and $L_\flat$ and $\tau_\flat$, respectively.

Since the pants graph of S is connected (§2.6), there is a simplicial path in the graph connecting $M_\sharp$ and $M_\flat$. Let $(M_i)_{i=0}^n$ be the corresponding sequence of maximal multiloops on S with $M_0 = M_\sharp$ and $M_n = M_\flat$, so that M_i and M_{i+1} are adjacent vertices of the pants graph for all $i = 0, \ldots, n-1$.

Each connected component P of $S \setminus M_i$ is a pair of pants. Pick a maximal geodesic lamination on P. Then it consists of three isolated geodesics, and each geodesic ray in the lamination (*half-leaf*) is asymptomatic to a boundary component of P, spiraling towards it. We can, in addition, assume that such half-leaves spiral towards left (with respect to the orientation of S) when they approach towards boundary components and that, on each leaf of the lamination, the rays in the opposite directions are asymptotic to different boundary components of P. For each i, let ν_i be the maximal lamination of S that is the union of the maximal multiloop M_i and the above maximal laminations on all connected components P of $S \setminus M_i$.

The lamination ν_i is obtained as the Hausdorff limit of the iteration of the left Dehn twist along M_i of some multiloop N_i on S (Figure 3). Indeed we can take the multiloop N_i so that the restriction of N_i to each connected component P of $S \setminus M_i$ is a union of three non-parallel arcs connecting all pairs of boundary components of P. Furthermore, for every $k \in \mathbb{Z}_{>0}$, by taking k parallel copies of the arcs on all P, we can also take N_i such that the number of the arcs of $N_i|P$ is $3k$ for all connected components P of $S \setminus M_i$.

The following lemma guarantees that ν_i is realized by a unique ρ-equivariant pleated surface.

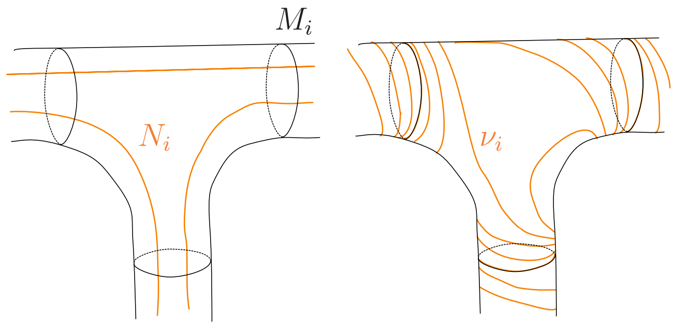


FIGURE 3. The maximal lamination ν_i is obtained by twisting N_i along M_i "infinitely many" times.

Lemma 4.3. *Suppose that* $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ *is purely loxodromic. Let* ν *be a geodesic lamination on* S *such that*

- ν *is maximal, and*
- *every half-leaf of* ν *accumulates to a closed leaf of* ν*.*

Then there is a unique ρ*-equivariant pleated surface realizing* ν*.*

Proof. Let Δ be a connected component of $S \setminus |\nu|$, which is an ideal triangle. Let $\tilde{S}$ be the universal cover of S, and let $\tilde{\nu}$ be the total lift of ν to $\tilde{S}$. Let $\tilde{\Delta}$ be a lift of Δ to $\tilde{S}$. Then $\tilde{\Delta}$ is an ideal triangle property embedded in $\tilde{S}$, and its vertices are at distinct points on the circle at infinity $\partial_\infty S$.

By the second assumption, each (ideal) vertex of Δ corresponds to a closed leaf of ν. This loop lifts to a unique leaf of $\tilde{\nu}$ whose endpoint is the corresponding vertex of $\tilde{\Delta}$ in the boundary circle of $\tilde{S}$ at infinity. Different vertices of $\tilde{\Delta}$ correspond to different leaves of $\tilde{\nu}$ that cover closed leaves of ν. Then the vertices are naturally fixed points of different elements of $\pi_1(S) \setminus \{id\}$. Since ρ is purely loxodromic, by Lemma 2.6, for different elements of $\pi_1(S) \setminus \{id\}$, their ρ-images are loxodromics sharing no fixed points. Then the vertices of $\tilde{\Delta}$ correspond to different points on $\hat{\mathbb{C}}$ fixed by different loxodromics, and they spans a unique ideal triangle in $\mathbb{H}^3$.

This correspondence defines a ρ-equivariant map β from $\tilde{S} \setminus |\tilde{\nu}|$ to $\mathbb{H}^3$. Note that every geodesic lamination is uniquely decomposed into isolated biinfinite leaves, closed leaves, and minimal irrational laminations (see [CEG87, I.4.2]). Then, if a leaf of $\tilde{\nu}$ is an isolated leaf, then either it separates adjacent complementary ideal triangles or it covers a closed leaf of ν by the second assumption and the decomposition theorem. Each leaf ℓ of $\tilde{\nu}$ either separates adjacent ideal triangles or descends to a closed leaf of ν on S. Clearly the ρ-equivariant map continuously extends to the leaves of $\tilde{\nu}$ of the first type. If a leaf ℓ cover a closed leaf of ν, then there is a sequence $\{\Delta_i\}$ of ideal triangles of $\tilde{S} \setminus |\tilde{\nu}|$

that converges to ℓ uniformly on compacts (in the Hausdorff topology). Then, by the second assumption, if $i \in \mathbb{N}$ is sufficiently large, a vertex of Δ_i must coincide with an endpoint of ℓ. Noting that $\tilde{S} \setminus |\tilde{\nu}|$ has only finitely many components up to $\pi_1(S)$, since β is ρ-equivariant, $\beta(\Delta_i)$ must converge to the geodesic axis of the loxodromic corresponding to ℓ. Therefore we can continuously extend β to the leaves of $\tilde{\nu}$ covering closed leaves of ν and obtain a desired a ρ-equivariant pleated surface $\mathbb{H}^2 \to \mathbb{H}^3$ realizing λ. □

4.1. **Bi-infinite Sequence of geodesic laminations connecting ν_i to ν_{i+1}.** Recall that for each $i \in \{0, \cdots, n-1\}$, M_i and M_{i+1} are maximal multiloops on S that are adjacent vertices on the pants graph of S. Then let m_i and m_{i+1} be the loops of M_i and M_{i+1}, respectively, such that $M_i \setminus m_i = M_{i+1} \setminus m_{i+1}$. Then let F_i denote the minimal subsurface of S containing both m_i and m_{i+1}, which is either a one-holed torus or a four-holed sphere (§2.6).

Case One. First suppose that F_i is a once-holed torus. Let $\hat{F}_i$ be the once-punctured torus obtained by pinching the boundary component of F_i to a point. Then, every geodesic lamination on F_i descends a unique geodesic lamination on $\hat{F}_i$. In particular, the geodesic laminations ν_i and ν_{i+1} on S restrict to geodesic laminations on F, then further to unique laminations $\hat{\nu}_i$ and $\hat{\nu}_{i+1}$, respectively, on $\hat{F}_i$. Since ν_i and ν_{i+1} are maximal, $\hat{\nu}_i$ and $\hat{\nu}_{i+1}$ are maximal on $\hat{F}_i$. (By pinching a boundary component to a point, we can eliminate the twisting direction v_i towards boundary component, as the direction is not the focus of the following argument.)

Let T be the trivalent tree dual to the Farey tessellation (see for example [Bon09]). Then the vertices of T bijectively correspond to the ideal triangulations of $\hat{F}_i$ and the edges to *diagonal exchanges* of the ideal triangulations — a *diagonal exchange* removes a diagonal of an (ideal) quadrangle and add the other diagonal of the quadrangle.

Pick a maximal lamination $\hat{\nu}_{i,0}$ of $\hat{F}_i$ such that

- for each leaf of $\hat{\nu}_{i,0}$, its endpoints are at the puncture of $\hat{F}_i$, and
- every leaf of $\hat{\nu}_{i,0}$ intersects each of m_i and m_{i+1} at most in a single point (Figure 4).

Then $\hat{\nu}_i$ is obtained by the infinite iteration of the Dehn twist of $\hat{\nu}_{i,0}$ along m_i and similarly $\hat{\nu}_{i+1}$ by the infinite iteration of the Dehn twist along m_{i+1}. The Dehn twists along m_i and m_{i+1} each correspond to a composition of two diagonal exchanges of $\hat{\nu}_{i,0}$. Then we obtain a bi-infinite path in T connecting $\hat{\nu}_i$ to $\hat{\nu}_{i+1}$. Then, there is a corresponding bi-infinite sequence $(\hat{\nu}_{i,j})_j$ of maximal laminations on $\hat{F}_i$ that converges to $\hat{\nu}_i$ as $j \to -\infty$ and to $\hat{\nu}_{i+1}$ as $j \to \infty$. Let $\nu_{i,j}\,(j \in \mathbb{Z})$ be the corresponding maximal laminations on S such that

(i) the restriction of $\nu_{i,j}$ to F_i descends to $\hat{\nu}_{i,j}$ on $\hat{F}_i$, and

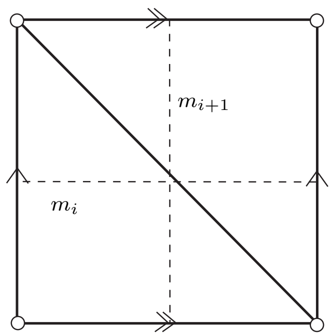


FIGURE 4. An example of $\hat{\nu}_{i,0}$, an ideal triangulation of the once-punctured torus $\hat{F}_i$. The arrows indicate the identification of edges of the square, so that the vertices correspond the puncture of $\hat{F}_i$.

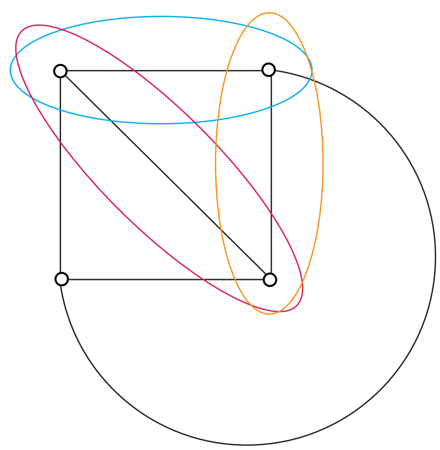


FIGURE 5. An ideal triangulation of a four-punctured sphere give the edges of a tetrahedron (black). The colored loops separate opposite edges of the tetrahedron.

(ii) $\nu_{i,j}$ is isomorphic to ν_i (and ν_{i+1}) in some neighborhood, in S, of the closure of $S \setminus F_i$.

Note that, in (ii), we need to take a neighborhood so that $\nu_{i,j}$ spirals to the left towards ∂F_i (as ν_i and ν_{i+1} do). Then $\nu_{i,j}$ is a bi-infinite sequence of maximal laminations of S that converges to ν_i as $i \to -\infty$ and to ν_{i+1} as $i \to \infty$.

Case Two. Suppose that F_i is a four-holed sphere.o Similarly let $\hat{F}_i$ be the four-punctured sphere obtained by pinching the boundary components of F_i. Then, consider the graph T associated with $\hat{F}_i$ defined by:

- The vertices of T bijectively correspond to the ideal triangulations of $\hat{F}_i$ isomorphic to the triangulation of the boundary of a tetrahedron (Figure 5).
- There is a (unique) edge between two vertices of T if and only if the triangulation corresponding to one vertex is obtained form the other by simultaneous diagonal exchanges of opposite edges of the tetrahedron.

Similarly to Case One, we have

Lemma 4.4. *This graph T is dual to the Farey tessellation.*

Proof. Each vertex of T corresponds, uniquely, to a ideal triangulation of $\hat{F}_i$. Then, for each pair of opposite edges of the triangulation, there is an unique essential loop on $\hat{F}_i$ disjoint from the edges. There are exactly three pairs of opposite edges and their corresponding loops are maximal mutually adjacent vertices of the curve graph of $\hat{F}_i$ (Figure 5). The curve graph of the four-punctures sphere is the Farey graph (for example, see [Sau, §5]). Thus, each ideal triangulation of $\hat{F}_i$ naturally corresponds to a unique triangle of the Farey tessellation, and therefore T is isomorphic to the graph dual to the Farey tessellation. □

The maximal laminations $\hat{\nu}_i$ and $\hat{\nu}_{i+1}$ are distinct endpoints of the graph T at infinity. Then, similarly, there is a unique bi-infinite sequence $(\hat{\nu}_{i,j})_{j\in\mathbb{Z}}$ of adjacent vertices of T connecting $\hat{\nu}_i$ to $\hat{\nu}_{i+1}$. Indeed, there is a vertex of T such that its corresponding triangulation of $\hat{F}_i$ contains two edges disjoint from m_i and two edges disjoint from m_{i+1}; then $\hat{\nu}_i$ is obtained by the (infinite) iteration of the left Dehn twist along m_i, and $\hat{\nu}_{i+1}$ by the iteration of the left Dehn twists along m_{i+1}. Then, similarly, let $\nu_{i,j}$ be the lamination on S satisfying (i) and (ii) in *Case One*.

In either Case One or Two, we have constructed the bi-infinite sequence $\{\nu_{i,j}\}_{j\in\mathbb{Z}}$ of maximal geodesic laminations on S connecting ν_i to ν_{i+1}, that is, $\{\nu_{i,j}\}$ converges to ν_i as $j \to -\infty$ and ν_{i+1} as $j \to \infty$. Then, by Lemma 4.3, for every $i \in \{0, \dots, n-1\}$ and $j \in \mathbb{Z}$, there is a unique ρ-equivariant pleated surface $\beta_{i,j} \colon \tilde{S} \to \mathbb{H}^3$ realizing $\nu_{i,j}$. Then, by Theorem 2.4, the convergence of $\nu_{i,j}$ immediately implies

Proposition 4.5. *For each $i = 0, 1, \dots, n-1$, the pleated surface $\beta_{i,j}$ converges to β_{i+1} as $j \to \infty$ and to β_i as $j \to -\infty$ (in term of the closeness defined in Theorem 2.4).*

Recall that ν_i is obtained by the infinite iteration of the left Dehn twist of a multiloop N_i along M_i. Recalling that $M_i \cap M_{i+1}$ is the set of common loops of M_i and M_{1+i}, we similarly have

Lemma 4.6. *For all $i \in \{1, \ldots, n-1\}$ and $j \in \mathbb{Z}$, the lamination $\nu_{i,j}$ is obtained by the infinite iteration of the left Dean twist of some multiloop on S along $M_i \cap M_{i+1}$.*

Proof. Recall that, for every $i \in \{1, \ldots, n-1\}$ and every positive integer k, we can take the multiloop N_i inducing ν_i such that, if P is a complementary pants of M_i in S, then, for each pair of different boundary components of P, the multiarc $N_i \cap P$ contains exactly k parallel arcs of connecting the components. Thus, for each for every $i \in \{1, \ldots, n-1\}$, we fix such a multiloop N_i given by $k = 3$ in Case One and $k = 2$ for Case Two. Then for each $j \in \mathbb{Z}$, we can construct a multiloop $N_{i,j}$ such that $N_{i,j} = N_i$ in $S \setminus F_i$ and the restriction of $N_{i,j}$ to F_i induces $\hat{\nu}_{i,j}$ on $\hat{F}_i$ by the integration of the left Dehn twist along M_i: See Figure 6 for an example of $N_{i,j}$ with $k = 2$ in Case Two. □

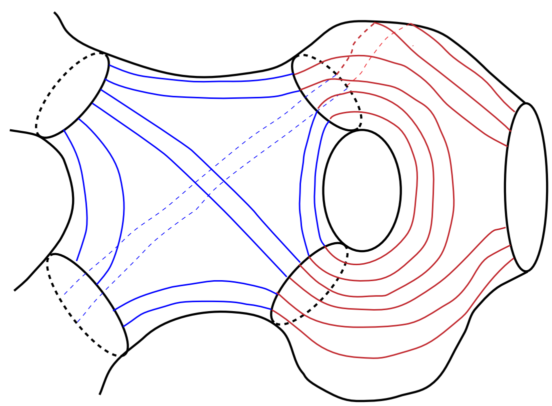


Figure 6.

5. Existence of admissible loops

Given a ρ-equivariant pleated surface $\beta \colon \mathbb{H}^2 \to \mathbb{H}^3$, the following theorem gives a way of finding an admissible loops on a projective structure C with holonomy ρ when its pleated surface is "close" to β.

Theorem 5.1 (c.f. §7 in [Bab15]). *Let $\rho \colon \pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$ be a homomorphism. Let $\beta \colon \mathbb{H}^2 \to \mathbb{H}^3$ be a ρ-equivariant pleated surface realizing $(\sigma, \nu) \in \mathcal{T} \times \mathcal{GL}$. Then, there exists $\delta > 0$ such that, if a projective structure $C \cong (\tau, L)$ with holonomy ρ satisfies $\angle_\sigma(L, \nu) < \delta$ and a loop ℓ on S satisfies $\angle_\tau(\ell, |L|) < \delta$, then there is an admissible loop on C isotopic to ℓ.*

Remark 5.2. *For every $\epsilon > 0$, if $\delta > 0$ is sufficiently small, then σ and τ are ϵ-close. We use $\angle_\sigma$ and $\angle_\tau$ because it seems to be natural choices. However (recall that) the choices of the hyperbolic structures on S are not important for such angles as long as it is fixed or bounded in $\mathcal{T}$.*

Proof. Suppose that Theorem 5.1 fails. Then there exists a sequence of projective structures $C_i \cong (\tau_i, L_i)$ with holonomy ρ such that, letting $\lambda_i = |L_i|$, we have $\angle(\lambda_i, \nu) \to 0$ as $i \to \infty$ and a sequence of geodesic loops ℓ_i on τ_i with $\angle(\ell_i, \lambda_i) \to 0$ as $i \to \infty$ such that there is no admissible loop on C_i homotopic to ℓ_i. Then, by Theorem 2.4, there are marking-preserving bilipschitz maps $\psi_i \colon \sigma \to \tau_i$ converging to an isometry as $i \to \infty$ and β is the limit of the pleated surfaces β_i corresponding to C_i via ψ_i.

Since $\mathcal{GL}$ is compact in the Chabauty topology, by taking a subsequence if necessary, we can in addition assume that λ_i converges to a geodesic lamination λ_∞ on S. Since $\beta_i \to \beta$, thus ν is a sublamination of λ_∞. We can also assume that the sequence of geodesic loops ℓ_i converges to a geodesic lamination ℓ_∞, taking a subsequence if necessary. Then $\angle(\ell_\infty, \lambda_\infty) = 0$. Thus $\ell_\infty \cup \lambda_\infty$ is a geodesic lamination on σ.

There is a constant $K > 0$ depending on $(\sigma, \ell_\infty \cup \lambda_\infty)$, such that for every $\epsilon > 0$, there is an (ϵ, K)-nearly straight traintrack neighborhood T_ϵ of $\ell_\infty \cup \lambda_\infty$ ([Bab15, Lemma 7.10]). Then, if $i \in N$ is sufficiently large, by the above convergences, there is a sufficiently small isotopy of $\psi_i(T_\epsilon)$ into an (ϵ, K)-straight traintrack T_i on τ_i that carries both λ_i and ℓ_i.

For each $i \in \mathbb{N}$, let $\kappa_i \colon C_i \to \tau_i$ denote the collapsing map, and let $\mathcal{L}_i$ be the circular measured lamination on C that descends to L via κ_i. By [Bab15, Proposition 7.12], for sufficiently large i, there is a corresponding traintrack $\mathcal{T}_i$ on C_i diffeomorphic to T_i such that

- each branch of $\mathcal{T}_i$ is supported on a round cylinder on $\hat{\mathbb{C}}$,
- $\kappa_i(|\mathcal{T}_i|) = T_i$, and
- $\kappa_i(\mathcal{T}_i)$ is ϵ-close to T_i, i.e. the κ_i-image of each branch $\mathcal{T}_i$ is ϵ-close to a corresponding branch of T_i in the hausdorff metric on τ_i.

Then $\mathcal{T}_i$ carries the measured lamination $\mathcal{L}_i$. Since T_i carries ℓ_i, thus $\mathcal{T}_i$ carries a corresponding loop m_i so that a small homotopy transforms $\kappa_i | m_i$ into ℓ in $|T_i|$. Since its branches are supported on cylinders, $\mathcal{T}_i$ is foliated by vertical circular arcs (§2.5). Then, since m_i is carried by $\mathcal{T}_i$, we can (further) isotope m_i through loops carried by $\mathcal{T}_i$, so that m_i is transversal to this foliation of $\mathcal{T}_i$. Then by Lemma 2.5, m_i is admissible which is a contradiction. 5.1

Let $C \cong (\tau, L)$ be a projective structure on S with holonomy ρ. Let $\beta \colon \mathbb{H}^2 \to \mathbb{H}^3$ be the ρ-equivariant pleated surface induced by (τ, L). Then, since $\angle(L, L) = 0 < \delta$, Theorem 5.1 immediately implies

Corollary 5.3. *There is $\delta > 0$ such that, if a geodesic loop ℓ on τ satisfies $\angle_\tau(\lambda, \ell) < \delta$, then there is an admissible loop on C isotopic to ℓ.*

In addition

Corollary 5.4. *For every a projective structure $C \cong (\tau, L)$ on S, there exists sufficiently small $\delta > 0$ such that, if a geodesic multiloop M on τ satisfies*

(1) $\angle_\tau(M, L) < \delta$ *and*
(2) $|L|$ *is contained in the δ-neighborhood of M in τ,*

then every loop ℓ on C satisfying $\angle_\tau(\ell, M) < \epsilon$ is isotopic to an admissible loop.

Proof. Let $L = (\lambda, \mu)$ denote the measured lamination, where $\lambda \in \mathcal{GL}$ and $\mu \in \mathcal{TM}(\lambda)$. Then, for every $\epsilon > 0$, there exists $\delta > 0$, such that, if a multiloop M on τ satisfies (1) and (2) for this δ, then $\angle_\tau(\ell, \lambda) < \epsilon$ for every loop ℓ on τ with $\angle_\tau(\ell, M) < \delta$. Thus if $\epsilon > 0$ is sufficiently small, then by Corollary 5.3, ℓ is isotopic to an admissible loop on C. □

5.1. **Admissible loops close to $\nu_{i,j+1}$ on projective surfaces close to $\nu_{i,j}$ in Thurston coordinates.** We carry over the notations from §4. Then

Proposition 5.5. *For all $i \in \{0, 1, \dots, n-1\}$ and $j \in \mathbb{Z}$, there exists $\delta > 0$, such that, if a projective structure $C \cong (\tau, L)$ on S satisfies L and $\nu_{i,j}$ are δ-hausdorff close, for every loop ℓ on C such that the hausdorff distance between ℓ and $\nu_{i,j+1}$ are δ-close, there is an admissible loop on C isotopic to ℓ; indeed such a loop ℓ exists.*

Proof. Let $\tau_{i,j}$ be the hyperbolic structure on S such that $(\tau_{i,j}, \nu_{i,j})$ is realized by the ρ-equivariant pleated surface $\beta_{i,j}$. Given a projective structure $C \cong (\tau, L)$ on S, let $L = (\lambda, \mu)$ denote the measured lamination, where $\lambda \in \mathcal{GL}$ and $\mu \in \mathcal{TM}(\lambda)$. For every $\epsilon > 0$, if $\angle(\nu_{i.j}, \lambda) > 0$ is sufficiently small, then there is a marking preserving ϵ-rough isometry $\psi_{i,j} \colon \tau_{i,j} \to \tau$ given by Theorem 2.4.

Let K be any positive number less than one third of the shortest closed leaf of (the geodesic representative of) $\nu_{i,j+1}$ on $\tau_{i,j}$. Then, for every $\epsilon > 0$, there is an (ϵ, K)-nearly straight traintrack $T_{i,j+1}$ on $\tau_{i,j}$ that carries $\nu_{i,j+1}$ on $\tau_{i,j}$ (Proposition 7.11 in [Bab15]). In addition, it is easy to show that, for every $H > 0$, we can in addition assume, if a branch $T_{i,j+1}$ intersects no closed leaf of $\nu_{i,j}$, then its length is at least H (since K is determined by the lengths of closed leaves of $\nu_{i,j+1}$).

If $\epsilon > 0$ is sufficiently small, each branch of $T_{i,j+1}$ is close to a geodesic segment of length at least K. Recall that $\nu_{i,j}$ and $\nu_{i,j+1}$ differ by a diagonal exchange or two simultaneous diagonal exchanges. Therefore, we can naturally assume that, for every leaf $d \in \nu_{i,j}$ with $d \notin \nu_{i,j+1}$, there is a unique branch R of $T_{i,j+1}$ such that d transversally intersects each horizontal edge of R exactly once (so that $d \cap R$ is a single geodesic segment) and such that no other leaf of $\nu_{i,j}$ intersects R.

We show that, if $H > 0$ is sufficiently large and $\epsilon > 0$ are sufficiently small, then the traintrack $T_{i,j+1}$ (on $\tau_{i,j}$) is *admissible* on C. That is, for every loop ℓ carried by $T_{i,j+1}$, there is an admissible loop on C isotopic to it. The proof is similar to that of Theorem 5.1, except more careful arguments for branches corresponding to leaves d of $\nu_{i,j+1}$ with $d \notin \nu_{i,j}$, where d and L are not (necessarily) close to being parallel.

Case One. First suppose that F_i is a one-holed torus. Let d_j be the leaf of $\nu_{i,j}$ removed by the diagonal exchange of $\nu_{i,j}$ yielding $\nu_{i,j+1}$, and let d_{j+1} be the leaf of $\nu_{i,j+1}$ added by the exchange. Then there is a unique branch R_0 of $T_{i,j+1}$ such that d_j is the only leaf of $\nu_{i,j}$ intersecting R_0. Since d_{j+1} is not a closed leaf, we can assume that R_0 has length at least $H > 0$. For every $\zeta > 0$, if $\delta > 0$ is sufficiently small, then the distance between τ and $\tau_{i,j}$ is less than ζ. Thus, since $T_{i,j+1}$ is (ϵ, K)-straight, we can assume that there is an $(2\epsilon, K)$-straight traintrack T also on τ obtained by a small perturbation of $\psi_{i,j}(T_{i,j+1})$. Let $\mathcal{T} = \cup_{k=0}^{m} \mathcal{R}_k$ be the traintrack on C that descends to T via the collapsing map $\kappa\colon C \to \tau$. We can assume that $\mathcal{R}_0$ is the branch of $\mathcal{T}$ corresponding to R_0. For every branch R of T that does not corresponding to R_0, $\mathsf{R} \cap L$ is a union of geodesic segments connecting the vertical edges of R. Then, since T is $(2\epsilon, K)$-nearly straight with sufficiently small $\epsilon > 0$, as in [Bab15, Proposition 7.12], by a small isotopy of $\mathcal{T}$ on C without changing $|\mathcal{T}|$, we may assume that $\mathcal{R}_k$ is a rectangle supported on a round cylinder for each $k = 1, \dots, m$.

To complete the proof, we show there is an isotopy of T on C such that

- this isotopy is *supported* on R_0 (i.e. it fixes $T \setminus R_0$), and
- after the isotopy, $\mathcal{R}_0$ can be subdivided into three branches which are supported on three (consecutive) round cylinders.

After such an isotopy, the traintrack $\mathcal{T}$ is admissible on C by Lemma 2.5; moreover, by Lemma 4.6, there are many distinct loops carried by $\mathcal{T}$.

Let $\sigma_{i,j}$ be the subsurface of $\tau_{i,j}$ with geodesic boundary which is isotopic to the subsurface F_i of S. Then the boundary component of $\sigma_{i,j}$ is a closed leaf of $\nu_{i,j}$, and the lamination $\nu_{i,j}$ decomposes $\sigma_{i,j}$ into two ideal triangles. Let $\tilde{\sigma}_{i,j}$ be the universal cover of $\sigma_{i,j}$. Then $\tilde{\sigma}_{i,j}$ is a convex subset of $\mathbb{H}^2$ bounded by the geodesics which cover the boundary component of $\sigma_{i,j}$. Then the total lift $\tilde{\nu}_{i,j}$ yields an ideal triangulation of $\tilde{\sigma}_{i,j}$. Let $\tilde{d}_j$ be a lift of d_j to $\tilde{\sigma}_{i,j}$. Then $\tilde{d}_j$ separates adjacent ideal triangles Δ_1 and Δ_2 of $\tilde{\sigma}_{i,j} \setminus \tilde{\nu}_{i,j}$. Then $\Delta_1 \cup \Delta_2 (=: Q_{i,j})$ is a fundamental domain of $\tilde{\sigma}_{i,j}$, and it is an ideal quadrangle. Different vertices of $Q_{i,j}$ are endpoints of different boundary geodesics of $\tilde{\sigma}_{i,j}$, and different boundary geodesics ℓ_1, ℓ_2 of $\tilde{\sigma}_{i,j}$ are preserved by different elements $\gamma_1, \gamma_2 \in \pi_1(S) \setminus \{id\}$. Then, since ρ is purely loxodromic, by Lemma 2.6, the axes of the loxodromic elements $\rho(\gamma_1)$ and $\rho(\gamma_2)$ share no endpoint. Then, since the pleated surface $\beta_{i,j}\colon \mathbb{H}^2 \to \mathbb{H}^3$ is ρ-equivariant, $\beta_{i,j}$ takes different boundary geodesics of $\tilde{\sigma}_{i,j}$ to distict geodesics in $\mathbb{H}^3$ without any common endpoint. In particular, $\beta_{i,j}$ takes the vertices of $Q_{i,j}$ to distinct points on $\hat{\mathbb{C}}$.

Since the branches $\mathcal{R}_1, \cdots, \mathcal{R}_m$ are supported on round cylinders, the vertical edges of $\mathcal{R}_0$ are circular.

Since d_{j+1} is the only leaf of $\nu_{i,j+1}$ intersecting $R_0 \subset \sigma_{i,j}$, there is a unique lift $\tilde{R}_0$ of R_0 contained in $Q_{i,j}$. Then, if $H > 0$ is sufficiently large and $\epsilon > 0$ is sufficiently small, then the opposite vertical edges of $\tilde{R}_0$ are sufficiently far in $Q_{i,j}$. Let $\tilde{\mathcal{R}}_0$ be the branch of $\tilde{\mathcal{T}}$ corresponding to $\tilde{R}_0$. Then the vertical edges of $\tilde{\mathcal{R}}_0$ are contained in a vertical edge of another branch of $\tilde{\mathcal{T}}$, which is supported on a round cylinder.

In the quadrangle $Q_{i,j}$, the diagonals $\tilde{d}_j$ and $\tilde{d}_{j+1}$ intersect in a single point. Let v_1, v_2, v_3, v_4 denote the vertices of $Q_{i,j}$ so that v_1 is the vertex of Δ_1 opposite of the diagonal $d_{i,j}$, v_2 is the vertices of Δ_2 opposite of the diagonal $d_{i,j}$, and v_3, v_4 are the endpoints of $d_{i.j}$.

Then take disjoint round disks D_1, D_2 in $\hat{\mathbb{C}}$ so that D_1 contains $\beta_{i,j}(v_1), c_1$ and $\beta_{i,j}(v_3)$ and D_2 contains $\beta_{i,j}(v_2), c_2$ and $\beta_{i,j}(v_4)$ (see Figure 7). For each $i = 1, 2$, let r_i be the round circle on $\hat{\mathbb{C}}$ bounding D_i.

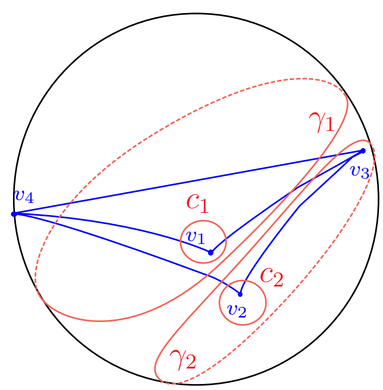


FIGURE 7. The $\beta_{i,j}$-images of the vertices v_i and the round circles separating them.

Claim 5.6. *If $\delta > 0$ is sufficiently small, then there is an isotopy of $\mathcal{T}$ on C which only moves $\mathcal{R}_0$ so that r_1 and r_2 decompose $\mathcal{R}_0$ into three branches supported on three consecutive round cylinders: the round cylinder $\mathcal{A}_1$ bounded by c_1 and r_1, the round cylinder $\mathcal{A}$, and the round cylinder bounded $\mathcal{A}_2$ by c_2 and r_2.*

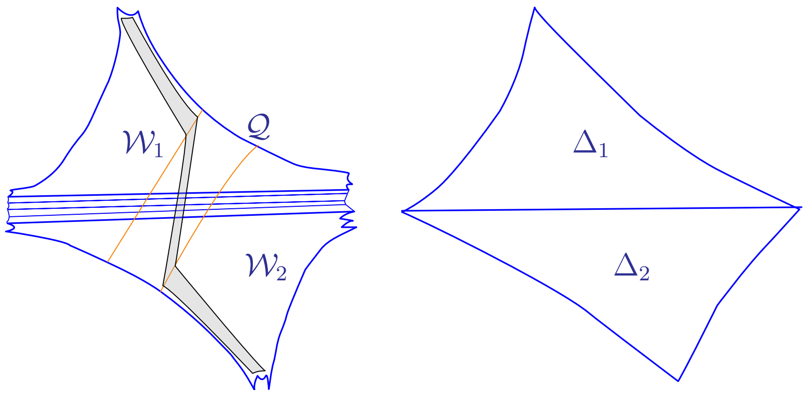


FIGURE 8. Left: $\mathcal{R}_0$ after isotopy, Right: the ideal quadrangle $Q_{i,j}$

Proof. Let P be a stratum of $(\mathbb{H}^2, \tilde{L})$ that intersects $\tilde{R}_0$. Then, as L and $\nu_{i,j}$ are sufficiently Hausdorff-close, P is Hausdorff close to either Δ_1, Δ_2 or $d_{i,j}$ (as a subset of the disk $\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$). Therefore the ideal points $\partial_\infty P$ of P map into $D_1 \cup D_2$ by $\beta_{i,j}$, and there is at least one point of $\partial_\infty P$ maps into D_i for each $i = 1, 2$. Let $\mathcal{P}$ be a stratum of $(\tilde{C}, \tilde{\mathcal{L}})$ that corresponds to P by the collapsing map. Let f be the developing map of C. Then the f-preimage of $r_1 \sqcup r_2$ decomposes $\mathcal{P}$ into three connected regions mapping into the different complementary regions D_1, D_2 and $\mathcal{A}$ of $r_1 \sqcup r_2$.

In particular, the preimage of $\mathcal{A}$ is either an arc or a rectangle supported in the round cylinder $\mathcal{A}$, and it depends on whether $\mathcal{P}$ is one-dimensional or two-dimensional. Let $\mathcal{Q}$ be the union, over all strata $\mathcal{P}$ of $(\tilde{C}, \tilde{L})$ whose $\tilde{\kappa}$-images intersect $\tilde{R}_0$; then $\mathcal{Q}$ is topologically a closed disk. Let U be the f-preimage of $\mathcal{A}$ in $\mathcal{Q}$. Then U is a union of a rectangles and arcs supported $\mathcal{A}$, and U is a projective structure on a rectangle supported on $\mathcal{A}$.

Let $\mathcal{W}_i$ be the stratum of $(\tilde{C}, \tilde{L})$ which map to a stratum of $(\mathbb{H}^2, \tilde{L})$ Hausdorff close to to Δ_i. Then $\mathcal{W}_i$ is bounded by circular arcs, and there is a unique boundary circular arc a_i intersecting both r_i and c_i transversally. Moreover, a_i is transversal to the circular foliation on $\mathcal{A}_i$, and $f^{-1}(\mathcal{A}_i) \cap \mathcal{W}_i$ contains a rectangle $\mathcal{R}_{0,i}$ supported on $\mathcal{A}_i$ such that its vertical edge on c_i matches with the vertical edge of $\mathcal{R}_0$ on c_i.

In addition, there is a rectangle $\mathcal{R}_{0,0}$ in U supported on $\mathcal{A}$ such that its vertical edges are the vertical edge of $\mathcal{R}_{0,1}$ on r_1 and the vertical edge of $\mathcal{R}_{0,2}$ on r_2.

Then it is easy to make an isotopy of $\mathcal{R}_0$ to $\mathcal{R}_{0,0} \cup \mathcal{R}_{0,1} \cup \mathcal{R}_{0,2}$ (Figure 8).

Case Two. Suppose that F_i is a four-holed sphere. Then the proof is similar to Case One, and we leave the proof to the readers. The only difference is that $\nu_{i,j}$ and $\nu_{i,j+1}$ differ by two diagonal exchanges (intend of one). We accordingly deal with two branches of the traintracks more carefully than the other branches. 5.5

6. Sequence of grafts traveling in $\mathcal{GL}$.

In this section, we prove our main theorem (Theorem 1.2) modulo Corollary 7.2. This corollary will be independently proved in Part 2. Recall that, in §4, starting with two projective structures on S sharing purely loxodromic holonomy $\rho\colon \pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$, we have constructed finitely many geodesic laminations $\nu_0, \nu_1, \ldots, \nu_n$ on S and, for each $0 \leq i \leq n$, a family of geodesic laminations $\nu_{i,j}$ ($j \in \mathbb{Z}$) on S "connecting" ν_i to ν_{i+1}. By Proposition 5.5, given a projective structure whose measured lamination is sufficiently Hausdorff close to $\nu_{i,j}$, one can find an admissible loop which is sufficiently close to $\nu_{i,j+1}$; then, by grafting along the admissible loop sufficiently many time, we obtain a projective structure whose measured lamination is Hausdorff close to $\nu_{i,j+1}$. Namely

Proposition 6.1. *Given $0 \leq i < n$ and $j \in \mathbb{Z}$, there exists $\delta_{i,j} > 0$ such that, if a projective structure $C \cong (\tau, L)$ with purely loxodromic holonomy $\rho\colon \pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$ satisfies $\angle(\lambda, \nu_{i,j}) < \delta_{i,j}$, then, for every $\epsilon_{i,j} > 0$, there is an admissible loop ℓ on C such that, letting $\mathrm{Gr}_\ell^k(C) \cong (\tau_k, L_k)$ in Thurston coordinates, we have $\angle(L_k, \nu_{i,j+1}) < \epsilon_{i,j}$ for sufficiently large k.*

Next, using grafting obtained by Proposition 6.1, we can transform a projective structure close to ν_i to a projective structure close to ν_{i+1}:

Proposition 6.2. *For $i = 0, 1, \cdots, n-1$, there exists $\delta_i > 0$, such that if $C \cong (\tau, L)$ is a projective structure in $\mathcal{P}_\rho$ with $\angle(\lambda, \nu_i) < \delta_i$, then, for every $\epsilon > 0$, there is a finite composition of grafts starting from C,*

$$C = C_0 \xrightarrow{Gr_{\ell_1}} C_1 \xrightarrow{Gr_{\ell_2}} C_2 \to \ldots \xrightarrow{Gr_{\ell_k}} C_k$$

so that the last projective structure $C_k \cong (\tau_k, L_k)$ satisfies $\angle(L_k, \nu_{i+1}) < \epsilon$.

Proof. Let $C \cong (\tau, L) \in \mathcal{P}_\rho$ be such that $\angle(L, \nu_i) > 0$ is sufficiently small. Recall that $\nu_{i,j} \to \nu_i$ as $j \to -\infty$ in the Chabauty topology. Thus, if $j \in \mathbb{Z}$ is sufficiently small and ℓ is a loop on C (whose geodesic representative) is sufficiently close to $\nu_{i,j}$ on $\tau_{i,j}$, then $\angle(\ell, \nu_i) > 0$ is also sufficiently small. Then, by Theorem 5.1, a loop ℓ on C is admissible up to an isotopy. Set $\mathrm{Gr}_\ell^h(C) \cong (\tau_h, L_h)$ for each $h \in \mathbb{Z}_{>0}$. Therefore, for every $\delta > 0$, if $j \in \mathbb{Z}$ is sufficiently small and ℓ is sufficiently close to $\nu_{i,j}$, then by Corollary 7.2, we have $\angle(L_h, \nu_{i,j}) < \delta$ for sufficiently large $h \in \mathbb{N}$.

Since $\nu_{i,j} \to \nu_{i+1}$ as $j \to \infty$, for every $\epsilon > 0$, we can pick sufficiently large $j_\epsilon \in \mathbb{N}$ such that $\angle(\nu_{i+1}, \nu_{i,j_\epsilon}) < \epsilon$. Then let $\delta_{i,j_\epsilon} > 0$ be such that, if a loop ℓ satisfies $\angle(\ell, \nu_{i,j_\epsilon}) < \delta_{i,j_\epsilon}$, then $\angle(\ell, \nu_{i+1}) < \epsilon$.

For every $j \in \mathbb{Z}$ with $j < j_\epsilon$, inductively define $\delta_{i,j} > 0$ inductively so that $\delta_{i,j}$ is the constant obtained by applying Proposition 6.1 to $\delta_{i,j+1}$. Thus, for $j < j_\epsilon$, if there is a projective structure $C' \cong (\tau', L')$ satisfies that $\angle(\nu_{i,j}, L') < \delta_{i,j}$, then we can inductively construct a composition of grafts starting from C',

$$C' = C_0 \xrightarrow{Gr_{\ell_1}} C_1 \xrightarrow{Gr_{\ell_2}} C_2 \to \ldots \xrightarrow{Gr_{\ell_k}} C_k = (\tau_k, L_k),$$

such that $\angle(L_k, \nu_{i+1}) < \epsilon$. In addition, it follows from the first paragraph in this proof that, if $j \in \mathbb{Z}$ is sufficiently small, then such a projective structure C' can be obtained by a finite iteration of grafts of C along a fixed admissible loop. This completes the proof. □

Recall that we started with arbitrary projective structures $C_\sharp \cong (\tau_\sharp, L_\sharp)$ and $C_\flat \cong (\tau_\flat, L_\flat)$ on S sharing purely loxodromic holonomy $\rho\colon \pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$ and that $L_\sharp = (\lambda_\sharp, \mu_\sharp)$ and $L_\flat = (\lambda_\flat, \mu_\flat)$ are measured laminations. Then

Proposition 6.3. *For every $\epsilon > 0$, there exists a finite composition of grafts starting from $C_\sharp$*

$$C_\sharp = C_0 \xrightarrow{Gr_{\ell_1}} C_1 \xrightarrow{Gr_{\ell_2}} C_2 \to \ldots \to C_n$$

such that the last projective structure $C_n \cong (\tau_n, L_n)$ satisfies $\angle(\lambda_\flat, L_n) < \epsilon$.

Proof of Proposition 6.3. Recall that the multiloops $M_\sharp$ and $M_\flat$ are taken to be sufficiently close to $\lambda_\sharp$ and $\lambda_\flat$, respectively. In other words, given $\zeta > 0$, we can assume that $\angle(M_\flat, \lambda_\flat) < \zeta$, $\angle(M_\sharp, \lambda_\sharp) < \zeta$ and that $\lambda_\flat$ and $\lambda_\sharp$ are contained in the ζ-neighborhoods of $M_\flat$ and $M_\sharp$, respectively. Since ν_n contains $M_n = M_\flat$, for every $\epsilon > 0$, if $\zeta > 0$ is sufficiently small, there exists $\delta_n > 0$ such that if a geodesic lamination λ on S satisfies $\angle(\lambda, \nu_n) < \delta_n$, then $\angle(\lambda, \lambda_\flat) < \epsilon$. For $i = 0, 1, \ldots, n-1$, let $\delta_i > 0$ be the constant given by Proposition 6.2.

Claim 6.4. *If $\zeta > 0$ is sufficiently small, then, for loops ℓ sufficiently close to ν_0 with the Hausdorff metric on $\tau_\sharp$,*

- *ℓ is admissible on $C_\sharp$ (up to an isotopy), and*
- *for $k \in \mathbb{N}$ letting $\mathrm{Gr}_\ell^k(C_\sharp) \cong (\tau_k, L_k)$ in Thurston coordinates, we have $\angle(L_k, \nu_0) < \delta_0$ for sufficiently large k.*

Proof. Since ℓ is sufficiently (Hausdorff-)close to ν_0, then $\angle(\nu_0, \ell)$ is sufficiently small. Since $\angle(\lambda_\sharp, \nu_0) < \zeta$ is sufficiently small and ν_0 is maximal, $\angle(\lambda_\sharp, \ell)$ is also sufficiently small. Thus, by Corollary 5.3, ℓ is admissible.

By Corollary 7.2, $\angle(L_k, \ell) \to 0$ as $k \to \infty$. Thus, since $\angle(\ell, \nu_0)$ is sufficiently small, $\angle(L_k, \nu_0) < \delta_0$ for sufficiently large i. □

Let C_{k_0} be $Gr_\ell^{k_0}(C) \cong (\tau_{k_0}, L_{k_0})$, given by Claim 6.4, with a sufficiently large $k_0 \in \mathbb{N}$. Then, since $\angle(L_{k_0}, \nu_0) < \delta_0$, by Proposition 6.2, there is a composition of grafts from C_{k_0}

$$C_{k_0} \to C_1 \to C_2 \to \ldots \to C_{k_1} \cong (\tau_{k_1}, L_{k_1}),$$

such that, $\angle(\nu_1, L_{k_1}) < \delta_1$. By inductively applying Proposition 6.2, for each $i = 0, 1, \cdots, n$, we can extend this composition of grafts to

$$C_{k_0} \to \cdots \to C_{k_1} \to \cdots \to C_{k_2} \to \cdots \to C_{k_i}$$

so that the last projective structure $C_{k_i} \cong (\tau_{k_i}, L_{k_i})$ satisfies $\angle(\nu_i, L_{k_i}) < \delta_i$. In particular, when $i = n$, we have $\angle(\nu_n, L_{k_n}) < \delta_n$. Hence $\angle(\lambda_\flat, L_{k_n}) < \epsilon$. 6.3

Proof of Theorem 1.2. Let $\delta > 0$ be the constant obtained by applying Theorem 2.8 to $C_\flat \cong (\tau_\flat, L_\flat)$. Then, by Proposition 6.3, there is a composition of grafts along loops,

$$C_\sharp = C_0 \xrightarrow{Gr_{\ell_1}} C_1 \xrightarrow{Gr_{\ell_2}} C_2 \to \ldots \to C_n,$$

such that, letting $C_n \cong (\tau_n, L_n)$, we have $\angle(L_n, L_\flat) < \delta$. Then we can graft C_n and $C_\sharp$ along multiloops to a common projective structure. Hence there are admissible loops M_n on C_n and $M_\flat$ on $C_\flat$ such that $\mathrm{Gr}_{M_n}(C_n) = \mathrm{Gr}_{M_\flat}(C_\flat)$. Since the grafting Gr_{M_n} of C_n is naturally a composition of grafts along loops of M_n, which completes the proof.

Part 2. Iteration of grafting along a loop

7. Limit in Thurston coordinates

We prove

Theorem 7.1. *Let ℓ be an admissible loop on a projective surface C. Let $C \cong (\tau, L) \in \mathcal{T} \times \mathcal{ML}$, and let $L = (\lambda, \mu)$ with $\lambda \in \mathcal{GL}$ and $\mu \in \mathcal{TM}(\lambda)$. For each $i \in \mathbb{N}$, let $C_i = Gr_\ell^i(C)$, the $2\pi i$-graft of C. Similarly, let $C_i \cong (\tau_i, L_i)$ and $L_i = (\lambda_i, \mu_i)$. Let $\beta_i \colon \mathbb{H}^2 \to \mathbb{H}^3$ be the pleated surface corresponding to C_i. Then*

- (i) *τ_i converges to a hyperbolic surface $\tau_\infty \in \mathcal{T}$ as $i \to \infty$.*
- (ii) *β_i converges to a ρ-equivariant pleated surface realizing $(\tau_\infty, \lambda_\infty)$ for some $\lambda_\infty \in \mathcal{GL}$ containing ℓ.*
- (iii) *L_i converges to an (heavy) measured lamination L_∞ supported on λ_∞ such that ℓ is the only leaf with weight infinity.*

Theorem 7.1 (i) (ii) immediately imply

Corollary 7.2. *$\angle(L_i, \ell)$ converges to 0 as $i \to \infty$.*

Arguments in Part 2 are independent of the rest of paper, and this corollary is an important ingredient for the proof of Theorem 1.2, proved in Part 1.

8. Limit of the complement of the admissible loop

Let C be a projective structure on S with holonomy $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$, and let ℓ be an admissible loop on C.

Definition 8.1. *Let $\tilde\ell$ be a lift of ℓ to the universal cover $\tilde C$, and γ_ℓ be the element of $\pi_1(S)$ that represents ℓ and preserves $\tilde\ell$. Normalize the developing map f of C by an element of $\mathrm{PSL}(2,\mathbb{C})$ so that the loxodromic $\rho(\gamma_\ell) \in \mathrm{PSL}(2,\mathbb{C})$ fixes 0 and ∞ in $\hat{\mathbb{C}}$. Pick a parametrization $\tilde\ell\colon \mathbb{R} \to \tilde C$ of $\tilde\ell$. Then its f-image can be written in polar coordinates $(e^{r(t)}, \theta(t))$, $t \in \mathbb{R}$, so that*

$$f \circ \tilde\ell(t) = \exp[r(t) + i\theta(t)]$$

where $r\colon \mathbb{R} \to \mathbb{R}_{>0}$ and $\theta\colon \mathbb{R} \to \mathbb{R}$ are continuous functions. Then we say that ℓ spirals if θ is an unbounded function. Otherwise it is called roughly circular.

Indeed, an admissible loop ℓ is roughly circular if and only if there is a homotopy (or an isotopy) between $f \circ \tilde\ell$ and a circular arc on $\hat{\mathbb{C}}$ connecting the fixed points of $\rho(\gamma_\ell)$ such that the homotopy is equivariant under the restriction of $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ to the infinite cyclic group generated by γ_ℓ.

In the setting of Theorem 7.1, since $C_i = \mathrm{Gr}^i_\ell(C)$, then $C \setminus \ell$ isomorphically embeds into C_i so that the complement of $C \setminus \ell$ in C_i is the cylinder inserted by the grafting Gr^i_ℓ. This cylinder is naturally cut, along parallel isomorphic copies of ℓ, into i isomorphic copies of a grafting cylinder (of "length 2π"). Let M_i be the union of $i+1$ parallel copies of ℓ on C_i that decomposes C into the i cylinders and $C \setminus \ell$. Then let ℓ_i be, if $i+1$ is odd, the middle loop of M_i and, if i is odd, a boundary component of the middle grafting cylinder. Let $\mathcal{C}_i = C_i \setminus \ell_i$. Then, there is a natural isomorphic embedding of $\mathcal{C}_i$ into $\mathcal{C}_{i+1}$. Therefore, we let

$$\mathcal{C}_\infty = \lim_{i\to\infty} (C_i \setminus \ell_i).$$

Then $\mathcal{C}_\infty$ is a projective structure on $S \setminus \ell$, and its holonomy is the restriction of ρ to $\pi_1(S \setminus \ell)$. (Equivalently $\mathcal{C}_\infty$ is obtained by attaching a half-infinite grafting cylinder along each boundary component of $C \setminus \ell$.)

Recall that the holonomy ρ of C is non-elementary [GKM00]. Noting that $C \setminus \ell$ has either one or two connected components, we have

Lemma 8.2. *Let P be a connected component of $C \setminus \ell$. Then, the restriction of ρ to $\pi_1(P)$ is non-elementary.*

Proof. Suppose, to the contrary, that $\rho(\pi_1(P))$ is elementary. Then, since $\rho(\ell)$ is loxodromic, the limit set Λ of $\rho(\pi_1(P))$ contains only the two fixed points of the loxodromic $\rho(\ell)$. Then the domain of discontinuity of $\rho(\pi_1(P))$ is identified with $\mathbb{C} \setminus \{0\}$, and it admits a complete Euclidean metric given by the exponential map $\exp\colon \mathbb{C}(\cong \mathbb{R}^2) \to \mathbb{C} \setminus \{0\}$, so that the $\rho(\pi_1(P))$-action on the domain is isometric.

Let $\mathcal{C}_P$ denote the connected component of $\mathcal{C}_\infty$ corresponding to P. Consider the inverse image of Λ under $dev(\mathcal{C}_P)$. Since Λ is in particular a discrete subset of $\hat{\mathbb{C}}$, the $dev(\mathcal{C}_P)$-inverse image of Λ is a discrete subset preserved by $\pi_1(P)$, and it descends to a set of finitely many points on $\mathcal{C}_P$. By pulling back the Euclidean metric via the developing map, $\mathcal{C}_P$ minus the finitely many points carries a complete Euclidean metric. This is a contradiction since the Euler characteristic of P is negative. □

Let $\tilde{\mathcal{C}}_i$ and $\tilde{\mathcal{C}}_\infty$ denote the universal covers of $\mathcal{C}_i$ and $\mathcal{C}_\infty$, respectively. Then, by Lemma 8.2, the holonomy of every connected component of $\mathcal{C}_\infty$ is non-elementary. Thus, by Corollary 3.3, let $\mathcal{C}_\infty \cong (\sigma_\infty, N_\infty)$ be the Thurston coordinates, where σ_∞ is a convex hyperbolic surface with geodesic boundary whose interior is homeomorphic to $S \setminus \ell$ and N_∞ is a (possibly heavy) measured geodesic lamination on σ_∞. (Recall from Theorem 3.1, each boundary component of σ_∞ may be either open or closed, i.e. entirely contained in σ_∞ or not contained in σ_∞ at all; moreover a closed boundary is must be a leaf of the Thurston lamination and its weight is ∞, which corresponds to a half infinite cylinder.) We have

Proposition 8.3. *The boundary of σ_∞ is the union of two geodesic loops corresponding the boundary circles of $S \setminus \ell$. The lengths of boundary components are the translation length of $\rho(\ell)$. Furthermore*

(i) *Suppose that ℓ is roughly circular. Then σ_∞ contains both boundary geodesic loops (i.e. closed boundary), and they are isolated leaves of N_∞ with weight infinity.*

(ii) *Suppose that ℓ spirals. Then σ_∞ contains no boundary geodesic (i.e. open boundary). Leaves of the lamination N_∞ spiral towards each boundary component of σ_∞ in the same direction with respect to the orientation on the boundary components of τ (induced by the orientation of S); see Figure 9. In particular N_∞ contains no heavy leaves.*

Remark 8.4.

- *In (ii), the metric completion of σ_∞ is the union of σ_∞ and the boundary loops. Then N_i naturally extends to yet a heavy measured lamination on the completion, so that both boundary loops are leaves of weight infinity.*
- *Let $\zeta_\infty\colon \tilde{\sigma}_\infty \to \mathbb{H}^3$ be the pleated surface associate with $\mathcal{C}_\infty$, where $\tilde{\sigma}_\infty$ is the universal cover of σ_∞ (note that, if ℓ is separating, $\tilde{\sigma}_\infty$ has two connected components). Then, let m be a boundary geodesic of $\tilde{\sigma}_\infty$ and let γ_m be a non-trivial deck transformation preserving m. Then, in both Case (i) and (ii), ζ_∞*

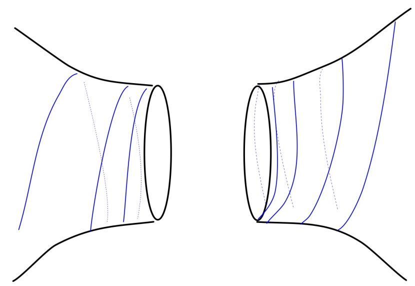


FIGURE 9. Geodesics spiraling to the left towards both boundary components (when you stand on the surface facing toward boundary).

isometrically takes m onto the axis of the loxodromic element $\rho(\gamma_m)$.

Proof of Proposition 8.3. Case (i). Suppose that ℓ is roughly circular. Let $\mathcal{A}$ be a component of $\mathcal{C}_\infty \setminus \mathcal{C}_0$, which is a half infinite (grafting) cylinder attached a boundary component of $\mathcal{C}_0$. As ℓ is roughly circular, the developing map of $\mathcal{A}$ is the restriction of the exponential map $\exp\colon \mathbb{C} \to \mathbb{C}^*$ to a region $\tilde{\mathcal{A}}$ bounded by a property embedded curve in $\mathbb{C}$ whose imaginary coordinate is bounded form below and above. Then we can in addition assume that $\pi_1(\mathcal{A}) \cong \mathbb{Z}$ acts on $\tilde{\mathcal{A}}$ by translations by integers (conjugating by an element of $\mathrm{PSL}(2, \mathbb{C})$). Then we can find a circular loop α in $\tilde{\mathcal{A}}$, which lifts to a horizontal line in $\mathbb{R}$. Then α bounds a (smaller) half-infinite cylinder $\mathcal{A}'$ isotopic to $\mathcal{A}$, and $\mathcal{A}'$ is uniquely foliated by circular loops. Let ℓ' be such a circular loop in $\mathcal{A}'$.

Let $\mathcal{N}_\infty$ be the circular measured lamination on $\tilde{\mathcal{C}}_\infty$, which descends to N_∞. In $\tilde{\mathcal{A}}$, one can easily find two horizontal parallel lines of distance π apart $\tilde{\mathcal{A}}$. Then the regions bounded by such two lines is a maximal ball, and its core is the horizontal line in the middle of them. The core descends to a closed leaf ℓ' of $\mathcal{N}_\infty$. In addition we can assume that $\mathcal{A}'$ is a maximal cylinder in $\mathcal{C}_\infty$ that is isotopic to $\mathcal{A}$ such that $\mathcal{A}'$ is foliated by closed leaves of $\mathcal{N}_\infty$. (Then $\mathcal{A}'$ may not be contained in $\mathcal{A}$ anymore.) Then $\mathcal{A}'$ is still half-infinite and the total transversal measure of $\mathcal{N}_\infty$ on $\mathcal{A}'$ is infinite. Let $\iota_\infty\colon \mathcal{C}_\infty \to \sigma_\infty$ be the collapsing map. The ι_∞ takes $\mathcal{A}'$ to a boundary geodesic loop of σ_∞ of infinite weight. Conversely the inverse-image of the geodesic loop is $\mathcal{A}'$ since $\mathcal{A}'$ is maximal. Since $\mathcal{N}_\infty$ has infinite measure on $\mathcal{A}'$, the boundary component is a leaf of N_∞ with infinite weight. Then, since the transversal measure of $\mathcal{N}_\infty$ is locally finite, no leaf of $\mathcal{N}_\infty$ spirals towards the boundary loop of $\mathcal{A}'$. Thus the boundary geodesic of σ_∞ is an isolated leaf of N_∞.

(ii). Suppose that ℓ spirals. Since ℓ is admissible, the restriction of $dev(C)$ to a lift $\tilde{\ell}$ to $\tilde{S}$ is a simple curve on $\hat{\mathbb{C}}$, and we can assume that it connects 0 and ∞. Then, as in Definition 8.1, it lifts to a curve

$$\begin{array}{rcl} \mathbf{l}\colon \mathbb{R} & \to & \mathbb{R}^2 \\ t & \mapsto & (\theta(t), r(t)) \end{array}$$

through $\exp\colon \mathbb{C} \cong \mathbb{R}^2 \to \mathbb{C} \setminus \{0\}$, where $\theta\colon \mathbb{R} \to \mathbb{R}$ and $r\colon \mathbb{R} \to \mathbb{R}$ are continuous functions, so that $\exp(r(t) + i\theta(t))$ is the curve $dev(C)|\tilde{\ell}$. Since ℓ is admissible, $\mathbf{l}$ is a simple curve. Since $dev(C)|\tilde{\ell}$ is preserved by the loxodromic $\rho(\ell)$, accordingly $\mathbf{l}$ is preserved by a nontrivial translation of $\mathbb{R}^2$ (along a geodesic). Since $\rho(\ell)$ is loxodromic and ℓ is spiraling, the axis of this translation intersects both θ and r-axes transversally. Thus one connected component of $\mathbb{R}^2 \setminus \mathbf{l}$ lies *above* $\mathbf{l}$, i.e. it contains $\{0\} \times [R, \infty)$ for sufficiently large $R > 0$, and the other component lies *below*. Let c be a boundary component of $\mathcal{C}_0$, which is isomorphic to ℓ. Let $\tilde{c}$ be a lift of c to the universal cover $\tilde{\mathcal{C}}_0$, so that $\tilde{c}$ is isomorphic to $\tilde{\ell}$. Then we can assume that $dev(\mathcal{C}_0)$ takes the small neighborhood of $\tilde{c}$ into the region below $\mathbf{l}$ in $\mathbb{R}^2$, if necessary, by exchanging 0 and $\infty \in \hat{\mathbb{C}}$ by an element of $\mathrm{PSL}(2, \mathbb{C})$.

Cearly $\tilde{\mathcal{C}}_0$ is isomorphically embedded in $\tilde{\mathcal{C}}_\infty$. Then in particular $\tilde{c}$ is embedded in $\tilde{\mathcal{C}}_\infty$ and we can regard the endpoints of $\tilde{c}$ as distinct ideal points of both $\tilde{\mathcal{C}}_0$ and $\tilde{\mathcal{C}}_\infty$. Then let $\mathtt{p}^+$ and $\mathtt{p}^-$ be the ideal points corresponding to ∞ and 0, respectively, via $\mathbf{l}$.

Lemma 8.5. *There is a maximal ball B in $\tilde{\mathcal{C}}_\infty$ such that $\mathtt{p}^+$ is an ideal point of B.*

Proof. Let A be the connected component of $\mathcal{C}_\infty \setminus \mathcal{C}_0$ bounded by c, so that A is a half-infinite grafting cylinder. Let $\tilde{A}$ be the corresponding connected component of $\tilde{\mathcal{C}}_\infty \setminus \tilde{\mathcal{C}}_0$ bounded by $\tilde{c}$, so that $\tilde{A}$ covers A. Then $dev(A)$ lifts, through exp, to an embedding onto the component of $\mathbb{R}^2 \setminus \mathbf{l}$ above $\mathbf{l}$. A *round ball* on $\hat{\mathbb{C}}$ is a ball bounded by a round circle. Thus we can find a round ball B contained in $\tilde{A}$ such that $\mathtt{p}^+$ is an ideal point of B ; see Figure 10. Since $\tilde{A} \subset \tilde{\mathcal{C}}_\infty$, there is a desired maximal ball in $\tilde{\mathcal{C}}_\infty$ that contains B. □

Lemma 8.6. *There is no maximal ball B in $\tilde{\mathcal{C}}_\infty$ such that $\mathtt{p}^+$ and $\mathtt{p}^-$ are both its ideal points.*

Proof. Suppose, to the contrary, that there is a maximal ball B in $\tilde{\mathcal{C}}_\infty$ such that $\partial_\infty B$ contains both $\mathtt{p}^+$ and $\mathtt{p}^-$. Then its core $\mathrm{Core}(B)$ contains a circular arc α connecting $\mathtt{p}^+$ and $\mathtt{p}^-$. The surface $\tilde{C}_\infty$ is of hyperbolic type (topologically). Then since α and $\tilde{c}$ share their ideal end points, they must project to isotopic loops on C_∞. Thus α covers a circular loop on $\mathcal{C}$ isotopic to c. This contradicts that c spirals. □

Next we show that σ_∞ has an open boundary component that is a closed geodesic homotopic to c. Let B be a maximal ball of $\tilde{\mathcal{C}}_\infty$ given

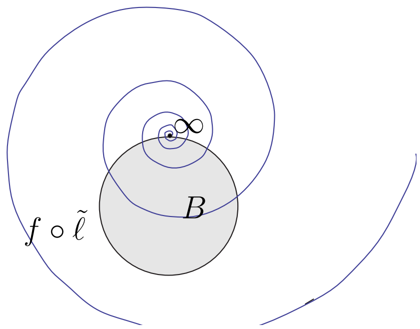


FIGURE 10.

by Lemma 8.5 so that $\partial_\infty B \ni \mathtt{p}^+$. Then the collapsing map $\tilde{\iota}_\infty \colon \tilde{\mathcal{C}}_\infty \to \tilde{\sigma}_\infty$ projects Core(B) onto a convex subset X of $\tilde{\sigma}_\infty$. Moreover $\tilde{\iota}_\infty$ continuously extends to a map from the ideal boundary of $\tilde{\mathcal{C}}_\infty$ to the ideal boundary of $\tilde{\sigma}_\infty$. Since c is an essential loop, there are points p^+ and p^- on the ideal boundary of $\tilde{\sigma}_\infty$ corresponding to $\mathtt{p}^+$ and $\mathtt{p}^-$, respectively. Then, by Lemma 8.6, $\partial_\infty X\, (\subset \partial_\infty \mathbb{H}^2)$ contains p^+ but not p^-.

By regarding $\tilde{\sigma}_\infty$ as a convex subset of $\mathbb{H}^2$, there is a unique geodesic g connecting p^+ and p^- in $\mathbb{H}^2$. Let γ_c be a deck transformation corresponding to c so that γ_c preserves g. Then we see that $(\gamma_c)^j X$ converges to g uniformly on compacts as $j \to \infty$, if necessary, changing γ_c to its inverse (Figure11). By Lemma 8.6, the geodesic g is not contained in $\tilde{\sigma}_\infty$. Thus g descends to a desired open boundary component of σ_∞. Let $\zeta_\infty \colon \tilde{\sigma}_\infty \to \mathbb{H}^3$ be the pleated surface for C_∞. Then since ζ_∞ is equivariant and 1-Lipschitz, the continuous extension of ζ_∞ takes g isometrically onto the axis of the loxodromic $\rho(c)$. Therefore the translation length of $\rho(c)$ is the length of the boundary component of σ_∞ homotopic to c. In addition the convergence $(\gamma_c)^j X \to g$ implies that leaves of N_∞ spiral towards c.

Recall that a small neighborhood of the boundary component c of C_0 in $\mathcal{A}$ develops above $\mathbf{l}$, which is used to distinguish p^+ and p^-. Letting c' be the other boundary component of C_0, then $dev(C_0)$ takes a small neighborhood of c to the region below $\mathbf{l}$ (with respect to $c = c'$ on C). Then it follows that the labels p^+ and p^- are opposite for c' and c. However, since the normal directions of c and c' of C_0 are the opposite on C, leaves of N_∞ spiral towards both boundary components in the same direction with respect to the normal directions of the boundary components. 8.3

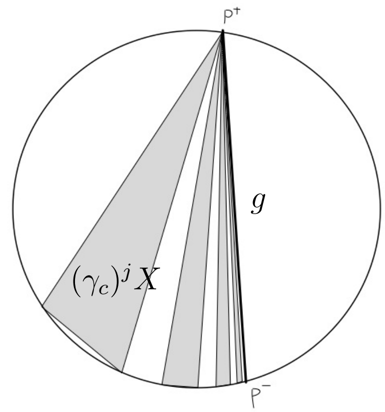


FIGURE 11.

9. Identification of boundary components of the limit structure

We have obtained the Thurston coordinates $(\sigma_\infty, N_\infty)$ of $\mathcal{C}_\infty$ (Proposition 8.3). In particular the boundary components of σ_∞ are two closed geodesic whose lengths are equal to the translation length of the loxodromic $\rho(\ell)$ corresponding to the admissible loop ℓ on C. Thus we can identify the boundary components of σ_∞ and obtain a hyperbolic structure τ_∞ on S and a heavy measured lamination L_∞ with a unique heavy leaf homotopic to ℓ. Although this (isometric) identification is a priori unique up to sharing along the heavy leaf, in fact

Lemma 9.1. *There is a unique identification of the boundary components of σ_∞ so that the resulting pair (τ_∞, L_∞) is realized by a ρ-equivariant pleated surface that coincides, in the complement of ℓ, with the pleated surface ι_∞ corresponding to $\mathcal{C}_\infty$.*

Proof. Let $\tilde{\ell}$ be a lift of ℓ to $\tilde{S}$. Let $\hat{\ell}$ be the total lift of ℓ to $\tilde{S}$. Let γ_ℓ be the element of $\pi_1(S)$ that corresponds to ℓ and preserves $\tilde{\ell}$. Let P_1 and P_2 be the connected components of $\tilde{S} \setminus \hat{\ell}$ that are adjacent along $\tilde{\ell}$.

For each $k = 1, 2$, let $\iota_{P_k} \colon X_k \to \mathbb{H}^3$ denote the pleated surface for the connected component of $\mathcal{C}_\infty$ corresponding to P_k so that ι_{P_k} is equivariant under the restriction of ρ to the subgroup $\pi_1(S)$ that preserves P_k. Let g_k be the boundary geodesic of X_k corresponding to $\tilde{\ell}$. Then ι_{P_k} isometrically takes g_k to the axis of $\rho(\gamma_\ell)$. Thus there is a unique identification of g_1 and g_2 so that ι_{P_1} and ι_{P_2} continuously extends the union $X_1 \cup X_2$ given by the identification. Then, by quotienting out $X_1 \cup X_2$ by the infinite cyclic group generated by γ_ℓ, the identification of g_1 and g_2 descends to a unique isometry between the boundary components of σ_∞.

Since $\mathcal{C}_\infty$ is obtained by grafting C and $dev(C)$ is ρ-equivariant, the identification of the boundary components of σ_∞ is independent of the choice of the lift $\tilde{\ell}$. Then, applying the identification of boundary components for all adjacent components of $\tilde{S} \setminus \hat{\ell}$, we obtain a ρ-equivariant pleated surface from $\mathbb{H}^2$ to $\mathbb{H}^3$ realizing (τ_∞, L_∞). □

10. Convergence of canonical neighborhoods under grafting

Recall, from §8, that $C_i = \mathrm{Gr}_\ell^i(C)$, where ℓ is an admissible loop on a projective structure C on S, and $\mathcal{C}_i = C_i \setminus \ell_i$ where ℓ_i is an isomorphic copy of ℓ which sits in the "roughly middle"' of the cylinder inserted by Gr_ℓ^i. For each $i \in \mathbb{N}$, let $e_i \colon \mathcal{C}_i \to \mathcal{C}_\infty$ denote the canonical isomorphic embedding. Then the embeddings e_i give an exhaustion of $\mathcal{C}_\infty$,

$$\mathcal{C}_1 \subset \mathcal{C}_2 \subset \mathcal{C}_3 \subset \cdots (\subset \mathcal{C}_\infty).$$

Let p be a point on $\mathcal{C}_\infty$, and let $\tilde{p}$ be a lift of p to the universal cover $\tilde{\mathcal{C}}_\infty$. Let $U_\infty(\tilde{p}) \subset \tilde{\mathcal{C}}_\infty$ be the canonical neighborhood (§3.3) of $\tilde{p}$.

For each $i \in \mathbb{N}$, let $\tilde{C}_i$ be the universal cover of C_i. If ℓ is non-separating, then let $\hat{C}_i$ be the quotient of $\tilde{C}_i$ by $\pi_1(S \setminus \ell)$. If ℓ is separating, for each connected component F of $S \setminus \ell$, quotient $\tilde{C}_i$ by $\pi_1(F)$, and let $\hat{C}_i$ be the disjoint union of both quotients. Then $\mathcal{C}_i$ is isomorphically embedded in $\hat{C}_i$.

$$\begin{array}{ccccc} \hat{C}_i & & & & \\ \downarrow & \nwarrow & & & \\ C_i & \hookleftarrow & \mathcal{C}_i & \xhookrightarrow{e_i} & \mathcal{C}_\infty \end{array}$$

For sufficiently large i, we have $p \in \mathcal{C}_i \subset C_i$. Accordingly $\tilde{p} \in \tilde{\mathcal{C}}_i \subset \tilde{C}_i$. Let U_i and $\mathcal{U}_\infty$ be the canonical neighborhoods of the point $\tilde{p}$ in $\tilde{C}_i$ and $\tilde{\mathcal{C}}_\infty$, respectively. Fix any metric on $\hat{\mathbb{C}}$ inducing the (standard) topology of $\hat{\mathbb{C}}$ (e.g. a spherical metric). Note that canonical neighborhoods embed into $\hat{\mathbb{C}}$ by developing maps. We consider a version of the Hausdorff metric: For two proper subsets X and Y of $\hat{\mathbb{C}}$, X and Y are *ϵ-close* if

- the Hausdorff distance of X and Y is less than ϵ and
- the Hausdorff distance of ∂X and ∂Y is less than ϵ.

With this distance on the subsets on $\hat{\mathbb{C}}$, we have

Proposition 10.1. *U_i converges to $\mathcal{U}_\infty$ as $i \to \infty$.*

Proof of Proposition 10.1. Let $\mathcal{U}_i$ be the canonical neighborhood of $\tilde{p}$ in $\tilde{\mathcal{C}}_i$. Then, since $\tilde{\mathcal{C}}_i$ embeds into $\tilde{C}_i$ and $\tilde{\mathcal{C}}_\infty$, canonically $\mathcal{U}_i \subset U_i$ and $\mathcal{U}_i \subset \mathcal{U}_\infty$.

Lemma 10.2. *$\mathcal{U}_i$ converges to $\mathcal{U}_\infty$ as $i \to \infty$.*

Proof. For every $\epsilon > 0$, there exits finitely many closed round balls $B_1, B_2 \ldots, B_n$ in $\tilde{\mathcal{C}}_\infty$ containing $\tilde{p}$ such that $\cup_{k=1}^n B_k$ is ϵ-close to $\mathcal{U}_\infty$ in $\hat{\mathbb{C}}$. Since $\{\tilde{\mathcal{C}}_i\}$ exhausts $\tilde{\mathcal{C}}_\infty$, thus each B_k is also contained in $\tilde{\mathcal{C}}_i$ for sufficiently large i. Thus $\tilde{\mathcal{C}}_i$ contains $\cup_{k=1}^n B_k$, and therefore $\cup_{k=1}^n B_k$ is contained in $\mathcal{U}_i$. □

Since canonical neighborhoods are topologically open balls, by $\mathcal{U}_i \subset U_i$ and Lemma 10.2, it suffices to show that for every $\epsilon > 0$, if i is large enough, then ∂U_i is contained in the (honest) ϵ-neighborhood of $\partial \mathcal{U}_\infty$.

Lemma 10.3. *Given any $x \in \partial_\infty \mathcal{U}_\infty$ and any neighborhood $\mathcal{V}_x$ of x in $\hat{\mathbb{C}}$, then $\mathcal{V}_x$ is not a subset of U_i for sufficiently large $i \in \mathbb{N}$.*

Proof. Suppose that the assertion fails; then there is a neighborhood $\mathcal{V}_x$ of x in $\hat{\mathbb{C}}$, such that, for every $n \in \mathbb{N}$, there is $i > n$ with $\mathcal{V}_x \subset U_i$. Let $\mathcal{N}_\infty$ be the circular lamination of $\mathcal{C}_\infty$ that descends to N_∞, and let $\tilde{\mathcal{N}}_\infty$ be the total lift of $\mathcal{N}_\infty$ to the universal cover $\tilde{\mathcal{C}}_\infty$.

Since $\mathcal{N}_\infty$ is nonempty, the endpoints of leaves of $\tilde{\mathcal{N}}_\infty$ is dense in the ideal boundary $\partial_\infty \mathcal{U}_\infty (\subset \partial_\infty \tilde{\mathcal{C}}_\infty)$. Therefore we can in addition assume that x is an endpoint of a leaf of $\tilde{\mathcal{N}}_\infty$. Then the leaf contains a ray $\tilde{r} \colon [0, \infty) \to \tilde{\mathcal{C}}_\infty$ ending at x. Let r be the projection of $\tilde{r}$ to $\mathcal{C}_\infty$.

For every $s > 0$, since $r|[0, s]$ is a compact subset of $\mathcal{C}_\infty$, thus, for sufficiently large i, it is also a circular curve in $\mathcal{C}_i$ and thus in C_i. Accordingly $\tilde{r}|[0, s]$ is a circular arc embedded in $\tilde{C}_i$. Since $\tilde{r}$ ends at x, if $s > 0$ is sufficiently large, $\tilde{r}|[s, \infty)$ is contained in $\mathcal{V}_x$. Therefore $\tilde{r}$ is contained in a compact subset of U_i for sufficiently large i. Thus we induce a contradiction, showing that x is an ideal point of $\tilde{C}_i$ for sufficiently large i.

First suppose that r stays in the compact subset of $\mathcal{C}_\infty$. Then, since every compact subset of $\mathcal{C}_\infty$ naturally embeds into C_i for sufficiently large i, accordingly $\tilde{r}$ is naturally a circular ray in $\tilde{C}_i$ limiting to a point of $\partial_\infty \tilde{C}_i$.

Next suppose that no compact subset of $\mathcal{C}_\infty$ contains r. Then the admissible ℓ loop must spiral— otherwise ℓ is roughly circular, and every leaf of $\mathcal{N}_\infty$ is contained in a compact subset of $\mathcal{C}_\infty$. The projection of r to σ_∞ is an (eventually simple) geodesic ray spiraling towards a boundary component c of σ_∞. Then x is a fixed point of the corresponding loxodromic element. For each $j \in \mathbb{N}$, let b_j be the boundary component of $\mathcal{C}_j$ isotopic to c, so that b_j isomorphic to ℓ. Then b_j are parallel in $\mathcal{C}_\infty$ and $r \in \mathcal{C}_\infty$ intersects b_i for sufficiently large i. Let $\tilde{b}_j \colon \mathbb{R} \to \tilde{\mathcal{C}}_j$ be a (parametrized) lift of b_j so that $\tilde{b}_j(t)$ limits to x as $t \to \infty$. Then $\mathcal{V}_x$ contains $b_j(t)$ for all $t > t_j$ with some t_j. Therefore U_i must contain $b_j(t)$ $(t > t_j)$ as well. Since x is also an ideal point of $\tilde{C}_i$.

□

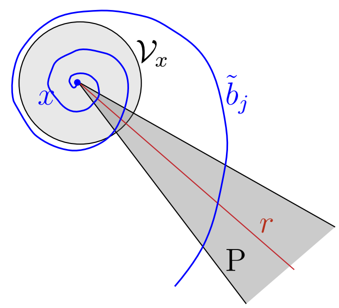


FIGURE 12.

Then Lemma 10.3 implies that

Corollary 10.4. *For every $\epsilon > 0$, if $i \in \mathbb{N}$ is large enough, then the ϵ-neighborhood of $\mathcal{U}_\infty$ contains all maximal balls B_i in $\tilde{C}_i$ containing $\tilde{p}$.*

Proof. The point $\tilde{p}$ is contained in $\hat{\mathbb{C}} \setminus \partial_\infty \mathcal{U}_\infty$. Note $\partial_\infty \mathcal{U}_\infty$ is a compact subset of $\hat{\mathbb{C}}$. Then $\hat{\mathbb{C}} \setminus \partial_\infty \mathcal{U}_\infty$ carries a canonical projective structure, and $\mathcal{U}_\infty$ naturally is isomorphic to the canonical neighborhood of $\tilde{p}$ in the complement.

For every $\delta > 0$, take finitely many points $x_1, \dots, x_{n(\delta)}$ in $\partial_\infty \mathcal{U}_\infty$ so that their δ-neighborhoods $\mathcal{V}_1, \dots, \mathcal{V}_{n(\delta)}$ cover $\partial_\infty \mathcal{U}_\infty$. Then their union $\mathcal{V}_1 \cup \dots \cup \mathcal{V}_{n(\delta)}$ converges to $\partial_\infty \mathcal{U}_\infty$ as $\delta \to 0$ in the Hausdorff topology. By Lemma 10.3, if i is sufficiently large, there is a point y_k in $\mathcal{V}_k$ that is not contained in U_i for each $k = 1, \dots, n(\delta)$. Then U_i is contained in the canonical neighborhood of $\tilde{p}$ in the punctured sphere $\hat{\mathbb{C}} \setminus \{y_1, \dots, y_{n(\delta)}\}$. Since $\{y_1, \dots, y_{n(\delta)}\}$ converges $\partial_\infty \mathcal{U}_\infty$ as $\delta \to 0$, for every $\epsilon > 0$, if i is sufficiently large, U_i is contained in the ϵ-neighborhood of $\mathcal{U}_\infty$. □

Corollary 10.4 immediately implies that U_i is contained in the ϵ-neighborhood of $\mathcal{U}_\infty$. Then ∂U_i is contained in the ϵ-neighborhood of $\partial \mathcal{U}_i$, since $\mathcal{U}_i \subset U_i$ and $\mathcal{U}_i \to \mathcal{U}_\infty$. [10.1]

11. CONVERGENCE OF DOMAINS IN $\hat{\mathbb{C}}$ AND THURSTON COORDINATES

Let R be a subset of $\mathbb{C}$ homeomorphic to an open disk. By Theorem 3.1, the projective structure on R has Thurston coordinates $(\mathbb{H}^2, L)$, where L is a measured lamination on $\mathbb{H}^2$ (note that L contains no heavy leaf since R is embedded in $\hat{\mathbb{C}}$). Let $\beta\colon \mathbb{H}^2 \to \mathbb{H}^3$ denote the corresponding pleated surface. Let $\mathcal{L} = (\nu, \omega)$ be the circular measured lamination on R that descends to $L = (\lambda, \mu)$ via the collapsing map $\kappa\colon R \to \mathbb{H}^2$, where $\lambda = |L|$ and $\mu \in \mathcal{TM}(\lambda)$.

Fix a conformal identification of $\hat{\mathbb{C}}$ with $\mathbb{S}^2$ in oder to fix a spherical Riemannian metric on $\hat{\mathbb{C}}$. Then let $\{R_i\}$ be a sequence of regions in $\hat{\mathbb{C}}$ homeomorphic to an open disk, such that $\partial R_i \to \partial R$ and $\hat{\mathbb{C}} \setminus R_i \to \hat{\mathbb{C}} \setminus R$ in the Hausdorff topology. For each i, we similarly let $(\mathbb{H}^2, L_i)$ denote Thurston coordinates of the projective structure on R_i; let $\beta_i \colon \mathbb{H}^2 \to \mathbb{H}^3$ be the corresponding pleated surface; let $\mathcal{L}_i = (\nu_i, \omega_i)$ be the circular measured lamination on R_i that descends to $L_i = (\lambda_i, \mu_i)$ via the collapsing map $\kappa_i \colon R_i \to \mathbb{H}^2$.

Every compact subset of R is also a compact subset of R_i for sufficiently large i. In particular, for every compact subset K of the target $\mathbb{H}^2$ of β, take i is large enough, so that R_i contains the compact set $\kappa^{-1}(K)$ of R. For each $x \in K$, pick a point in $y_x \in \kappa^{-1}(x)$. (Note that if x in on a leaf of L with atomic measure, then $\kappa^{-1}(x)$ is a circular arc.) Then define $\psi_i \colon K \to \mathbb{H}^2$ by $\psi_i(x) = \kappa_i(y_x)$. Note that this map ψ_i is not necessarily unique or continuous.

Theorem 11.1. (i) *$\mathcal{L}_i$ converges to $\mathcal{L}$, uniformly on compacts, via the convergence of R_i to R.*
(ii) *L_i converges to L pointwise.*
(iii) *ψ_i converges to an isometry uniformly on compacts; $\beta_i \circ \psi_i \colon \mathbb{H}^2 \to \mathbb{H}^3$ converges to $\beta \colon \mathbb{H}^2 \to \mathbb{H}^3$ uniformly on compacts.*

Remark 11.2. *In (i), by the uniform convergence, we mean that, for every $\epsilon > 0$ and every compact subset K of R, if i is sufficiently large, then given any $p, q \in K$, $\omega(p, q)$ is ϵ-close to $\omega_i(p, q)$, where $\omega_i(p, q)$ and $\omega(p, q)$ denote the transversal measures of the geodesic segments connecting p to q on R_i and R, respectively, in the Thurston metric. In (ii), by the pointwise convergence, for any $p, q \in R$ not on leaves with positive weight, $\mu_i(p, q) \to \mu(p, q)$ as $i \to \infty$. In (iii), for every compact subset K of $\mathbb{H}^2$, ψ_i is ϵ_i-rough isometry with the sequence $\epsilon_i > 0$ converging to 0. The convergence, $\beta_i \circ \psi_i \to \beta$ is with respect to the sup norm.*

Note that (i) implies (ii) by the definition of ψ_i. The rest of §11 is the proof of Theorem 11.1. For each point $x \in R$, let $B(x)$ be the maximal ball in R centered at x. For sufficiently large i, we have $x \in R_i$. Thus let $B_i(x)$ be the maximal ball in R_i centered at x.

Proposition 11.3. *(i) For every $\epsilon > 0$ and every compact subset K of R, if $i \in \mathbb{N}$ is sufficiently large, then $B_i(x)$ is ϵ-close to $B(x)$ for every $x \in K$. (ii) For every $\epsilon > 0$ and $x \in R$, there is a neighborhood U_x of x in R, such that, if i is sufficiently large, then the ideal boundaries $\partial_\infty B_i(y)$ and $\partial_\infty B(y)$ are contained in the ϵ-neighborhood of $\partial_\infty B(x)$ in $\hat{\mathbb{C}}$ for all $y \in U_x$.*

Proof of 11.3. (See also the proof of Theorem 4.4 in [KP94].) For every compact subset X of the Euclidean plane $\mathbb{R}^2$, there is a unique closed round ball $D = D(X)$ of least radius containing X. Let $\partial_X(D)$ be the intersection of X with the boundary circle of D. Then the minimality implies that the convex hull of $\partial_X(D)$ (for the Euclidean metric) contains the center of D. In addition, the uniqueness of D implies that D changes continuously when X changes continuously in the Hausdorff metric. Therefore, for every $\epsilon > 0$, there exists $\delta > 0$ such that if Y is a compact subset of $\mathbb{R}^2$ that is δ-close to X, then, letting D_Y be the round ball of least radius containing Y, the ϵ-neighborhood of $\partial_X(D)$ contains $\partial_Y(D_Y)$.

If x be a point in R, regarding $\hat{\mathbb{C}} = \mathbb{R}^2 \cup \{\infty\}$, we can assume $x = \{\infty\}$ by the $\mathrm{PSL}(2,\mathbb{C})$-action on $\hat{\mathbb{C}}$. Note that the round $B(x)$ is the complement of $D(\hat{\mathbb{C}} \setminus R)$ in $\hat{\mathbb{C}}$. In addition $\partial_\infty B(x) = \partial_X D(\hat{\mathbb{C}} \setminus R)$. If U is a neighborhood of the identity element in $\mathrm{PSL}(2,\mathbb{C})$, then Ux and $U^{-1}x$ are neighborhoods of $x = \{\infty\}$ in $\hat{\mathbb{C}}$. Thus for every $\delta > 0$, if the neighborhood U is sufficiently small, for every $\gamma \in U$, R and γR are δ-close. Therefore, it follows from the preceding paragraph that, for $\epsilon > 0$, if U is sufficiently small, then, letting $y = \gamma^{-1}x$, $B(x)$ and $B(y)$ are ϵ-close and the ϵ-neighborhood of $\partial_\infty B(x)$ contains $\partial_\infty B(y)$.

Since $\hat{\mathbb{C}} \setminus R_i \to \hat{\mathbb{C}} \setminus R$ and γR_i changes continuously in $\gamma \in U$, for every $\delta > 0$, if i is sufficiently large and U is sufficiently small, then $\mathbb{R}^2 \setminus \gamma R_i$ is δ-close to $\mathbb{R}^2 \setminus R$. Therefore, for every $\epsilon > 0$, we can assume that the maximal balls $B(y)$ and $B_i(y)$ are ϵ-close to $B(x)$ and the ϵ-neighborhood of $\partial_\infty B(x)$ contains $\partial_\infty B(y)$, which proves (ii).

Thus $B(y)$ and $B_i(y)$ are 2ϵ-close for all y in the small neighborhood U of x. Since K is compact, this implies (i). 11.3

Recalling $\hat{\mathbb{C}} \setminus R_i$ converges to $\hat{\mathbb{C}} \setminus R$ in $\hat{\mathbb{C}}$ as $i \to \infty$, let $e\colon R \cap R_i \to R$ and $e_i\colon R \cap R_i \to R_i$ be the trivial embeddings. Let $\phi = \beta \circ \kappa \circ e\colon R \cap R_i \to \mathbb{H}^3$ and $\phi_i = \beta_i \circ \kappa_i \circ e_i\colon R_i \cap R \to \mathbb{H}^3$.

$$\begin{array}{ccccccc}
R \cap R_i & \xrightarrow{e} & R & \xrightarrow{\kappa} & \mathbb{H}^2 & & \\
& \searrow^{e_i} & & & \downarrow{\psi_i} & \searrow^{\beta} & \\
& & R_i & \xrightarrow{\kappa_i} & \mathbb{H}^2 & \xrightarrow{\beta_i} & \mathbb{H}^3
\end{array}$$

Note that ν and ν_i are circular and we can measure their intersection angle with respect to the spherical Riemannian metric on $\hat{\mathbb{C}}$. Then

Corollary 11.4. *ϕ_i converges to ϕ uniformly on compacts in R as continuous maps; therefore, $\beta \circ \psi_i$ converges to β_i uniformly on compacts as $i \to \infty$ in the sup norm.*

Proof. By Proposition 11.3 (i), for every $\epsilon > 0$ and every compact subset K of R, if $i \in \mathbb{N}$ is sufficiently large, then for every $p \in K$, the maximal balls $B_i(p)$ and $B(p)$ of R_i and R, respectively, at p are ϵ-close. Thus, for sufficiently large i, the orthogonal projection of p into the totally geodesic hyperplane in $\mathbb{H}^3$ bounded by ∂B_i is ϵ-close to that into the hyperplane bounded by ∂B_i for all $p \in K(\subset \hat{\mathbb{C}})$. Since these projections of p are $\phi_i(p)$ and $\phi(p)$, the first assertion holds. Then the second assertion immediately follows from the definition of ψ_i. □

Proposition 11.5. *Let K be an arbitrary compact subsurface in R. Then $\angle_K(\nu_i, \nu) \to 0$ as $i \to \infty$.*

Proof. Since K is compact, it suffices to show that, for every $\epsilon > 0$ and $x \in R$, if an open neighborhood U_x of x in R is sufficiently small, then $\angle_{U_x}(\nu_i, \nu) < \epsilon$ for sufficiently large i.

Suppose that x is contained in a leaf ℓ_x of ν. Let ℓ and ℓ_i be leaves of ν and ν_i, respectively, that intersect in U_x. Let $B(\ell)$ be the maximal ball in R whose core contains ℓ, and let $B_i(\ell_i)$ be the maximal ball in R_i whose core contains ℓ_i. Then, it follows from Proposition 11.3 (ii) that, for every $\epsilon > 0$, if U_x is small enough and i is sufficiently large, then the endpoints of ℓ and ℓ_i are sufficiently close to the end points of ℓ_x so that $\angle_{U_x}(\ell_i, \ell) < \epsilon$. Hence $\angle_{U_x}(\nu_i, \nu) < \epsilon$.

Suppose that $x \in R \setminus |\nu|$. Then take U_x disjoint from ν. Then $\angle_{U_x}(\nu_i, \nu) = 0$ □

Let $\mathsf{d}\colon R \times R \to \mathbb{R}_{\geq 0}$ be the continuous map obtained by pulling back of the hyperbolic distance $\mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{R}_{\geq 0}$ via $\kappa\colon R \to \mathbb{H}^2$. Then d is a pseudometric on R. In fact d coincides with the Thurston metric on each stratum of $(R, \mathcal{L})$. On the other hand, d does not measure the part of Thurston metric corresponding to the transversal measure of $\mathcal{L}$. In particular, the Euclidean region of R, circular arcs orthogonal to ν have "length" zero since they map to single points on $\mathbb{H}^2$. Similarly $\mathsf{d}_i\colon R_i \times R_i \to \mathbb{R}_{\geq 0}$ be the pseudometric on R_i obtained via $\kappa_i\colon R_i \to \mathbb{H}^2$.

Proposition 11.6. *Let K be a compact subset of R. Then, for every $\epsilon > 0$, if $i \in \mathbb{N}$ is sufficiently large, then d and d_i are ϵ-close on $K \times K$, i.e. $|\mathsf{d}(x, y) - \mathsf{d}_i(x, y)| < \epsilon$ for all x, y in K.*

In the sense of Remark 11.2, we have

Corollary 11.7. *ψ_i converges to an isometry uniformly on compacts; Theorem 11.1 (iii) holds.*

Proof of Proposition 11.6. First suppose that $\kappa(K)$ and $\kappa_i(K)$ do not intersect the Euclidean regions. Then $\kappa|K$ and $\kappa_i|K$ are C^1-diffeomorphism onto their images, and d and d_i are both hyperbolic metrics on K. Thus, by Proposition 11.3 (i), for every $\epsilon > 0$, if i is sufficiently large, then for every unit tangent vector v at a point x in K, the length of the derivative $\kappa'(v)$ is ϵ-close to that of $\kappa_i'(v)$. Thus d and d_i are ϵ-close in K. This special case extends to general Thurston metrics by:

Proposition 11.8. *Let P be a topological open disk in $\hat{\mathbb{C}}$ with $P \not\cong \mathbb{C}$, so that P has Thurston coordinates (by Proposition 3.1). Then, for every $\epsilon > 0$ and every compact subset K of P homeomorphic to a closed disk, there is another topological open disk Q in $\hat{\mathbb{C}}$ containing K such that*

(i) *P and Q are ϵ-close on $\hat{\mathbb{C}}$ in the Hausdorff metric,*
(ii) *in Thurston coordinates, its lamination (on $\mathbb{H}^2$) has no leaf with atomic transversal measure, and*
(iii) *letting $\mathsf{d}_P\colon P\times P\to\mathbb{R}_{\geq 0}$ be the pseudo metric and $\mathsf{d}_Q\colon Q\times Q\to\mathbb{R}_{\geq 0}$ be the metric defined as above, then d_P and d_Q are ϵ-close in $K\times K$.*

Indeed, by Proposition 11.8, we can take Q and Q_i for each i that are close to R and R_i on $\hat{\mathbb{C}}$, respectively, so that d_Q and d_{Q_i} are sufficiently close to d_R and d_{R_i} on $K\times K$. Then, by (ii), the general case is reduced to the case that the Thurston laminations have no atomic measure.

Proof of Proposition 11.8. For every $\delta_1>0$, pick another open topological disk Q contained in P such that

(1) when projected to $\mathbb{C}$ by stereographic projection, ∂Q is a smooth loop embedded in $\hat{\mathbb{C}}$ and the sign of its curvature changes at most finitely many times in the Euclidean metric, and
(2) the Hausdorff distance of $\hat{\mathbb{C}}\setminus P$ and $\hat{\mathbb{C}}\setminus Q$ is less than δ_1 ((i)).

We can in addition assume that Q also contains K by taking sufficiently small δ_1. Let $(\mathbb{H}^2, L_Q)$ be the Thurston coordinates of the projective surface Q. Set $\mathcal{L}_Q=(\nu_Q,\omega_Q)$ to be the circular measured lamination on Q descending to L_Q. By the smoothness in (1), for each point of ∂Q, there is a unique maximal ball in Q tangent, at the point, to ∂Q. Thus every leaf of L_Q has no atomic measure ((ii)). Moreover

Lemma 11.9. *The two-dimensional strata of $(Q,\mathcal{L}_Q)$ are isolated. Therefore two-dimensional strata of $(\mathbb{H}^2, L_Q)$ are isolated.*

Proof. Let R be a two-dimensional stratum of $\mathcal{L}_Q$, and let B_R be the maximal ball in Q whose core is R. Then, by the curvature condition, B_R has only finitely many ideal points $p_1,\dots,p_n$. Let $I_1,\dots,I_n$ be sufficiently small neighborhoods of $p_1,\dots,p_n$ in ∂Q. Since ∂Q is smooth, at every point x of ∂Q, there is a unique maximal B_x ball in Q such that x is an ideal point of B_x. Suppose that x is in $I_i\setminus p_i$ for some $i\in\{1,\dots,n\}$. Then as I_i is sufficiently small, x is the unique ideal point of B_x in I_i. Let P_x be the connected component of $Q\setminus \mathrm{Core}(R)$ whose boundary contains x. Let ℓ be the circular boundary segment of R bounding P_x, so x is close to one of the end point of ℓ (Figure 13). Then the ideal points of B_x must be in $\partial P_x\setminus\ell$. In addition they must be contained in a small neighborhood of ℓ by the continuity. Thus, there is exactly one more ideal point of P_x near the other endpoint of ℓ.

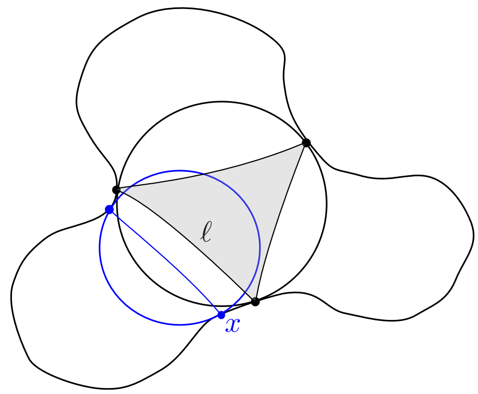


FIGURE 13.

□

Let $\mathrm{pr}_P\colon \mathbb{H}^3 \to \mathrm{Conv}(\hat{\mathbb{C}} \setminus P)$ denote the nearest point projection onto the convex hull of $\hat{\mathbb{C}} \setminus P$ in $\mathbb{H}^3$. Let $(P, \mathcal{L}_P) \to (\mathbb{H}^2, L_P)$ denote the collapsing map of P. Then $\partial\,\mathrm{Conv}(\hat{\mathbb{C}} \setminus P)$ is the pleated surface induced by $(\mathbb{H}^2, L_P)$.

For $\delta_2 > 0$, consider the δ_2-neighborhood of $\mathrm{Conv}(\hat{\mathbb{C}} \setminus P)$. Then its boundary surface S_{δ_2} is C^1-smooth and it carries an intrinsic Riemannian metric induced from $\mathbb{H}^3$ (see [EM87, II.1.3.6, II.1.5]). Similarly let $\mathrm{pr}_{\delta_2}\colon P \to S_{1\delta_2}$ denote the orthogonal projection along geodesics in $\mathbb{H}^3$; then pr_{δ_2} is a C^1-diffeomorphism. Consider the Riemannian metric on P obtained by pulling back the Riemannian metric on S_{δ_2} via pr_{δ_2}, let $d_{\delta_2}\colon P \times P \to \mathbb{R}_{\geq 0}$ be the associated distance function of P. Then (iii) follows from:

Claim 11.10. *For every $\epsilon > 0$, if $\delta_1 > 0$ and $\delta_2 > 0$ are sufficiently small then*

(1) *d_{δ_2} and d_Q are $(1+\epsilon)$-bilipschitz on $K \times K$, i.e.*

$$1 - \epsilon < d_{\delta_2}(x, y)/\mathsf{d}_Q(x, y) < 1 + \epsilon$$

for all distinct x, y in K.

(2) *d_{δ_2} and d_P are ϵ-close on $K \times K$.*

Proof. (1) For each point $x \in P$, let $H_{\delta_2}(x)$ be the unique hyperbolic plane in $\mathbb{H}^3$ tangent to S_{δ_2} at $\mathrm{pr}_{\delta_2}(x)$. Then the boundary circle $\partial H_{\delta_2}(x)$ is contained in P. The boundary of the maximal ball $B_P(x)$ bounds another hyperbolic plane supporting, at $\mathrm{pr}_P(x)$, the pleated surface bounding $\mathrm{Conv}(\hat{\mathbb{C}} \setminus P)$. Those two hyperbolic planes are perpendicular to the geodesic through at $\mathrm{pr}_{\delta_2}(x)$ and $\mathrm{pr}_P(x)$. Then the distance between the planes is exactly δ_2.

Let $B_{\delta_2}(x)$ be the round open ball in P bounded by $\partial H_{\delta_2}(x)$. Then $B_{\delta_2}(x)$ contains x. Let $B_P(x)$ and $B_Q(x)$ be the maximal balls in P and Q, respectively, centered at x. Then, for every $\epsilon > 0$, if $\delta_2 > 0$ is sufficiently small, then $B_P(x)$ is ϵ-close to $B_{\delta_2}(x)$ on $\hat{\mathbb{C}}$ for every $x \in P$. In addition, by Proposition 11.3 (i), if $\delta_1 > 0$ is sufficiently small then, $B_P(x)$ and $B_Q(x)$ are ϵ-close for all $x \in K$. Then $B_Q(x)$ and $B_{\delta_2}(x)$ are 2ϵ-close. Therefore if $\delta_1 > 0$ and $\delta_2 > 0$ are sufficiently small then, for every unite tangent vector v at a point in K, the derivatives $d\,\mathrm{pr}_{\delta_2}(v)$ and $d\,\mathrm{pr}_P(v)$ are tangent vectors in $\mathbb{H}^3$, that are ϵ-close. Thus d_{δ_2} and d_Q are ϵ-bilipschitz on K.

(2) Let H be the subsurface of P where the Thurston metric is negatively curved. Then pr_P takes H isometrically onto its image in $\partial\,\mathrm{Conv}(\hat{\mathbb{C}} \setminus P)$ with the intrinsic metric induced by $\mathbb{H}^3$. By identifying L_P and its image on $\partial\,\mathrm{Conv}(\hat{\mathbb{C}} \setminus P)$, then, $x \in H$ if and only if $\mathrm{pr}_P(x)$ is not on a leaf of L_P with positive weight. Then pr_P is C^1-smooth on H. Thus similarly to (1), for every $\epsilon > 0$, if $\delta_2 > 0$ is sufficiently small, then d_{δ_2} and d_P are ϵ-bilipschitz on each connected component of H.

Each connected component E_ℓ of the Euclidean subsurface of P corresponds to a leaf ℓ of L_P with positive weight, so that $E_\ell = \mathrm{pr}_P^{-1}(\ell)$. we show that, for every $\epsilon > 0$, if $\delta_2 > 0$ is sufficiently small, then, for every leaf ℓ of L_P with weight $w(\ell) > 0$, d_P and d_{δ_2} are $(1+\epsilon, w(\ell)\delta_2)$-quasi isometric on E_ℓ.

Then E_ℓ is, in the Thurston metric, an infinite Euclidean strip with width $w(\ell)$. Thus we may regard E_ℓ as a subset of $\mathbb{R}^2$ so that it is infinite in the vertical direction. On $\hat{\mathbb{C}}$, the strip E_ℓ is regarded as a *wedge*, i.e. a region bounded by two circular arcs sharing both endpoints. Consider the δ_2-neighborhood M, in $\mathbb{H}^3$, of the geodesic m connecting the vertices of E_ℓ — it is an infinite solid cylinder invariant under any hyperbolic translation along m. Then d_{δ_2} on E_ℓ is given by pulling back the intrinsic metric on ∂M by pr_{δ_2}. The boundary of M is foliated by round loops bounding (geometric) disks of radius δ_2 orthogonal to m in $\mathbb{H}^3$. Then, by the nearest point projection $\mathbb{H}^3 \to m$, each loop map to a single point on m. In addition there is another foliation of ∂M by straight lines (with its intrinsic metric) that are orthogonal to the round loops. Then each straight line diffeomorphically projects onto m by the projection $\mathbb{H}^3 \to m$.

For different points p, q in E_ℓ, a geodesic connecting p to q with d_P can be realized as a union of a vertical geodesic segment and horizontal geodesic segment. On vertical lines in E_ℓ, for every $\epsilon > 0$, if $\delta_2 > 0$ is sufficiently small, then d_P and d_{δ_2} are $(1+\epsilon)$-bilipschitz. On the other hand, on the horizontal lines, d_P and d_{δ_2} are $w(\ell)\delta_2$ rough isometric. Therefore for even $\epsilon > 0$, if $\delta_2 > 0$ is small, then for every leaf ℓ of L_P of positive weight, the projection d_P and d_{δ_2} are $(1+\epsilon, w(\ell)\delta_2)$-quasiisometric on E_ℓ.

The total transversal measure on K given by $\mathcal{L}_P$ is finite. Therefore, for every $\epsilon > 0$, if $\delta_2 > 0$ is sufficiently small, then d_P and d_{δ_2} are $(1+\epsilon, \epsilon)$-quasiisometric on K. Since K is a compact subset, we can in addition assume that they are δ_2-rough isometric. □

11.8

In the rest of this section, we show the convergence of the transversal measures for Theorem 11.1 (i).

Proposition 11.11. *Let p_0, p_1 be (distinct) points in a single stratum of $(R, \mathcal{L})$. The $\omega_i(p_0, p_1) \to \omega(p_0, p_1) = 0$ as $i \to \infty$, where $\omega(p_0, p_1)$ and $\omega_i(p_0, p_1)$ are the transversal measures of, in Thurston metrics, the geodesic segment from p_0 to p_1 on R and R_i, respectively.*

Proof. Let P be the strata of $(\mathcal{R}, \mathcal{L})$ containing p_0 and p_1. Then let $\alpha\colon [0,1] \to P$ be the geodesic segment from p_0 to p_1, i.e. $\alpha(0) = p_0$ and $\alpha(1) = p_1$. Let $Q = \kappa(P)$, the corresponding strata of $(\mathbb{H}^2, L)$. We can naturally identify Q and $\beta(Q)$. For each $j = 1, 2$, let $q_j = \kappa(p_j)$ and $q_{i,j} = \kappa_i(p_j)$ for all sufficiently large $i \in \mathbb{N}$. Then, for each j, the point $\beta_i(q_{i,j})$ converges to the point $\beta(q_j)$ as $i \to \infty$ (Corollary 11.4). Let N be the totally geodesic hyperplane in $\mathbb{H}^3$ orthogonally intersecting $\beta(Q)$ in the geodesic segment from $\beta(q_0)$ to $\beta(q_1)$.

For each $t \in [0,1]$, considering the maximal ball in R_i centered at $\alpha(t)$, let $H_{i,t}$ be the totally hyperbolic plane in $\mathbb{H}^3$ bounded by the boundary of the maximal ball. By Proposition 11.3, $\sup_{t\in[0,1]} \angle(N, H_{i,t}) \to \pi/2$ as $i \to \infty$. Thus, for every $\epsilon > 0$, if i is sufficiently large, then $H_{i,s}$ and $H_{i,t}$ intersect and $\angle(H_{i,s}, H_{i,t}) < \epsilon$ for all $s, t \in [0,1]$. This implies that $\mu_i(q_0, q_1) < \epsilon$ (see [EM87]). Therefore $\omega_i(p_0, p_1) < \epsilon$. □

Let d_R denote Thurston metric on R. Then

Proposition 11.12. *For every $\epsilon > 0$, there exists $\delta > 0$ such that, if $p_0, p_1 \in R$ are points contained in different strata of $(R, \mathcal{L})$ satisfying $d_R(p_0, p_1) < \delta$ and $\angle([p_0, p_1], \mathcal{L}) < \delta$, then*

$$1 - \epsilon < \frac{\omega(\,p_0, p_1\,)}{\omega_i(p_0, p_1)} < 1 + \epsilon,$$

for all sufficiently large $i \in \mathbb{N}$.

Proof of 11.12. Let p_0, p_1 be points in R satisfying the assumptions. Let $q_0 = \kappa(p_0)$ and $q_1 = \kappa(p_1)$. Then for every $\epsilon > 0$, $\delta > 0$ is sufficiently small, then there is a hyperbolic plane N in $\mathbb{H}^3$ passing $\beta(q_0), \beta(q_1)$ so that N is ϵ-nearly orthogonal to $\beta(Q)$ for all strata Q of $(\mathbb{H}^2, L)$ intersecting $[q_0, q_1]$.

Let $\alpha\colon [0,1] \to R$ be the geodesic connecting p_0 to p_1. Then, for each $t \in [0,0]$, the maximal ball in R centered at $\alpha(t)$ shares its boundary circle with a unique hyperbolic plane H_t in $\mathbb{H}^3$. Note that, if $d_R(p_0, p_1) > 0$ is sufficiently small, then H_s an H_t must intersect for all $s, t \in [0,1]$.

Let θ be a subdivision of $[0,1]$ as $0 = t_0 < t_0 < \cdots < t_{n_\theta} = 1$. Let $|\theta|$ be the maximal width of the subintervals $[t_k, t_{k+1}]$ $(0 \leq k < n_\theta - 1)$. Then the transversal measure $\omega([p_0, p_1])$ is the limit of

$$\Sigma_{k=1}^{n_\theta} \angle_{\mathbb{H}^3}(H_{t_k}, H_{t_{k+1}})$$

as $|\theta| \to 0$, where $\angle_{\mathbb{H}^3}(H_{t_k}, H_{t_{k+1}})$ be the angle taking its value in $[0, \pi/2]$ between the hyperbolic planes H_{t_k} and $H_{t_{k+1}}$ in $\mathbb{H}^3$. Note that this summation decreases when the subdivision θ is refined ([EM87, II.1.10]).

For $s, t \in [0,1]$, the geodesic $H_s \cap N$ intersect the geodesic $H_t \cap N$; let $\angle_N(H_s, H_t) \in [0, \pi/2]$ denote their intersection angle in N. Since $\angle_{\mathbb{H}^3}(H_t, N)$ is ϵ-close to $\pi/2$ for all $t \in [0,1]$, by taking sufficiently small $\delta > 0$, we can assume that

$$1 - \epsilon < \frac{\angle_{\mathbb{H}^3}(H_s, H_t)}{\angle_N(H_s, H_t)} < 1 + \epsilon,$$

for all $s, t \in [0,1]$. Thus

$$1 - \epsilon < \frac{\Sigma_{k=1}^{n_\theta} \angle_{\mathbb{H}^3}(H_{t_k}, H_{t_{k+1}})}{\Sigma_{k=1}^{n_\theta} \angle_N(H_k, H_{k+1})} < 1 + \epsilon.$$

Since R_i contains the geodesic segment α for sufficiently large i, similarly let $H_{i,t}$ be a copy of $\mathbb{H}^2$ such that $\partial_\infty H_{i,t}$ bounds the maximal ball in R_i centered at $\alpha(t)$. Then the transversal measure $\omega_i(p_0, p_1)$ is the limit of

$$\Sigma_{k=1}^{n_\theta} \angle_{\mathbb{H}^3}(H_{i,t_k}, H_{i,t_{k+1}})$$

as $|\theta| \to 0$. For every $\epsilon > 0$, if i is sufficiently large, the hyperbolic planes H_t and $H_{i,t}$ are ϵ-close for all $t \in [0,1]$. Thus, if $\delta > 0$ is sufficiently small and i is sufficiently large, then $H_{i,t}$ intersects N at an angle ϵ-close to $\pi/2$. Thus we can in addition assume that

$$1 - \epsilon < \frac{\Sigma_{k=1}^{n_\theta} \angle_{\mathbb{H}^3}(H_{t_k}, H_{t_{k+1}})}{\Sigma_{k=1}^{n_\theta} \angle_N(H_{t_k}, H_{t_{k+1}})} < 1 + \epsilon,$$

for any subdivision θ.

Therefore it remains to show that, if $|\theta|$ is sufficiently small, then

$$-\epsilon < \Sigma_{k=1}^{n_\theta} \angle_N(H_{t_k}, H_{t_{k+1}}) - \Sigma_{k=1}^{n_\theta} \angle_N(H_{t_k}, H_{t_{k+1}}) < \epsilon. \tag{4}$$

Consider the convex subset X_θ of $\mathbb{H}^3$ bounded by the hyperbolic planes $H_{t_1}, \ldots, H_{t_n}$ so that X_θ contains $\mathrm{Conv}(\hat{\mathbb{C}} \setminus R)$. Then N intersects ∂X_θ nearly orthogonally, and the intersection is a piecewise geodesic that is a convex bi-infinite curve through $\beta(q_0)$ and $\beta(q_1)$, and its non-smooth points are between $\beta(q_0)$ and $\beta(q_1)$. Pick a segment η_θ of this curve that is slightly larger than the segment from $\beta(q_0)$ to $\beta(q_1)$ so that the interior of η_θ contains $\beta(q_0)$ and $\beta(q_1)$. Then $\Sigma_{k=1}^{n_\theta} \angle_N(H_{t_k}, H_{t_{k+1}})$ is equal to the sum of the exterior angles of η_θ.

Similarly let $X_{i,\theta}$ be the convex subset of $\mathbb{H}^3$ bounded by $H_{i,t_1}, \dots, H_{t_n}$, such that $X_{i,\theta}$ contains $\mathrm{Conv}(\hat{\mathbb{C}} \setminus R_i)$. Then $\partial X_{i,\theta} \cap N$ is a piecewise geodesic convex curve in N, which converges to the convex curve $\partial X_\theta \cap N$ above as $i \to \infty$. For sufficiently large i, each endpoint η_θ has a unique closest point on $\partial X_{i,\theta} \cap N$. Then those closest points cut off a segment $\eta_{i,\theta}$ of $\partial X_{i,\theta} \cap N$ that contains all non-smooth points. Then $\Sigma_{k=1} \angle_N(H_{i,t_k}, H_{i,t_{k+1}})$ is the sum of the exterior angles of $\eta_{i,\theta}$.

Consider the loop ℓ_i that obtained by connecting the corresponding endpoints of η_θ and $\eta_{i,\theta}$ by geodesic segments. Then, since $\eta_{i,\theta}$ converges to η_θ as $i \to \infty$, the area in N bounded by ℓ_i converges to 0 as $i \to \infty$. By applying Gauss-Bonnet Theorem to ℓ_i in the hyperbolic plane N, we obtain (4). 11.12

Proposition 11.13. *For all $p, q \in R$, $\omega_i(p, q) \to \omega(p, q)$ as $i \to \infty$.*

Proof. For every $\delta > 0$, pick a simple piecewise geodesic path $\eta = \cup_{k=1}^n [p_k, p_{k+1}]$ in R connecting p to q, where p_k are points in R, such that $d_R(p_k, p_{k+1}) < \delta$ and $\pi/2 - \delta < \angle(\mathcal{L}, [p_k, p_{k+1}]) < \pi/2 - \delta$ for all $k = 0, 1, \dots, n-1$. By Proposition 11.12, if $\delta > 0$ is sufficiently small, then if p_k and p_{k+1} are in different strata of $(R, \mathcal{L})$, then

$$1 - \epsilon < \frac{\omega(p_k, p_{k+1})}{\omega_i(p_k, p_{k+1})} < 1 + \epsilon$$

For sufficiently large i. If p_k and p_{k+1} are in a single stratum of $(R, \mathcal{L})$, then by Proposition 11.11, $\omega_i(p_k, p_{k+1}) \to 0 = \omega(p_k, p_{k+1})$. Clearly $\omega_i(p, q) = \Sigma_{k=1}^n \omega_i(p_k, p_{k+1})$ and $\omega(p, q) = \Sigma_{k=1}^n \omega(p_k, p_{k+1})$. Thus for every $\epsilon > 0$, if $\delta > 0$ is sufficiently small, then $|w(p, q) - w_i(p, q)| < \epsilon$ for sufficiently large i. □

Corollary 11.14. *Let K be a compact subset of R. Then for every $\epsilon > 0$, if $i \in \mathbb{N}$ is sufficiently large, then $-\epsilon < \omega(p, q) - \omega_i(p, q) < \epsilon$ for all $p, q \in K$.*

Proof. For every point $x \in K$, there is a neighborhood U_x such that it follows from Proposition 11.3 that, for every $\epsilon > 0$, if i is sufficiently large, then $-\epsilon < \omega(y, z) - \omega_i(y, z) < \epsilon$ for all $y, z \in U_x$. Since K is compact, there are finitely many points $x_1, \dots, x_n$ such that $U_{x_1}, \dots, U_{x_n}$ cover K. Applying Proposition 11.13 to all pairs of points in $x_1, \dots, x_n$, we have $-\epsilon < \omega(x_j, x_k) - \omega_i(x_j, x_k) < \epsilon$ for all $0 < j, k \leq n$. Then the Triangle Inequality implies the corollary. □

12. Proof of Theorem 7.1

Let $\mathcal{K}$ be a compact connected surface π_1-injectively embedded in $\mathcal{C}_i$. Recalling the natural embedding $e_i \colon \mathcal{C}_i \to \mathcal{C}_\infty$, we let $\mathcal{K}_i = e_i^{-1}(\mathcal{K})$ for each $i \in \mathbb{N}$. Since $\mathcal{C}_1 \subset \mathcal{C}_2 \subset \dots$ exhausts $\mathcal{C}_\infty$, if i is sufficiently large, $\mathcal{K}_i$ is isomorphic to $\mathcal{K}$ by e_i, and thus $\mathcal{K}_i$ is a compact subsurface of $\mathcal{C}_i$. Since $\mathcal{C}_i \subset C_i$, naturally $\mathcal{K}_i \subset C_i$. Recall that τ_i and τ_∞ are

$$\begin{array}{ccc} \mathcal{C}_\infty \supset \mathcal{K} & \xrightarrow{e_i^{-1}} & \mathcal{K}_i \subset C_i \\ \Big\downarrow \iota_\infty & & \Big\downarrow \kappa_i \\ \sigma_\infty & \xrightarrow{\quad \psi_i \quad} & \tau_i \end{array}$$

FIGURE 14.

homeomorphic to S and that τ_∞ is obtained by identifying the boundary geodesics of σ_∞. Then, since $\kappa_i\colon C_i \to \tau_i$ and $\iota_\infty\colon \mathcal{C}_\infty \to \sigma_\infty$ are collapsing maps, when $\mathcal{K}_i$ is isomorphic to $\mathcal{K}$, then $\kappa_i|\mathcal{K}_i$ and $\iota_\infty|\mathcal{K}$ are homotopic as maps to S. Let $\tilde{\kappa}_i(\tilde{\mathcal{K}}_i)$ and $\tilde{\iota}_\infty(\tilde{\mathcal{K}})$ denote the universal covers of $\kappa_i(\mathcal{K}_i)$ and $\iota_\infty(\mathcal{K})$, respectively. Then, recalling that $\mathcal{N}_\infty$ is the canonical circular lamination on $\mathcal{C}_\infty$, we have

Proposition 12.1. *There exists a sequence of (not necessarily continuous) maps $\psi_i\colon \iota_\infty(\mathcal{K}) \to \kappa_i(\mathcal{K}_i)$ for $i \in \mathbb{N}$, such that, letting $\tilde{\psi}_i\colon \tilde{\iota}_\infty(\tilde{\mathcal{K}}) \to \tilde{\kappa}_i(\tilde{\mathcal{K}}_i)$ be the lift of ψ_i, which commutes with deck transformations, we have*

(i) *$\mathcal{L}_i$ on $\mathcal{K}_i$ converges to $\mathcal{N}_\infty$ on $\mathcal{K}$ uniformly,*
(ii) *ψ_i converges to an isometry uniformly as $i \to \infty$,*
(iii) *the sup distance between $\kappa_i \circ e_i^{-1}$ and $\psi_i \circ \iota_\infty$ converges to zero on $\mathcal{K}$ as $i \to \infty$ (Figure 14),*
(iv) *the sup distance between $\beta_i \circ \tilde{\psi}_i$ and β_∞ converges to 0 on $\iota_\infty(\mathcal{K}_\infty)$ as $i \to \infty$,*

and therefore

(v) *for $x, y \in \iota_\infty(\mathcal{K})$ not on leaves of positive atomic measure, let $[x, y]$ be a geodesic segment connecting x to y in σ_∞ and let $[\psi_i(x), \psi_i(y)]$ be the geodesic segment on τ_i that is homotopic to $\psi_i([x, y])$ with its endpoints fixed; then the transversal measure on $[\psi_i(x), \psi_i(y)]$ by L_i converges to the transversal measure on the geodesic segment $[x, y]$ by N_∞.*

More precisely, in (i), we mean that for every $\epsilon > 0$, if i is sufficiently large, then for all $x, y \in \mathcal{K}$, then the transversal measure of $[x, y]$ given by $\mathcal{L}_i$ is ϵ-close to that given by $\mathcal{N}_\infty$. By (ii), we mean that for every $\epsilon > 0$, if i is sufficiently large, then

$$-\epsilon < \mathrm{dist}_{\mathbb{H}^2}(\tilde{\psi}_i(x), \tilde{\psi}_i(y)) - \mathrm{dist}_{\mathbb{H}^2}(x, y) < \epsilon$$

for every $x, y \in \tilde{\iota}_\infty(\tilde{\mathcal{K}})$.

Proof of 12.1. It suffices to show that, for every p in $\tilde{\mathcal{C}}_\infty$, there is a compact neighborhood of p with the desired properties. Consider all maximal balls in $\tilde{\mathcal{C}}_\infty$ containing p; then the union of their cores is a neighborhood of p contained in the canonical neighborhood of p.

Thus we can assume that $\mathcal{K}$ is a simply connected region contained this union.

For sufficiently large $i \in \mathbb{N}$, let $p_i \in \tilde{\mathcal{C}}_i$ such that $\tilde{e}_i(p_i) = p$. Let U_i be the canonical neighborhood of p_i in $\tilde{C}_i$. By Proposition 10.1, U_i converges to $\mathcal{U}_\infty$ and ∂U_i converges to $\partial \mathcal{U}_\infty$ on $\hat{\mathbb{C}}$ as $i \to \infty$ in the Hausdorff metric for $\hat{\mathbb{C}}$ (by fixing a natural metric on $\hat{\mathbb{C}}$). Hence, by Theorem 11.1 (including the definition of ϕ_i) and Proposition 3.6, we have (i) - (iv). 12.1

Let l_∞ and l_i be the geodesic representatives of ℓ in τ_∞ and τ_i, respectively. We first show that $\tau_i \to \tau_\infty$ and $\beta_i \to \beta_\infty$ as $i \to \infty$. Let σ_i be $\tau_i \setminus l_i$. Then, by Proposition 12.1 (ii), σ_i converges to $\sigma_\infty (= \tau_\infty \setminus l_\infty)$ as $i \to \infty$. In other words, τ_i converges to τ_∞ possibly up to a "twist" along ℓ_∞. By Proposition 12.1 (iii), the restriction of β_i to a lift of $\sigma_i (\subset \tau_i)$ to $\mathbb{H}^2$ converges to the restriction of β_∞ to the corresponding lift of $\sigma_\infty (\subset \tau_\infty)$ to $\mathbb{H}^2$. Since β_i and β are both ρ-equivariant, β_i must converge to β_∞ (c.f. §9) as $i \to \infty$, which proves (ii). Therefore τ_i must converge to τ_∞.

Last we show the convergence of L_i. By Proposition 12.1 (v), the restriction of L_i to σ_i converges to the restriction of L_∞ to σ_∞ as $i \to \infty$ uniformly on compacts. Thus it is left to show that the transversal measure of L_i near ℓ_i must diverges to ∞. Each connected component of $\mathcal{C}_\infty \setminus \mathcal{C}_0$ is a half-infinite grafting cylinder. Then this cylinder has infinite total transversal measure given by $\mathcal{N}_\infty$. Thus, by Proposition 12.1 (iv), for any fixed $j \in \mathbb{N}$, the total transversal measure on $C_i \setminus \mathcal{C}_j$ given by $\mathcal{L}_i$ diverges to ∞ as $i \to \infty$. Let α_∞ be a smooth arc on τ_∞ transversal to L_∞ such that α_∞ intersects ℓ_∞ in a single point. Then the transversal measure of α_∞ by L_∞ is infinite. By the convergence $\tau_i \to \tau$, we have $|L_i| \to |L_\infty|$ (in $\mathcal{GL}$) as $i \to \infty$. Thus let (α_i) be a sequence of arcs α_i on τ_i smoothly converges to an arc α_∞, so that α_i is transversal to L_i for sufficiently large i. Since the total transversal measure of $C_i \setminus \mathcal{C}_j$ diverges as $i \to \infty$ as above, the divergence, accordingly the transversal measure of α_i given by L_i must diverge to ∞. Therefore L_i converges to L_∞ as $i \to \infty$.

Part 3. Appendix: density of holonomy map fibers $\mathcal{P}_\rho$ in $\mathcal{PML}$.

Recall Thurston coordinates $\mathcal{P} \cong \mathcal{T} \times \mathcal{ML}$ (§3) on the space $\mathcal{P}$ of all (marked) projective structures on S. This gives an obvious projection from $\mathcal{P}$ to $\mathcal{ML}$. Then the obvious projection $\mathcal{ML} \setminus \{\emptyset\} \to \mathcal{PML}$ extends to $\Phi\colon \mathcal{ML} \to \mathcal{PML} \sqcup \{\emptyset\}$ so that the empty lamination $\emptyset$ maps to $\emptyset$.

Recall from §1 that $\mathcal{P}_\rho$ is the set of all projective structures with fixed holonomy $\rho\colon \pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$ and that $\mathcal{P}_\rho$ is a discrete subset of $\mathcal{P}$. If ρ is fuchsian, letting $\tau \in \mathcal{T}$ be the corresponding hyperbolic structure, we have

$$\mathcal{P}_\rho \cong \{\, (\tau, M) \mid \text{multiloops M with } 2\pi\text{-multiple weights}\},$$

in Thurston coordinates ([Gol87], c.f. [Bab15]). Thus $\Phi(\mathcal{P}_\rho)$ is the union of $\emptyset$ and a dense subset of $\mathcal{PML}$. Note that a projective structure $C \in \mathcal{P}$ maps to $\emptyset$ via Φ if and only if C is a hyperbolic structure ([Gol87]). Thus, for almost all $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$, we have $\Phi(\mathcal{P}_\rho) \not\ni \emptyset$. Then

Theorem 12.2. *Given arbitrary $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$, if $\mathcal{P}_\rho$ is non-empty, then $\Phi(\mathcal{P}_\rho) \setminus \{\emptyset\}$ is a dense subset of $\mathcal{PML}$.*

A *Schottky decomposition* of a representation $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ is a decomposition of S into pairs of pants P_k along a (maximal) multiloop M such that the restriction of ρ to $\pi_1(P_k)$ is an isomorphism onto a Schottky group for each pants P_k. A *Schottky decomposition* of a projective structure $C = (f, \rho)$ is a decomposition of C into pairs of pants along a multiloop M on consisting of admissible loops such that M realizes a Schottky decomposition of ρ.

Proposition 12.3. *Let $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ be the holonomy representation of some projective structure on S. Then for every uniquely ergodic measured lamination L, there is a sequence of projective structures C_i with holonomy ρ such that there is, for each i, a Schottky decomposition of C_i along some admissible multiloop containing a loop ℓ_i and $[\ell_i] \to [L]$ in $\mathcal{PML}$ as $i \to \infty$.*

Proof. Given a non-elementary representation $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ that lifts to $\pi_1(S) \to \mathrm{SL}(2,\mathbb{C})$, Gallo Kapovich and Marden gave a Schottky decomposition of ρ along a multiloop M and then constructed a projective structure C with holonomy ρ that admits a Schottky decomposition along M ([GKM00, §4, 5]). We sketch their construction and explains how it implies the Proposition, following the notations in [GKM00].

Let a be an arbitrary element of $\pi_1(S)$ representing an essential loop on S; then modify a in several steps to another loop (namely $d^n x$ in [GKM00, p.650]) that represents to a loop d'. Then d' extends to a multiloop realizing a Schottky decomposition of ρ. Thus, for a sequence of $a_i \in \pi_1(S)$ representing simple loops α_i with $[\alpha_i] \to [L]$ as $i \to \infty$, letting d'_i be the loop given by appropriately applying the above contraction to d_i, we claim that d'_i also converges to $[L]$ because each modification step in the construction preserves the convergence property.

We can assume that α_i are non-separating loops, replacing α_i by a non-separating loop disjoint from α_i. Then the convergence $[\alpha_i] \to [L]$ still holds. By abusing notation, we let elements of $\pi_1(S)$ also denote their corresponding loops on S. Then given $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$, a *handle* is a pair of elements $a, b \in \pi_1(S)$ such that

- a and b are simple loops on S intersecting in a single point, and

- $\rho(a), \rho(b)$ are loxodromic, and they generate a non-elementary subgroup of $\mathrm{PSL}(2, \mathbb{C})$.

By modifying $a_i \in \pi_1(S)$, possibly in a few steps, we obtain a handle H_i by Proposition 3.1.1 in [GKM00] for each i. if necessary after changing a_i, let $H_i = \langle a_i, b_i \rangle$ with some $b_i \in \pi_1(S)$. By the proof of the proposition, we can assume that the projective classes $[a_i]$ and $[b_i]$ also converge to $[L]$ as $i \to \infty$ in $\mathcal{PML}$: Basically each modification is given by chaining a loop ℓ to another loop with a bounded intersection number with ℓ or to a loop "twisted" along ℓ many times.

For each handle H_i, we let $\alpha_i = \rho(a_i)$ and $\beta_i = \rho(b_i) \in \mathrm{PSL}(2, \mathbb{C})$, which are loxodromic elements. Then we may in addition assume that β_i does not take a fixed point of α_i to the other ([GKM00],§4.2). Indeed this modification is done by, if necessary, replacing $\langle a_i, b_i \rangle$ by a new handle of the form either $\langle a_i b_i^q, b_i \rangle$ or $\langle b_i, a_i b_i^q \rangle$. This modification also preserves the convergence to $[L]$ since a_i and b_i intersects in a single point.

Pick another pair of non-separating loops x_i, y_i in $\pi_1(S)$ such that x_i, y_i intersect in a single point and they are disjoint from a_i and b_i([GKM00],§4.3). Clearly $[x_i]$ and $[y_i]$ converge to $[L]$ as $i \to \infty$. Then the induced multiloop for the Schottky decomposition of ρ contains a loop of the form $d_i^{n_i} x_i$, where $d_i = y_i b_i a_i^{k_i}$, for some $k_i, n_i \in \mathbb{Z}$. Then $d_i^{n_i} x_i$ is a non-separating loop disjoint from $b_i a_i^{k_i}$ ([GKM00],§4.5). Since $\langle a_i, b_i \rangle$ is a handle, the loop $b_i a_i^{k_i}$ intersects the loop a_i in a single point, and thus the projective class $[b_i a_i^{k_i}]$ also converges to $[L]$ as $i \to \infty$. Hence $[d_i]$ and thus $[d_i^{n_i} x_i]$ converge to $[L]$ as $i \to \infty$.

□

Proof of Theorem 12.2. Let L be a uniquely ergodic measured lamination on S. By Proposition 12.3, there are sequences of projective structures C_i with holonomy ρ and admissible loops ℓ_i on C_i converging to $[L]$ in $\mathcal{PML}$.

For $n_i \in \mathbb{N}$, consider the projective structure $\mathrm{Gr}_{\ell_i}^{n_i}(C_i)$ obtained by n_i times grafting C_i along ℓ_i. Then its measured lamination L_{i,n_i}, in Thurston coordinates, converges to $[\ell_i]$ as $n_i \to \infty$ in $\mathcal{PML}$ by Theorem 7.1. We can pick sufficiently large n_i for each i so that $[L_{i,n_i}]$ converges to $[L]$ as $i \to \infty$. Therefore $[L]$ is an accumulation point of $\Phi(\mathcal{P}_\rho)$. Almost all measured laminations are uniquely ergodic lamination and in particular they are dense in $\mathcal{PML}$. Thus $\Phi(\mathcal{P}_\rho)$ is dense in $\mathcal{PML}$. 12.3

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