NECK-PINCHING OF $\mathbb{C}\mathrm{P}^1$-STRUCTURES IN THE $\mathrm{PSL}_2\mathbb{C}$-CHARACTER VARIETY

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Abstract. We characterize a certain neck-pinching degeneration of (marked) $\mathbb{C}\mathrm{P}^1$-structures on a closed oriented surface S of genus at least two. In a more general setting, we take a path of $\mathbb{C}\mathrm{P}^1$-structures C_t ($t \geq 0$) on S which leaves every compact subset in its deformation space, such that the holonomy of C_t converges in the $\mathrm{PSL}_2\mathbb{C}$-character variety as $t \to \infty$. Then it is well known that the complex structure X_t of C_t also leaves every compact subset in the Teichmüller space of S. In this paper, under an additional assumption that X_t is pinched along a loop m on S, we describe the limit of C_t from different perspectives: namely, in terms of the developing maps, holomorphic quadratic differentials, and pleated surfaces.

The holonomy representations of $\mathbb{C}\mathrm{P}^1$-structures on S are known to be non-elementary (i.e. strongly irreducible and unbounded). We also give a rather exotic example of such a path C_t whose limit holonomy is the trivial representation.

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Date: 2024/02/15.

1. Introduction

Let S be a (connected) closed oriented surface of genus at least two, throughout this paper. For a (marked) $\mathbb{CP}^1$-structure C on S, the holonomy of C is a homomorphism $\pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ uniquely determined up to conjugation by $\mathrm{PSL}_2\mathbb{C}$; see §2.2. This correspondence yields the *holonomy map*

$$\mathrm{Hol}\colon \mathsf{P} \to \mathcal{X},$$

where $\mathsf{P}\,(\cong \mathbb{R}^{12g-12})$ is the deformation space of all $\mathbb{CP}^1$-structures on S and χ is the $\mathrm{PSL}_2\mathbb{C}$-character variety of S. Note that there are many $\mathbb{CP}^1$-structures whose holonomy is not discrete.

Hejhal [Hej75] proved that Hol is a local homeomorphism (moreover, it is a local biholomorphic map [Hub81], [Ear81]). However, it is not a covering map onto its image ([Hej75]). Thus it is a natural question to ask how the path-lifting property fails:

Problem 1.1. *(Kapovich* [Kap95, Problem 1]*, see also* [GKM00, Problem 12.5.1]*.) Let C_t $(t > 0)$ be a path of $\mathbb{CP}^1$-structures on S such that*

(1) C_t leaves every compact subset in P at $t \to \infty$, and
(2) the holonomy $\eta_t \in \mathcal{X}$ of C_t converges to $\eta_\infty \in \mathcal{X}$ as $t \to \infty$.

What is the asymptotic behavior of C_t?

In this paper, we give various limiting behaviors to answer Question 1.1 in the "neck-pinching" case.

1.1. Pinching loops on Riemann surfaces.

For each $t \geq 0$, let X_t denote the complex structure on S induced by C_t. Then, by the work of Kapovich ([Kap95], see also [GKM00, Dum17]), the conditions (1) and (2) imply that X_t must also leave every compact subset in the Teichmüller space T (see Corollary 2.3).

We focus on the following basic type of degeneration of X_t. Given a path $X_t \in \mathsf{T}$, X_t is *pinched along a loop m* if

- $\mathrm{length}_{X_t} m \to 0$, and
- if an essential loop ℓ in $S \setminus m$ is *not* homotopic to m, then $\mathrm{length}_{X_t} \ell$ is bounded between two positive numbers for all $t \geq 0$.

Here "length_{X_t}" is either the extremal length of X_t or the hyperbolic length of the uniformization of X_t. (In the augmented Teichmüller space, this definition of pinching is equivalent to saying that X_t accumulates to a compact subset of the boundary stratum corresponding to m being pinched.)

A multiloop is a union of disjoint finitely many essential simple closed curves. Then, similarly, we say that X_t is *pinched along a multiloop M* on S, if,

- for each loop m of M, $\mathrm{length}_{X_t} m \to 0$ as $t \to \infty$, and
- for each loop ℓ in $S \setminus M$ not homotopic to a loop of M, $\mathrm{length}_{X_t} \ell$ is bounded between two positive numbers for all $t \geq 0$.

The *quasi-Fuchsian representation* $\pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ is a discrete faithful representation whose limit set is a Jordan curve in $\mathbb{CP}^1$, the quasi-Fuchsian Space QF is an open subset of the character variety $\mathcal{X}$. There is no path C_t in Problem 1.1, whose limit holonomy η_∞ is in QF. On the other hand, a dense subset of the boundary of QF consists of holonomy representations of $\mathbb{C}\mathrm{P}^1$-structures pinched along loops ([McM91]), and it has been quite important to study such degeneration for the study of Klein groups.

1.2. **Asymptotic behaviors.** One of our main results is that $\operatorname{tr}\eta_\infty(m)$ must be ± 2. In other words, the holonomy along m at $t = \infty$ corresponds to either (i) a parabolic element (which is not the identity) or (ii) the identity of $\mathrm{PSL}_2\mathbb{C}$. We will describe, in both Cases (i) and (ii), the asymptotic behavior of C_t from three different perspectives of $\mathbb{C}\mathrm{P}^1$-structures:

(A) A holomorphic quadratic differential on a marked Riemann surface homeomorphic to S (*Schwarzian parameters*).
(B) A hyperbolic structure on S and a measured lamination, which induces an equivariant pleated surface $\mathbb{H}^2 \to \mathbb{H}^3$ (*Thurston parameters*).
(C) A developing map $f\colon \tilde{S} \to \mathbb{C}\mathrm{P}^1$ and a holonomy representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$. (Developing pair)

The *residue* of a meromorphic quadratic differential q at a pole is the integral of $\pm\sqrt{q}$ around the pole, which is well-defined up to sign (see [GW19]). Given a pole of order two, letting r be its residue, q is expressed as $r^2/z^{-2}dz^2$ for an appropriate parametrization in a neighborhood of the pole (see [Str84, Theorem 6.3]).

Let X be a nodal Riemann surface, and let $\mathring{X}$ be the smooth part of X. Then the *normalization* $\overline{X}$ *of* X is the smooth Riemann surface together with a continuous map $\xi\colon \overline{X} \to X$ such that ξ is a biholomorphic in $\xi^{-1}(\mathring{X})$ and for each node p of X, $\xi^{-1}(p)$ consists of exactly two points. A *regular quadratic differential* on X is a meromorphic quadratic differential $\bar{q}$ on $\overline{X}$ such that

- every pole of $\bar{q}$ has an order at most two and it maps to a node of X, and
- if z_1, z_2 on $\overline{Z}$ map to the same node on X, then the residue around z_1 is equal to that of z_2

(see [Ber74] [LZ]).

For Perspective (A), the path C_t corresponds to a path of pairs (X_t, q_t), $t \geq 0$ in Schwarzian coordinates, where X_t is a marked Riemann surface homeomorphic to S and q_t is a holomorphic quadratic differential q_t on X_t for all $t \geq 0$.

Theorem A.
- *Suppose that X_t is pinched along a loop m. Then, exactly one of the following holds:*
  - *(i) X_t converges to a nodal Riemann surface X_∞ with a single node, and q_t converges to a regular quadratic differential on X_∞ such that the node is at worst a pole of order one (Theorem 10.12.)*
  - *(ii) For every diverging sequence $0 \leq t_1 < t_2 < \dots$, up to a subsequence, X_{t_i} converges to a nodal Riemann surface X_∞ with a single node and q_{t_i} converges to a regular quadratic differential q_∞ on X_∞ such that the residue of each pole is a non-zero integral multiple of $\sqrt{2}\pi$. (Theorem 13.20.)*
- *Suppose that X_t is pinched along a multiloop M consisting of n loops. Then, for every diverging $t_1 < t_2 < \dots$, there is a subsequence such that X_{t_i} converges a nodal Riemann surface X_∞ with n nodes and q_t converges to a meromorphic quadratic differential q_∞ on X_∞ such that each node of X_∞ is, at most, a pole of order two. (Corollary 7.6.)*

The convergence of the holomorphic quadratic differential in Theorem A is normal convergence, and in particular, the $\mathbb{C}\mathrm{P}^1$-structure C_t converges to the $\mathbb{C}\mathrm{P}^1$-structure corresponding to (X_∞, q_∞) minus the node, uniformly on every compact subset.

The space of homomorphisms $\pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ is called the *representation variety*, and the *character variety* $\mathcal{X}$ is the GIT-quotient of the representation variety (see §3). In order to obtain an equivariant object as a limit of C_t, we pick a (continuous) lift $\rho_t \colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ of $\eta_t \in \mathcal{X}$, such that ρ_t converges, as $t \to \infty$, to a homomorphism $\rho_\infty \colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ which maps to η_∞. In fact, we prove the existence of such a lift in Proposition 3.2, since it is not obvious when η_∞ is an elementary representation.

Note that for every discrete faithful representations $\pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$, there is a unique equivariant continuous map $\partial_\infty \pi_1(S) \cong \mathbb{S}^1 \to \mathbb{C}\mathrm{P}^1$ called the Cannon-Thurston map ([Mj14]). This map is closely related to the question which we consider, by identifying the ideal boundary of $\tilde{S}$ with $\mathbb{S}^1$.

Let N be a regular neighborhood of the loop m in S. For $t \geq 0$, let $C_t \cong (\tau_t, L_t)$ be Thurston parameters, where τ_t is a path of marked hyperbolic structures on S and L_t is a path of measured laminations on S (§2.2.2). Fixing a marking $\iota_t \colon S \to \tau$ in its isotopy class, (τ_t, L_t) yields to a ρ_t-equivariant pleated surface $\beta_t \colon \tilde{S} \cong \mathbb{H}^2 \to \mathbb{H}^3$, which changes continuously in $t \geq 0$. Then, in fact, β_t converges to a continuous equivariant map:

Theorem B. *Suppose that X_t is pinched along a loop m. Then, by taking an appropriate path of markings $\iota_t \colon S \to \tau_t$ $(t \geq 0)$, exactly one of the following holds:*

(i) $\rho_\infty(m) \in \mathrm{PSL}_2\mathbb{C}$ is a parabolic element, and $\beta_t \colon \tilde{S} \to \mathbb{H}^3$ converges to a ρ_∞-equivariant continuous map $\beta_\infty \colon \tilde{S} \to \mathbb{H}^3 \cup \mathbb{C}\mathrm{P}^1$ uniformly on compact subset, such that $\beta_\infty^{-1}(\mathbb{C}\mathrm{P}^1)$ is a $\pi_1(S)$-invariant multicurve on $\tilde{S}$ which is $\pi_1(S)$-equivariantly homotopic to $\phi^{-1}(m)$, where $\phi \colon \tilde{S} \to S$ is the universal covering map. (Theorem 10.5).

(ii) $\rho_\infty(m)$ is the identity in $\mathrm{PSL}_2\mathbb{C}$, and, for every sequence $0 \leq t_1 < t_2 < \dots$ diverging to ∞, up to a subsequence, $\beta_{t_i} \colon \tilde{S} \to \mathbb{H}^3$ converges to a ρ_∞-equivariant continuous map $\beta_\infty \colon \tilde{S} \to \mathbb{H}^3 \cup \mathbb{C}\mathrm{P}^1$ such that $\beta_\infty^{-1}(\mathbb{C}\mathrm{P}^1)$ descends either to the loop m or to a subsurface isotopic to one or two components of $S \setminus N$ (§13.0.1.)

Let $f_t \colon \tilde{S} \to \mathbb{C}\mathrm{P}^1$ be the developing map of C_t which is a ρ_t-equivariant local homeomorphism. As C_t changes continuously in t, we may assume that f_t also changes continuously in $t \geq 0$. Such a family (f_t) is unique up to a path of isotopies $S \to S$ in $t \geq 0$ homotopic to the identity.

Pick a regular neighborhood $N$of m. Pick a component $\tilde{N}$ of $\phi^{-1}(N)$. By abuse of notation, we regard the loop m also as the element of $\pi_1(S)$ which preserves $\tilde{N}$. We show that the developing map f_t converges in the complement of $\phi^{-1}(N)$, and the asymptotic behavior on $\partial\phi^{-1}(N)$ is well controlled by the holonomy $\rho_t(m)$. Hyperbolic structures are in particular $\mathbb{C}\mathrm{P}^1$-structures. If a hyperbolic surface has a cusp, it has a neighborhood obtained by quotienting a horodisk in $\mathbb{H}^2$ by the cyclic group generated by a parabolic holonomy around the puncture.

Theorem C. *Suppose that X_t is pinched along a loop m. Then, by an appropriate isotopy of S in $t \geq 0$ homotopic to the identity, exactly one of (i) and (ii) holds.*

(i) • *$\rho_\infty(m)$ is parabolic;*

- *the cusps of C_∞ have horodisk quotient neighborhoods;*
- $f_t\colon \tilde{S} \to \mathbb{C}\mathrm{P}^1$ *converges to a ρ_∞-equivariant continuous map $f_\infty\colon \tilde{S} \to \mathbb{C}\mathrm{P}^1$ uniformly on compact subset, and moreover, there is a multiloop M which is a union of finitely many parallel copies of m such that f_∞ is a local homeomorphism exactly on $\tilde{S}\setminus\phi^{-1}(M)$, and f_∞ takes each component $\tilde{m}$ of $\phi^{-1}(M)$ to its corresponding parabolic fixed point (Theorem 10.9).*

(ii) $\rho_\infty(m) = I$, and for every diverging sequence $t_1 < t_2 < \dots$, up to a subsequence,

- *the restriction of f_{t_i} to $\tilde{S}\setminus\phi^{-1}(N)$ converges to a ρ_∞-equivariant continuous map $f_\infty\colon \tilde{S}\setminus\phi^{-1}(N) \to \mathbb{C}\mathrm{P}^1$, and*
- $\mathrm{Axis}(\rho_{t_i}(m))$ *converges to a geodesic in $\mathbb{H}^3$ or a point in $\mathbb{C}\mathrm{P}^1$ so that f_∞ takes the boundary components of $\tilde{N}$ onto the ideal points (in $\mathbb{C}\mathrm{P}^1$) of $\lim_{i\to\infty}\mathrm{Axis}(\rho_{t_i}(m))$ (Theorem 13.1), where $\mathrm{Axis}(\rho_{t_i}(m))$ is the convex hull of the fixed point on $\mathbb{C}\mathrm{P}^1$ (Definition 3.6).*

Remark 1.2. *If a general $\mathbb{C}\mathrm{P}^1$-structure has a cusp with parabolic peripheral holonomy, there is its cusp neighborhood isomorphic to either a horodisk quotient or a grafting of a horodisk quotient. (See Proposition 5.2.)*

A (2π-)grafting is a cut-and-paste operation of a $\mathbb{C}\mathrm{P}^1$-structure, and it yields a new $\mathbb{C}\mathrm{P}^1$-structure with the same holonomy, by inserting an appropriate cylinder along an (admissible) loop ([Gol87], see also [Kap01, Bab20]). Let n be the number of parallel copies of m constituting M in (i). Then there is another diverging path C'_t of $\mathbb{C}\mathrm{P}^1$-structure on S with holonomy ρ_t and a path of admissible loops m'_t on C'_t for $t \gg 0$ such that C_t is obtained by $2\pi(n-1)$-grafting of C'_t.

In fact, Cases (i) and (ii) in Theorem A, Theorem B, and Theorem C correspond. In particular, the Type (i) degeneration occurs on the boundary of the quasi-Fuchsian space, by pinching a loop on a Bers slice.

On the other hand, Type (ii) degeneration is new indeed. In particular, η_t must be a non-discrete representation for all sufficiently large $t > 0$, possibly except at $t = \infty$ (Theorem 13.21). Notice that if the peripheral loop of a cusp of a $\mathbb{C}\mathrm{P}^1$-structure has trivial holonomy, then the $\mathbb{C}\mathrm{P}^1$-structure can be deformed without changing its holonomy (of the entire surface), by moving the cusp (c.f. Theorem 5.6). Then, since $\rho_\infty(m) = I$, therefore it is necessary to take a subsequence. In §14, we give examples of Type (ii) degenerations.

Next, we explain a certain uniform bound of C_t, which yields the convergence of C_t away from the pinched loop m. This uniform bound holds for a more general path C_t with a multiloop being pinched. The integration of $\sqrt{q_t}$ along paths on X_t yields a singular Euclidean structure E_t on X_t such that a zero of order d of q_t is the singular point of cone angle $(d/2+1)\pi$ of E_t (see for example, [FM12, Str84]). Recall that the upper injectivity radius of E_t is the supremum of the injectivity radii over all points in E_t (as E_t is compact it is indeed maximum).

Theorem D. *(Theorem 6.1) Suppose that X_t is pinched along a multiloop. Then the upper injectivity radius of E_t for all $t \ge 0$ is bounded from above.*

It is a classical theorem that the holonomy map Hol is a local homeomorphism for the closed surface S. In the limit of C_t, we have a $\mathbb{C}\mathrm{P}^1$-structure with cusps, such that cusp points are at most poles of order two in the Schwarzian coordinates. The holonomy theorem is proved for such $\mathbb{C}\mathrm{P}^1$-surfaces cusps by Luo ([Luo93]) if punctures have non-trivial peripheral holonomy. In this paper, we prove a more general holonomy theorem (Theorem 5.6) for the developing pairs of $\mathbb{C}\mathrm{P}^1$-structures allowing trivial holonomy around punctures. We apply this holonomy theorem for the convergence on C_t in every thick part as $t \to \infty$. This holonomy theorem is given by appropriately enlarging the character variety, and this enlargement is a certain ramification of the framed representation space introduced by Fock and Goncharov ([FG06]). (For recent developments on $\mathbb{C}\mathrm{P}^1$-structure corresponding to higher order poles, see [GM21, AB20].)

Gallo, Kapovich, and Marden algebraically characterized the image of Hol; in particular, it is almost onto one of the two components of the character variety χ ([GKM00]). To be more precise, $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \in \operatorname{Im}\mathrm{Hol}$ if and only if $\operatorname{Im}\rho$ is non-elementary and ρ lifts to a homomorphism from $\pi_1(S)$ into $\mathrm{SL}(2,\mathbb{C})$. As an example of Type (ii) degeneration, we construct a path C_t whose holonomy limits to an elementary representation, or even to the trivial representation in the representation variety (§14).

If the holonomy of a $\mathbb{C}\mathrm{P}^1$-structure around a puncture is trivial, as stated above, the $\mathbb{C}\mathrm{P}^1$-structure can be deformed around the puncture without changing the holonomy of the entire surface. A non-elementary subgroup of $\mathrm{PSL}_2\mathbb{C}$ has a non-trivial stabilizer, a similar difficulty occurs when the limit holonomy of a component of $S \setminus m$ is elementary. As a result of such flexibility, we have rather exotic degenerations described in Case (ii) of Theorem B and Theorem C.

One may certainly hope that some of the results extend to a more general setting of Problem 1.1. In particular, Theorem D may hold in general:

Conjecture E. *In the setting of Problem 1.1 (without the neck-pinching assumption), let E_t be the singular Euclidean structure on X_t given by the Schwarzian parameters of C_t. Then the upper injectivity radius of E_t is bounded from above uniformly in $t \geq 0$.*

Recall that ρ_t $(t \geq 0)$ is a topological path in the character variety χ which converges to ρ_∞ as $t \to \infty$ without any regularity assumption. It is plausible that Cases (ii) in Theorem A, Theorem B and Theorem C do not occur if ρ_t has a one-side derivative at $t = \infty$ (in the ambient affine space of χ).

Conjecture F. *Suppose that X_t is pinched along a loop m. If the path ρ_t is tangential at $t = \infty$, Then $\eta_\infty(m) \in \mathrm{PSL}_2\mathbb{C}$ is a parabolic element (not equal to the identity I).*

1.3. **Outline of this paper.** In §2, we recall $\mathbb{C}\mathrm{P}^1$-structures, the Schwarzian parameters, Thurston parameters, and the Epstein surfaces for $\mathbb{C}\mathrm{P}^1$-structures. In §3, we prove a lifting property of paths in the character variety to paths in the representation variety. In §4, we give some estimates of the Epstein surfaces, based on Dumas' work [Dum17]. In §5, we prove a holonomy theorem for the space of developing pairs of $\mathbb{C}\mathrm{P}^1$-structures on surfaces with punctures, where punctures are at most poles of order two. In §6, we show that there is an upper bound for the upper injectivity radius of E_t for all $t \geq 0$.

In §7, we show that C_t converges on every thick part as $t \to \infty$, so that C_t converges to a $\mathbb{C}\mathrm{P}^1$-structure on a surface with two punctures homeomorphic to $S \setminus m$. In §8, we state our main theorems and prove some properties of developing maps of a surface with punctures. The limit holonomy around m can only be parabolic or the identity. This will be shown, in §11 and §12. In §10, we determine the asymptotic behavior of C_t when $\rho_\infty(m)$ is parabolic. In §13, we give the asymptotic behavior of C_t when $\rho_\infty(m) = I$.

In §14, we give new examples realizing (ii) in Theorem A, Theorem B, Theorem C.

1.4. **Acknowledgements.** I thank Dick Canary, David Dumas, Ko Honda, Misha Kapovich, and Joan Porti. The work done here is partially supported by German

Research Foundation (BA 5805/1-1), GEAR Network (U.S. National Science Foundation grants DMS 1107452, 1107263, 1107367), and a JSPS Grant-in-Aid for Research Activity start-up (18H05833).

I deeply appreciate the anonymous referees for carefully and patiently reading through my original manuscript and giving various comments which significantly improved this paper.

2. Preliminaries

2.1. **Hyperbolic geometry.** Let τ be a hyperbolic structure on S. Let L be a geodesic measured lamination on τ. Given a geodesic loop m on τ, for a point x in the intersection of m and L, let $\angle_x(L, m) \in [0, \pi)$ denote the intersection angle of of the leaf L and m intersecting at x. Then, the *angle* $\angle_\tau(m, L) \in [0, 1)$ between L and m be the maximum of $\angle_x(L, m)$ over all intersection points $x \in L \cap m$ if $L \cap m \neq \emptyset$, and $\angle_\tau(m, L) = 0$ if $L \cap m = \emptyset$.

Let $\phi\colon \mathbb{H}^2 \to \tau$ denote the universal covering map. Then the ϕ-inverse image $\tilde{L}$ of L is a $\pi_1(S)$-invariant measured lamination on $\mathbb{H}^2$. The pair (τ, L) induces a *bending map* $\beta\colon \mathbb{H}^2 \to \mathbb{H}^3$ which is equivariant via an associated homomorphism $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$. This mapping β is defined by bending the universal cover $\mathbb{H}^2$ of τ along L, where the bending angle is given by the transversal measure of $\tilde{L}$ ([EM87]). Then the pair (τ, L) determines $\beta\colon \mathbb{H}^2 \to \mathbb{H}^3$ uniquely up to $\mathrm{PSL}_2\mathbb{C}$; thus the pair (β, ρ) is identified with $(\alpha \circ \beta, \alpha\rho\alpha^{-1})$ for $\alpha \in \mathrm{PSL}_2\mathbb{C}$.

It follows from Corollary 4.3 in [Bab15] (see also Theorem 5.1 in [Bab17]) that, if a geodesic loop on τ intersects the lamination in a small angle, then the holonomy along the loop must be hyperbolic.

Theorem 2.1. *There is a universal constant $\delta > 0$ such that if $\angle_\tau(L, m) < \delta$, then $\rho(m)$ is hyperbolic.*

Proof. Let $\tilde{m}$ be a lift of m to the bi-infinite geodesic in the universal cover $\tilde{\tau} = \mathbb{H}^2$. Then, the restriction of β to $\tilde{m}$ is a $(1 + \epsilon)$-bilipschitz embedding (Corollary 4.3 in [Bab15]). Since β is ρ-equivariant, $\rho(m)$ is a hyperbolic element whose axis connects the ideal point of the bilipschitz embedding $\beta(\tilde{m})$. □

2.2. $\mathbb{C}\mathrm{P}^1$**-structures.** (General references of $\mathbb{C}\mathrm{P}^1$-structures are found in [Dum09, Kap01].)

A $\mathbb{C}\mathrm{P}^1$*-structure* C, or a *complex projective structure*, on S is a $(\mathbb{C}\mathrm{P}^1, \mathrm{PSL}_2\mathbb{C})$-structure, i.e. an atlas of charts embedding into $\mathbb{C}\mathrm{P}^1$ with transition maps given by $\mathrm{PSL}_2\mathbb{C}$.

Let $\tilde{S}$ be the universal cover of S. Then, equivalently, a $\mathbb{C}\mathrm{P}^1$-structure is a pair (f, ρ) of a local homeomorphism $f\colon \tilde{S} \to \mathbb{C}\mathrm{P}^1$ and a homomorphism $\pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ such that f is ρ-equivariant. The map f is called the *developing map* and ρ is called the *holonomy representation* of C.

The pair is defined up to $\mathrm{PSL}_2\mathbb{C}$, i.e. $(f, \rho) \sim (\alpha f, \alpha\rho\alpha^{-1})$ for all $\alpha \in \mathrm{PSL}_2\mathbb{C}$. Thus the holonomy is in the character variety $\chi = \mathrm{Hom}(\pi_1(S), \mathrm{PSL}_2\mathbb{C}) /\!\!/ \mathrm{PSL}_2\mathbb{C}$.

2.2.1. *Schwarzian parametrization.* Each $\mathbb{C}\mathrm{P}^1$-structure corresponds to a holomorphic quadratic differential q on a marked Riemann surface X. Thus the deformation space P of $\mathbb{C}\mathrm{P}^1$-structures is an (affine) vector bundle over the Teichmüller space T, such that a fiber over a Riemann surface X is the vector space $Q(X)$ of holomorphic quadratic differentials on X (in fact, it is the cotangent bundle). In this paper, considering the projection map $\Pi\colon \mathsf{P} \to \mathsf{T}$ given by the uniformization, we regard the space of marked hyperbolic structures on S as our real analytic zero section.

Although $\mathrm{Hol}\colon \mathsf{P} \to \mathcal{X}$ is a highly non-proper map ([Hej75]), for each $X \in \mathsf{T}$, the restriction of Hol to the space $Q(X)$ is a proper embedding onto a complex analytic subvariety of $\mathcal{X}$ (see [GKM00, Theorem 11.4.1] and its proof). Moreover

Theorem 2.2 ([Kap95, Tan99])**.** *For every compact subset K of T, the restriction of* Hol *to $\Pi^{-1}(K)$ is a proper map.*

Corollary 2.3. *Suppose that $C_t \in \mathsf{P}$ leaves every compact subset in P and its holonomy ρ_t converges in $\mathcal{X}$. Then the complex structure X_t of C_t also leaves every compact subset in T as $t \to \infty$.*

2.2.2. *Thurston's parametrization of* $\mathbb{C}\mathrm{P}^1$*-structures.* ([KP94a, KT92], see also [Bab20].) Thurston gave a homeomorphism

$$\mathsf{P} \cong \mathsf{T} \times \mathsf{ML},$$

where T is the space of marked hyperbolic structures on S and ML is the space of measured laminations on S.

A pair $(\tau, L) \in \mathsf{T} \times \mathsf{ML}$ yields a pleated surface $\mathbb{H}^2 \to \mathbb{H}^3$ equivariant under the holonomy $\pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ of its corresponding $\mathbb{C}\mathrm{P}^1$-structure on S. Given a $\mathbb{C}\mathrm{P}^1$-structure C on S, its associated *collapsing map* $\kappa\colon C \to \tau$ is a marking preserving continuous map which relates the developing map and the bending map of C. First, there is a measured lamination $\mathcal{L}$ on C consisting of circular leaves, such that topologically $\mathcal{L}$ is obtained by replacing each periodic leaf ℓ of L by cylinder foliated circumferences so that the weight of ℓ is equal to the total transversal measure of the foliated cylinder. The collapsing map κ, conversely, collapses such foliated cylinders of $\mathcal{L}$ to their corresponding periodic leaves of L, and κ takes the strata of $\mathcal{L}$ to the strata of L.

Moreover, κ relates the developing map $f\colon \tilde{S} \to \mathbb{C}\mathrm{P}^1$ and the pleated surface $\beta\colon \mathbb{H}^2 \to \mathbb{H}^3$ in an equivariant manner: For each $z \in \tilde{S}$, let B_z be the *maximal ball* in $\tilde{C}$ whose core contains z. Let $\Psi_z\colon B_z \to \mathrm{Conv}\partial_\infty B_z \subset \mathbb{H}^3$ denote the orthogonal projection, where $\mathrm{Conv}\partial_\infty B_z$ is the hyperbolic plane (*support plane*) bounded by the boundary circle. Then, in fact, the commutativity

$$\beta \circ \tilde{\kappa}(z) = \Psi_z f(z),$$

holds equivariantly, where $\tilde{\kappa}\colon \tilde{C} \cong \mathbb{H}^2 \to \tilde{\tau}$ be the lift of κ to a map between universal covers. Note that there is a canonical normal direction of the support plane $\mathrm{Conv}\partial B_z$ at $\Psi_z f(z)$ toward $f(z)$.

2.3. **Epstein maps.** Let $C = (X, q)$ be a $\mathbb{C}\mathrm{P}^1$-structure on S in the Schwarzian coordinates, where X is the complex structure of X, and q is a holomorphic quadratic differential on X. Then, the integration of $\sqrt{q}$ along paths yields a singular Euclidean metric E on X in the same conformal class (see for example [FM12]). In the complex plane, the lines parallel to the real axis give a foliation of $\mathbb{C}$, and it has a transversal measure induced by the vertical length (*horizontal measured foliation*). Similarly, the lines parallel to the imaginary axis give a foliation of $\mathbb{C}$, and it has a transversal measure induced by the horizontal length (*vertical measured foliation*). Then, by pulling back the vertical and the horizontal foliations of $\mathbb{C}$, we obtain a vertical singular measured foliation V and a horizontal singular measured foliation H on E, where the singular points are the zeros of the differential q. Moreover H and V are orthogonal, and the vertical and the horizontal foliation of $\mathbb{C}$ are orthogonal.

Given a point $x \in \mathbb{H}^3$, we can normalize the unit disk model of $\mathbb{H}^3$ so that x is the center of the disk; then the ideal boundary of $\mathbb{H}^3$ has the spherical metric uniquely determined by $x \in \mathbb{H}^3$.

Theorem 2.4 (Epstein [Eps]). *Given a $\mathbb{C}\mathrm{P}^1$-structure $C = (f, \rho)$ on S, there is a unique continuous ρ-equivariant map* $\mathrm{Ep}\colon \tilde{X} \to \mathbb{H}^3$*, such that, for every point $z \in \tilde{X}$, the Euclidean metric of $\tilde{E}$ at z agrees with the spherical metric at $f(z) \in \mathbb{C}\mathrm{P}^1$ when $\mathbb{C}\mathrm{P}^1$ is identified with $\mathbb{S}^2$ so that* $\mathrm{Ep}(z) \in \mathbb{H}^3$ *is at the center of the disk model of $\mathbb{H}^3$.*

Moreover $\mathrm{Ep}\colon \tilde{X} \to \mathbb{H}^3$ *is smooth away from the singular points of $\tilde{E}$ (see Equation (3.1) in* [Dum17]*).*

Let $U\mathbb{H}^3$ denote the unit tangent bundle of $\mathbb{H}^3$. Then Ep lifts to a (Lagrangian) immersion $\mathrm{Ep}_*\colon T\tilde{E} \to U(\mathbb{H}^3)$ ([Dum17, Lemma 3.2]) which is a unit normal vector of the surface $\mathrm{Ep}\colon \tilde{X} \to \mathbb{H}^3$ in the complement of the singular points of E. For $z \in \tilde{X}$, let $d(z)$ denote the Euclidean distance from z to the set Z of the zeros of the differential q.

Lemma 2.5 (Lemma 2.6, Lemma 3.4 in [Dum17]). *Let $h'(z)$ and $v'(z)$ be the horizontal and vertical unit tangent vectors at $z \in \tilde{X} \setminus \tilde{Z}$. If $\frac{6}{d(z)^2} < \frac{3}{4}$, then*

(1) $\| \mathrm{Ep}_*\, h'(z)\| < \frac{6}{d(z)^2}$,
(2) $\sqrt{2} < \| \mathrm{Ep}_*\, v'\| < \sqrt{2} + \frac{6}{d(z)^2}$,
(3) $h'(z), v'(z)$ *are the principal directions of* Ep *at z, and*
(4) $|k_v| < \frac{6}{d(z)^2}$, *where k_v is the curvature of* Ep *in the v-direction.*

3. A lifting property of paths in the character variety

Definition 3.1. *A representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ is* elementary *if $\mathrm{Im}\,\rho$ fixes a point in $\mathbb{H}^3 \cup \mathbb{C}\mathrm{P}^1$ or preserves two points on $\mathbb{C}\mathrm{P}^1$. Equivalently, ρ is elementary if $\mathrm{Im}\,\rho$ is strongly irreducible and $\mathrm{Im}\,\rho$ is unbounded in $\mathrm{PSL}_2\mathbb{C}$. Otherwise ρ is called non-elementary.*

Let $\mathcal{R}$ denote the $\mathrm{PSL}_2\mathbb{C}$-*representation variety* of S, the space of representations $\pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$. By fixing a generating set $\gamma_1, \dots, \gamma_n$, the topology of $\mathcal{R}$ is the restriction of the product topology on $\mathrm{PSL}_2\mathbb{C}^n$, which is independent on the choice of $\gamma_1, \dots, \gamma_n$. The Lie group $\mathrm{PSL}_2\mathbb{C}$ acts on $\mathcal{R}$ by conjugation, and its GIT-quotient

$$\Psi\colon \mathcal{R} \to \chi = \{\pi_1(S) \to \mathrm{PSL}_2\mathbb{C}\} /\!\!/ \mathrm{PSL}_2\mathbb{C}$$

is called the $\mathrm{PSL}_2\mathbb{C}$-character variety of S.

Each fiber of this GIT-quotient is an *extended orbit equivalence* class: Namely, for $\rho_1, \rho_2\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$, $\rho_1 \sim \rho_2$ if and only if the closure of the $\mathrm{PSL}_2\mathbb{C}$-orbit of ρ_1 intersects that of ρ_2 in $\mathcal{R}$. In fact, equivalently $\rho_1 \sim \rho_2$ if and only if $\mathrm{tr}^2 \rho_1(\gamma) = \mathrm{tr}^2 \rho_2(\gamma)$ for all $\gamma \in \pi_1(S)$ [HP04]. In particular, for a non-elementary representation $\pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$, its $\mathrm{PSL}_2\mathbb{C}$-orbit is a closed subset of $\mathrm{PSL}_2\mathbb{C}$ and form a single equivalence class ([New]). For $\rho \in \mathcal{R}$, let $[\rho]$ denote it equivalent class $\Psi(\rho)$ in χ.

Proposition 3.2. *Suppose that C_t $(t \geq 0)$ is a one-parameter family of $\mathbb{C}\mathrm{P}^1$-structures on S, such that its holonomy $\eta_t \in \chi$ converges to $\eta_\infty \in \chi$. Then η_t lifts a path $\rho_t \in \mathcal{R}$ which converges to $\rho_\infty \in \mathcal{R}$ as $t \to \infty$, so that $[\rho_\infty] = \eta_\infty$.*

Remark 3.3. *The limit η_∞ can be an elementary representation (§14), and thus this proposition is nontrivial. In addition, there is $\eta \in \mathcal{R}$ with $[\eta] = \rho_\infty$ such that there is no lift η_t of ρ_t ending at η.*

Proof of Proposition 3.2. Fix a generating set $\gamma_1, \dots, \gamma_n$ of $\pi_1(S)$. We divide the proof into three cases:

(1) η_∞ is non-elementary.
(2) η_∞ is elementary and there is $\gamma \in \mathrm{PSL}_2\mathbb{C}$ such that $\eta_\infty(\gamma)$ is hyperbolic, i.e. $\mathrm{tr}^2(\gamma) \in \mathbb{C} \setminus [0,4]$.
(3) η_∞ is elementary and there is no hyperbolic element in its image, i.e. $\mathrm{tr}^2\,\eta_\infty(\gamma) \in [0,4]$ for all $\gamma \in \pi_1(S)$.

Case 1.

Lemma 3.4. *Suppose that η_∞ is non-elementary. For every lift $\rho_\infty \in \mathcal{R}$ of $\eta_\infty \in \mathcal{X}$, there is a lift $\rho_t \in \mathcal{R}$ of the path $\eta_t \in \mathcal{X}$ such that $\rho_t \to \rho_\infty$ as $t \to \infty$.*

Proof. Over non-elementary representations, Ψ is a fiber bundle with fibers $\mathrm{PSL}_2\mathbb{C}$. This implies the lemma. □

Case 2. Suppose that η_∞ is elementary and there is $\gamma \in \pi_1(S)$ such that $\eta_\infty(\gamma)$ is hyperbolic. Then, if $\rho \in \Psi^{-1}(\eta_\infty)$, letting ℓ be the axis of the hyperbolic element $\rho(\gamma)$, we have either:

(i) $\mathrm{Im}\,\rho$ preserves ℓ and contains an elliptic element which reverses the orientation of ℓ, or
(ii) $\mathrm{Im}\,\rho$ pointwise fixes the endpoints of ℓ on $\mathbb{C}\mathrm{P}^1$.

Case (i). Suppose that $\rho \in \Psi^{-1}(\eta_\infty)$ contains an elliptic element which exchanges the endpoints of ℓ.

Claim 3.5. *There are generators $\gamma_1, \gamma_2, \ldots, \gamma_n$ of $\pi_1(S)$, such that, for each $i = 1, \ldots, n$,*

(1) $\rho(\gamma_i)$ is a hyperbolic element for $i = 1, \ldots, n-1$, and
(2) $\rho(\gamma_n)$ is an elliptic element of order two about a geodesic orthogonal to ℓ.

Proof. By the hypothesis, one can pick generators $\gamma_1, \gamma_2, \ldots, \gamma_n$ of $\pi_1(S)$, such that $\rho(\gamma_1)$ is a (nontrivial) hyperbolic element. Then we can, in addition, assume that $\rho(\gamma_2), \ldots, \rho(\gamma_n)$ are not I, by composing $\gamma_i\ (i \geq 2)$ with γ_1 if necessary. If $\rho(\gamma_i)$ is an elliptic element preserving the orientation of ℓ, then $\rho(\gamma_1\gamma_i)$ is hyperbolic— thus without loss of generality, we can assume that if $\rho(\gamma_i)$ is an elliptic element, it must reverse the orientation of ℓ. Suppose that $\rho(\gamma_i)$ and $\rho(\gamma_j)$ are both elliptic elements reversing the orientation of ℓ; then $\rho(\gamma_i\gamma_j)$ preserves the orientation of ℓ. Thus, by replacing γ_j with $\gamma_i\gamma_j$, we can reduce the number of the generators which map to elliptic elements reversing the orientation of ℓ. We can repeat such replacements of generators, we obtain a desired generating set. □

Let $\gamma_1, \gamma_2, \ldots, \gamma_n$ be the generating set of $\pi_1(S)$ obtained by Claim 3.5. We show that there is indeed a lift ρ_t in $\mathcal{R}$ of η_t converging to ρ as $t \to \infty$.

One can easily find a lift $\rho_t\,(t \geq 0)$ so that $\rho_t(\gamma_1)$ converges to $\rho(\gamma_1)$. Then $\mathrm{Axis}(\rho_t(\gamma_1))$ must converge to ℓ. For all $1 \leq i \leq n-1$, $\mathrm{Axis}(\rho_t(\gamma_i))$ and $\mathrm{Axis}(\rho_t(\gamma_n))$ are asymptotically orthogonal, as η_∞ is an equivalence class of some elementary representation. In particular, we can in addition assume that $\rho_t(\gamma_n)$ converges to $\rho(\gamma_n)$, so that $\mathrm{Axis}(\rho_t(\gamma_n))$ converges to a geodesic m orthogonal to ℓ. Then, for $1 < i < n$, $\mathrm{Axis}(\rho_t(\gamma_i))$ converges ℓ, since it is asymptotically orthogonal to m and η_∞ is elementary. Thus ρ_t converges to ρ as $t \to \infty$.

Case (ii). Next suppose that $\rho \in \Psi^{-1}(\eta_\infty)$ preserves the endpoints of ℓ. Then, similarly to Claim 3.5, we can find a generating set $\gamma_1, \ldots, \gamma_n$ such that $\eta_\infty(\gamma_1), \ldots, \eta_\infty(\gamma_n)$ are all hyperbolic elements (i.e. $\mathrm{tr}^2\,\eta_\infty(\gamma_i) \in \mathbb{C} \setminus [0,4]$).

Pick any lift ρ_t of η_t for $t \geq 0$ (which may not converge as $t \to \infty$).

Fix a $\mathrm{PSL}_2\mathbb{C}$-invariant metric on the projectivized unit tangent bundle $\mathrm{PT}^1\mathbb{H}^3$ of $\mathbb{H}^3$. Then, given two geodesics ℓ_1, ℓ_2 in $\mathbb{H}^3$, we can measure their distance by embedding ℓ_1 and ℓ_2 into the bundle. Thus, similarly, for all $1 \leq i, j \leq n$, the distance between $\mathrm{Axis}(\rho_t(\gamma_i))$ and $\mathrm{Axis}(\rho_t(\gamma_j))$ goes to zero as $t \to \infty$, since otherwise η_∞ is an equivalent class of some non-elementary representations due to the limit of $\rho_t(\gamma_i)$ and $\rho_t(\gamma_j)$. Thus we can continuously conjugate ρ_t by elements of $\mathrm{PSL}_2\mathbb{C}$ so that all axes of $\rho_t(\gamma_1), \ldots, \rho_t(\gamma_n)$ converge to geodesics sharing an endpoint. Therefore ρ_t converges as $t \to \infty$ by this normalization.

Case 3. Suppose that $\mathrm{Im}\,\eta_\infty$ contains no hyperbolic elements. Given an elliptic element and a parabolic element in $\mathrm{PSL}_2\mathbb{C}$ sharing a fixed point on $\mathbb{C}\mathrm{P}^1$ then their product is an elliptic element. Therefore we can pick generators $\gamma_1, \ldots, \gamma_n$ of $\pi_1(S)$, such that $\eta_\infty(\gamma_i)$ are either all elliptic or all parabolic: In fact, given a generating set $\gamma_1, \ldots, \gamma_n$, if the η_∞-image of at least one γ_i is elliptic, then by replacing γ_j with parabolic $\eta_\infty(\gamma_j)$ with $\gamma_i\gamma_j$, we obtain a generating set with elements whose η_∞-images are all elliptic. Pick any lift $\rho_t \in \mathcal{R}$ of the path $\eta_t \in \mathcal{X}$ for $t \geq 0$, which may not converge as $t \to \infty$.

Definition 3.6. *For $\gamma \in \mathrm{PSL}_2\mathbb{C}$, the axis of γ is the convex hull of the fixed point set in $\mathbb{H}^3 \cup \mathbb{C}\mathrm{P}^1$ of γ, and we denote it by $\mathrm{Axis}(\gamma) \subset \mathbb{H}^3 \cup \mathbb{C}\mathrm{P}^1$.*

In particular, if γ is hyperbolic or elliptic, $\mathrm{Axis}(\gamma)$ is a geodesic in $\mathbb{H}^3$ plus its endpoints in $\mathbb{C}\mathrm{P}^1$, and if γ is parabolic, $\mathrm{Axis}(\gamma)$ is a single point on $\mathbb{C}\mathrm{P}^1$. Clearly an ideal point of $\mathrm{Axis}(\gamma)$ is a fixed point of γ on $\mathbb{C}\mathrm{P}^1$.

Suppose that $\gamma, \omega \in \mathrm{PSL}_2\mathbb{C}$ be hyperbolic or elliptic elements with axes ℓ_γ, ℓ_ω. As above, we measure the distance between ℓ_γ, ℓ_ω by embedding them into the projective unit tangent bundle of $\mathbb{H}^3$.

Lemma 3.7.
1. *Suppose that $\eta_\infty(\gamma_i)$ and $\eta_\infty(\gamma_j)$ are both elliptic for distinct $1 \leq i, j \leq n$. Then the distance between $\mathrm{Axis}(\rho_t(\gamma_i))$ and $\mathrm{Axis}(\rho_t(\gamma_j))$ in $\mathrm{PT}^1(\mathbb{H}^3)$ limits to zero as $t \to \infty$.*
2. *Suppose that $\eta_\infty(\gamma_i)$, $\eta_\infty(\gamma_j)$, $\eta_\infty(\gamma_k)$ are all elliptic for distinct $1 \leq i, j, k \leq n$. Then there is a lift $\rho_t \in \mathcal{R}$ of η_t for $t \geq 0$, such that $\mathrm{Axis}(\rho_t(\gamma_i))$, $\mathrm{Axis}(\rho_t(\gamma_j))$, and $\mathrm{Axis}(\rho_t(\gamma_k))$ converge to geodesics sharing a common endpoint on $\mathbb{C}\mathrm{P}^1$.*

Proof. (1) If there is a diverging sequence $0 < t_1 < t_2 < \ldots$ such that the distance between $\mathrm{Axis}(\rho_t(\gamma_i))$ and $\mathrm{Axis}(\rho_t(\gamma_j))$ in $\mathrm{PT}^1(\mathbb{H}^3)$ is bounded from below by a positive number, then η_∞ is non-elementary. This is a contradiction.

(2) By (1), if the assertion of (2) fails, there is a lift ρ_t such that $\mathrm{Axis}(\rho_t(\gamma_i))$, $\mathrm{Axis}(\rho_t(\gamma_j))$, and $\mathrm{Axis}(\rho_t(\gamma_k))$ converge to the distinct edges of an ideal triangle in $\mathbb{H}^3$. Then, η_∞ is non-elementary against the hypothesis. □

Corollary 3.8. *Suppose that there is a generating set $\{\gamma_1, \ldots, \gamma_n\}$ of $\pi_1(S)$, such that $\eta_\infty(\gamma_1), \ldots, \eta_\infty(\gamma_n)$ are all elliptic. Then, there is a lift $\rho_t \in \mathcal{R}$ of η_t such that $\rho_t(\gamma_1), \ldots, \rho_t(\gamma_n)$ converge to elliptic elements whose axes share an endpoint on $\mathbb{C}\mathrm{P}^1$.*

Last we suppose that $\eta_\infty(\gamma_1), \ldots, \eta_\infty(\gamma_n)$ are all parabolic, and we show that there is a lift of ρ_t of η_t to $\mathcal{R}$ such that ρ_t converges to the trivial representation.

Pick a base point $O \in \mathbb{H}^3$. For each $t \geq 0$, let $\delta_{t,i} = d_{\mathbb{H}^3}(O, \rho_t(\gamma_i)O)$. Let $i_t \in \{1, \ldots, n\}$ be such that

$$\delta_{t,i_t} = \max_{1 \leq i \leq n} \delta_{t,i}.$$

Lemma 3.9. *Let $t_1 < t_2 < \ldots$ be a sequence diverging to ∞, such that, at t_{t_k}, the indices $i_{t_k} \in \{1, \ldots, n\}$ $(k = 1, 2, \ldots)$ defined above are a fixed constant h. Suppose that* *there is a sequence* $\omega_{t_k} \in \mathrm{PSL}_2\mathbb{C}$ *such that the conjugation* $\omega_{t_k}\rho_{t_k}(\gamma_h)\omega_{t_k}^{-1} =: \omega_{t_k}\cdot\rho_{t_k}(\gamma_h)$ *converges in* $\mathrm{PSL}_2\mathbb{C}$*, as* $k \to \infty$*, to a parabolic element* $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ *in* $\mathrm{PSL}_2\mathbb{C}$ *with* $u \neq 0$*. Then, for every* $j = 1, \ldots, n$*, the conjugation* $\omega_{t_k}\cdot \rho_{t_k}(\gamma_j)$ *accumulates to a bounded subset of* $\begin{pmatrix} 1 & \mathbb{C} \\ 0 & 1 \end{pmatrix}$ *which has a diameter less than* $|u|$ *in* $\mathbb{C}$*.*

Proof. First we show that, unless $\omega_{t_k}\cdot\rho_{t_k}(\gamma_j) \to I$, the limit of the fixed point set of $\omega_{t_k}\cdot\rho_{t_k}(\gamma_j) \subset \mathbb{C}\mathrm{P}^1$ must converge to $\{\infty\}$. Suppose, to the contrary, that this assertion fails. Then, up to a subsequence, the limit set of the fixed point set of $\omega_{t_k}\cdot\rho_{t_k}(\gamma_j) \subset \mathbb{C}\mathrm{P}^1$ converges to a point on $\mathbb{C}\mathrm{P}^1$ not equal to ∞. For sufficiently large positive integers p, $\omega_{t_k}\cdot\rho_{t_k}(\gamma_h\gamma_i^p)$ are hyperbolic elements and their translation lengths diverge to ∞ as $p \to \infty$([GKM00, Lemma 2.1.1 (iii)]). This contradicts that $\mathrm{Im}\,\eta_\infty$ consists of only parabolic elements.

For each $k = 1, 2, \ldots,$ set

$$\omega_{t_k}\cdot \rho_{t_k}(\gamma_j) = \begin{pmatrix} a_{j,k} & b_{j,k} \\ c_{j,k} & d_{j,k} \end{pmatrix} \tag{1}$$

Thus $c_{j,k} \to 0$ and $a_{j,k}, d_{j,k} \to 1$ as $k \to \infty$. Then the definition of h implies that $b_{t_k,h} - \max_{1\le i\le n} b_{t_k,i} \to 0$. Hence we have the upper bound on the image in $\mathbb{C}$. □

By a straight computation, we obtain the following.

Corollary 3.10. *For every* $j = 1, \ldots, n$*, let* $s_{j,k} > 0$ *be a sequence in* k*, such that* $s_{j,k} \to 0$ *and* $\frac{\sqrt{|c_{j,k}|}}{s_{j,k}} \to 0$*. Then, using the notation from (1), we have*

$$\begin{pmatrix} s_{j,k} & 0 \\ 0 & s_{j,k}^{-1} \end{pmatrix} \begin{pmatrix} a_{j,k} & b_{j,k} \\ c_{j,k} & d_{j,k} \end{pmatrix} \begin{pmatrix} s_{j,k} & 0 \\ 0 & s_{j,k}^{-1} \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2}$$

as $k \to \infty$*.*

Moreover, Corollary 3.10 implies that the sequence $\max_{j=1,\ldots,n} s_{j,k}$ in k yields the convergence (2) for all $j = 1, \ldots, n$. Therefore we have the following.

Proposition 3.11. *There is a continuous path* $\omega_t \in \mathrm{PSL}_2\mathbb{C}$ *such that* $\omega_t\cdot \rho_t(\gamma_i)$ *accumulates to a bounded subset of parabolic elements in* $\begin{pmatrix} 1 & \mathbb{C} \\ 0 & 1 \end{pmatrix}$ *for each* i*. Therefore there is a continuous path* $\omega_t \in \mathrm{PSL}_2\mathbb{C}$ *such that* $\omega_t \cdot \rho_t$ *converges to the trivial representation in* $\mathcal{R}$*.*

We have completed the proof for all cases. 3.2

3.1. **Approximation of moduli.** Let E be a singular Euclidean surface induced by a holomorphic quadratic differential on a Riemann surface X. A *regular annulus* A_E is a cylinder embedded in E such that there is a closed geodesic loop ℓ on E and the annulus A_E is foliated by loops equidistant from ℓ. Minsky gave a useful approximation of the modulus of cylinders.

Theorem 3.12 ([Min92], Theorem 4.6; see also [Ser12], Theorem 6.2)**.** *Let* E *be a singular Euclidean surface induced by a holomorphic quadratic differential on a Riemann surface* X*. There are constant* $0 < c < 1$ *depending on the topology of the surface, such that, for every essential annulus* A *embedded in* X*, there is a regular annulus* A_E *in* E *homotopy equivalent to* A *satisfying* $\mathrm{Mod}(E_A) > c\,\mathrm{Mod}(A)$*.*

4. Holonomy estimates away from zeros

In this section, based on Dumas' work on Epstein surfaces [Dum17], we give some further analysis of the Epstein surfaces in the horizontal direction. We use Dumas' notations as below. Let $g_1 = e^{\alpha_1}|dz|, g_2 = e^{\alpha_2}|dz|$ be two conformal metrics on a Riemann surface; then the Schwarzian derivative of g_2 relative to g_1 is the quadratic differential

$$B(g_1, g_2) = [(\alpha_1)_{zz} - {\alpha_2}_z^2 - (\alpha_1)_{zz} + (\alpha_1)_z^2]dz^2.$$

Let $C = (X, q)$ be a $\mathbb{C}\mathrm{P}^1$-structure on S. Then, we set the following notations associated with C:

- Let τ be the hyperbolic metric on S uniformizing X;
- let $|\sqrt{q}|$ denote the singular Euclidean metric on X obtain by integrating $\sqrt{q}$ along paths;
- let $g_{\mathbb{C}\mathrm{P}^1}$ be the spherical metric on $\mathbb{C}\mathrm{P}^1$ given by some conformal identification $\mathbb{C}\mathrm{P}^1 \cong \mathbb{S}^2$;
- let $f\colon \tilde{X} \to \mathbb{C}\mathrm{P}^1$ be the developing map of C, and $f^*(g_{\mathbb{C}\mathrm{P}^1})$ be the pull back of the conformal metric $g_{\mathbb{C}\mathrm{P}^1}$ by f to the universal cover $\tilde{X}$.

Then set

$$\begin{aligned} \omega &= 2B(\tau, f^*(g_{\mathbb{C}\mathrm{P}^1})), \\ \hat{\omega} &= 2B(|\sqrt{q}|, f^*(g_{\mathbb{C}\mathrm{P}^1})), \\ \nu &= 2B(\sigma, \sqrt{q}), \end{aligned}$$

which are holomorphic quadratic differentials on $\tilde{X}$.

4.1. Curvature of Epstein surfaces in the horizontal direction.

Let k_h and k_v be the principle curvatures of $\mathrm{Ep}\colon \tilde{X} \to \mathbb{H}^3$ in the horizontal and the vertical directions, respectively. First by Equation 3.7 in [Dum17]

$$k_v = \frac{|\hat{\omega}| - |\omega|}{|\hat{\omega}| + |\omega|}.$$

As the Gaussian curvature $\kappa_h \kappa_v = 1$ ([Dum17, p448]), we have

$$k_h = \frac{|\hat{\omega}| + |\omega|}{|\hat{\omega}| - |\omega|}.$$

In addition, recalling that h' denotes a unit tangent vector in the horizontal direction at a non-singular point, we have

$$\| \mathrm{Ep}_* h' \|^2 = \frac{(|\hat{\omega}| - |\omega|)^2}{2|\omega\hat{\omega}|}$$

(Equation 3.6 in [Dum17, p448]). Therefore

$$\begin{aligned} (k_h \| \mathrm{Ep}_*(h') \|)^2 &= (\frac{|\hat{\omega}| + |\omega|}{|\hat{\omega}| - |\omega|})^2 \cdot \frac{(|\hat{\omega}| - |\omega|)^2}{2|\omega\hat{\omega}|} \\ &= 1 + \frac{(|\hat{\omega}|^2 + |\omega|^2)}{2|\omega\hat{\omega}|} \\ &= 1 + \frac{1}{2}(\frac{|\hat{\omega}|}{|\omega|} + \frac{|\omega|}{|\hat{\omega}|}). \end{aligned}$$

Since $\hat{\omega} = \omega - \nu$ ([Dum17, p447]), we have

$$\left|\frac{\hat{\omega}}{\omega}\right| = \left|1 - \frac{\nu}{\omega}\right| \quad \text{([Dum17, p449]).}$$

By [Dum17, Lemma 2.6], we have

$$\left|\frac{\nu(z)}{\omega(z)}\right| \leq \frac{6}{d(z)^2}.$$

Thus, recalling that $d(z)$ is the distance from the singular points, we have

$$\begin{aligned} \frac{|\hat{\omega}(z)|}{|\omega(z)|} &= 1 + O(d(z)^{-2}), \text{ and} \\ (k_h(z)\| \operatorname{Ep}_*(h'(z))\|)^2 &= 2 + O(d(z)^{-2}). \end{aligned}$$

Therefore, we have the following.

Lemma 4.1. *For all nonzero $z \in \tilde{X}$ of the differential $\tilde{q}$,*

$$k_h(z)\| \operatorname{Ep}_*(h'(z))\| = \sqrt{2} + O(d(z)^{-1}).$$

4.2. **Holonomy estimates of long flat cylinders.** Let E be a singular Euclidean surface. A *flat cylinder* in E is a cylinder foliated by closed geodesics. A cylinder A in E is *expanding* if there is a geodesic loop ℓ or a puncture p on E, such that A is foliated by a one-parameter family of circles equidistant from ℓ or p, respectively, whose length strictly increases as the distance to ℓ or p increases.

Let $\operatorname{Ep}\colon \tilde{X} \to \mathbb{H}^3$ be the Epstein surface of a projective structure $C = (X, q)$ on S. Let $\alpha\colon [0,1] \to \tilde{C} \cong \tilde{X}$ be an arc such that $\alpha(0)$ and $\alpha(1)$ are in $\tilde{X} \setminus \tilde{Z}$ and α differentiable at both endpoints. Then the curve $\operatorname{Ep} \circ \alpha\colon [0,1] \to \mathbb{H}^3$ is differentiable at both endpoints. Let $\zeta(\alpha) \in \mathrm{PSL}_2\mathbb{C}$ be such that $\zeta(\alpha)$ takes the unit tangent vector $\alpha'(0)$ to $\alpha'(1)$ on Ep and the unit normal $\operatorname{Ep}_* \alpha(0)$ to the unit normal $\operatorname{Ep}_* \alpha(1)$. We call $\zeta(\alpha) \in \mathrm{PSL}_2\mathbb{C}$ the *holonomy (of* Ep*) along* α.

Definition 4.2. *For $\alpha \in \mathrm{PSL}_2\mathbb{C}$, the rotation angle in $[0, \pi]$ is the (unsigned) rotation angle of the tangent plane of $\mathbb{C}\mathrm{P}^1$ at a fixed point of α.*

In the case that α has two fixed points on $\mathbb{C}\mathrm{P}^1$, then the "signed" rotation angle of α which takes a value in $[-\pi, \pi]/(\pi \sim -\pi)$ at a fixed point is -1 times the "signed" rotation angle at the other fixed point, where the sign is determined by the orientation from $\mathbb{C}\mathrm{P}^1$; thus the unsigned rotation angle is well-defined in Definition 4.2.

Let (E, V) be the singular Euclidean surface given by $C = (X, q)$.

Definition 4.3. *Let $\alpha\colon [0,1] \to \mathbb{H}^3$ be a C^1-smooth arc on the Epstein surface $\tilde{E} \to \mathbb{H}^3$. Let $v(t)$ and $h(t)$ denote the (unit) vector fields along α tangent to the vertical and horizontal foliations of E, respectively.*

Let ℓ be a geodesic in $\mathbb{H}^3$. Let $\mathcal{H}$ be the foliation of $\mathbb{H}^3$ by the totally geodesic hyperbolic planes H orthogonal to ℓ. Note that these hyperbolic planes are isometrically identified by parallel transport along ℓ, and thus their ideal boundary circles are also identified diffeomorphically.

Suppose that $v(t)$ is transversal to the foliation $\mathcal{H}$. Let H_t be the leaf of $\mathcal{H}$ containing $\alpha(t)$. The translation length *of α along ℓ is the distance between H_0 and H_1 (i.e. the length of the segment of ℓ between H_0 and H_1).*

As $v(t)$ is transversal to $\mathcal{H}$, then, by the orthogonal projection $\mathbb{H}^3 \to H_t$, the horizontal tangent vector $h(t)$ projects to a non-zero vector at the tangent space $T_{\alpha(t)}H_t$.

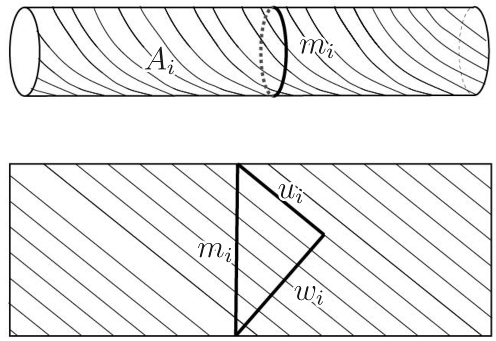

Figure 1. Isotope m_i to a union of a vertical and horizontal segment.

This non-zero tangent vector determines a geodesic ray in $\mathbb{H}^3$ by being its initial tangent direction. Let $\theta(t) \in \partial_\infty H_t$ be the endpoint of the geodesic ray in H_t given by the tangent vector. As all ideal boundaries $\partial_\infty H_t$ are identified, $\theta(t) \in \mathbb{S}^1$ lifts to $\tilde{\theta}(t) \in \mathbb{R}$. The rotation angle *of α about ℓ is the total increase of $\tilde{\theta}(t)$, which takes a value in $\mathbb{R}$.*

Proposition 4.4. *Let $C_i = (f_i, \rho_i)$ be a sequence of $\mathbb{C}\mathrm{P}^1$-structures on S, and let (E_i, V_i) be the pair of a singular Euclidean structure E_i and a vertical foliation V_i on E_i induced by the Schwarzian parameters of C_i. Suppose that there are a loop m on S, a geodesic representative m_i of m on E_i for each i, and a flat cylinder A_i in E_i contains m_i, such that*

- *m_i is in the middle of A_i, so that $A_i \setminus m_i$ is a union of two isometric flat cylinders,*
- *$\mathrm{Mod}(A_i) \to \infty$ as $i \to \infty$, and*
- *the height a_i of A_i diverges to ∞ as $i \to \infty$.*

Let $\bar{m}_i$ be a segment on the universal cover $\tilde{E}_i$ obtained by lifting the simple closed curve m_i. Then, by parametrizing $\bar{m}_i$ by arc length $s \in [0, \mathrm{length}(m_i)]$, for every $\epsilon > 0$, if $i > 0$ is sufficiently large, then

(1) the translation length of $\mathrm{Ep}_i\, \bar{m}_i(s)$ along ℓ_i is $(1+\epsilon)$-bi-Lipschitz to $\sqrt{2}\,(\mathrm{Re} \int_{m_i} \sqrt{q_t})$,

(2) the total rotation angle of $\mathrm{Ep}\, \bar{m}_i$ about ℓ_i is $(1+\epsilon, \epsilon)$-bi-Lipschitz to to $\sqrt{2}\,(\mathrm{Im} \int_{m_i} \sqrt{q_t})$, and

Proof. Isotope m_i in A_i, fixing a point on m_i, so that m_i is a union of a vertical segment u_i and a horizontal segment w_i (Figure 1). Then m_i remains close to the middle of A_i.

We first analyze the vertical segment $\mathrm{Ep}_i\,|u_i$. In the principal direction, the normal vector is preserved by parallel transports. Thus, the parallel transport along the curve $\Sigma_i|u_i$ yields the holonomy $\zeta_i(u_i(s)) \in \mathrm{PSL}_2\mathbb{C}$. By the hypotheses, the distance from the loop $u_i \cup w_i$ and the set Z_i of zeros of the differential q_i diverges to ∞. Therefore, by Lemma 2.5 (4), the curvature along $\mathrm{Ep}_i\,|u_i$ limits to zero, and it asymptotically has a constant speed $\sqrt{2}$ by Lemma 2.5 (2), so that its length is $\sqrt{2}$ times the Euclidean length of u_i, which yields (1).

To analyze the total rotation angle in the vertical direction, we next consider the total curvature. In a more general setting, the following holds.

Lemma 4.5. *For every $\epsilon > 0$, if $R > 0$ is sufficiently large, then, if a vertical segment u on a $\mathbb{C}\mathrm{P}^1$-surface C has Euclidean length less than R/ϵ, then total curvature of the curve $\mathrm{Ep}\,|u$ is less than ϵ, where $\mathrm{Ep}\colon \tilde{C} \to \mathbb{H}^3$ is the Epstein surface of C.*

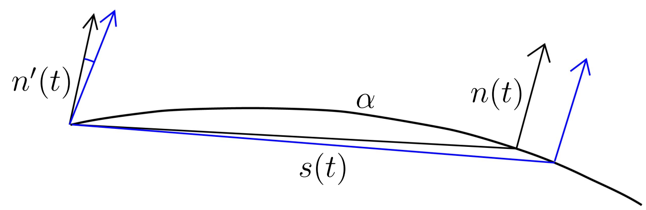

Figure 2. Infinitesimal change of the rotation angle $n'(t)$.

Proof. The curvature of the curve $\mathrm{Ep}\,|u$ at every point on u is bounded from $\frac{6}{R^2}$ by Lemma 2.5 (4). Since, by the hypothesis, the length of u is bounded from above by $\frac{R}{\epsilon}$, the total curvature along u is bounded from above by

$$\frac{R}{\epsilon} \cdot \frac{6}{R^2} = \frac{6}{\epsilon R}.$$

Therefore, if $R > \frac{6}{\epsilon^2}$, then the total curvature along u is bounded from above by ϵ. □

In our current setting, as $a_i \to \infty$ and $\mathrm{Mod}(A_i) \to \infty$, one can easily show that, for every ϵ, the vertical segment u_i satisfies the conditions of Lemma 4.5 when i is sufficiently large. Thus the following corollary holds.

Corollary 4.6. *The total (principal) curvature of the vertical segment* $\mathrm{Ep}_i\,|u_i$ *limits to zero as* $i \to \infty$.

We next show that the rotational holonomy along u_i asymptotically vanishes as $i \to \infty$.

Lemma 4.7. *For every* $\epsilon > 0$*, if* $R > 0$ *is sufficiently large, then, if a vertical segment* v *on a* $\mathbb{C}\mathrm{P}^1$*-surface* C *has length less than* R/ϵ *and a distance at least* R *from the singular set w.r.t. the singular Euclidean structure of* C*, then, letting* Ep *be its Epstein surface, the derivative of rotation of its* Ep*-image is bounded from above by* ϵ*. Moreover, the total rotation of its* Ep*-image bounded from above by* ϵ *with respect to the geodesic* ℓ *connecting the endpoints of* Ep.

Proof. Fix $\epsilon > 0$. Let v be a vertical segment on C of length less than R/ϵ. Let $\alpha\colon [0, \ell] \to \mathbb{H}^3$ be the curve $\mathrm{Ep} \circ v$, where ℓ is the Euclidean length of v. Let $s(t)$ be the geodesic segment in $\mathbb{H}^3$ connecting $\alpha(0)$ and $\alpha(t)$ for each $t \in [0, \ell]$. For $u \in [0, \ell]$, let $\mathrm{Ep}(u)$ be the surface which $s(t)$ sweeps out over $t \in [0, u]$, so that $\mathrm{Ep}(u)$ is bounded by $\alpha([0, u])$ and the geodesic segment $s(u)$ connecting its endpoints. Then, the intrinsic metric of $\mathrm{Ep}(u)$ is a hyperbolic surface. Then, if $R > 0$ is sufficiently large, then $\mathrm{Ep}(u)$ isometrically embeds into a hyperbolic plan $\mathbb{H}^2$ so that its image is bounded by a geodesic segment isometric to $s(u)$ and a curve isometric to $\alpha(u)$. The curvature of the second segment is bounded from above the curvature of $\alpha|[0, u]$ at every point.

Therefore, if $R > 0$ is sufficiently large, then the area of Ep is less than ϵ by the Gauss-Bonnet theorem to Ep, since the total curvature α is small. Let $n(t)$ denote the unit normal vector Ep_* at $u(t)$. Let $n'(t)$ be the parallel transport of $n(t)$ along the geodesic segment $s(t)$, so that $n'(t)$ be a tangent vector at $\alpha(0)$. By the Gauss-Bonnet theorem, the norm of the derivative $dn'(t)/dt$ is bounded from above by the curvature of α and the derivative area of $\mathrm{Ep}(t)$ (Figure 2). Thus, the total rotation of $n'(t)$ from $t = 0$ to $t = \ell$ is bounded from above by the sum of the total curvature of α and the total area of Ep. Therefore, by the combination of the small upper bounds above if $R > 0$ is sufficiently large, the total rotation is bounded by ϵ. □

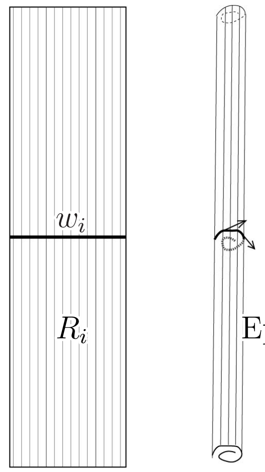

FIGURE 3.

Next we analyze the holonomy along the horizontal segment w_i. By Lemma 2.5 (1),

$$\mathrm{length}_{\mathbb{H}^3} \mathrm{Ep}_i\, w_i < \frac{6\, \mathrm{length}_{E_i} m_i}{(a_i/3)^2} \to 0,$$

as $i \to \infty$.

Proposition 4.8. *Let $v_i(t)$ denote the tangent vector of* Ep_i *at* $\mathrm{Ep}_i\, w_i(t)$ *in the direction of V_i. For every $\epsilon > 0$, if i is large enough, then along w_i,* $\mathrm{Ep}^*\, w_i(t)$ *is contained in an ϵ-ball in the unit tangent bundle* $\mathrm{T}^1\mathbb{H}^3$.

Proof. Let $\tilde{E}_i$ be the universal cover of E_i. Pick a lift $\tilde{u}_i$ of the vertical segment u_i in E_i to $\tilde{E}_i$. Let R_i be a Euclidean rectangle, in $\tilde{E}_i$, bounded by vertical and horizontal edges, such that w_i divides R_i into two isometric rectangles of half height (Figure 3, left). We may in addition assume that the height of R_i divided by the width of R_i goes to zero as $i \to \infty$.

The vertical foliation V_i and the horizontal foliation H_i of E_i induce a vertical and a horizontal foliation of R_i. By Lemma 2.5 (2), for every $\epsilon > 0$, if i is large enough, the restrictions of Ep_i to vertical leaves in R_i are $(\sqrt{2} - \epsilon, \sqrt{2} + \epsilon)$-bi-Lipschitz. By Lemma 2.5 (1), the Ep_i-images of the horizontal leaves in R_i have diameters less than ϵ. Therefore, for sufficiently large i, the images of vertical leaves of R_i are pairwise ϵ-close in the Hausdorff metric (Figure 3 below). As v_i is tangent to the image of such a vertical leaf, we have the lemma. □

We have already shown a good approximation of the holonomy along the vertical segment u_i. For every $\epsilon > 0$, if i is sufficiently large, then the translation length along u_i is $(1 + \epsilon)$-bilipschitz to $\sqrt{2}$ times the Euclidean length of u_i and the rotation is less than ϵ (Lemma 4.7). On the other hand, by Proposition 4.8 and Lemma 4.1, if i is sufficiently large, then the total rotation along the horizontal segment w_i is $(1 + \epsilon, \epsilon)$-bi-Lipschitz to $\sqrt{2}$ times the Euclidean length of w_i and the translation is less than ϵ. Thus we obtained, (1) and (2). 4.4

4.3. **The exponential map and Epstein surfaces.** Recall that, given a $\mathbb{C}\mathrm{P}^1$-structure $C = (X, q)$ on S, for $x \in C$, $d(x)$ is the Euclidean distance from the singular set of the singular Euclidean structure E induced by the holomorphic quadratic differential

q. Note that, if $x \in C$ is not a singular point of E, then there is a neighborhood U of x in E so that U is isometrically embedded in the Euclidean plane $\mathbb{C} \cong \mathbb{E}^2$ so that vertical leaves of E in U map into horizontal lines of $\mathbb{C}$, and horizontal leaves map into vertical lines.

Consider the $\exp\colon \mathbb{C} \to \mathbb{C} \setminus \{0\}$. Its domain $\mathbb{C}$ is isometrically identified with the Euclidean plane $\mathbb{E}^2$, and the codomain $\mathbb{C} \setminus \{0\}$ admits a push-forward Euclidean metric. Note that this induced Euclidean metric on $\mathbb{C} \setminus \{0\}$ is invariant under the dilations $\mathbb{C} \to \mathbb{C} : z \mapsto kz$ for all $k \in \mathbb{C} \setminus \{0\}$. Therefore, given, any two distinct points p, q in $\mathbb{CP}^1$, by a conformal mapping from $\mathbb{CP}^1 \setminus \{p, q\}$ to $\mathbb{C} \setminus \{0\}$, the complement $\mathbb{CP}^1 \setminus \{p, q\}$ has the push-forward Euclidean metric. By abuse of notation, we denote this composition by $\exp\colon \mathbb{C} \to \setminus \{p, q\}$ and call it the *normalized exponential map.*

Let (p, q) be the geodesic in $\mathbb{H}^3$ connecting p to q. Recalling that $\mathbb{CP}^1$ is the ideal boundary of $\mathbb{H}^3$, let $\Psi\colon \mathbb{CP}^1 \setminus \{p, q\} \to (p, q)$ be the orthogonal projection along a geodesic rays in $\mathbb{H}^3$. Let $\Psi_*\colon \mathbb{CP}^1 \setminus \{p, q\} \to T^1\mathbb{H}^3$ be the map taking $z \in \mathbb{CP}^1 \setminus \{p, q\}$ to the unit tangent vector at $\Psi(z) \in \mathbb{H}^3$ which is tangent to the geodesic ray from $\Psi(z)$ to $z \in \partial\mathbb{H}^3$.

For $r > 0$, let $Q_r(z)$ be a r-neighborhood of a point z of the singular Euclidean surface E in the L^∞-metric (w.r.t. the vertical and the horizontal directions). If $Q_r(z)$ contains no singular point, then it is a square with horizontal and vertical edges of length $2r$.

Proposition 4.9. *For every $\epsilon > 0$, there is $R > 0$ such that, if $z \in \tilde{C}$ satisfies $d(z) > R$, then we have a normalized exponential map $\exp\colon \mathbb{C} \to \mathbb{CP}^1 \setminus \{p, q\}$ and can isometrically embed the $\frac{1}{\epsilon}$-neighborhood $Q_{1/\epsilon}(z)$ of z in $\mathbb{C}$ exchanging the horizontal and the vertical directions, such that, in the C^0-metric,*

(1) the restriction of the Epstein surface Σ to $Q_{1/\epsilon}(z)$ is ϵ-close to $w \mapsto \Psi_* \exp(\frac{w}{\sqrt{2}})$,

(2) the restriction of Σ_* to $Q_{1/\epsilon}(z)$ of z is ϵ-close to $w \mapsto \Psi_* \exp(\frac{w}{\sqrt{2}})$, and

(3) the restriction of the developing map f to $Q_{1/\epsilon}(z)$ is ϵ-close to the normalized exponential map.

Proof. we prove the desired approximations by showing them along all leaves of the restriction of the vertical foliation V and the horizontal foliation H to the square $Q_{1/\epsilon}(z)$.

For every $\epsilon' > 0$, by Lemma 2.5 and Lemma 4.7, if $R > 0$ is sufficiently large, then

(i) the restriction of Σ to each leaf of the vertical foliation V in $Q_{\frac{1}{\epsilon'}}(z)$ is a smoothly $(\sqrt{2} - \epsilon', \sqrt{2} + \epsilon')$-bilipschitz embedding,

(ii) the restriction of Σ to each leaf of the horizontal foliation H in $Q_{\frac{1}{\epsilon'}}(z)$ has derivative less than ϵ', and

(iii) the derivative of the rotation of Σ_* along a vertical leaf in $Q_{\frac{1}{\epsilon'}}(z)$ is bounded from above by ϵ', and the total rotation along the leaf is also bounded from above by ϵ'.

Pick a vertical leaf v_0 in $Q_{\frac{1}{\epsilon}}(z)$, and let ℓ be the geodesic in $\mathbb{H}^3$ passing through the endpoints of the $(\sqrt{2} - \epsilon', \sqrt{2} + \epsilon')$-bilipschitz curve $\Sigma|v_0$. We normalize the exponential map with respect to the endpoints of this geodesic. Then (i) and (ii) implies (1) with this normalization.

We next show (3). We first analyze f on each vertical leaf. By (i) and (iii), the restriction of the developing map f to v_0 is ϵ'-close to the normalized exponential map, by isometrically embedding e onto $\mathbb{C} \cong \mathbb{E}^2$ in the scaled Euclidean metric $\sqrt{2}E$ (i.e. the metric on v_0 is scaled by $\sqrt{2}$).

The Σ-images of horizontal segments are very short curves in $\mathbb{H}^3$. Therefore, for every $\epsilon' > 0$, if $R > 0$ is sufficiently large, then for each vertical leaf v of $Q_{\frac{1}{\epsilon}}(z)$, the restriction of f to v is ϵ'-close to the restriction of the normalized exponential map to a vertical segment in $\mathbb{C}$ by isometrically embedding v w.r.t. $\sqrt{2}E$.

Next, we analyze f on horizontal leaves. Let h be a horizontal leaf in $Q_{\frac{1}{\epsilon}}(z)$. Consider the vector field along h consisting of the unit vectors in the vertical direction. Then, for every $\epsilon' > 0$, if $R > 0$ is sufficiently large, then, as in the proof of Proposition 4.8, the image of the tangent vectors are ϵ'-close to each other in the C^0-topology. By the curvature estimate along the horizontal direction in Lemma 4.1, for every $\epsilon' > 0$ if $R > 0$ is large enough, the amount of the total rotation of f along every horizontal segment in $Q_{\frac{1}{\epsilon}}(z)$ is close to the horizontal length times $\sqrt{2}$. Therefore, a restriction of f to every horizontal segment h is ϵ'-close to the restriction of exp when h is isometrically embedded onto a horizontal segment after scaling the length of h by $\sqrt{2}$. Therefore, a restriction of Σ_* to every horizontal segment h is ϵ'-close to the restriction of $\Psi_* \exp$ when h is isometrically embedded onto a horizontal segment w.r.t the $\sqrt{2}E$-metric.

We proved that the restrictions of f to horizontal and vertical leaves in $Q_{\frac{1}{\epsilon}}(z)$ are ϵ'-close to the normalized exponential map when $Q_{\frac{1}{\epsilon}}(z)$ is isometrically embedded in $\mathbb{C}$. This immediately implies (3).

Finally (1) and (3) immediately imply (2), since $f(z)$ and $\Psi(z)$ determines $\Psi_*(z)$. $\boxed{4.9}$

5. Holonomy maps for surfaces with punctures

5.1. Classification of cusps of $\mathbb{C}\mathrm{P}^1$-structures.

Definition 5.1. *Let F be a surface with punctures. A $\mathbb{C}\mathrm{P}^1$-structure on F is a pair (X, q) of a Riemann surface structure X on F and a holomorphic quadratic differential q, such that at each puncture of X, q is at most a pole of order two.*

This class is a natural class to consider, especially in our setting due to the upper injectivity radius bound (see Theorem 6.1).

Proposition 5.2. *Let F be a closed surface with at least one puncture c such that the Euler characteristic of F is negative. Let $C = (f, \rho)$ denote a $\mathbb{C}\mathrm{P}^1$-structure on F expressed by a developing pair. Denote by ℓ_c the peripheral loop around c. Let $C \cong (\tau, L)$ denote Thurston parameters, and (E, V) be the singular Euclidean structure E with the vertical foliation V given by the Schwarzian parameters of C.*

(1) Suppose that a cusp neighborhood of c in E is an expanding cylinder of infinite modulus shrinking towards c. Then
- *$\rho(\ell_c)$ is parabolic,*
- *c has a horodisk quotient neighborhood, and*
- *in Thurston parameters (τ, L), c also has a horodisk quotient neighborhood where the lamination L is the empty lamination.*

(2) Suppose that a cusp neighborhood of c in E is a (half-infinite) flat cylinder F of infinite modulus. Then exactly one of the following holds.

(a) *The circumferences of F are not orthogonal to V, $\rho(\ell_c)$ is hyperbolic, and $\sqrt{2}\int_{\ell_c}\sqrt{q}$ is its complex translation length.*
In Thurston parameters, the cusp c corresponds to boundary component b of τ whose length is the real part of the translation length (in $\mathbb{C}/2\pi i\mathbb{Z}$).

(b) *The circumferences of F are orthogonal to V.*
- *If $\sqrt{2}\,V(\ell_c)$ is not a 2π-multiple, then $\rho(\ell_c)$ is an elliptic element of angle $\sqrt{2}\int_{\ell_c}\sqrt{q}$ $(\in\mathbb{R})$. In Thurston parameters, c is a cusp of τ and the total weight of leaves of L around c (counted with multiplicity) is, modulo 2π, equal to the rotation angle of $\rho(\ell_c)$.*
- *If $\sqrt{2}\,V(\ell_c)$ is a 2π-multiple, then $\rho(\ell_c)$ is either the identity I or a parabolic element. In Thurston parameters, c is a cusp of τ and the total weight of L around c is the 2π-multiple.*

In (2b), by "counted with multiplicity", we mean that, if a single leaf of L has both endpoints at c, the weight of the leaf is counted twice.

Proof. (1) We first describe an intuition, and then make it precise. As the Euclidean distance to the cusp is finite in E, in the hyperbolic metric on X, the quadratic differential q vanishes asymptotically towards the cusp c. A Riemann surface with the zero differential (in our parametrization) corresponds to a hyperbolic structure.

To make it precise, for $t > 0$, let D_t be the punctured disk of radius t centered at c. Note that the c may be the zero of the quadratic differential q induced by C. Thus, if $t > 0$ is small enough, D_t is a union of the Euclidean semi-disks of radius t foliated by geodesics parallel to the diameter segment. Consider the restriction of q to D_t. Then, by conformally identifying a once-punctured unit disk with D_t, the holomorphic quadratic differential on D_t, the differential goes to zero uniformly on every compact subset as $t \to \infty$.

The solution of the Schwarzian equation depends continuously in the differential. As a punctured disk with the zero differential corresponds to a hyperbolic structure h with a cusp at the puncture, and the holonomy around the cusp is parabolic. Therefore, the developing map of D_t converges to the developing map of the hyperbolic cusp-neighborhood structure h, which is a quotient of horodisk by the infinite cyclic group generated by a parabolic element. By the equivariance property of the developing maps, the holonomy of D_t around the cusp must converge to a parabolic element, and as the holonomy of D_t around the cusp is independent of $t > 0$, the holonomy is genuinely parabolic. Moreover, if one deforms a little bit the hyperbolic structure h on the punctured disk to any other $\mathbb{C}\mathrm{P}^1$-structure on the punctured disk keeping the holonomy around the cusp parabolic, it still contains a horodisk quotient as a cusp neighborhood. Therefore c has a horodisk quotient neighborhood in C.

In Thurston parameters, c is a cusp of τ, and L is the empty lamination in a sufficiently small neighborhood of c.

(2) By Proposition 4.9, the developing map of the half-infinite flat cylinder becomes closer and closer to the exponential map $\exp\colon \mathbb{C} \to \mathbb{C}^*$ as a point in the domain approaches the cusp, where, in the domain $\mathbb{C}$, the vertical direction corresponds to the real direction and the horizontal direction corresponds to the imaginal direction (to be precise, the exp is composed with the calling to the domain $\mathbb{C}$ by $\sqrt{2}$). Thus the assertions about the holonomy along ℓ_c hold.

It remains only to show the description in Thurston parameters.

(2a) By Proposition 4.4, outside of a large compact set of F, all circumferences of F are admissible loops. Therefore an appropriate neighborhood of c corresponds to an infinite grafting cylinder. By [Bab17, Proposition 8.3], the hyperbolic surface τ has a (possibly open) boundary component corresponding to c, and its boundary length is indeed the translation length of the hyperbolic element $\rho(\ell_c)$.

(2b) The developing map in an appropriate cusp neighborhood is the exponential map $\exp\colon \mathbb{C} \to \mathbb{C}^*$ so that the deck transformation corresponds to the translation in the imaginary direction in the domain.

Therefore, c is a cusp of τ and the total weight of leaves of L near c must be the length of the circumference times $\sqrt{2}$ (Proposition 4.4 (2)). 5.2

5.2. $\mathrm{PSL}_2\mathbb{C}$ **and fixed points on** $\mathbb{C}\mathrm{P}^1$**.** In order to construct an appropriate holonomy map for a surface with punctures, we will make $\mathrm{PSL}_2\mathbb{C}$ slightly bigger as a topological space, by carefully pairing its elements with their fixed points on $\mathbb{C}\mathrm{P}^1$. Let $(\mathbb{C}\mathrm{P}^1)^2/\mathbb{Z}_2$ denote the set of unordered pairs of points on $\mathbb{C}\mathrm{P}^1$. Let $\widehat{\mathrm{PSL}_2\mathbb{C}}$ be the set of all pairs $(\gamma, \Lambda) \in \mathrm{PSL}_2\mathbb{C} \times ((\mathbb{C}\mathrm{P}^1)^2/\mathbb{Z}_2)$ such that

- if γ is a hyperbolic element with zero rotation (i.e. $\operatorname{tr}\gamma \in \mathbb{R} \setminus [-2, 2]$ when γ is lifted to $\mathrm{SL}(2, \mathbb{C})$), then Λ is a pair of (not necessarily distinct) fixed points of γ, and
- otherwise, Λ is the pair (a, a) of identical fixed points $a \in \mathbb{C}\mathrm{P}^1$ of γ.

We call the pair Λ a *framing*. In particular, if $\gamma = I$, then Λ can be (a, a) for any $a \in \mathbb{C}\mathrm{P}^1$. The second case also includes the case where γ is a hyperbolic element with non-zero torsion. (By abuse of notation, if Λ is a pair (a, a) of identical points on $\mathbb{C}\mathrm{P}^1$, for simplicity, we may regard Λ as a single point a.)

Fock and Goncharov introduced a framing of a representation, which equivariantly assigns a single fixed point to each peripheral element ([FG06]). What is new here is that we are assigning a pair of fixed points in the first case.

Next we define a (non-Hausdorff) topology on $\widehat{\mathrm{PSL}_2\mathbb{C}}$ by the following open base of neighborhoods at each $(\gamma, \Lambda) \in \widehat{\mathrm{PSL}_2\mathbb{C}}$.

- If γ is hyperbolic, then, for every (small) connected neighborhood U of γ in $\mathrm{PSL}_2\mathbb{C}$ consisting of hyperbolic elements, the set of all pairs $(\gamma', \Lambda') \in \widehat{\mathrm{PSL}_2\mathbb{C}}$ such that
  - if $\operatorname{tr}\gamma$ is real and $\sharp\Lambda = 2$, then for $\gamma' \in U$ with $\operatorname{tr}\gamma'$ real, $\sharp\Lambda' = 2$, and
  - otherwise $\gamma' \in U$ and, Λ' is a pairs of identical points identified with Λ by identifying the fixed points of γ with those of γ' by a path connecting γ to γ' in U.
- If γ is *not* hyperbolic, then the topology near (γ, Λ) is given by the product topology of $\mathrm{PSL}_2\mathbb{C} \times (\mathbb{C}\mathrm{P}^1)^2/\mathbb{Z}_2$ equipped with the Hausdorff topology on $(\mathbb{C}\mathrm{P}^1)^2/\mathbb{Z}_2$.

Remark 5.3. *Let $C = (f, \rho)$ be a $\mathbb{C}\mathrm{P}^1$-structure on a surface with punctures. Let $\alpha \in \pi_1(S)$ be such that its free homotopy class is the peripheral loop around a cusp c of C. Then γ corresponds to a unique element in $(\mathbb{C}\mathrm{P}^1)^2/\mathbb{Z}_2$ as follows: As the universal cover $\tilde{C}$ of C is conformally identified with $\mathbb{H}^2$ by the uniformization, let $\tilde{c}$ be the point on the ideal boundary of $\partial\mathbb{H}^2$ fixed by $\alpha \in \pi_1(S)$. Let $(\tau, L) \in \mathsf{T} \times \mathsf{ML}$ be the Thurston parametrization of C, and let $\mathcal{L}$ be the circular measured lamination on C which descends to L. For each leaf ℓ of $\tilde{\mathcal{L}}$ ending at $\tilde{c}$, the corresponding endpoint of the circular arc $f(\ell)$ on $\mathbb{C}\mathrm{P}^1$ is a fixed point of $\rho(\alpha)$. If $\mathcal{L}$ is non-empty in a small neighborhood of the cusp, let Λ be the set of such half leaves of $\tilde{\mathcal{L}}$ ending at c. Then α corresponds to a unique element $(\rho(\alpha), \Lambda)$ in $\widehat{\mathrm{PSL}_2\mathbb{C}}$. If $\mathcal{L}$ is empty near the cusp,*

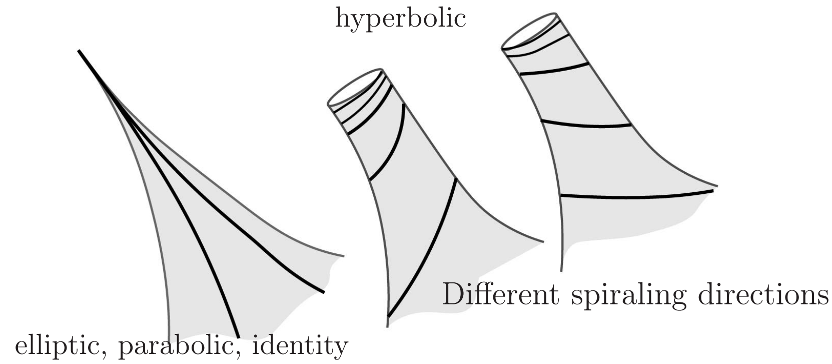

Figure 4. Cusp neighborhoods in Thurston parameters

an appropriate cusp neighborhood of c is a horodisk quotient, and α corresponds to $(\rho(\alpha), \Lambda)$, where Λ is the parabolic fixed point of $\rho(\alpha)$.

5.3. **Cusp neighborhoods in Thurston parameters.** The following lemma determines the isomorphism classes of cusp neighborhoods of $\mathbb{C}\mathrm{P}^1$-structures in Thurston coordinates.

Lemma 5.4. *Let $C = (f, \rho)$ be a $\mathbb{C}\mathrm{P}^1$-structure on a surface F with cusps. Let $C \cong (\tau, L)$ be Thurston parameters of C. Then, for each cusp c of C, its small neighborhood (i.e. its germ) in C is determined by*

- *the holonomy $h \in \mathrm{PSL}_2\mathbb{C}$ around c,*
- *the transversal measure of a peripheral loop around c given L, and*
- *if h is hyperbolic, the direction in which the leaves of L spirals towards the boundary component.*

(See Figure 4.)

Proof. Let (E, V) be the pair of a singular Euclidean structure E on F, and V be a vertical foliation on E induced by C.

Hyperbolic Case. First suppose that $h \in \mathrm{PSL}_2\mathbb{C}$ is hyperbolic. Then, by Proposition 5.2, its cusp neighborhood, in (E, V), corresponds to a half-infinite cylinder A, and the complex translation length is $\sqrt{2}\int_{\ell_c}\sqrt{q}$, where ℓ_c is a peripheral loop of c.

The developing map f of a small neighborhood of c is a restriction of the exponential map $\mathbb{C} \to \mathbb{C}^*$. Thus the complex translation length determines the deck transformation on the domain $\mathbb{C}$ by $\mathbb{Z} \cong \langle \ell_c \rangle$, which determines the $\mathbb{C}\mathrm{P}^1$-structure of a small cusp neighborhood.

The cusp c corresponds to the geodesic boundary circle b of τ whose length is equal to the translation length of h. By the properties of bending maps, one can show that the total weight of L along ℓ_c times $\sqrt{2}$ is the rotational angle of h and the direction of rotation in which leaves of L spiral towards b determines the orientation of the angle (Figure 5).

Parabolic Case. Suppose that h is parabolic.

If a neighborhood of a cusp c in E is an expanding cylinder shrinking towards c, then a neighborhood of c in (τ, L) is a hyperbolic cusp with the empty lamination (Proposition 5.2 (1)).

Next suppose that the cusp neighborhood of c in (E, V) is a half-infinite flat cylinder A in E. Then the circumferences of A are orthogonal to V, and $\sqrt{2}V(\ell_c)$ is a positive 2π-multiple.

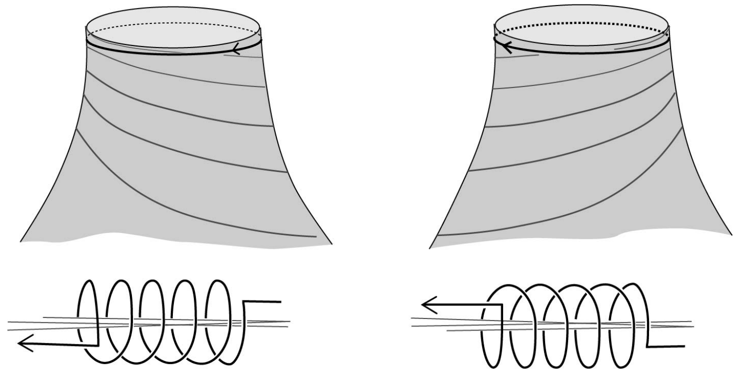

FIGURE 5. The opposite spiral directions give the holonomy the opposite rotational directions.

Let $\mathrm{Ep}\colon \tilde{X} \to \mathbb{H}^3$ be the Epstein map associated with $C = (X, q)$. Let $\tilde{V}$ be the pull-back of V to the universal cover of E, and let $\tilde{c}$ be the lift of c to the ideal boundary of $\tilde{X} \cong \mathbb{H}^2$. Let $\gamma \in \pi_1(F)$ be the element which fixes $\tilde{c}$ such that its free homotopy class is ℓ_c. Then, for every leaf ℓ of $\tilde{V}$ ending at $\tilde{c}$, its image $\mathrm{Ep}(\ell)$ is indeed a quasi-geodesic limiting to the parabolic fixed point of $\rho(\gamma)$ on $\mathbb{C}\mathrm{P}^1$, and its curvature of $\mathrm{Ep}(\ell)$ converges to zero as it approaches the fixed point by Lemma 2.5. Therefore c corresponds to a cusp of τ. By Proposition 4.4, the total weight of the leaves must be $\sqrt{2}V(\ell_c)$.

Elliptic Case. The proof when h is elliptic is similar to the parabolic case. 5.4

Let D be the unit closed disk in $\mathbb{C}$ centered at the origin O. Let $D^* = D \setminus \{O\}$, and let ℓ be the peripheral loop around the origin. Let $\mathcal{P}(D^*)$ denote the space of all developing pairs (f, h) for the $\mathbb{C}\mathrm{P}^1$-structures on $D \setminus \{O\}$ (not up to $\mathrm{PSL}_2\mathbb{C}$) so that O is a cusp and the boundary circle is smooth, where $f\colon \tilde{D^*} \to \mathbb{C}\mathrm{P}^1$ is the developing map and $h \in \mathrm{PSL}_2\mathbb{C}$ is the holonomy along ℓ. Recall from Remark 5.3 that each cusp corresponds to a unique element (γ, Λ) in $\widehat{\mathrm{PSL}_2\mathbb{C}}$. Let $\check{D}^*$ be a subsurface of D^* obtained by removing a regular neighborhood of the boundary circle of D^*.

By the following proposition, the deformation of the $\mathbb{C}\mathrm{P}^1$-structures of the cusp neighborhoods is locally modeled on $\widehat{\mathrm{PSL}_2\mathbb{C}}$.

Proposition 5.5. *Let F be a closed surface minus finitely many points, and let C be a $\mathbb{C}\mathrm{P}^1$-structure on F, and pick its developing pair (f, ρ). Then, each cusp c of C has a disk neighborhood $\Sigma = (f, \gamma) \in \mathcal{P}(D^*)$ of c in C with the following properties:*

(1) *Let $(\gamma, \Lambda) \in \widehat{\mathrm{PSL}_2\mathbb{C}}$ be the element corresponding to the peripheral loop around c. Then, for every $\epsilon > 0$ and every compact subset K of the universal cover $\tilde{\Sigma}$, there is a subset $U = U(K, \epsilon)$ of (γ, Λ) in $\widehat{\mathrm{PSL}_2\mathbb{C}}$, such that, for every $(\gamma', \Lambda') \in U$,*
   (a) *if $\sharp\Lambda = 1$, then there is $\Sigma' = \Sigma'(\gamma', \Lambda') \in \mathcal{P}(D^*)$ with holonomy γ' and the framing Λ, such that its developing map f' of $\Sigma(\gamma', \Lambda')$ is ϵ-close, in C^1-topology, to the developing map f of Σ in K,*
   (b) *if $\sharp\Lambda = 2$, then there is a neighborhood W of γ in $\mathrm{PSL}_2\mathbb{C}$, such that, for every $\gamma' \in W$, there is $\Sigma' = \Sigma'(\gamma', \Lambda') \in \mathcal{P}(D^*)$ with holonomy γ' and a unique framing Λ, such that its developing map f' of $\Sigma(\gamma', \Lambda')$ is ϵ-close, in C^1-topology, to the developing map f of Σ in K.*

(2) *Moreover, Σ' is uniquely determined on $\check{D}^*$ by an isotopy of D^* (uniqueness near the cusp).*

Proof of Proposition 5.5. We divide the proof by the isometry type of γ. In each case, we construct a deformation of Σ in a small neighborhood in $\widehat{\mathrm{PSL}_2\mathbb{C}}$ by specifying the deformation of a fundamental membrane.

Elliptic Case. First, suppose that $\gamma = I$ or γ is an elliptic element. Then the puncture O corresponds to a unique point $f(O)$ on $\mathbb{C}\mathrm{P}^1$ by continuously extending f. Then pick a cusp neighborhood Σ biholomorphic to a punctured disk, such that the development of the boundary circle is a round circle α on $\mathbb{C}\mathrm{P}^1$ and there is a unique Lie subgroup of $\mathrm{PSL}_2\mathbb{C}$ isomorphic to $SO(2)$ which preserves α and $f(O)$. We identify $\mathbb{C}\mathrm{P}^1$ with $\mathbb{C} \cup \{\infty\}$ so that the puncture $f(O)$ is at the origin and α is the unit circle of $\mathbb{C}$ centered at the origin $f(O)$.

Pick a "fan-shaped fundamental domain" in $\tilde{D}^*$ bounded by three circular arcs e_1, e_2, e_3 such that

- $f|e_1$ and $f|e_2$ are radii of α connecting $f(O)$ to points on α, so that $\gamma f(e_1) = f(e_2)$ are orthogonal to α, and
- $f|e_3$ immerses into α, and it connects the endpoints of e_1 and e_2

(Figure 6, left). Let q be the endpoint of the arc $f(e_1)$ on r.

If the neighborhood U of $(\gamma, f(O))$ is sufficiently small, then given $(\gamma', \Lambda') \in U$, one can easily construct a $\mathbb{C}\mathrm{P}^1$-structure $\Sigma' = (f', \gamma')$ close to Σ on D^* realizing (γ', Λ'). Indeed, we pick $z \in \Lambda'$, we can construct a fundamental membrane bounded by e_1', e_2', e_3' such that,

(1) $f'(e_1')$ is a straight line on $\mathbb{C}$ connecting z and fq,
(2) $f'(e_2')$ is $\gamma'(f'(e_1'))$ (which is a circular arc connecting z and $\gamma(q)$),
(3) $f'(e_3')$ is an arc connecting q to $\gamma(q)$ so that $f(e_3')$ is a segment of a trajectory under a one-dimensional Lie subgroup of the affine transformations of $\mathbb{C}$ preserving z, and
(4) $f'(e_i)$ is close to $f(e_i)$ in the Hausdorff topology on $\mathbb{C}\mathrm{P}^1$.

(see Figure 6, right). (The choice of z may not be unique if r is identity and $\mathrm{tr}\, r' \in \mathbb{R} \setminus [-2, 2]$, i.e. hyperbolic without screw motion)

On the other hand, one can easily see that, for every small deformation Σ' of Σ, there is a "fan-shaped" fundamental membrane satisfying all conditions (1) - (4) such that the fundamental membranes coincide on $\check{D}^*$. Therefore, we have the uniqueness property of Σ' near the cusp.

Generic hyperbolic case. Let (τ, L) be the Thurston parametrization of C, and let $\mathcal{L}$ be the Thurston lamination on C. Let ℓ be the peripheral loop around O. Suppose that γ is hyperbolic and $L(\ell) \neq 0$, so that Λ is a single point. Then τ has a geodesic boundary loop b corresponding to the cusp c and, as $L(\ell) > 0$, leaves of L spiral towards b. Let $\tilde{b}$ be a lift of b to the universal cover $\tilde{\tau}$ of τ, so that $\tilde{b}$ is a boundary geodesic of $\tilde{\tau}$. Then those spiraling leaves lift to geodesics in $\tilde{\tau}$ having a common endpoint at an endpoint of $\tilde{b}$; by the bending map $\beta\colon \tilde{\tau} \to \mathbb{H}^3$, the endpoint maps to the point Λ. Accordingly, the leaves of $\mathcal{L}$ near the cusp O develop onto circular arcs ending at Λ.

Normalize $\mathbb{C}\mathrm{P}^1 = \mathbb{C} \cup \{\infty\}$ by an element of $\mathrm{PSL}_2\mathbb{C}$, so that $0 = \Lambda$ and the other fixed point of γ is at ∞. Let (E, V) be the foliated singular Euclidean structure given by C. Then, there is a half-infinite flat cylinder A in E which corresponds to a cusp neighborhood of c; then each circumference has a positive transversal measure given by the horizontal foliation. Therefore, one can take a cusp neighborhood Σ bounded

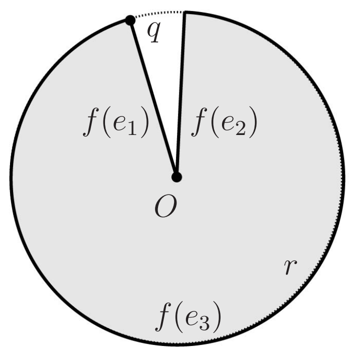

FIGURE 6. Perturbing a fundamental membrane of a cusp with elliptic holonomy.

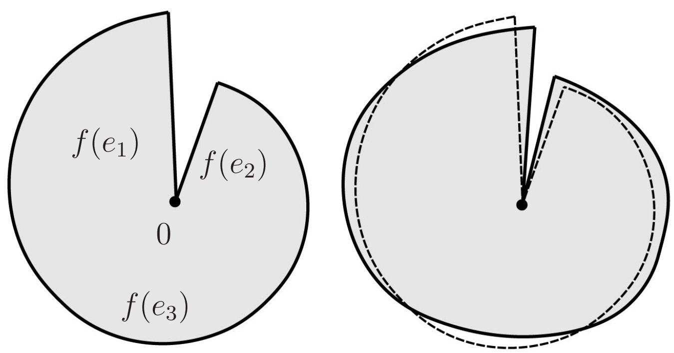

FIGURE 7. Perturbing a fundamental membrane of a cusp with a hyperbolic holonomy.

by a loop m such that m develops onto a *spiral* on $\mathbb{C}\mathrm{P}^1$, i.e. a curve invariant under a one-parameter subgroup in $\mathrm{PSL}_2\mathbb{C}$ which contains γ.

Take, similarly, a "fan-shaped" fundamental domain F in the universal cover $\tilde{\Sigma}$ which is bounded by three smooth segments e_1, e_2, e_3 such that

- e_1 and e_2 are half-leaves of $\tilde{\mathcal{L}}$ such that $\gamma e_1 = e_2$ and the circular arcs $f(e_1)$ and $f(e_2)$ end at $0 \in \mathbb{C}$,, and
- $f(e_3)$ is in a segment of the spiral which connects the other endpoints of $f(e_1)$ and $f(e_2)$

(Figure 7, left). Then $\gamma(f(e_1)) = f(e_2)$ by the equivariant property.

Take a sufficiently small neighborhood U of (γ, Λ) such that the subset $W \subset \mathrm{PSL}_2\mathbb{C}$ of holonomy elements of pairs in U consists of only hyperbolic elements closed to γ; then, for all $(\gamma', \Lambda') \in U$, the fixed point Λ' of the hyperbolic element γ' uniquely corresponds to the fixed point of γ in Λ by every short path connecting γ' to γ in W. Then, similarly to the elliptic case, one can easily find a $\mathbb{C}\mathrm{P}^1$-structure on D^* close to Σ which realizes (γ', Λ'), by constructing a fundamental membrane close to F (Figure 7).

On the other hand, for every small deformation Σ'' of Σ realizes (γ', Λ'), one can easily find a fundamental membrane of Σ'' so that it coincides, on $\check{D}^*$ with that of Σ' constructed above.

Special hyperbolic case ($\sharp\Lambda = 2$). Suppose that γ is hyperbolic and $L(\ell) = 0$ (in particular $\operatorname{tr}\gamma \in \mathbb{R}$). Then the boundary component b of τ is a leaf of L with weight infinity ([Bab17, Proposition 8.3]).

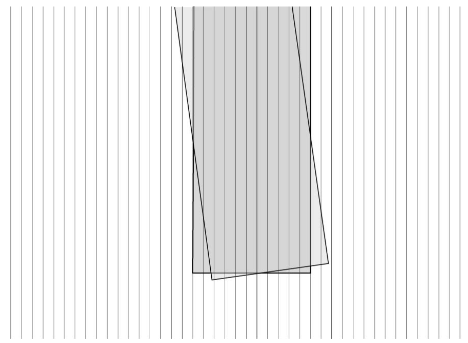

Figure 8. Deformation of a hyperbolic cusp neighborhood.

Let $\kappa\colon C \to \tau$ be the collapsing map. Then $\kappa^{-1}(b) =: F$ is a half-infinite cylinder. The developing map of F is the restriction of $\exp\colon \mathbb{C} \to \mathbb{C}^*$ to a half-space bounded by a horizontal line in $\mathbb{C}$. Then we identify the universal cover $\tilde{F}$ of F with the half-space, so that γ acts as a horizontal translation t_γ. Take a fundamental domain Q in $\tilde{F}$ such that Q is a vertical half-infinite strip in $\mathbb{C}$ bounded by two vertical rays and one horizontal segment (Figure 8).

If W is a small neighborhood of γ in $\mathrm{PSL}_2\mathbb{C}$ consisting of hyperbolic elements, for every $\gamma' \in W$, there is a translation $t_{\gamma'}$ of $\mathbb{C}$ (close to the horizontal translation t_γ), such that $t_{\gamma'}$ descends to γ' by the exponential map up to $\mathrm{PSL}_2\mathbb{C}$. Therefore, there is a small deformation of Σ realizing (γ', Λ'), and (1) holds.

On the other hand, arbitrary deformations of the cusp neighborhood F contain such a deformation of such a half-infinite strip fundamental domain on D. Moreover, if U is sufficiently small, then if there are two ϵ-small deformations of F with the same framed holonomy (γ', Λ'), up so isotopy, the structures on D^* coincide by the ϵ-closeness to F. Thus the uniqueness holds (2).

Parabolic case. Suppose that γ is parabolic. Then in Thurston parameters, the puncture corresponds to a cusp of the hyperbolic surface τ, and the total weight of L along the peripheral loop ℓ is a non-negative 2π-multiple. Then, similarly to the case that $\gamma = I$, we can show the claim by finding a cusp neighborhood and a fundamental domain in its universal cover which is bounded by circular arcs. 5.5

5.3.1. *Holonomy maps of* $\mathbb{C}\mathrm{P}^1$*-structures with cusps.* Let F be a closed surface minus finitely many points $p_1, \dots, p_n$. Recall that $\mathcal{P}(F)$ denotes the space of all developing pairs (f, ρ) for $\mathbb{C}\mathrm{P}^1$-structures on F. Let $(f, \rho) \in \mathcal{P}(F)$. Then (f, ρ) gives a $\mathbb{C}\mathrm{P}^1$-structure on F, and we let X be its induced complex structure on F. Identify the universal cover $\tilde{X}$ of X with $\mathbb{H}^2$; then for each $i = 1, \dots n$, pick a lift $\tilde{p}_i$ of p_i to a point on the ideal boundary of $\tilde{X}$. Then, by Remark 5.3, for every $(f, \rho) \in \mathcal{P}(F)$ and a puncture p_i, we have a corresponding element in $(\gamma_i, \Lambda_i) \in \widehat{\mathrm{PSL}_2\mathbb{C}}$. Thus, by the definition of the topology of $\widehat{\mathrm{PSL}_2\mathbb{C}}$, we have a continuous mapping from $\mathrm{hol}\colon \mathcal{P}(F) \to (\widehat{\mathrm{PSL}_2\mathbb{C}})^n \times \mathcal{R}(F)$ taking $(f, \rho) \in \mathcal{P}(F)$ to $((\gamma_i, \Lambda_i)_{i=1}^n, \rho)$. In fact, hol yields a holonomy theorem in our setting.

Theorem 5.6. *Every* $(f, \rho) \in \mathcal{P}(F)$ *has a neighborhood* W *such that*

$$\mathrm{hol}\,|W$$

is a local homeomorphism onto its image. Moreover, for any $(f, \rho) \in \mathcal{P}(F)$*, if there is a path* ρ_t $(t > 0)$ *converging to* ρ *in* $\mathcal{R}(F)$ *as* $t \to \infty$*, then there is a lift of* ρ_t *to a path in* $\mathcal{P}(F)$ *for* $t \gg 0$ *converging to* (f, ρ)*.*

Remark 5.7. *The image of* $\mathrm{hol}(W)$ *is contained in*

$$\{\,((\gamma_i,\Lambda_i)_{i=1}^n,\rho)\mid \rho\in W, \rho(\alpha_i)=\gamma_i\,(i=1,2,\dots,n)\,\}.$$

Furthermore, its subset cut by the condition on the framing given by Proposition 5.5 (1) determines the local image $\mathrm{hol}(W)$*.*

Proof. Let $(f,\rho)\in\mathcal{P}(F)$, and let C be the $\mathbb{C}\mathrm{P}^1$-structure on F given by the developing pair (f,ρ). Applying Proposition 5.5 to a small $\epsilon>0$, we obtain, for each $i=1,\dots,n$, a (small) cusp neighborhood C_i of the puncture p_i of C, and a neighborhood U_i of (γ_i,Λ_i) in $\widehat{\mathrm{PSL}_2\mathbb{C}}$ modeling the deformation of C_i. Let N_i be the underlying topological cusp neighborhood of the punctured surface F supporting C_i. Without loss of generality, we can assume $C_1,\dots,C_n$ are disjoint in C. Let C_i' be an open cusp neighborhood of p_i smaller than C_i and U_i be a subset of $\widehat{\mathrm{PSL}_2\mathbb{C}}$ containing $\mathrm{hol}((f,\rho))$ given by Proposition 5.5(2), such that the small deformation of C_i on C_i' is parametrized the framed holonomy in U_i.

Let N_i'' be a (even smaller) cusp neighborhood of p_i whose closure is contained in the interior of N_i'. Let $\check{F}$ be $F\setminus\sqcup_i N_i''$, and let $\check{C}$ be the restriction of C to $\check{F}$. For every $(\gamma_i',\Lambda_i')\in U_i$, let $C_i(\gamma_i',\Lambda_i')$ denote the unique $\mathbb{C}\mathrm{P}^1$-structure on N_i' with the framed holonomy $(\gamma_i',\Lambda_i')\in U_i$ such that $C_i(\gamma_i',\Lambda_i')$ is sufficiently close to C_i.

We shall regard (f,ρ) as a smooth section Σ of a $\mathbb{C}\mathrm{P}^1$-bundle B over F such that Σ is transversal to the horizontal foliation H_ρ associated with ρ (see for example [Gol22]) Let $\check{\Sigma}$ be the restriction of Σ to the bundle over the subsurface $\check{F}$. Then, there is a neighborhood U of ρ in the representation variety $\mathcal{R}(F)$ such that, for each $\xi\in U$, letting H_ξ be the horizontal foliation of B associated with ξ, $\check{\Sigma}$ is still transversal to H_ξ by the openness of transversality; then $\check{\Sigma}$ yields a projective structure $\check{C}_\xi$ on $\check{F}$ with holonomy ξ. In this way, we obtain a unique $\mathbb{C}\mathrm{P}^1$-structure on $\check{F}$ close to (f,ρ) on $\check{F}$. This new structure is unique in a compact subset of $\check{F}$ whose interior contains the closure of $F\setminus\sqcup_{i=1}^n N_i'$.

For each i, pick any Λ_i in $\mathrm{Fix}\,\xi_i(\gamma_i)\in\mathbb{C}\mathrm{P}^1$ so that $(\xi_i(\gamma_i),\Lambda_i)\in U_i$. Then $C_i(\xi_i(\gamma_i),\Lambda_i)$ is its associated deformation. Then we can glue $\check{C}_\xi$ and $C_i(\xi_i(\gamma_i),\Lambda_i)$ in the overlapping region, and obtain a desired developing pair for a $\mathbb{C}\mathrm{P}^1$-structure on F. Consider the subset W in $\Pi_{i=1}^n U_i\times\mathcal{R}(F)$ consisting $(\gamma_i,\Lambda_i)_{i=1}^n,\rho)$ satisfying $\rho(\alpha_i)=\gamma_i\,(i=1,2,\dots,n)$; clearly W contains $\mathrm{hol}(f,\rho)$. In this way, given a sufficiently small neighborhood of $\mathrm{hol}((f,\rho))$ in this subset W, for every element in this neighborhood, we construct a developing pair realizing it. This new $\mathbb{C}\mathrm{P}^1$-structure on F is unique by the uniqueness of the thick part $\check{C}_\xi$ on $F\setminus\sqcup N_i'$ and the uniqueness of the cusp neighborhoods $C_i(\xi_i(\gamma_i),\Lambda_i)$ on N_i'.

Notice that W projects to a neighborhood of ρ in $\mathcal{R}$. The path lifting along a path in $\mathcal{R}$ easily follows from the construction as U is a neighborhood of ρ in $\mathcal{R}(F)$. 5.6

6. Bound on the upper injectivity radius

Recall that $C_t=(f_t,\rho_t)$ is a path of $\mathbb{C}\mathrm{P}^1$ structures on S such that C_t diverges to ∞ and the equivalence class $[\rho_t]=:\eta_t$ converges in the character variety as $t\to\infty$. Recall also that $C_t=(X_t,q_t)$ is the expression in the Schwarzian parameters.

Let E_t be the singular Euclidean structure on X_t given by $|q^{\frac12}|$. Let $R(E_t)\geq 0$ denote that the *upper injectivity radius* of E_t. In this section we show

Theorem 6.1. *Suppose that* X_t *is pinched along a multiloop* M*. Then the upper injectivity radius* $R(E_t)$ *of* E_t *is bounded from above for all* $t\geq 0$*.*

Immediately we have the following.

Corollary 6.2. *There is an upper bound for the area of the expanding cylinders in* E_t *for all* $t \geq 0$.

The rest of this section is a proof of Theorem 6.1. We suppose, to the contrary, that $\limsup R(E_t) = \infty$ and show that ρ_t cannot converge. Let M_t be a geodesic representative of M on E_t (in the Euclidean metric) such that, for every $\epsilon > 0$ if $t > 0$ is sufficiently large, then M_t is contained in the ϵ-thin part of X_t. We will find a conformally thick part which is, in the Euclidean metric, bigger than its adjacent thick parts:

Lemma 6.3. *Suppose that there is a diverging sequence* $(0 <)\, t_1 < t_2 < \dots$ *such that* E_{t_i} *contains a flat cylinder* A_{t_i} *homotopy equivalent to a fixed loop* m *of* M *such that*

(1) $\operatorname{Mod} A_{t_i} \to \infty$ *as* $i \to \infty$ *, and*
(2) the circumference of A_{t_i} *limits to* ∞ *(equivalently* $\operatorname{Area} A_{t_i} \to \infty$*) as* $i \to \infty$.

Then, leaves of the vertical foliation V_{t_i} *must be asymptotically orthogonal to the circumferences of* A_{t_i}.

Proof. Suppose, to the contrary, that V_{t_i} is *not* asymptotically orthogonal to circumferences. Then, up to a subsequence, we may assume that there is a limiting angle $\theta_\infty \in [0, \pi/2)$ between the angle between V_{t_i} and the circumferences of A_{t_i}. Let m_{t_i} be a geodesic representative of m which sits in the middle of A_{t_i}. Since $\theta_\infty \neq \pi/2$, Hypotheses (1) and (2) imply that the transversal measure of the horizontal foliation H_{t_i} along m_{t_i} diverges to infinity as $i \to \infty$. By Proposition 4.9, the translation length of $\rho_{t_i}(m_{t_i})$ is asymptotically $\sqrt{2}$ times the transversal measure. Therefore, the translation length of $\rho_{t_i}(m_{t_i})$ must diverge to infinity, which contradicts the convergence of $[\rho_t]$ as $t \to \infty$. □

Proposition 6.4. *Suppose that there are a component* F *of* $S \setminus M$ *and a diverging sequence* $(0 <)\ t_1 < t_2 < \dots$ *such that, letting* F_{t_i} *be the component of* $E_{t_i} \setminus M_{t_i}$ *homotopic to* F *on* S,

- $\operatorname{Area}_{E_{t_i}} F_{t_i} \to \infty$ *as* $i \to \infty$*, and.*
- *for each boundary component* ℓ *of* F*, there is an expanding cylinder* B_{ℓ, t_i} *in* F_{t_i} *bounded by the boundary component* ℓ_i *of* F_{t_i} *homotopic to* ℓ *on* S *such that*
  - B_{ℓ,t_i} *shrinks toward* ℓ_i*, i.e.* ℓ_i *is the shorter boundary component of* B_{ℓ,t_i}*, and*
  - $\operatorname{Mod} B_{\ell,t_i} \to \infty$ *as* $i \to \infty$.

Then $[\rho_{t_i}] | \pi_1 F$ *diverges to* ∞ *in* χ *as* $i \to \infty$.

Proof. Let $k_i > 0$ be such that $k_i \operatorname{Area}(F_{t_i}) = 1$ for each $i = 1, 2, \dots$. Then, as $\operatorname{Area} F_{t_i} \to \infty$, thus $k_{t_i} \to 0$ as $i \to \infty$. All ends of F_{t_i} have conformally long expanding cylinders shrinking towards adjacent components. Take a base point in the thick part of F_{t_i}. Let $\hat{F}$ denote the compact surface with finitely many punctures, obtained by pinching the boundary loops of F to puncture points. Then the space of all holomorphic quadratic differentials on Riemann surfaces structures on $\hat{F}$ with Euclidean area one is a sphere of finite dimension. Then, by compactness, up to a subsequence

- $k_i E_{t_i}$ converges, in the Gromov-Hausdorff topology, to a compact singular Euclidean surface minus finitely many points, E_∞, which is homeomorphic to F, and
- the restriction of $k_i V_{t_i}$ to $k_i E_{t_i}$ converges to a measured foliation V_∞ on E_∞.

Take a piecewise geodesic loop ℓ on E_∞ such that

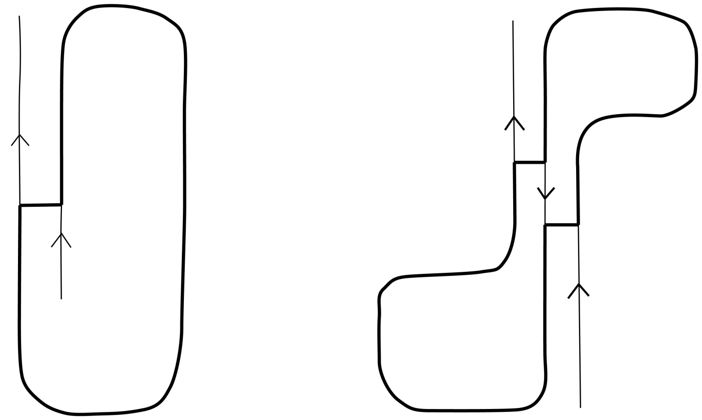

FIGURE 9. Staircase closed loops ℓ consisting of long vertical segments and short horizontal segments

(1) ℓ does not cross any singular point of E_∞,
(2) each segment of ℓ is either vertical or horizontal, and ℓ contains at least one vertical segment, and
(3) ℓ is a geodesic in the L^∞-metric, so that at adjacent singular points, ℓ bends in the different direction by an angle $\pi/2$.

In fact, if V_∞ contains a periodic leaf, then take it as ℓ, which obviously satisfies the conditions. Otherwise, V_∞ contains a minimal irrational subfoliation, using the density of each leaf in the subfoliation, a standard closing lemma gives a desired loop ℓ as in Figure 9 (see [CEG87, I.4.2.15]). By the convergence $k_i E_i \to E_\infty$, for i large enough, we pick a piecewise geodesic loop ℓ_i on E_i satisfying the properties (1), (2), (3) such that ℓ_i has the same number of horizontal and vertical segments as ℓ has, and $k_i \ell_i$ on $k_i E_i$ converges to ℓ on E_∞ smoothly on each segment as $i \to \infty$. Then the distance from ℓ_i to the singular set of E_i goes to ∞ as $k_i \to 0$. Therefore, by Proposition 4.9, $\rho_{t_i}(\ell)$ is a hyperbolic element of translation length close to $\sqrt{2}$ times the total length of the vertical segment of ℓ_i. Then, as $k_i \to 0$, the total vertical length of ℓ_i on E_i goes to infinity, and therefore $\operatorname{tr} \rho_i(\ell)$ must diverge to infinity. □

Let $m_1, \dots, m_p$ be the loops of the multiloop M.

Proposition 6.5. *For every (large) $T > 0$, there are $t > T$ and $k \in \{1, \dots, p\}$ such that*

$$\frac{1}{2} < \frac{\operatorname{length}_{E_t}(m_k)}{\max_{i=1,\dots,p} \operatorname{length}_{E_t}(m_i)} \leq 2,$$

and $\operatorname{Mod}_{E_t}(m_k)$ is $\frac{1}{3}$-dominated by an expanding cylinder $B_{k,t}$ homotopic to m_k, i.e.

$$\frac{\operatorname{Mod} B_{k,t}}{\operatorname{Mod}_{E_t} m_k} > \frac{1}{3}.$$

Proof. For $u > T$, let m_{k_u} be the loop realizing $\max_{i=1,\dots,q} \operatorname{length}_{E_u}(m_i)$. We may assume that $\max_{i=1,\dots,p} \operatorname{length}_{E_t}(m_i) \to \infty$ as $t \to \infty$: in fact, otherwise, since $\limsup R(E_t) = \infty$, Proposition 6.4 implies that $[\rho_t]$ diverges in $\mathcal{X}$.

We first show that if a long flat cylinder persists, then its circumference must stay almost the same. Namely

Claim 6.6. *For every $\epsilon > 0$, there is $K > 0$ such that, if there are $w > u > K$ and a flat cylinder in E_t of height at least K homotopic to m, then, for every $t \in [u, w]$, then*

$$1 - \epsilon < \frac{\operatorname{length}_{E_t} m}{\operatorname{length}_{E_u} m} < 1 + \epsilon$$

for all $t \in [u, w]$.

Proof. By Lemma 6.3, for every $\epsilon > 0$, if $K > 0$ is sufficiently large, then the vertical foliation V_t is ϵ-almost orthogonal to circumferences of the flat cylinder homotopic to m. Then, by Proposition 4.4, for every $\epsilon > 0$, if $K > 0$ is sufficiently large, then the total rotation angle along m is $(1+\epsilon)$-bi-Lipschitz to $\sqrt{2} \operatorname{length}_{E_t} m$ for $t \in [u, w]$. As the holonomy of $\rho_t(m)$ converges as $t \to \infty$, for every $\epsilon > 0$, if K is sufficiently large, then the total rotation along m must be ϵ-almost constant for all $t \in [u, w]$. Thus, if K is sufficiently large, then the ratio of $\operatorname{length}_{E_t} m$ and $\operatorname{length}_{E_u} m$ is ϵ-close to 1. □

By Claim 6.6, for every $\epsilon > 0$, if $K > 0$ is sufficiently large, then, if a flat cylinder $\frac{1}{4}$-dominates $\operatorname{Mod} m_{k_u}$ for all $t \in [u, v]$ for some $u > K$; then $1 - \epsilon < \frac{\operatorname{length}_{E_t} m_{k_u}}{\operatorname{length}_{E_u} m_{k_u}} < 1 + \epsilon$ for all $t \in [u, w]$. Suppose, in addition, that there is a loop m_h of M not m_{k_u}, such that m_h on E_t becomes exactly twice as long as m_{k_u} on E_u for the first time at $t = w < v$ after $t = u$. Then, by applying Claim 6.6 to m_h, we can show that there is $t \in [u, w)$ such that $\operatorname{Mod}_{E_t} m_h$ is 1/3-dominated by an expanding cylinder: Indeed, otherwise, $\max_{i=1,\dots,p} \operatorname{length}_{E_t}(m_i)$ must bounded from above by $\frac{3}{2} \operatorname{length}_{E_t}(m_{k_u})$ for all $t \in [u, w]$. 6.5

Corollary 6.7. *There are a component F of $S \setminus M$ and a diverging sequence $0 < t_1 < t_2 < \dots$ such that the corresponding component F_{t_i} of $E_{t_i} \setminus M_{t_i}$ satisfies the assumptions of Proposition 6.4.*

Proof. By Proposition 6.5, there is a loop m of M and a diverging sequence $t_1 < t_2 < \dots$ such that

- $\operatorname{length}_{E_{t_i}} m \to \infty$ as $i \to \infty$,
- $$\frac{1}{2} < \frac{\operatorname{length}_{E_{t_i}} m}{\max\{\operatorname{length}_{E_{t_i}} m_1, \dots, \operatorname{length}_{E_{t_i}} m_p\}} < 2$$
  for all $i = 1, 2, \dots$, and
- there is an expanding cylinder B_{t_i} homotopic m which $\frac{1}{3}$-dominates $\operatorname{Mod}_{E_{t_i}} m$.

Then, up to a subsequence, we may in addition assume that B_{t_i} is expanding in the same direction. Then, let F be the connected component of $S \setminus M$ such that m is a boundary component of F and B_{t_i} expands towards F. As the size of F_{t_i} becomes bigger and bigger than the length of $\operatorname{length}_{E_{t_i}} m$, the first assumption of Proposition 6.4 holds. Thus, by the second condition on the loop m and the sequence $\{t_i\}$, the second assumption of Proposition 6.4 is satisfied. □

By this corollary, we obtained a contradiction by Proposition 6.4 against the convergence of ρ_t. Hence we obtain Theorem 6.1.

7. Convergence of $\mathbb{C}P^1$-structures away from pinched loops

We continue to suppose that X_t is pinched along a multiloop. We will first see that the holonomy $\rho_\infty(m)$ determines the type of a conformally long Euclidean cylinder in E_t which is homotopic to m for $t \gg 0$.

Lemma 7.1. *(1) Suppose that there are a sequence $t_1 < t_2 < \dots$ diverging to ∞ and a sequence of expanding cylinders B_{t_i} in E_{t_i} homotopic to m at time t_i, such that $\operatorname{Mod}_{E_{t_i}} B_{t_i} \to \infty$ as $t \to \infty$. Then $\rho_\infty(m)$ is parabolic.*

(2) Suppose that there is a sequence of flat cylinders A_{t_i} in E_{t_i} homotopic to a fixed loop m on S such that $\operatorname{Mod} A_{t_i}$ diverges to ∞ and the circumference of A_{t_i} is bounded from below and above by positive numbers. Let $w \in \mathbb{C}$ be such that the Möbius transformation $z \mapsto (\exp w)z$ conjugates to $\rho_\infty(m)$. Then, $\sqrt{2}\int_m \sqrt{q_{t_i}}$ converges to $w \mod 2\pi i$ up to a sign.

Proof. (1) If a puncture of a $\mathbb{C}\mathrm{P}^1$-structure corresponds to a regular point of its holomorphic quadratic differential, its peripheral holonomy is parabolic. Suppose that there are a sequence $t_1 < t_2 < \dots$ and an expanding cylinder B_{t_i} in E_{t_i} homotopic to m such that $\operatorname{Mod} B_{t_i} \to \infty$ as $t \to \infty$. Then, by Corollary 6.2, the length of the shorter boundary component of B_{t_i} goes to zero as $i \to \infty$, and it asymptotically corresponds to, at most, a pole of order one of the quadratic differential. (A pole of order at least two corresponds to an infinite area end.) Therefore $\rho_\infty(m)$ is parabolic, against the hypothesis.

(2) follows immediately from Proposition 4.4. □

Given a compact surface F with boundary, let $\hat{F}$ denote the surface with punctures obtained by pinching each boundary component of F to a (puncture) point.

Proposition 7.2. *Let $\epsilon > 0$ be a number less than the Bers constant. Let F be a component of $S \setminus M$, and let F_t^ϵ be the component of the conformally ϵ-thick part of E_t isotopic to F for $t \gg 0$. Then, if*

$$\liminf_{t\to\infty} \operatorname{Area}_{E_t}(F_t^\epsilon) > 0,$$

there is a path of $\mathbb{C}\mathrm{P}^1$-structures $\hat{F}_t$ on the punctured surface $\hat{F}$ such that

(1) for every $\epsilon > 0$, if $t > 0$ is sufficiently large, then F_t^ϵ isomorphically embeds into $\hat{F}_t$,
(2) for each boundary component ℓ of F, there is a cylinder $A_{\ell,t}$ in E_t homotopic to ℓ such that
- *$\operatorname{Mod} A_{\ell,t} \to \infty$ as $t \to \infty$;*
- *$A_{\ell,t}$ is either a flat cylinder for all $t \gg 0$ or an expanding cylinder shrinking towards the adjacent component of $S \setminus M$ across m for all $t \gg 0$;*

(3) $\hat{F}_t$ contains $A_{\ell,t}$ for every boundary component ℓ of F.

Proof. We first show that, for each boundary component ℓ of F, there is a cylinder A_ℓ homotopic to ℓ, such that

(i) $\operatorname{Mod} A_{\ell,t} \to \infty$ as $t \to \infty$, and
(ii) $A_{\ell,t}$ remains either a flat cylinder for all sufficiently large $t > 0$ or an expanding cylinder shrinking forwards ℓ for all sufficiently large $t > 0$,

Let Y_t, Z_t, W_t be disjoint cylinders homotopic to ℓ, such that Z_t is a maximal flat cylinder, Y_t is the maximal expanding cylinder expanding towards the thicker part of F_t and W_t is the maximal expanding cylinder expanding towards the adjacent component across the geodesic representative ℓ_t of ℓ.

As X_t is pinched along M, by Theorem 3.12, $\max\{\operatorname{Mod} Y_t, \operatorname{Mod} Z_t, \operatorname{Mod} W_t\} \to \infty$ as $t \to \infty$. Let $\operatorname{diam} W_t$ and $\operatorname{diam} Y_t$ denote the diameters of W_t and Y_t, respectively, in the Euclidean metric E_t. Then, by $\liminf_{t\to\infty} \operatorname{Area}_{E_t}(F_t^\epsilon) > 0$ and the upper injectivity radius bound (Theorem 6.1), the ratio $\frac{diam W_t}{diam Y_t + 1}$ is bounded from above for all $t > 0$. Thus $\frac{\operatorname{Mod} W_t}{\operatorname{Mod} Y_t + 1}$ is bounded from above for all $t > 0$. Therefore $\operatorname{Mod} Y_t + \operatorname{Mod} Z_t$ diverges to ∞ as $t \to \infty$. We claim, moreover, that either $\lim \operatorname{Mod} Y_t = \infty$ or $\lim \operatorname{Mod} Z_t = \infty$ holds.

Lemma 7.3. *Suppose that* $\limsup_{t\to\infty} \operatorname{Mod} Y_t = \infty$. *Then* $\operatorname{Mod} Y_t \to \infty$ *as* $t \to \infty$.

Proof. Let $t_1 < t_2 < \dots$ be a sequence with $\lim_{i\to\infty} \operatorname{Mod} Y_{t_i} = \infty$. Then the circumference of Z_{t_i} limits to zero, and by Lemma 7.1 (1), $\rho_\infty(\ell)$ is parabolic.

Suppose to the contrary that there is a sequence $s_1 < s_2 < \dots$ diverges to ∞ such that $\operatorname{Mod} Y_{s_i}$ is bounded from above by some constant $b > 0$. Then $\operatorname{Mod} Z_{s_i} \to \infty$, and the circumference of Z_{s_i} is bounded from below $c > 0$. On the other hand, since $\operatorname{Mod} Y_{t_i} \to \infty$, the circumference of Z_{t_i} goes to zero as $i \to \infty$. We can assume that $s_1 < t_1 < s_2 < t_2 < \dots$ by taking subsequences of s_i and t_j if necessary.

Therefore, for every $r \in (0, c)$, for every sufficiently large i, there is $u_i \in [s_i, t_i]$, such that the circumference of Z_{u_i} is r. Then, as $\operatorname{Mod} Y_{u_i}$ is bounded from above, $\operatorname{Mod} Z_{u_i} \to \infty$ as $i \to \infty$.

Then, by Lemma 7.1 (2), the limit holonomy of $\rho_{u_i}(m)$ is determined by the complex length of the circumference. For almost all $r \in (0, c)$, $\rho_{u_i}(m)$ is not parabolic. This contradicts the convergence of ρ_t as $\rho_\infty(\ell)$ is parabolic. □

Then, Y_t satisfies (i) and (ii).

Next suppose that $\limsup_{t\to\infty} \operatorname{Mod} Y_t < \infty$. Then $\operatorname{Mod} Z_t$ diverges to ∞ as $t \to \infty$, and the circumference of Z_t converges to a positive number. Then Z_t satisfies (i) and (ii).

We shall construct $\hat{F}_t$ satisfying (3) as follows. Suppose that, for a boundary component ℓ of F, $\lim_{t\to\infty} \operatorname{Mod} Y_t = \infty$. Let $\hat{Y}_t$ be an expanding cylinder of infinite modulus, obtained by extending the expanding cylinder Y_t only in the shrinking direction, so that $\hat{Y}_t$ is conformally a punctured disk. Then replace Y_t by $\hat{Y}_t$ in E_t by gluing $E_t \setminus Y_t$ and $\hat{Y}_t$ along the boundary component of $\hat{Y}_t$. Then the boundary component ℓ of F corresponds to the puncture of $\hat{Y}_t$.

Next suppose that $\limsup_{t\to\infty} \operatorname{Mod} Y_t < \infty$. Then, since $\operatorname{Mod} Z_t \to \infty$ and the circumference $\operatorname{Circ}(Z_t)$ converges to a positive number as $t \to \infty$, we extend the flat cylinder Z_t, in the direction of W_t, to the half-infinite flat cylinder $\hat{Z}_t$; then $\hat{Z}_t$ is conformally a punctured disk. Then replace Z_t in E_t with $\hat{Z}_t$ so that it has a puncture corresponding to ℓ.

By applying, such a replacement for all boundary component ℓ of F, we obtains a desired complete singular Euclidean surface $\hat{F}_t$ satisfying (1), (2), (3), as (2) follows from (i) and (ii). 7.2

Theorem 7.4. *Let F be a component of $S \setminus M$. Let $\epsilon > 0$ be less than the Bers constant of S. For every $t > 0$ large enough, let F_t^ϵ be the component of the ϵ-thick part of C_t isotopic to F.*

(1) Suppose that

$$\liminf_{t\to\infty} \operatorname{Area}_{E_t}(F_t^\epsilon) = 0.$$

Then, there is a continuous function $\epsilon_t > 0$ in t with $\lim_{t\to\infty} \epsilon_t = 0$, such that $F_t^{\epsilon_t}$ converges (in the Gromov-Hausdorff topology) to a complete hyperbolic structure on a closed surface with finitely many punctures, denoted by $\hat{F}_\infty$, which is homeomorphic to F, as $t \to \infty$.

(2) Suppose that

$$\liminf_{t\to\infty} \operatorname{Area}_{E_t}(F_t^\epsilon) > 0.$$

Then, $\hat{F}_t$ accumulates to a bounded subset on the space of $\mathbb{C}\mathrm{P}^1$-structures on $\hat{F}$. Moreover, if $\rho_\infty(m) \neq I$ for each boundary component m of F, then $\hat{F}_t$ converges to a $\mathbb{C}\mathrm{P}^1$-structure on $\hat{F}$ as $t \to \infty$.

Remark 7.5. *In Case (2), similarly to (1), one can take a sequence $t_1 < t_2 < \dots$ diverging to ∞ so that $\hat{F}_t$ converges to a $\mathbb{C}\mathrm{P}^1$-structure $\hat{F}_\infty$ on $\hat{F}$. Then, for every $\epsilon > 0$ less than the Bers' constant, the ϵ-thick part $F^\epsilon_{t_i}$ converge to a subsurface of $\hat{F}_\infty$. If, in addition, the $\rho_\infty(m) \neq I$ for every boundary component of F, then F^ϵ_t converge to a subsurface of $\hat{F}_\infty$.*

Proof. (1) Let $t_1 < t_2 < \dots$ be a diverging sequence such that $\mathrm{Area}(F_{t_i}) \to 0$ as $t \to \infty$. Then the holomorphic quadratic differential on F_{t_i} asymptotically vanishes. Thus, for every small $\epsilon > 0$, $F^\epsilon_{t_i}$ and $X_{t_i}|F^\epsilon_{t_i}$ asymptotically identical, where X_{t_i} is regarded as a hyperbolic surface by the uniformization theorem for each i. Here, by asymptotically identical, we mean that, for every $\upsilon > 0$ and every compact set K in the universal cover $\mathbb{H}^2$ of X_{t_i}, if i is sufficiently large, the developing map of $F^\epsilon_{t_i}$ is υ-close to the developing map of the hyperbolic structure $X_{t_i}|F^\epsilon_{t_i}$ on K.

The holonomy representations of $F^{\epsilon_i}_{t_i}$ and $X_{t_i}|F^\epsilon_{t_i}$ are asymptotically identical in the character variety. As the holonomy of $F^{\epsilon_i}_{t_i}$ converges in the representation variety, the holonomy of $X_{t_i}|F^\epsilon_{t_i}$ must converge in the representation variety. Thus $X_{t_i}|F^\epsilon_{t_i}$ converges to a complete hyperbolic structure σ_∞ on F. Therefore $F^{\epsilon_t}_t$ must genuinely converge to σ_∞ (without taking a subsequence). In particular $\mathrm{Area}_{E_t} F^{\epsilon_t}_t \to 0$ as $t \to \infty$.

(2) Suppose that $\liminf_{t\to\infty} \mathrm{Area}\, F^\epsilon_t > 0$ for sufficiently small $\epsilon > 0$. Then let $\hat{F}_t$ denote the singular Euclidean structure on $\hat{F}$ obtained from F_t by Proposition 7.2. Then $\hat{F}_t$ induces a $\mathbb{C}\mathrm{P}^1$-structure on $\hat{F}$. Let (Y_t, w_t) be the Schwarzian parameterization of $\hat{F}_t$. Then, indeed, every puncture of Y_t is, at most, a pole of order two.

As X_t is pinched along a multiloop M, Y_t is bounded in the Teichmüller space $\mathsf{T}(\hat{F})$. By Theorem 6.1, the upper injectivity radius of $\hat{F}_t$ is also bounded from above, and (Y_t, w_t) is also bounded in the parameter space. Thus, the $\mathbb{C}\mathrm{P}^1$-structures $\hat{F}_t$ are contained in a compact subset of the deformation space of $\mathbb{C}\mathrm{P}^1$-structures on $\hat{F}$. Therefore $\hat{F}_t$ accumulates to a bounded subset in the deformation space of $\mathbb{C}\mathrm{P}^1$-structures on $\hat{F}$.

Moreover, if each peripheral loop has non-trivial holonomy at $t = \infty$, by Theorem 5.6, the convergence of the holonomy of $\hat{F}_t$ implies the convergence in $(\widehat{\mathrm{PSL}_2\mathbb{C}})^n \times \mathcal{R}(F)$. Therefore $\hat{F}_t$ has a unique limit in $\mathcal{P}(\hat{F})$. □

Theorem 7.4 immediately implies

Corollary 7.6. *Suppose that X_t is pinched along a multiloop M. Then, for every sequence $t_1 < t_2 < \dots$ diverging to ∞, up to a subsequence, X_{t_i} converges to a nodal Riemann surface X_∞ and q_{t_i} converges to a regular quadratic differential on X_∞.*

8. Degeneration by neck-pinching

In this section, we summarize our main theorems on asymptotic behavior under neck-pinching.

Let $C_t = (f_t, \rho_t),\ t \geq 0$ be a path of $\mathbb{C}\mathrm{P}^1$-structures which diverges to ∞ in the deformation space, such that its holonomy $[\rho_t] =: \eta_t$ converges in the character variety χ. By Proposition 3.2, we can assume that the holonomy $\rho_t \in \mathcal{R}$ also converges in the representation variety. Let X_t be the complex structure of C_t.

Theorem 8.1. *Suppose that X_t is pinched along a loop m. Then $\rho_\infty(m)$ is either I or a parabolic element. Moreover $\rho_t(m) \neq I$ for large enough $t > 0$.*

Recall that $\phi\colon \tilde{S} \to S$ is the universal covering map. Let N_m be a regular neighborhood of m in S. Regard the loop m also as a fixed element of $\pi_1(S)$ representing m, and let $\tilde{N}_m$ be the component of $\phi^{-1}(N_m)$ preserved by $m \in \pi_1(S)$.

Theorem 8.2 (Convergence of developing maps)**.** *Suppose that X_t is pinched along a loop m. Then, exactly one of the following two holds.*

(1)
- *$\rho_\infty(m)$ is parabolic;*
- *the cusp neighborhoods of C_∞ are horodisk quotients;*
- *$f_t\colon \tilde{S} \to \mathbb{C}\mathrm{P}^1$ converges a ρ_∞-equivariant continuous map $f_\infty\colon \tilde{S} \to \mathbb{C}\mathrm{P}^1$ uniformly on compact subsets;*
- *there is a multiloop M on S consisting of finitely many parallel copies of m, such that f_∞ is a local homeomorphism on $\tilde{S} \setminus \phi^{-1}(M)$ and it takes each component of $\phi^{-1}(M)$ to its corresponding parabolic fixed point.*

(2) $\rho_\infty(m) = I$, and, for every sequence $t_1 < t_2 < \ldots$ diverging to ∞, up to a subsequence, there is a path of markings $S \to C_t$ such that, as $i \to \infty$,
- *$C_{t_i}|S \setminus N_m$ converges to a $\mathbb{C}\mathrm{P}^1$-structure on a surface with punctures homeomorphic to $S \setminus m$;*
- *the axis a_i of $\rho_{t_i}(m)$ converges to a point on $\mathbb{C}\mathrm{P}^1$ or a geodesic in $\mathbb{H}^3$;*
- *the restriction of f_{t_i} to $\tilde{S} \setminus \phi^{-1}(N_m)$ converges to a continuous map, and each boundary component of $\tilde{N}_m$ maps to an ideal point of $\lim_{i\to\infty} a_i$.*

For each $t \geq 0$, let $(\tau_t, L_t) \in \mathsf{T} \times \mathsf{ML}$ be the Thurston parameterization of C_t, and let $\beta_t\colon \mathbb{H}^2 \to \mathbb{H}^3$ be the ρ_t-equivariant pleated surface. In fact, β_t converges a continuous map to $\mathbb{H}^3 \cup \mathbb{C}\mathrm{P}^1$:

Theorem 8.3. *Suppose that X_t is pinched along a loop m on S. Let N_m be a regular neighborhood of m on S. Then, by taking an appropriate path of markings $\iota_t\colon S \to \tau_t$, exactly one of the following two holds:*

(1) $\rho_\infty(m) \in \mathrm{PSL}_2\mathbb{C}$ is parabolic, and $\beta_t\colon \tilde{S} \to \mathbb{H}^3$ converges to a ρ_∞-equivariant continuous map $\beta_\infty\colon \tilde{S} \to \mathbb{H}^3 \cup \mathbb{C}\mathrm{P}^1$ uniformly on compact subsets as $t \to \infty$, such that $\beta_\infty^{-1}(\mathbb{C}\mathrm{P}^1)$ is a $\pi_1(S)$-invariant multicurve which is $\pi_1(S)$-equivariantly homotopic to the multicurve $\phi^{-1}(m)$.

(2) $\rho_\infty(m) = I \in \mathrm{PSL}_2\mathbb{C}$, and for every diverging sequence $t_1 < t_2 < \ldots$, up to a subsequence, $\beta_{t_i}\colon \tilde{S} \to \mathbb{H}^3$ converges to a ρ_∞-equivariant continuous map $\beta_\infty\colon \tilde{S} \to \mathbb{H}^3 \cup \mathbb{C}\mathrm{P}^1$ as $i \to \infty$ and the axis a_i of $\rho_{t_i}(m)$ converges to a point $\mathbb{C}\mathrm{P}^1$ or a geodesic of $\mathbb{H}^3$ such that
- *if $\lim_{i\to\infty} a_i$ is a point on $\mathbb{C}\mathrm{P}^1$, then $\beta_\infty^{-1}(\mathbb{C}\mathrm{P}^1) = \phi^{-1}(m)$, and*
- *if $\lim_{i\to\infty} a_i$ is a geodesic a_∞ in $\mathbb{H}^3$, then β_∞ takes each component of $\phi^{-1}(N_m)$ to its corresponding limit geodesic a_∞ and each component of $\tilde{S} \setminus \phi^{-1}(N_m)$ to either a pleated surface in $\mathbb{H}^3$ or a single point on $\mathbb{C}\mathrm{P}^1$.*

In order to prove Theorem 8.1, Theorem 8.2 and Theorem 8.3, we carefully observe the behavior of C_t, fixing the isometry type of $\rho_\infty(m)$. In particular, for Theorem 8.1, we will show that, supposing, to the contrary, that $\rho_\infty(m)$ is hyperbolic (§11) or elliptic (§12), then ρ_t cannot converge. The convergence when $\rho_\infty(m) = I$ is given in §13 and the convergence when $\rho_\infty(m)$ is parabolic is given in §10.

9. $\mathbb{C}\mathrm{P}^1$-structures on punctured surfaces with elementary holonomy

Lemma 9.1. *Let F be a closed surface with finitely many punctures, such that the Euler characteristic of F is negative. Let $C = (f, \rho)$ be a $\mathbb{C}\mathrm{P}^1$-structure on F such that*

- *ρ is an elementary representation, and*
- *for each puncture of C, its peripheral holonomy is non-hyperbolic (so that its developing image is a single point on $\mathbb{C}\mathrm{P}^1$).*

Let Λ be the subset in $\mathbb{C}\mathrm{P}^1$ of cardinality $0, 1,$ or 2 which $\mathrm{Im}\,\rho$ preserves as a set. Then, there is at least one puncture of C which maps to a point in the complement $\mathbb{C}\mathrm{P}^1 \setminus \Lambda =: \Omega$ by f.

Proof. The discrete subset $f^{-1}(\Lambda)$ in $\tilde{F}$ descends a finite subset D on F.

We can assume that Λ is a non-empty set, since if Λ is the empty set, then the assertion is obvious. First, suppose that the cardinality of Λ is two, then Ω admits a complete Euclidean metric invariant under $\mathrm{Im}\,\rho$. Then, if all cups of F map to Λ, $F \setminus D$ admits a complete Euclidean metric, which is a contradiction against the Euler characteristic of F.

Next, suppose that the cardinality of Λ is one. Suppose, to the contrary, that all cups of C map to the point Λ. Then $C \setminus D$ has a complex affine structure.

We claim that $C \setminus D$ is *complete*, i.e. the developing map of $C \setminus D$ is a diffeomorphism onto $\mathbb{C}$, when we normalize $\mathrm{dev}\, C$ so that $\{\infty\}$ corresponds to the punctures. Suppose, to the contrary, that $C \setminus D$ is incomplete. As the cardinality of Λ is not two, $\mathrm{Im}\,\rho$ does not preserve an incomplete point of $C \setminus D$ in $\mathbb{C}$. Thus C admits Thurston's parametrization (τ, L) where τ is a finite area hyperbolic structure on F and L is a measured lamination on τ (Theorem [KP94b, Theorem 11.6], cf [Bab17, Theorem 3.1]). Since F is incomplete and the cardinality of Λ is not two, there is a maximal ball B of $\mathrm{dev}\, F$ such that its ideal point set contains two distinct points in $\mathbb{C}$. Then the holonomy of F must contain a hyperbolic element in $\mathrm{PSL}_2\mathbb{C}$ whose fixed points are in $\mathbb{C}$, whose endpoints are close to those two points in $\mathbb{C}$. This leads to a contradiction to all cups mapping to the same point. Therefore, the $C \setminus D$ is complete.

Thus, the holonomy of F consists of parabolic elements fixing ∞. Then the Euler characteristic of $F \setminus D$ is zero, since $F \setminus D$ admits Euclidean structure. Therefore F has a positive Euler characteristic, which is a contradiction. □

Proposition 9.2. *Let F be a closed surface with two punctures p and q such that the Euler characteristic of F is negative. Suppose that $C = (f, \rho)$ is a $\mathbb{C}\mathrm{P}^1$-structure on F such that*

- *the holonomy of C is elementary, and the stabilizer of $\mathrm{Im}\,\rho$ (in $\mathrm{PSL}_2\mathbb{C}$) is non-discrete, and*
- *the degrees of f around the two punctures are the same.*

Then, no cusp of F maps to the subset Λ defined in Lemma 9.1.

Proof. By Lemma 9.1, we can assume that p does not develop to Λ. As the Euler characteristic of F is negative, we let $C \cong (\tau, L)$ be the Thurston parameters of C; then by the assumption of the holonomy, p and q correspond to cusps of τ. Then, as the degrees at p and q agree, the total weights of leaves of L around the punctures are the same.

Suppose, to the contrary, that a puncture q develops to a point of Λ. Then f takes all lifts of q to the same point r of Λ: Otherwise, as Λ has cardinality two, $\mathrm{Im}\,\rho$ contains hyperbolic elements, and it also contains an elliptic element exchanging the points of Λ; then the stabilizer of $\mathrm{Im}\,\rho$ must be discrete against the hypothesis.

Let ℓ be a leaf of L initiating from q. Then its lift $\tilde{\ell}$ to the universal cover of τ maps, by the bending map $\beta\colon \mathbb{H}^2 \to \mathbb{H}^3$, to a geodesic in $\mathbb{H}^3$ initiating from q. As all lifts of p map to r, the other endpoint of $\beta(\tilde{\ell})$ is the image of a lift of q. Therefore all leaves of L initiating from q must end at p. For every complementary region R of $\tau \setminus L$, letting $\tilde{R}$ be the universal cover of R (in $\tilde{\tau} = \mathbb{H}^2$), at most, one ideal point of $\tilde{R}$ maps to q by the pleated surface.

Moreover, every leaf of L initiating from p must end at q, since the total weights of L around p and q agree. Let $L_{p,q}$ be the sublamination of L consisting of the isolated leaves of L connecting p and q. This implies that each component σ of $\tau \setminus L_{p,q}$ has a negative Euler characteristic. Since no leaves of $L \setminus L_{p,q}$ has an endpoint on the boundary of $\tau \setminus L_{p,q}$, the restriction of ρ to $\pi_1(\sigma)$ is non-elementary, which is a contradiction. □

10. Parabolic limit

In this section, we assume that $\rho_\infty(m)$ is parabolic, and analyze the limit of C_t as $t \to \infty$ in terms of its bending map and developing map. First, by Theorem 7.4, for each component F of $S \setminus m$, by taking an appropriate base point b_t in the thick part of C_t homotopic to F, (C_t, b_t) converges to a $\mathbb{C}\mathrm{P}^1$-structure F_∞ on a compact surface with one or two punctures, such that F_∞ is homeomorphic to F. Let C_∞ be the disjoint union of all such geometric limits F_∞ over all thick parts. Then C_∞ is a $\mathbb{C}\mathrm{P}^1$-structure on a closed surface with two cusps homeomorphic to $S \setminus m$. Note that C_∞ is not connected if and only if m is separating. Then the limit holonomy has the following algebraic property.

Lemma 10.1. *Suppose that $\rho_\infty(m)$ is parabolic. Then, for each component F of $S \setminus m$, $\rho_\infty(F)$ is non-elementary.*

Proof. Since S is a closed oriented surface of genus at least two, each component of $S \setminus m$ is also of hyperbolic type. Thus let (σ, ν) be the Thurston parameterization of F_∞, where σ is a complete closed hyperbolic with one or two cusps homeomorphic to F and ν is a measured lamination on σ. Clearly, the cusps of F_∞ correspond to the cusps of σ. Then there is a bi-infinite simple geodesic ℓ properly embedded in σ such that ℓ is a leaf of ν or disjoint from ν (note that each endpoint of ℓ is at a cusp of σ).

Let $\beta\colon \mathbb{H}^2 \to \mathbb{H}^3$ be the bending map given by (σ, ν), such that β is equivariant via $\rho_\infty | \pi_1(F)$. Let $\tilde{\ell}$ be a lift of ℓ to the universal cover $\mathbb{H}^2$ of σ. Then the endpoints of $\tilde{\ell}$ are parabolic fixed points in the ideal boundary of $\mathbb{H}^2$. Let $\gamma_1, \gamma_2 \in \pi_1(F)$ be the peripheral elements fixing the endpoints. As ℓ does not cross ν, its image $\beta(\tilde{\ell})$ is a geodesic in $\mathbb{H}^3$. Moreover, as β is ρ_∞-equivariant, $\rho_\infty(\gamma_1)$ and $\rho_\infty(\gamma_2)$ are parabolic elements fixing the different endpoints of $\beta(\tilde{\ell})$. Therefore $\rho_\infty(\gamma_1)$ and $\rho_\infty(\gamma_2)$ are non-commuting parabolic elements in $\mathrm{PSL}_2\mathbb{C}$, and they generate a non-elementary subgroup of $\mathrm{PSL}_2\mathbb{C}$. □

Proposition 5.2 implies that the developing map extends to cups with parabolic holonomy.

Proposition 10.2. *Let $C = (f, \rho)$ be a $\mathbb{C}\mathrm{P}^1$-structure on a closed surface with finitely many punctures, denoted by F, such that the holonomy around each puncture is parabolic. Then the developing map $f\colon \tilde{F} \to \mathbb{C}\mathrm{P}^1$ extends continuously to the lift of cups so that they map to their corresponding parabolic fixed points.*

Proof. Set $C \cong (\tau, L)$ in Thurston's parameters, where τ is a hyperbolic surface homeomorphic to F and L is a measured lamination on τ. For each cusp c of C, by Proposition 5.2, as the holonomy ρ around c is parabolic element in $\mathrm{PSL}_2\mathbb{C}$, c corresponds to a cusp of τ and the total weight of leaves of L ending at the cusp is either 0 or a positive multiple of 2π. Let $\beta\colon \mathbb{H}^2 \to \mathbb{H}^3$ be the bending map, and let $\tilde{L}$ be the $\pi_1(F)$-invariant measured lamination on $\mathbb{H}^2$ by pulling back L by the universal covering map $\mathbb{H}^2 \to \tau$. Let r be a geodesic ray in the universal cover $\mathbb{H}^2$ ending at a parabolic fixed point p of a peripheral element of $\pi_1(S)$. Then r eventually does not cross the $\tilde{L}$. Thus the curve $\beta(r)$ is eventually a geodesic ray in $\mathbb{H}^3$ ending at p. By the correspondence between the developing map and the pleated surface, the assertion follows. □

Recall that $\phi\colon \tilde{S} \to S$ denotes the universal covering map. Then the above lemmas imply a good convergence of the developing map of C_t away from m.

Theorem 10.3. *Suppose $\rho_\infty(m)$ is parabolic. Then there is a regular neighborhood N of m such that $f_t|\tilde{S} \setminus \phi^{-1}(N)$ converges to a ρ_∞-equivariant continuous map $f_\infty\colon \tilde{S} \setminus \phi^{-1}(N) \to \mathbb{CP}^1$ uniformly on compact subsets, such that the developing image of each boundary component of $\tilde{S} \setminus \phi^{-1}(N)$ maps to its corresponding parabolic fixed point.*

Proof. By Theorem 7.4 (2), the restriction C_t to $S \setminus N$ converges to C_∞ as $t \to \infty$ by taking an appropriate isotopy of S uniformly. Since $\rho_\infty(F)$ is non-elementary (Lemma 10.1), the restriction of f_t to $\tilde{S} \setminus \phi^{-1}(N)$ converges to the developing map of C_∞ uniformly on compact subsets. By Proposition 10.2, each boundary component $\tilde{S} \setminus \phi^{-1}(N)$ converges to its corresponding parabolic fixed point uniformly on compact subsets. □

In the rest of this section, we show the convergence of the developing map of C_t on the entire surface. First we analyze the holonomy of C_t along m.

Proposition 10.4. *For sufficiently large $t > 0$, $\rho_t(m)$ is not the identity element of* $\mathrm{PSL}_2\mathbb{C}$. *Moreover, if the cusp neighborhoods of C_∞ are horodisk quotients. Then, for sufficiently large $t > 0$, $\rho_t(m)$ is hyperbolic.*

Proof. Set $C_t \cong (\tau_t, L_t) \in \mathsf{T} \times \mathsf{ML}$ in Thurston's parameters for $t > 0$. Similarly set $C_\infty \cong (\tau_\infty, L_\infty)$, where τ_∞ is a complete hyperbolic structure on $F \setminus m$ with finite volume, and L_∞ is a measured geodesic lamination on τ_∞.

Let m_t denote the geodesic representative of m on τ_t. Then, the length of m_t on τ_t converges to 0 as $t \to \infty$ since $\rho_\infty(m)$ is parabolic.

Suppose, to the contrary, that there is a sequence $t_1 < t_2 < \dots$ diverging to ∞ such that $\rho_{t_i}(m)$ is *not* hyperbolic. Then a leaf ℓ_i of L_{t_i} intersects the geodesic loop m_{t_i} for each $i = 1, 2 \dots$. Pick a point p_i on $m_{t_i} \cap L_{t_i}$. Pick a lift $\tilde{m}_t$ of m_t to the universal cover $\tilde{\tau}_i \cong \mathbb{H}^2$ which is preserved by an element γ_m in $\pi_1(S)$ whose free homotopy class is m. Then, for each i, let $p_{i,j}$ $(j \in \mathbb{Z})$ be the lifts of p_{t_i} on $\tilde{m}_{t_i}$ in $\mathbb{H}^2$ indexed linearly, so that $p_{i,j} = \gamma_m^j \cdot p_{i,0}$.

For $t > 0$, let $\beta_t\colon \mathbb{H}^2 \to \mathbb{H}^3$ be the ρ_t-equivariant bending map induced by (τ_t, L_t). Then, since $\{p_{i,j}\}_{j \in \mathbb{Z}}$ is an orbit of the infinite cyclic group generated by γ_m, its image $\{\beta_{t_i}(p_{i,j})\}_{j \in \mathbb{Z}}$ is an orbit of the cyclic group generated by $\rho_{t_i}(\gamma_m) \in \mathrm{PSL}_2\mathbb{C}$. Then, since $\rho_{t_i}(m)$ is elliptic or parabolic (possibly the identity), by basic hyperbolic geometry, the points $\beta_{t_i}(p_{i,j})$ is contained in a totally geodesic hyperbolic plane H_{t_i} in $\mathbb{H}^3$. (In comparison, if $\rho_{t_i}(m)$ is hyperbolic and its screw rotation angle is not a multiple of π, then most of its orbits do not lie in a totally geodesic plane.)

Note that H_{t_i} is uniquely determined by the choice of p_i and the lift $\tilde{m}_i$, unless $\rho_{t_i}(m)$ is the identity.

If ρ_{t_i} is the identity element in $\mathrm{PSL}_2\mathbb{C}$, then, letting $\tilde{\ell}_i$ be the leaf of $\tilde{L}_{t_i}$ intersecting $\tilde{m}_i$ in $p_{i,j}$, let H_{t_i} be the hyperbolic plane orthogonal to the geodesic $\beta_{t_i}(\tilde{\ell}_i)$ in the point

$\beta_{t_i}(p_{i,j})$ for some $j \in \mathbb{Z}$. Clearly H_{t_i} is independent on the choice of $j \in \mathbb{Z}$, as $\rho_{t_i}(m)$ is the identity.

The infimum of $\angle_{\tau_{t_i}}(m_{t_i}, L_{t_i}) \geq 0$ over $i = 1, 2, \dots$ is positive, since $\angle_{\tau_i}(m_{t_i}, L_{t_i})$ is close to zero, then ρ_{t_i} must be hyperbolic (Theorem 2.1). Then, there is $\delta > 0$, such that, if i is large enough, then, if a leaf ℓ of $\tilde{L}_{t_i}$ intersects $\tilde{m}_{t_i}$, then the angle between the geodesic $\beta_{t_i}(\ell)$ and the hyperbolic plane H_{t_i} is at least δ. Indeed, otherwise, $\lim_{i\to\infty} \angle_{\tau_i}(m_{t_i}, L_{t_i}) = 0$.

Recall that τ_∞ is a complete hyperbolic surface of finite volume homeomorphic to $S \setminus m$, so that each boundary component of $S \setminus m$ corresponds to a cusp of τ_∞. Pick a loop α on S such that

(1) α essentially intersects m in a single point if m is non-separating, and in two points if m is separating,
(2) each segment $\alpha \setminus m$ descends to a geodesic g on τ_∞ with endpoints at cusps, and g does not crossing L_∞.

Below we show that the translation length of $\rho_{t_i}(\alpha)$ diverges to ∞, which contradicts the convergence of ρ_t. We assume that m is non-separating, and one can similarly prove the case when m is separating.

For each $i = 1, 2 \dots$, let α_i be the piecewise geodesic loop on τ_{t_i} to homotopic to α, such that

- α_i is a union of two geodesic segments,
- one geodesic segment s_i of α_i has its interior contained in $\tau_{t_i} \setminus m_{t_i}$, and at each endpoint, s_i meets m_{t_i} orthogonally, and
- the other geodesic segment u_i contained in m_{t_i}.

Since τ_{t_i} is pinched along m as $i \to \infty$, the length of s_i goes to ∞. Let $\tilde{\alpha}_i$ be a lift of α_i to $\mathbb{H}^2$ which is a simple piecewise geodesic, and it is a bilipschitz curve.

For each $i = 1, 2, \dots$, let $\tilde{u}_i$ be a lift of u_i to a geodesic segment of $\tilde{\alpha}_i$. Then, let $\tilde{m}_i$ be the lift of m_{t_i} to $\mathbb{H}^2$ which contains $\tilde{u}_i$, and let $\gamma_{\tilde{u}_i} \in \pi_1(S)$ be the element preserving $\tilde{m}_{t_i}$. For every $\epsilon > 0$ if i is large, the $\beta_{t_i}(\tilde{u}_i)$ is contained in the ϵ-neighborhood the $\rho_{t_i}(\gamma_{\tilde{u}_i})$-invariant hyperbolic plane $H_{\tilde{u}_i}$ above, since $\text{length}_{\tau_{t_i}} m_{t_i}$ goes to 0.

Let $\tilde{s}_i$ be a lift of s_i to a segment of $\tilde{\alpha}_i$. Then, the length of $\tilde{s}_i$ goes to ∞ as $i \to \infty$. For every $\epsilon > 0$, by (2), the transversal measure of s_i by L_{t_i} in the ϵ-thick part of τ_{t_i} limits to 0 as $i \to \infty$. In addition, there is $r > 0$, such that, the intersection angle of L_{t_i} and s_i in the r-thin part of $\tau_{\tau_{t_i}}$ goes to zero as $i \to \infty$. Therefore, for every $\epsilon > 0$, if i is sufficiently large, then the restriction of β_{t_i} to $\tilde{s}_i$ is a $(1-\epsilon, 1+\epsilon)$-bilipschitz embedding. Let g_i be the bi-infinite geodesic in $\mathbb{H}^3$ passing through the endpoints of $\beta_{t_i}(\tilde{s}_i)$.

Let $u_{i,1}, u_{i,2}$ be the lifts of u_i to the geodesic segments of $\tilde{\alpha}_{t_i}$ which are adjacent to $\tilde{s}_i$. Then let $H_{i,1}$ and $H_{i,2}$ be the hyperbolic planes corresponding to $u_{i,1}$ and $u_{i,2}$, respectively. Then, g_i transversally intersects $H_{i,1}$ and $H_{i,2}$ at angle at least $\delta/2$. Moreover, for every $\epsilon > 0$, if i is large enough, then those intersection points are ϵ-close to the endpoints of $\beta_{t_i}(\tilde{s}_i)$. Therefore, the distance between the hyperbolic planes $H_{i,1}$ and $H_{i,2}$ goes to ∞ as $i \to \infty$ (Figure 10). Therefore the translation length of $\rho_t(\alpha)$ goes to ∞ as desired. This contradicts the hypothesis. Therefore $\rho_t(m)$ must be parabolic for sufficiently large $t > 0$. $\square$

Let $\phi \colon \tilde{S} \to S$ be the universal covering map. Let $\kappa_t \colon C_t \to \tau_t$ denote the collapsing map of C_t, and $\tilde{\kappa}_t \colon \tilde{C}_t \to \mathbb{H}^2$ denote its lift from the collapsing of the universal cover (§2.2.2). We next show the convergence of the bending map.

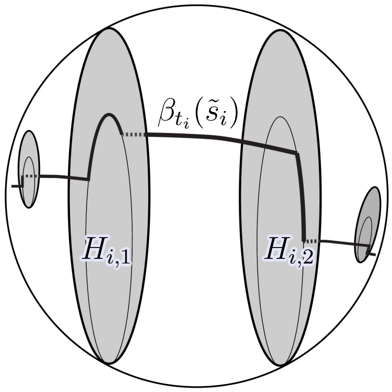

FIGURE 10. The quasi-geodesic $\beta_{t_i}(\tilde{\alpha}_i)$ preserved by the hyperbolic element $\rho_{t_i}(\alpha)$.

Theorem 10.5. *Suppose that $\rho_\infty(m)$ is parabolic. Then, up to an isotopy of S in t, $\beta_t \circ \tilde{\kappa}_t \colon \tilde{S} \to \mathbb{H}^3$ converges to a ρ_∞-equivariant continuous map $\alpha \colon \tilde{S} \to \mathbb{H}^3 \cup \mathbb{C}\mathrm{P}^1$ such that*

- *$\alpha^{-1}(\mathbb{C}\mathrm{P}^1)$ is a $\pi_1(S)$-invariant multicurve on $\tilde{S}$ isotopic to $\phi^{-1}(m)$ though $\pi_1(S)$-invariant multicurves, and*
- *for each component P of $\tilde{S} \setminus \phi^{-1}(m)$, the restriction $\beta_t \circ \tilde{\kappa}_t | P$ converges to the pleated surface for the component of C_∞ corresponding to P.*

Proof. The second assertion holds immediately from Theorem 10.3.

The axis a_t of $\rho_t(m)$ converges to the parabolic fixed point of $\rho_\infty(m)$. By Proposition 10.4, $\rho_t(m)$ is a hyperbolic element for sufficiently large $t > 0$. Let $D \subset \mathbb{H}^3$ be a horoball centered at the parabolic fixed point of $\rho_\infty(m)$. Then we pick a continuous path of $\rho_t(m)$-invariant subsets D_t in $\mathbb{H}^3$ bounded by the surface equidistant from the axis of $\rho_t(m)$ so that D_t converges to D as $t \to \infty$.

Pick a sufficiently small $\delta > 0$. For sufficiently large $t > 0$, let N_t^δ be the component of the δ-thin part of τ_t homotopic to m. Let $\tilde{N}_t^\delta$ be the lift of N_t^δ to the universal cover $\tilde{\tau} \cong \mathbb{H}^2$. If $\delta > 0$ is sufficiently small, by the convergence of ρ_t, the β_t-image of $\tilde{N}_t^\delta$ is eventually contained in D_t. This implies the first assertion. □

Next, we prove that cusp neighborhoods of the limit surface are isomorphic to cusp neighborhoods of a hyperbolic surface.

Proposition 10.6. *Suppose that $\rho_\infty(m)$ is parabolic. The cusps of C_∞ must be horodisk quotients.*

Proof. Suppose, to the contrary, that the cusp neighborhoods of C_∞ are *not* horodisk quotients.

Let $C_t \cong (\tau_t, L_t)$ denote the Thurston's parameters of C_t. Then, as $\rho_\infty(m)$ is parabolic, $L_t(m)$ converges to a non-negative integral multiple $2\pi n$ of 2π. As the limit cusp neighborhoods are assumed to be *not* horodisk quotients, n is a positive integer. Similarly, let $C_\infty \cong (\tau_\infty, L_\infty)$ denote Thurston parameters of C_∞. Thus the L_∞-transversal measure of each peripheral loop of C_∞ is $2\pi n$.

For sufficiently large $t > 0$, $\rho_t(m)$ is not the identity; let a_t be its axis (Definition 3.6). Pick $\delta > 0$ less than the two-dimensional Margulis constant. Let N_t be the δ-thin part of τ_t homotopic to m. Let $\tilde{N}_t$ be the lift of N_t to the universal cover $\mathbb{H}^2$. If $\delta > 0$ is sufficiently small, for all t large enough, each component of $N_t \cap L_t$

is a geodesic segment connecting one boundary component of N_t to the other. Since the transversal measure of each peripheral loop of L_t is close to $2\pi n > 0$ Thus, for $t \gg 0$, pick a fundamental domain F_t in $\tilde{N}_t$ bounded by two leaves of $\tilde{L}_t$ such that a component $F_{t,1}$ of $F_t \setminus \tilde{m}_t$ converges to a fundamental domain of the bending map $\beta_\infty \colon \mathbb{H}^2 \to \mathbb{H}^3 \cup \mathbb{CP}^1$ (Theorem 10.5) near a cusp of τ_∞.

Let ℓ_t be a leaf of $\tilde{L}_\infty$ bounding F_t, so that, for each component r_t of $\ell_t \setminus \tilde{m}_t$, the restriction of β_t converges to a bi-infinite geodesic in $\mathbb{H}^3$ as $i \to \infty$. Clearly the length of $\ell_t \cap \tilde{N}_t$ goes to ∞, and the length of each segment of $\ell_t \cap \tilde{N}_t \setminus \tilde{m}_t$ goes to ∞ as $t \to \infty$.

Let $F_{t,2}$ be the other component of $F_t \setminus \tilde{m}_t$. Then there is an element γ_t of G_t such that the restriction of β_t to $\gamma_t F_{t,2}$ converges to the fundamental domain of the other cusp of C_∞.

We first show that if $\rho_t(m)$ is hyperbolic, it must be "almost elliptic" for sufficiently large $t > 0$.

Claim 10.7. *Suppose that there is a sequence $t_1 < t_2 < \dots$ diverging to ∞, such that $\rho_{t_i}(m)$ is hyperbolic for each $i = 1, 2, \dots$. Then, the complex translation of $\rho_{t_i}(m)$ goes to zero from the imaginary direction as $i \to \infty$. In other words, the sequence $\operatorname{tr}^2 \rho_{t_i}(m) \in \mathbb{C}$ converges to 4 tangentially to the real ray $\{x \in \mathbb{R} \mid x \leq 4\}$.*

Proof. Suppose to the contrary that there is a sequence $t_1 < t_2 < \dots$ such that $\rho_{t_i}(m)$ is hyperbolic and the complex translation length converges to 0 from the non-imaginary direction. As $\rho_{t_i}(m)$ is hyperbolic, the axis is a geodesic and it converges to the parabolic fixed point of $\rho_\infty(m)$. Pick a point p_i on $\beta_{t_i}(F_{t_i})$ closest to a_{t_i} in $\mathbb{H}^3$. Let R_i be the set of points in $\mathbb{H}^3$ whose distance from a_{t_i} is at most the distance from p_i to the axis a_{t_i}.

For each i, let G_i be a one-dimensional Lie subgroup of $\mathrm{PSL}_2\mathbb{C}$ containing $\rho_{t_i}(m)$ such that the infinite cyclic group $\langle \rho_{t_i}(m) \rangle$ is asymptotically dense in G_i as $i \to \infty$ w.r.t. the path metric on G_i induced by the invariant metric on $\mathrm{PSL}_2\mathbb{C}$. Since the complex translation length of ρ_{t_i} converges to 0 from a non-imaginary direction, G_t converges to a one-dimensional subgroup in $\mathrm{PSL}_2\mathbb{C}$ consisting of only hyperbolic elements except the identity. For every i, let c_i be the G_i-invariant smooth curve in $\mathbb{H}^3$ passing p_i. Then c_i spirals on the boundary of R_i limiting to the endpoints of a_i. (See Figure 11.)

The β_{t_i}-image of the leaf ℓ_i is a geodesic in $\mathbb{H}^3$ tangent to R_i passing p_i. Then, moreover, the geodesic $\beta_{t_i}(\ell_i)$ and the curve c_i are asymptotically tangent to each other at p_i as $i \to \infty$, because of the convergence of the bending map β_{t_i} and the holonomy $\rho_{t_i}(m)$ as $i \to \infty$.

Let $s_{i,1}$ be the geodesic segment $\ell_i \cap F_{i,1}$, so that $\beta_{t_i}(s_{i,1})$ converges to a geodesic ray limiting to the fixed point of $\rho_\infty(m)$. Let $q_{1,i}$ be the endpoint of $s_{i,1}$ that is on the boundary of $\tilde{N}_i$, and let $q_{2,i}$ be the other endpoint of $\ell_i \cap \tilde{N}_i$. Then $\beta_{t_i}(q_{i,1})$ converges to a point in $\mathbb{H}^3$ as $i \to \infty$. Then $\beta_{t_i}(\gamma_i q_{i,2})$ also converges to a point on $\mathbb{H}^3$.

Since the length of each segment of $\ell_i \cap \tilde{N}_i \setminus \tilde{m}_i$ goes to infinity, and $\beta_i(\ell_i)$ is asymptotically tangent to the curve c_i, therefore the distance between $\beta_{t_i}(q_{i,1})$ and $\beta_{t_i}(q_{i,2})$ diverges to ∞ as $i \to \infty$. This is a contradiction against the convergence of the bending map β_{t_i} as $i \to \infty$. □

Next we show the convergence of ρ_t forces the convergence of twisting parameter along m.

Claim 10.8. *The Fenchel-Nielson twisting parameter of τ_t along m must converge (in $\mathbb{R}$) as $t \to \infty$.*

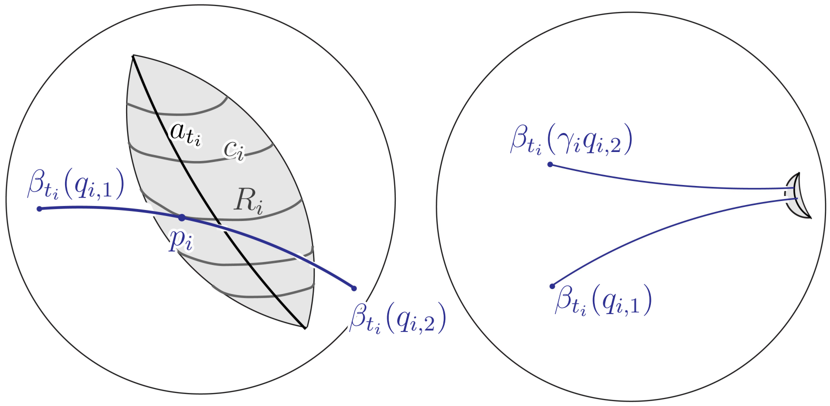

Figure 11. The left figure is the normalization of the right figure so that p_i is at the center

Proof. First, for each non-identity element of $\mathrm{PSL}_2\mathbb{C}$, we describe an associated foliation. For a hyperbolic isometry or an elliptic isometry of $\mathbb{H}^3$, the hyperbolic planes containing its axis give a foliation on $\mathbb{H}^3$ minus the axis. For a parabolic isometry $\alpha \in \mathrm{PSL}_2\mathbb{C}$, pick a hyperbolic plane H in $\mathbb{H}^3$ invariant under α, which contains the parabolic fixed point. Then there is a foliation of $\mathbb{H}^3$ by hyperbolic planes orthogonal to H and containing the parabolic fixed point; this foliation is independent of the choice of H. For sufficiently large $t > 0$, as $\rho_t(m)$ is not the identity (Proposition 10.4), let $\mathcal{F}_t$ denote such a foliation for $\rho_t(m)$.

Let $\tilde{m}$ be a lift of m to the universal cover $\tilde{S}$. Let P_1, P_2 be the connected components of $\tilde{S} \setminus \phi^{-1}(m)$ adjacent along $\tilde{m}$. For each $i = 1, 2$, given a point x_i in P_i near $\tilde{m}$, let $v_{\infty,i}$ be the tangent vector at the point $\beta_\infty \circ \tilde{\kappa}_\infty(x_i)$ in $\mathbb{H}^3$ orthogonal its support hyperbolic plane of x_i in the normal direction (§2.2.2). Since the L_t-transversal measure along m_t converges to $2\pi n > 0$, we can pick x_i so that $v_{\infty,i}$ is tangent to the foliation $\mathcal{F}_\infty$. Similarly, for each $t \gg 0$, pick a point $x_{t,i}$ in P_i such that, letting $v_{t,i}$ be the tangent vector of $\beta_t \circ \tilde{\kappa}_t$ at $x_{t,i}$ orthogonal to its support plane, $v_{t,i}$ is tangent to $\mathcal{F}_t$ and $v_{t,i}$ converges to $v_{\infty,i}$ as $t \to \infty$. (See Figure 12.)

Let $\mathcal{L}_t$ be the circular measured lamination on C_t which descends to the measured lamination of Thurston's parametrization by the collapsing map. Let e_t be the minimal transversal measure, given by $\mathcal{L}_t$, of arcs connecting x_1 to $\rho_t(\gamma_t)x_{t,2}$. Note that, since the isometry $\rho_t(m)$ preserves the foliation $\mathcal{F}_t$, the tangent vector $\rho_t(\gamma_t)v_{t,2}$ at $\rho_t(\gamma_t)x_{t,2}$ is also tangent to $\mathcal{F}_t$. By Claim 10.7, $\rho_t(m)$ is either parabolic, elliptic, or "almost elliptic" for $t \gg 0$. Therefore, for every $\epsilon > 0$, if $\delta > 0$ is sufficiently small, then, for $t \gg 0$, the transversal measure e_t is ϵ-close to a multiple of 2π. Thus the twisting parameter along m converges modulo 2π. By continuity, the twisting parameter of τ_t along m must converge as $t \to \infty$.

□

By Claim 10.8, the Fenchel-Nielsen twisting parameter of τ_t along m converges. For all $t > 0$, let $Q_{t,1}$ and $Q_{t,2}$ be the adjacent components of $\mathbb{H}^2 \setminus \psi^{-1}(m_t)$ corresponding to P_1 and P_2, respectively, so that $Q_{t,1}$ and $Q_{t,2}$ are separated by the geodesic $\tilde{m}_t$. Then, as the restriction of β_t of the component $Q_{t,1}$ converges, uniformly on compact subsets, to the bending map of the corresponding cusp neighborhood of C_∞ by Theorem 10.5. Then, since the length of the geodesic loop m_t goes to 0 as $t \to \infty$, the convergence of the twisting parameter implies that the restriction of β_t to $Q_{t,2}$ converges to the parabolic fixed point of $\rho_\infty(m)$ uniformly on compact subsets. This is a contradiction against the convergence of the bending map β_t of $Q_{t,2}$ uniformly on compact subsets

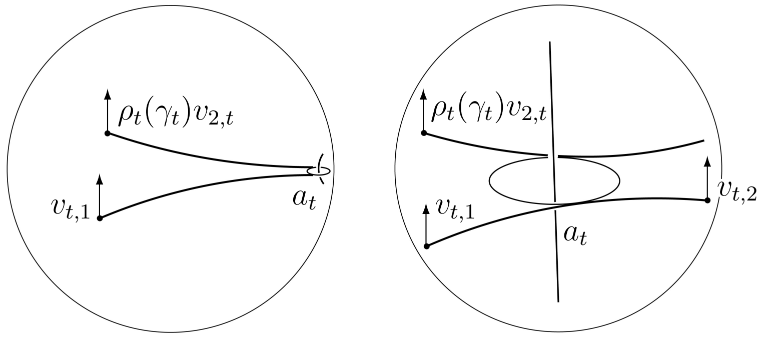

FIGURE 12. The right figure is a normalization of the left figure so that the axis a_t passes the center.

guaranteed by Theorem 10.5. 10.6

Theorem 10.9. *Suppose that $\rho_\infty(m)$ is parabolic. Then, by an appropriate isotopy of S in t, $f_t\colon \tilde S \to \mathbb{C}\mathrm{P}^1$ converges to a ρ_∞-equivariant continuous map $\tilde S \to \mathbb{C}\mathrm{P}^1$ such that, for some multiloop M on S consisting of finitely many parallel copies of m,*

- *f_∞ is a local homeomorphism on $\tilde S \setminus \phi^{-1}(M)$, and*
- *f_∞ takes each component of $\phi^{-1}(M)$ to its corresponding parabolic fixed point.*

Under the assumption of Theorem 10.9, each cusp of C_∞ is a horodisk quotient by Proposition 10.6. Thus, by Proposition 10.4, $\rho_t(m)$ is hyperbolic for all sufficiently large $t > 0$, and it converges to the parabolic element $\rho_\infty(m)$ as $t \to \infty$.

More generally, let $\gamma_t \in \mathrm{PSL}_2\mathbb{C},\ t \geq 0$ be a path of hyperbolic elements such that γ_t converges to a parabolic element γ_∞ in $\mathrm{PSL}_2\mathbb{C}$ as $t \to \infty$. Let G_t be the one-parameter subgroup of $\mathrm{PSL}_2\mathbb{C}$ containing γ_t such that the cyclic group generated by γ_t is asymptotically dense in G_t with respect to the path metric on G_t induced by the (left) invariant metric on $\mathrm{PSL}_2\mathbb{C}$.

Continuously conjugate γ_t by elements ω_t of $\mathrm{PSL}_2\mathbb{C}$ so that the axis of $\omega_t \cdot \gamma_t \coloneqq r_t w_t r_t^{-1}$ remains, for all t, to be the geodesic in $\mathbb{H}^3$ which connects 0 to ∞ in the ideal boundary $\mathbb{C}\mathrm{P}^1 = \mathbb{C} \cup \{\infty\}$.

Proposition 10.10. *Let A be a cylinder and homeomorphically identify A with $[-1,1]\times \mathbb{S}^1$, and let $\tilde A$ be the universal cover of A. Let A_t $(t > 0)$ be a path of $\mathbb{C}\mathrm{P}^1$-structures on a cylinder A, and let f_t be its developing map which changes continuously in t, such that*

- *the holonomy of A_t is the limit holonomy isomorphism $\pi_1(S) \cong \mathbb{Z} \to \langle \gamma_t \rangle$,*
- *each boundary of A_t develops onto a G_t-invariant curve on $\mathbb{C}\mathrm{P}^1$ for all $t > 0$.*
- *for each boundary circle b of A, the restriction of f_t to the lift $\tilde b$ to $\tilde A$ converges to a G_∞-invariant simple curve on $\mathbb{C}\mathrm{P}^1$ (which is a G_∞-invariant round circle minus the parabolic fixed point).*

Then, by an isotopy of A fixing the boundary, $\operatorname{dev} A_t\colon \tilde A \to \mathbb{C}\mathrm{P}^1$ converges to an continuous map $f_\infty\colon \tilde A \to \mathbb{C}\mathrm{P}^1$ such that

- *f_∞ is equivariant via the isomorphism $\mathbb{Z} \to \langle \gamma_\infty \rangle$;*
- *there is a multiloop M consisting of loops homotopy equivalent to A, such that f_∞ is a local homeomorphism on $\tilde A \setminus \tilde M$;*
- *f_∞ takes $\tilde M$ to the parabolic fixed point of γ_∞.*

Proof. We construct a path of fundamental membranes Z_t for the developing maps f_t which give the desired limit as $t \to \infty$.

The normalized developing map $\omega_t \circ f_t \colon \tilde{A} \to \mathbb{C} \cup \{\infty\}$ is identified with the restriction of $\exp \colon \mathbb{C} \to \mathbb{C}^*$ to a bi-infinite strip I_t bounded by parallel lines in $\mathbb{C} \cong \mathbb{E}^2$. Let b_1 and b_2 denote the boundary components of A. Regarding b_1, b_2 as simple closed curves, we can lift b_1 and b_2 to segments s_1 and s_2, respectively, of segments of the boundary components of $\tilde{A}$. For each $t > 0$ and $i = 1, 2$, let $s_{i,t}$ be the segment of the boundary line of I_t such that $\omega_t \circ f_t(s_i) = \exp(s_{i,t})$. Then $s_{1,t}$ and $s_{2,t}$ are parallel and have the same length. Thus $s_{2,t}$ is the Euclidean translation of $s_{1,t}$ by unique $z_t \in \mathbb{C} \setminus \{0\}$.

Claim 10.11. *(1) The length of $s_{i,t}$ goes to zero as $t \to \infty$, and*
(2) z_t converges to an integer multiple of $2\pi i$ as $t \to \infty$.

Proof. (1) As 0 and ∞ are the fixed points of $\omega_t \gamma_t \omega_t^{-1}$ and γ_t converges to γ_∞, both $\omega_t^{-1}(0)$ and $\omega_t^{-1}(\infty)$ converge to the parabolic fixed point of γ_∞ as $t \to \infty$. Since the development of b_i converges to a G_∞-invariant curve on $\mathbb{C}\mathrm{P}^1$, clearly the development of $s_{i,t}$ converges to a simple arc contained in the G_∞-invariant curve. Therefore, since $f_t = w_t^{-1} \exp$ on $\tilde{A}$, the norm of the derivative of f_t at each point on the segment s_i goes to infinity as $t \to \infty$. Hence the Euclidean length of $s_{i,t}$ must go to zero as $t \to \infty$.

(2) Since the Euclidean length of $s_{1,t}$ goes to zero on $I_t \subset \mathbb{C}$, translating I_t by a multiple of $2\pi i$, we may assume that $s_{1,t}$ converges to a point p on $\mathbb{C}$. Let $q \in \mathbb{C}\mathrm{P}^1$ be the parabolic fixed point of γ_∞. Let K be a compact subset K in $\mathbb{C}\mathrm{P}^1 \setminus \{q\}$ and U_p be a neighborhood of p in $\mathbb{C}$. Let U denote the union of translates of U_p by the integer multiples of $2\pi i$. Then, if t is sufficiently large, then $\omega_t^{-1} \exp(I_t \setminus U)$ is contained in $\mathbb{C}\mathrm{P}^1 \setminus K$. Therefore, as the developments of $s_{1,t}$ and $s_{2,t}$ converge to simple arcs in $\mathbb{C}\mathrm{P}^1 \setminus \{q\}$, their difference z_t must converge to a multiple of $2\pi i$. □

Let n be the integer such that z_t converges to $2\pi i n$. Pick a polygonal fundamental domain Z_t of A_t in I_t with following properties: Z_t is a union of $(n+1)$-rectangles $R_{t,1}, R_{t,2}, \ldots, R_{t,n+1}$ and n parallelograms $P_{t,1}, \ldots, P_{t,n}$ as in the figure (Figure 13) so that

- for each $i = 1, \ldots, n, n+1$, a pair of edges of $R_{t,i}$ are parallel to the boundary of the Euclidean strip I_t , the boundary segment $s_{1,t}$ is an edge of $R_{t,1}$, the boundary segment $s_{2,t}$ is an edge of $R_{t,n+1}$, and, for each $i = 2, \ldots, n-2$, the Euclidean translation of $s_{t,i}$ by $2\pi i$ decomposes $R_{t,i+1}$ into two isometric rectangles, and
- for each $i = 1, \ldots, n$, the parallelogram $P_{t,i}$ have edges parallel to the boundary of I_t which are an edge $R_{t,i}$ and an edge $R_{t,i+1}$.

In addition, we take $R_{t,1}, R_{t,2}, \ldots, R_{t,n+1}$ and n parallelograms $P_{t,1}, \ldots, P_{t,n}$ appropriately so that

- the development of $P_{t,i}$ by f_t converges to the parabolic fixed point of γ_∞ as $t \to \infty$;
- the f_t-images of $R_{t,1}$ and $R_{t,n+1}$ converge to horodisks bounded by the limit of $f_t(\tilde{b})$ in the hypothesis, and the restriction of f_t to $R_{t,1}$ and $R_{t,n+1}$ converge to a developing map of horodisk quotients;
- for $i = 2, \ldots, n$, the restriction of f_t to $R_{t,i}$ converges to a developing map of the Euclidean cylinder $(\mathbb{C}\mathrm{P}^1 \setminus \{p\}) / \langle \gamma_\infty \rangle$

(Figure 14). Let M be a multiloop on A consisting of n boundary parallel loops. Pick a path of regular neighborhood N_t of M so that N_t converges to M as $t \to \infty$. Isotope

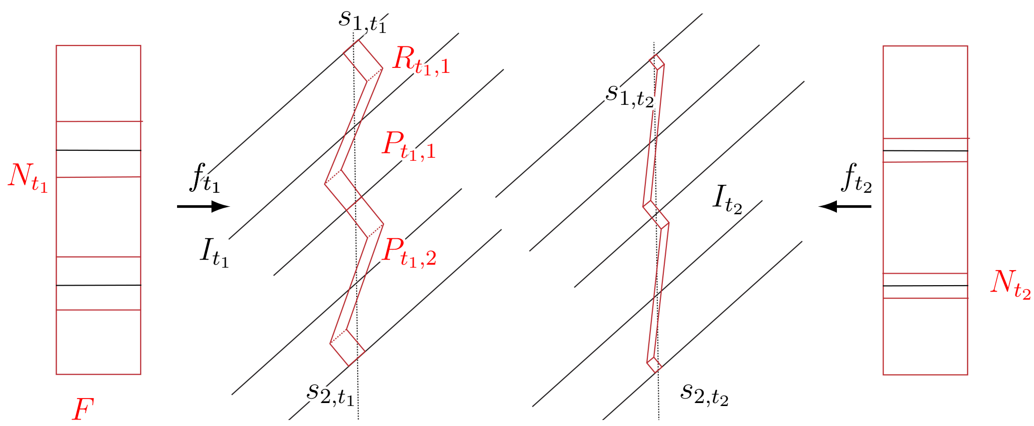

FIGURE 13. The limiting behavior of the fundamental membrane Z_t of A_t, where $n = 2$ and $t_1 < t_2$.

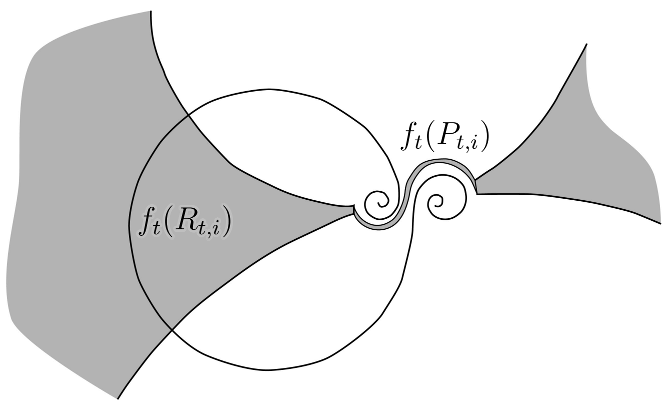

FIGURE 14.

A so that a fundamental domain F of $\tilde{A}$ maps to Z_t and that N_t are identified with $P_{t,1}, \ldots, P_{t,n}$. Then we a desired convergence.

□

Proof of Theorem 10.9. We already know the convergence of the developing map in every thick part by Theorem 10.3. There are two cusps c_1, c_1 of C_∞, which are horodisk quotients by Proposition 10.6. For each cusp c_i of C_∞, pick a simple closed curve ℓ_i which develops to a G_∞-invariant simple curve on $\mathbb{C}\mathrm{P}^1$. Then, for large $t > 0$, pick a simple closed curve $\ell_{i,t}$ on C_t such that $\ell_{i,t}$ develops onto a G_t-invariant curve on $\mathbb{C}\mathrm{P}^1$ and $\ell_{i,t}$ converges to ℓ_i as $t \to \infty$.

Let A_t be the cylinder in C_t bounded by $\ell_{1,t}$ and $\ell_{2,t}$. Then we can take such a path of cylinders A_t in C_t and a constant $\delta > 0$ such that A_t contains the δ-thin part of C_t for all sufficiently large t. Thus, by applying Proposition 10.10 to A_t, we obtain a multiloop for the desired convergence property of $\operatorname{dev} C_t$. □

10.1. **Convergence in holomorphic quadratic differential in the case of parabolic cusps.** Under the assumption that $\rho_\infty(m)$ is parabolic, we already have the limit C_∞ of C_t as $t \to \infty$ where C_∞ is a $\mathbb{C}\mathrm{P}^1$-structure on a Riemann surface X_∞ with two cusps homeomorphic to $S \setminus m$. Moreover, each cusp of C_∞ has a neighborhood which is a horodisk quotient (i.e. isomorphic, as a $\mathbb{C}\mathrm{P}^1$-structure, to a cusp neighborhood of a hyperbolic surface) by Proposition 10.6. Then the holomorphic quadratic differential ϕ_∞ on X_∞ representing C_∞ has, at worst, a first order pole at each cusp. Therefore we have the following convergence of the differential.

Theorem 10.12. *Suppose that $\rho_\infty(m)$ is parabolic. Then X_t converges to a nodal Riemann surface X_∞ such that X_∞ minus the node is homeomorphic to $S \setminus m$ and q_t converges to a quadratic differential q_∞ on X_∞ such that the node is at worst first order pole.*

11. $\rho_\infty(m)$ cannot be hyperbolic

In this section, we show that $\rho_\infty(m)$ cannot be a hyperbolic element.

Lemma 11.1. *Suppose that X_t is pinched along a loop m and $\rho_\infty(m)$ is hyperbolic. Then*

(1) C_t converges to a $\mathbb{C}\mathrm{P}^1$-structure C_∞ on a compact surface with two punctures, which is homeomorphic to $S \setminus m$, in the sense that, for every $\epsilon > 0$, the ϵ-thick part of C_t converges to the ϵ-thick part of C_∞ uniformly, and
(2) $\rho_\infty(F)$ is non-elementary for each component F of $S \setminus m$.

Proof. (1) is an immediate corollary of Theorem 7.4.

(2) Let F_∞ be the component of C_∞ corresponding to F. Let (σ, ν) denote the Thurston parametrization of F_∞. Then σ is a hyperbolic surface with geodesic boundary, such that the lengths of the boundary components are the translation length of $\rho_\infty(m)$ (see the proof of Lemma 5.4). Let $(\tilde\sigma, \tilde\nu)$ be the universal cover of (σ, ν) so that $\tilde\sigma$ is a convex subset of $\mathbb{H}^2$ bounded by geodesics and that $\tilde\nu$ is a $\pi_1(\sigma)$-invariant lamination on $\tilde\sigma$.

Let $\alpha\colon \tilde\sigma \to \mathbb{H}^3$ be its pleated surface equivariant by the holonomy of F_∞. Let ℓ be a boundary geodesic of $\tilde\sigma$. Then the endpoints of $\alpha(\ell)$ are in the limit set Λ of $\mathrm{Hol}\, F_\infty$, as $\alpha(\ell)$ is the axis of the hyperbolic $\rho_\infty(m)$. Every component R of $\tilde\sigma \setminus \tilde\nu$ has at least three ideal points. Then the ideal points of $\alpha(R)$ are in Λ (see [Bab20, Lemma 5.1]). Thus $\rho_\infty | F$ is non-elementary. □

Lemma 11.2. *For each cusp p of C_∞, there is a neighborhood of p foliated by isomorphic admissible loops which develop to simple curves on $\mathbb{C}\mathrm{P}^1$ invariant under a one-parameter subgroup in $\mathrm{PSL}_2\mathbb{C}$ containing $\rho_t(m)$.*

Proof. The developing map near a cusp neighborhood is the restriction of the exponential map $\exp\colon \mathbb{C} \to \mathbb{C}^*$; moreover, by taking an appropriate neighborhood, one can assume that the restriction is to a half-plane bounded by a straight line in $\mathbb{C}$ invariant under the deck transformation corresponding to the hyperbolic element $\rho_t(m)$.

The half-plane is foliated by straight lines parallel to the boundary, and this foliation descends to a desired foliation of the cusp neighborhoods by admissible loops. □

Proposition 11.3. *If $\epsilon > 0$ is sufficiently small, then, for every sufficiently large $t > 0$, there is a cylinder A_t in C_t homotopy equivalent to m such that*

- *A_t changes continuously in $t \gg 0$;*
- *A_t is foliated by admissible loops whose developments are invariant under a one-parameter subgroup G_t in $\mathrm{PSL}_2\mathbb{C}$ containing $\rho_t(m)$;*
- *A_t contains the conformally ϵ-thin part of C_t;*
- *$C_t \setminus A_t$ converges to a $\mathbb{C}\mathrm{P}^1$-structure on $S \setminus m$ whose boundary components are admissible loops.*

Proof. Consider the cusp neighborhoods of C_∞ foliated by admissible loops by Lemma 11.2. By the convergence of Lemma 11.1 and the stability of the admissible loops, for $t \gg 0$, there is a cylinder A_t foliated by admissible loops whose developments are invariant under G_t. Then it is easy to realize other desired properties. □

By Claim 11.1 (2), the developing map of $C_t \setminus A_t$ converges uniformly on compacts. By normalizing ρ_t by $\mathrm{PSL}_2\mathbb{C}$ continuously, so that, for sufficiently large $t > 0$, we can, in addition, assume that the axis of the hyperbolic element $\rho_t(m)$ connects 0 and ∞ of $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. Then the developing map of the cylinder A_t is the restriction of the exponential map $\exp\colon \mathbb{C} \to \mathbb{C}^*$ to the strip region R_t bounded by parallel lines, since the boundary components of A_t develop to G_t-invariant curves by Proposition 11.3. Since the boundary components of A_t converge to peripheral loops of C_∞, by the continuity of $\operatorname{dev} C_t$ in t, the region R_t converges to a strip in $\mathbb{C}$ with finite width. Therefore A_t must converge as $t \to \infty$. Thus C_t converges to a $\mathbb{CP}^1$-structure on S— this contradicts the divergence of C_t in the deformation space. Hence $\rho_\infty(m)$ cannot be hyperbolic.

12. $\rho_\infty(m)$ cannot be elliptic

In this section, similarly to the previous section (§11), we show that $\rho_\infty(m)$ cannot be elliptic. To show this, we assume, to the contrary, that $\rho_\infty(m)$ is elliptic and obtain a contradiction against the convergence of ρ_t as $t \to \infty$. By Theorem 7.4, we have

Proposition 12.1. *Suppose that $\rho_\infty(m)$ is elliptic. Then C_t converges to a $\mathbb{CP}^1$-structure C_∞ on a compact surface minus two points homeomorphic to $S \setminus m$, in the sense that, for every $\epsilon > 0$, the ϵ-thick part of C_t converges to the ϵ-thick part of C_∞.*

Lemma 12.2. *For each component F_∞ of C_∞, the stabilizer of $\rho_\infty(F_\infty)$ by conjugation is a discrete subgroup in $\mathrm{PSL}_2\mathbb{C}$.*

Proof. Let F_∞ be a component of C_∞. Then let (σ, ν) be the Thurston parametrization of F_∞, and let $(\tilde\sigma, \tilde\nu)$ be the universal cover of (σ, ν). Then the rotation angle of the elliptic element $\rho_\infty(m)$ is, modulo 2π, equal to the total weight, given by ν, of the leaves ending at a puncture (Proposition 5.2). Let $\beta_\infty\colon \tilde\sigma \to \mathbb{H}^3$ be the equivariant pleated surface. Pick a leaf ℓ of ν whose endpoints are at cusps of ν; then ℓ is an isolated leaf. Let $\tilde\ell$ be a leaf of $\tilde\nu$ which is a lift of ℓ. Then its image $\beta_\infty(\ell)$ is a geodesic in $\mathbb{H}^3$. Each endpoint of this geodesic is a fixed point of the parabolic element in the image $\rho_\infty(\pi_1(F))$ corresponding to its associated peripheral loop.

As the leaf ℓ is isolated, ℓ bounds a component P of $\tilde\sigma \setminus \tilde\nu$, and P has at least three ideal points. Then, for each ideal point p of P, let $\gamma \in \pi_1(F_\infty)$ be such that γ fixes p. Then $\beta_\infty(p)$ is fixed by the elliptic element $\rho_\infty(\gamma)$. Therefore, the stabilizer of $\rho_\infty(F_\infty)$ is a discrete subgroup of $\mathrm{PSL}_2\mathbb{C}$. □

Similarly to Proposition 11.3, the following follows from Lemma 12.1 and Lemma 12.2:

Proposition 12.3. *If $\epsilon > 0$ is sufficiently small, then for every sufficiently large $t > 0$, there is a cylinder A_t in C_t homotopy equivalent to m such that*

- *A_t changes continuously in $t \gg 0$;*
- *A_t is foliated by loops whose developments are invariant under the one-dimensional subgroup G_t of $\mathrm{PSL}_2\mathbb{C}$ containing $\rho_t(m)$, and G_t converges to a one-dimensional subgroup G_∞ of $\mathrm{PSL}_2\mathbb{C}$ containing $\rho_\infty(m)$;*
- *A_t contains the conformally ϵ-thin part of C_t homotopic to m;*
- *$C_t \setminus A_t$ converges to a $\mathbb{CP}^1$-structure on $S \setminus m$ such that the boundary components cover round circles on $\mathbb{CP}^1$.*

Proposition 12.4. *Suppose that $\rho_\infty(m)$ is elliptic. Then C_t converges to a $\mathbb{CP}^1$-structure on S, which is a contradiction as desired.*

Proof. Fix sufficiently small $\epsilon > 0$, and let A_t be a cylinder given by Proposition 12.3. Let $\phi_t\colon \tilde C_t \to C_t$ be the universal covering map. Then the developing map of

$\tilde{C}_t \setminus \phi_t^{-1}(A_t)$ converges uniformly on compacts. Let $\tilde{A}_t$ be the component of $\phi_t^{-1}(A_t)$ invariant under $m \in \pi_1(S)$, so that A_t changes continuously in t. We can normalize $\operatorname{dev} C_t$ by $\mathrm{PSL}_2\mathbb{C}$ continuously in t, such that, for sufficiently large $t > 0$, the geodesic axis of $\rho_t(m)$ connects 0 and ∞ of $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. Then, the restriction of $\operatorname{dev} C_t = f_t$ to $\tilde{A}_t$ is the restriction of the exponential map $\exp\colon \mathbb{C} \to \mathbb{C}^*$ to an infinite strip in $\mathbb{C}$. Since f_t converges on the boundary components of $\tilde{A}_t$, thus the restriction of f_t to $\tilde{A}$ converges as $t \to \infty$. Hence A_t must converge as $t \to \infty$ as a $\mathbb{CP}^1$-structure on a cylinder with boundary. Therefore C_t converges to a $\mathbb{CP}^1$-structure on S, which is a contradiction. □

13. Limit when $\rho_\infty(m) = I$

Let A be a regular neighborhood of a loop m on S. For $t \geq 0$, let (τ_t, L_t) be Thurston parameters of C_t. Let $\beta_t\colon \mathbb{H}^2 \to \mathbb{H}^3$ be its ρ_t-equivariant pleated surface. Let $\kappa_t\colon C \to \tau$ be the collapsing map, and $\tilde{\kappa}_t\colon \tilde{C} \to \mathbb{H}^2$ denote the lift of κ to the map between their universal covers. Let a_t denote the axis of $\rho_t(m) \in \mathrm{PSL}_2\mathbb{C}$ (Definition 3.6).

Note that a $\mathbb{CP}^1$-structure on S is defined up to an isotopy of the base surface S. Thus the developing map $f_t\colon \tilde{S} \to \mathbb{CP}^1$ of the path C_t of $\mathbb{CP}^1$-structures on S can be modified by an isotopy $\psi_t\colon S \to S$ in t without changing C_t. Finally, recall that $\phi\colon \tilde{S} \to S$ is the universal covering map.

Theorem 13.1. *Suppose that $\rho_\infty(m) = I$. Then the following hold:*

1. *$\rho_t(m) \neq I$ for sufficiently large $t > 0$.*
2. *The Fenchel-Nielsen twisting parameter (in $\mathbb{R}$) of X_t along m diverges to either ∞ or to $-\infty$.*
3. *For every diverging sequence $0 < t_1 < t_2 < \dots$, there is a subsequence such that*
   - (a) *the axis a_{t_i} converges to a point on $\mathbb{CP}^1$ or a geodesic in $\mathbb{H}^3$, denoted by a_∞;*
   - (b) *there is a $\mathbb{CP}^1$-structure in $\mathcal{P}(S \setminus m)$ such that, for every $\epsilon > 0$, the ϵ-thick part of C_{t_i} converges to the ϵ-thick part of C_∞ uniformly;*
   - (c) *up to an isotopy of S in t, the restriction of f_{t_i} to $\tilde{S} \setminus \phi^{-1}(A)$ converges to a ρ_∞-equivariant continuous map $f_\infty\colon \tilde{S} \setminus \phi^{-1}(A) \to \mathbb{CP}^1$ as $t_i \to \infty$ such that, for each component $\tilde{A}$ of $\phi^{-1}(A)$, its boundary components map onto the ideal points of a_∞.*
4. *the pleated surface $\beta_{t_i} \circ \tilde{\kappa}_{t_i}\colon \tilde{S} \to \mathbb{H}^3$ converges to a ρ_∞-equivariant continuous map $\tilde{S} \to \mathbb{H}^3 \cup \mathbb{CP}^1$, up to an isotopy of S.*

Notice that, by the surjectivity in (3c), if a_∞ is a geodesic, then the different boundary components of $\tilde{A}$ map to the different endpoints of a_∞.

We will prove (4) in the next subsection (§13.0.1). In this section, we will prove the other assertions: (1) will be proved in Lemma 13.8; (2) will be proved in Lemma 13.9; (3c) will be proved in Proposition 13.12. The proof of (3b) is similar to the proof of Theorem 7.4.

Let $C_\infty \cong (\sigma_\infty, \nu_\infty)$ denote the Thurston parameterization, where σ_∞ be a hyperbolic structure in the Teichmüller space $\mathsf{T}(S \setminus m)$ and ν_∞ be a measured lamination on σ_∞. Then σ_∞ has two cusps. At each cusp c of σ_∞, there are only finitely many leaves of ν_∞ ending at c by a basic property of geodesic laminations ([CEG87]). Then, since $\rho_\infty(m) = I$, the total weight of those leaves is a positive 2π-multiple.

Lemma 13.2. *If ν_∞ contains an irrational sublamination, then the holonomy of C_∞ is non-elementary.*

Proof. Suppose that ν_∞ contains an irrational sublamination. Then, there is a minimal irrational sublamination N of L, so that every leaf of N is dense in N. Let F be a (topologically) smallest subsurface of S containing N, such that $F \subset N$ is a π_1-injective. Let ℓ be a geodesic loop in σ_∞ which is a good approximation of N. Let $\beta_\infty \colon \mathbb{H}^2 \to \mathbb{H}^3$ be the equivariant pleated surface corresponding to $(\sigma_\infty, \nu_\infty)$. Then, for each component R of $F \setminus \ell$, the restriction of β_∞ to R is a quasi-isometric embedding ([Bab10]). Thus $\rho_\infty|\pi_1 R$ is non-elementary, immediately implying the lemma. □

Using the assumption that C_t is pinched along a single loop, we prove the following:

Proposition 13.3. *For each component F of $S \setminus m$, the restriction of ρ_∞ to $\pi_1 F$ is a non-trivial representation in the representation variety.*

Remark 13.4. *On the other hand, the restriction $\rho_\infty|\pi_1(F)$ may be the trivial representation in the character variety (see Theorem 14.5).*

Proof. If ν_∞ contains an irrational lamination, by Lemma 13.2, ρ_∞ is non-elementary. Then we can assume, without loss of generality, that ν_∞ contains only isolated leaves, and ν_∞ divides σ_∞ into ideal polygons.

Since each component of σ_∞ has one or two cusps, there is a leaf ℓ of ν_∞ whose endpoints are at a single cusp c of σ_∞. Let D be a small horodisk quotient neighborhood of c. Then $\ell \setminus D$ is a long geodesic segment, and by connecting its endpoints by a horocyclic simple arc in ∂D, we obtained a simple loop γ (which is a good approximation of ℓ); see Figure 15 (Left).

Pick a lift $\tilde{\ell}$ of ℓ to the universal cover $\mathbb{H}^2$ of σ_∞ and fix an orientation. Then there is $\alpha_\gamma \in \pi_1(S)$ representing γ which takes the oriented (bi-infinite) geodesic $\tilde{\ell}$ to an oriented geodesic starting from the endpoint of $\tilde{\ell}$; see Figure 15 (Right). Clearly $\beta_\infty(\tilde{\ell})$ is an oriented geodesic in $\mathbb{H}^3$. Then, by the equivariant proeprty, the holonomy along α takes the oriented geodesic $\beta_\infty(\tilde{\ell})$ to an oriented geodesic starting from the endpoint of $\beta_\infty(\tilde{\ell})$, and thus $\rho_\infty(\gamma) \neq I$.

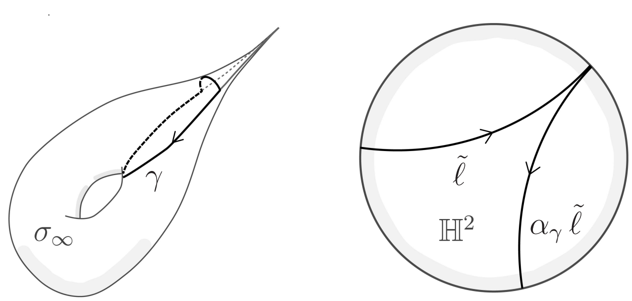

Figure 15.

□

Lemma 13.5. *Let G be a non-trivial subgroup of $\mathrm{PSL}_2\mathbb{C}$. Consider the (pointwise) stabilizer of the action $\mathrm{PSL}_2\mathbb{C} \curvearrowright G$ by conjugation. Suppose that the stabilizer is continuous. Then there is a set Λ of one or two points of $\mathbb{CP}^1$ fixed pointwise by the action of G.*

Proof. Suppose that G has a continuous stabilizer. Then, clearly, G is an elementary subgroup of $\mathrm{PSL}_2\mathbb{C}$. First suppose, in addition, that G contains a hyperbolic element h. Then no element in G exchanges the fixed points of h, as otherwise, the stabilizer cannot be continuous. Therefore Λ is the fixed point set of h, and all elements in $G \setminus \{I\}$ must be hyperbolic or elliptic elements with the same axis.

Next suppose that G contains a parabolic element p. Then there is no elliptic element or hyperbolic element in G, as otherwise, the stabilizer cannot be continuous. Then Λ must be the single fixed point of p, and all $G \setminus \{I\}$ are all parabolic elements with the same fixed point.

Suppose that G contains an elliptic element e and contains no hyperbolic element. Then, similarly, Λ must be the fixed point set of e, and G contains no parabolic element. Moreover $G \setminus \{I\}$ are all elliptic elements with a common axis. Then Λ is the set of the two endpoints of the axis. □

Given a $\mathbb{C}\mathrm{P}^1$-surface with a cusp such that the holonomy around the cusp is trivial, its developing map continuously extends to the cusp, so that it is a branched covering map near the cusp.

Lemma 13.6. *Let F be a compact surface with finitely many punctures, such that the Euler characteristic of F is negative. Let (f, ρ) be a developing pair of a $\mathbb{C}\mathrm{P}^1$-structure C on F such that*

- *$\rho\colon \pi_1(F) \to \mathrm{PSL}_2\mathbb{C}$ is not the trivial representation,*
- *the holonomy around each puncture is trivial, and*
- *the stabilizer of $\mathrm{Im}\,\rho$ in $\mathrm{PSL}_2\mathbb{C}$ is continuous; thus let $\Lambda \subset \mathbb{C}\mathrm{P}^1$ be the one- or two-point set in Lemma 13.5.*

Then, there is a cusp p of F such that $f(p)$ is not a point of Λ.

Proof. Notice that $\mathbb{C}\mathrm{P}^1$ minus Λ admits a complete Euclidean metric invariant under $\mathrm{Im}\,\rho$, which is unique up to scaling. Thus, if f takes all cusps of F into Λ, then the surface F minus finitely many points admits a complete Euclidean metric. This is a contradiction as the Euler characteristic of F is negative. □

The next proposition immediately follows from Proposition 9.2.

Proposition 13.7. *Let F be a compact connected surface with two punctures, such that the Euler characteristic of F is negative. Let $C = (f, \rho)$ be a $\mathbb{C}\mathrm{P}^1$-structure on F, such that*

- *$\mathrm{Im}\,\rho$ has a continuous stabilizer in $\mathrm{PSL}_2\mathbb{C}$;*
- *the holonomy around each puncture is trivial;*
- *the degrees around the two punctures are the same.*

Then no cusp of C maps to a point of Λ by f, where Λ is as in Lemma 13.5.

Let $\tilde{m}$ be a lift of m to $\tilde{S}$. Let Q and R be the adjacent components of $\tilde{S} \setminus \phi^{-1}(m)$ across $\tilde{m}$. Let $\mathrm{Stab}\,Q$ and $\mathrm{Stab}\,R$ denote the subgroups in $\pi_1(S)$ which setwise preserve Q and R, respectively. Let C_∞^Q, C_∞^R denote the component of C_∞ corresponding Q, R (if m is non-separating, $C_\infty^Q = C_\infty^R$).

We first prove (1) in Theorem 13.1.

Lemma 13.8. *For sufficiently large $t > 0$, $\rho_t(m) \neq I$.*

Proof. Suppose, to the contrary, that there is a diverging sequence $0 \leq t_1 < t_2 < \dots$ such that $\rho_{t_i}(m) = I$ for each i. We may, in addition, assume that C_{t_i} converges to C_∞ as $i \to \infty$ uniformly on compacts as $i \to \infty$. Then, as $\rho_{t_i}(m) = I$ and C_t is pinched along m, for $i \gg 0$, there is a cylinder A_i in C_{t_i} homotopic to m such that

- A_i is bounded by round circles (i.e. the development of each boundary component is a round circle on $\mathbb{C}\mathrm{P}^1$),
- $\mathrm{Mod}\,A_i \to \infty$, and

- $C_{t_i} \setminus A_i$ converges to C_∞ minus cusp neighborhoods bounded by round circles (in other words, for every $\epsilon > 0$, if i is sufficiently large, then A_i is contained in ϵ-thin part of C_{t_i}).

We can normalize ρ_{t_i} so that $\rho_{t_i}|\operatorname{Stab} R$ converges as $i \to \infty$ and the developing map $f_{t_i}|R$ also converges to a developing map of C_∞^R as $i \to \infty$. Then the development of $\tilde{m}$ converges to a point p on $\mathbb{C}\mathrm{P}^1$.

First suppose that the stabilizer of $\rho_\infty|\operatorname{Stab} Q$ is discrete. Then, there are elements α_1, α_2 of Stab Q with disjoint fixed point sets on $\mathbb{C}\mathrm{P}^1$. Pick a sequence $\gamma_i \in \mathrm{PSL}_2\mathbb{C}$ such that the restriction of the conjugation $\gamma_i \rho_{t_i} \gamma_i^{-1} =: \rho'_{t_i}$ to $\operatorname{Stab} Q$ converges as $i \to \infty$. Therefore, the properties of A_i imply that γ_i must leave every compact in $\mathrm{PSL}_2\mathbb{C}$. As α_1, α_2 have disjoint fixed point sets in $\mathbb{C}\mathrm{P}^1$, one of the fixed point sets does not contain the puncture point of C_∞^Q. Therefore either $\rho_{t_i}(\alpha_1)$ or $\rho_{t_i}(\alpha_2)$ diverges to infinity in $\mathrm{PSL}_2\mathbb{C}$ as $i \to \infty$ against the hypothesis.

Next suppose that the stabilizer of $\rho_\infty|\operatorname{Stab} Q$ is continuous. Then, by Proposition 13.7 and Lemma 13.6, with respect to the normalization ρ'_{t_i}, no cusp of C_∞^Q develops to a point of Λ for C_∞^Q. Let $\omega \in \operatorname{Stab} Q$ such that $\rho_\infty(w)$ is non-trivial (Proposition 13.3). Then, by the properties of A_i, $\rho_{t_i}(\omega)$ must diverges to ∞ since the continuous stabilizer preserves Λ.

This is a contradiction against the convergence of ρ_t. □

Lemma 13.9. *The Fenchel-Nielsen twist coordinate along m must diverge to ∞ or $-\infty$ as $t \to \infty$.*

Proof. The proof is similar to that of Lemma 13.8. Suppose to the contrary that there is a sequence $t_1 < t_2 < t_3 < \dots$ such that the Fenchel-Nielsen twist parameter of C_{t_i} along m converges as $i \to \infty$. We normalize ρ_{t_i} so that $\rho_{t_i}|\operatorname{Stab} R$ converges as $i \to \infty$ the developing map $f_{t_i}|R$ also converges to a developing map of C_∞^R as $i \to \infty$. Then, similarly to the proof of Lemma 13.8, one can show that $\rho_{t_i}|\operatorname{Stab} Q$ diverges to infinity, since the cylinder A_i becomes longer and longer and it pushes $\rho_{t_i}|\operatorname{Stab} Q$ farther and farther away; this contradicts the convergence of ρ_t as $t \to \infty$. □

Then, for each $t > 0$, let ι_t be some power of the Dehn twist of S along m such that the twist coordinates of $\iota_t C_t$ along m is bounded from above and below in $\mathbb{R}$ uniformly in $t > 0$. Then, by Lemma 13.9, the power must diverge to either ∞ or $-\infty$ as $t \to \infty$.

There is a diverging sequence $0 \leq t_1 < t_2 < \dots$ such that $C_{t_i} \to C_\infty$ as $i \to \infty$ uniformly on compact. Let F be a component of $S \setminus m$. Let $\tilde{F}$ be the universal cover of F.

First suppose that $\rho_\infty|F$ has a discrete stabilizer (in $\mathrm{PSL}_2\mathbb{C}$). Let F_∞ be a component of C_∞ which corresponds to F. Then $\operatorname{dev} F_\infty$ is the limit of $f_{t_i}|\tilde{F}$, so that $\lim_{i\to\infty} f_{t_i}$ takes each boundary component of $\tilde{F}$ to a single point corresponding to a cusp of C_∞.

Pick a fundamental domain D_i in $\tilde{F}$ with an arc s_i on $\partial D_i \cap \partial \tilde{F}$ such that s_i descends to a loop m_i isotopic to m, the loop m_i is contained in the ϵ_i-thin part of C_{t_i} with $\epsilon_i \searrow 0$ as $i \to \infty$, and the development of m_i is invariant under a one-dimensional subgroup G_i of $\mathrm{PSL}_2\mathbb{C}$ containing $\rho_i(m)$. As $\rho_{t_i}(m) \to I$, the image of s_i becomes more and more like a round circle c_i as $i \to \infty$.

Next suppose that $\rho_\infty(F)$ has a continuous stabilizer. Then $\rho_\infty(F)$ is elementary, and the restriction of f_{t_i} to $\tilde{F}$ may not converge to a local homeomorphism, even up to a subsequence. Nonetheless, as C_{t_i} converges to C_∞ in $\mathsf{P}(S \setminus m)$, clearly we can normalize ρ_{t_i} for the convergence of developing pairs:

Lemma 13.10. *Suppose that there is no subsequence of t_i such that $f_{t_i}|\tilde{F}$ converges to a developing map of F_∞. Then there is a sequence γ_i of $\mathrm{PSL}_2\mathbb{C}$ such that, up to a subsequence, $\gamma_i(f_{t_i}|\tilde{F}, \rho_{t_i}|\pi_1 F)$ converges to a developing pair (h_∞, ζ_∞) of F_∞.*

Next, without normalization, we show a convergence of the developing map as a continuous map.

Proposition 13.11. *Suppose that there is no subsequence such that the restriction $f_{t_i}|\tilde{F}$ converges to a developing map of F_∞ as $i \to \infty$. Then $f_{t_i}|\tilde{F}$ converges to a $\rho_\infty|\pi_1 F$-equivariant continuous map $f_{F,\infty}\colon \tilde{F} \to \mathbb{C}\mathrm{P}^1$ uniformly on compacts, such that each boundary component of $\tilde{F}$ maps to a single point. Moreover, either $f_{F,\infty}$ is a constant map to a fixed point of $\rho_\infty|F$ or there are open disks $D_1, \ldots, D_n$ on F such that $f_{F,\infty}$ takes $\tilde{F} \setminus \phi^{-1}(D_1 \sqcup \cdots \sqcup D_n)$ to a fixed point p of $\rho_\infty(F)$ and each lift $\tilde{D}_i$ of D_i to $\mathbb{C}\mathrm{P}^1 \setminus \{p\}$ for all $i = 1, \ldots, n$.*

Proof. Let $\gamma_i \in \mathrm{PSL}_2\mathbb{C}$ be the sequence and (h_∞, ζ_∞) be the normalized limit obtained by Lemma 13.10. By the non-subconvergence hypothesis, $\rho_\infty(\pi_1 F)$ is an elementary representation. We divide the proof into cases depending on the types of elementary subgroups.

First suppose that $\rho_\infty(\pi_1 F)$ contains a loxodromic or elliptic element. Then, let ℓ be the axis of the loxodromic or the elliptic element. Then, there is a corresponding loxodromic or elliptic element in $\mathrm{Im}\, h_\infty$, and let ℓ' be its axis. By the non-subconvergence hypothesis, there is $\omega \in \pi_1 F$ such that $h_\infty(\omega)$ is a parabolic element but $\rho_\infty(\omega)$ is the identity in $\mathrm{PSL}_2\mathbb{C}$. Thus γ_i must be a hyperbolic element for sufficiently large i such that as $i \to \infty$, the translation length of γ_i diverges to infinity. In addition, $\mathrm{Axis}(\gamma_i)$ converges to the ℓ' in $\mathbb{H}^3$. Let p be the limit of the repelling fixed point of γ_i, and let q be the limit of the attracting fixed point of γ_i, so that $\{p, q\}$ are the endpoints of ℓ'. Note that as $\rho_\infty(\pi_1(F))$ is elementary, $\rho_\infty \pi_1(F)$ preserves p and q point-wise.

Take a connected compact fundamental domain Q in $\tilde{F}$. We can assume that $Q \cap \partial\tilde{F}$ is disjoint from q, by perturbing the loop m_i on C_{t_i} if necessary. For simplicity, we first suppose that $h_\infty(Q)$ is disjoint from q. Then, letting $f_i = f_{t_i}$, the restriction $f_i|Q$ converges to the constant map to p uniformly, as $i \to \infty$, and thus $f_i\colon \tilde{F} \to \mathbb{C}\mathrm{P}^1$ converges to the constant map to p uniformly on compacts.

Suppose that $h_\infty(Q) \cap \{q\} \neq \emptyset$. Then, by the compactness of Q, there are finitely many points of $h_\infty^{-1}(q)$ in the interior of Q. Pick small disjoint open disk neighborhoods of the points in $h_\infty^{-1}(q)$ in Q. Then, as the disks are contained in a fundamental domain, their images $D_1, \ldots, D_n$ in F are disjoint. Then, as ζ_∞ preserves q, the restriction of f_i to $\tilde{F} \setminus \phi^{-1}(D_1 \sqcup \cdots \sqcup D_n)$ converges to the constant map to p uniformly on compacts. Moreover, for each lift $\tilde{D}_i$ of D_i to $\tilde{F}$, $\tilde{D}_i$ contains a unique point mapping to q. Thus up to an isotopy of S, we can in addition assume that $f_i|D_i$ converges to a homeomorphism to $\mathbb{C}\mathrm{P}^1 \setminus \{p\}$, as desired. By Lemma 9.1 and Proposition 9.2, the boundary components of $\tilde{F}$ all map to p.

Next, suppose that $\rho_\infty(F)$ contains a (non-trivial) parabolic element but no hyperbolic and elliptic element. Let $\omega \in \pi_1 F$ such that $\rho_\infty(\omega)$ is also a non-trivial parabolic element. Therefore ρ_∞ and ρ'_∞ are conjugate to each other, and $(f_{t_i}, \rho_{t_i}|\pi_1 F)$ converges to a developing pair of F_∞. This contradicts the non-subconvergence hypothesis.

Last, suppose that $\rho_\infty(\pi_1 F)$ is the trivial representation. This case will be similar to the case when $\rho_\infty(\pi_1 F)$ contains an elliptic or a hyperbolic element. Then the normalized holonomy ζ_∞ is a parabolic representation. Let p be the parabolic fixed point of ζ_∞. We can assume that γ_i is a hyperbolic element for i large, and the axis of γ_i converges to a geodesic ℓ starting from p. Let q be the other endpoint of ℓ. Pick

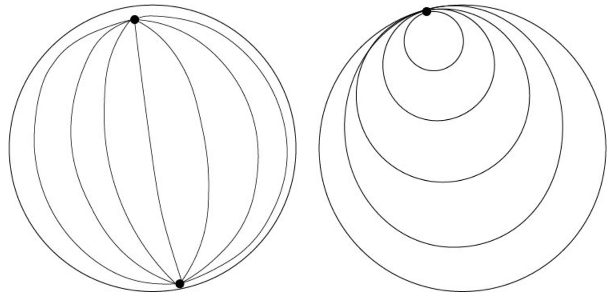

FIGURE 16.

a connected fundamental domain Q in $\tilde{F}$ so that $h^{-1}(p)$ is disjoint from ∂Q. Suppose in addition that no point of Q maps to p. Then, up to a subsequence, $f_i|\tilde{F}$ converges to a constant map to q uniformly on compacts. Suppose there are (finitely many) points of Q which map to p. Then, similarly to the case of a hyperbolic and an elliptic representation, take disjoint open ball neighborhoods of those points in Q, and let $D_1, D_2 \dots D_n$ be disjoint disks on F which lift to those open balls. Then the desired convergence follows similarly. □

By Proposition 13.11, the restriction of f_i to $\tilde{S} \setminus \phi^{-1}(A)$ converges to a ρ_∞-equivariant map $f_\infty \colon \tilde{S} \setminus \phi^{-1}(A) \to \mathbb{CP}^1$. We next prove the convergence of the boundary components to complete the proof of (3c).

Proposition 13.12. *For each component $\tilde{A}$ of $\phi^{-1}(A)$, let $\gamma \in \pi_1(S)$ be the representative of γ preserving $\tilde{A}$. Then, by taking a subsequence so that* $\mathrm{Axis}(\rho_{t_i}\gamma) =: a_i$ *converges to a subset $a_\infty \in \mathbb{H}^3 \cup \mathbb{CP}^1$, which is either a point on $\mathbb{CP}^1$ or a geodesic in $\mathbb{H}^3$, then f_∞ takes the boundary components of $\tilde{A}$ onto the ideal points of a_∞.*

Proof. By Lemma 13.8, $\rho_{t_i}(m) \neq I$ for sufficiently large $i \in \mathbb{Z}_{>0}$. Thus, by taking a subsequence, we may in addition assume that $\rho_{t_i}(\gamma)$ converges to I tangentially to a unit tangent vector of $\mathrm{PSL}_2\mathbb{C}$ at I. Let G_i be the one-parameter subgroup of $\mathrm{PSL}_2\mathbb{C}$ which contains $\rho_{t_i}(\gamma)$, such that the cyclic group generated by $\rho_{t_i}(\gamma)$ is asymptotically dense in G_i with respect to the intrinsic metric on G_i. Then the trajectories of G_i yields a unique foliation of $\mathbb{H}^3$ except that, if $\rho_{t_i}(\gamma)$ is elliptic, only of $\mathbb{H}^3 \setminus a_i$ (Figure 16). We have chosen a subsequence t_i so that $C_{t_i} \to C_\infty$ uniformly on every thick part and the axis a_i converges to a closed subset a_∞ of $\overline{\mathbb{H}^3}$. Let P, Q be the components of $\tilde{S} \setminus \phi^{-1}(A)$ adjacent across $\tilde{A}$.

Claim 13.13. *Let ℓ be the common boundary component of P and $\tilde{A}$. Suppose, to the contrary, that $\lim f_{t_i}(\ell)$ is not a point, in $\mathbb{CP}^1$, of the limit axis a_∞. Then $\rho_{t_i}|Q$ diverges to ∞ in χ.*

Proof. Let ι_i be some power of the Dehn twist of S along m so that the Fenchel-Nielsen twist parameter of the remarked Riemann surface $\iota_i X_{t_i}$ along m is bounded from above and below uniformly in i.

Let ℓ' be the common boundary component of $\tilde{A}$ and Q. By Proposition 13.3, there is $\gamma \in \pi_1(S)$ belonging to $\mathrm{Stab}\, Q$ such that $\rho_\infty(\gamma)$ is *not* the identity matrix. We may in addition assume the axis of $\rho_{t_i}(\iota_i \cdot \gamma)$ converges to the point $f_{F,\infty}(\ell)$ on $\mathbb{CP}^1$ (if $\rho_\infty(\mathrm{Stab}\, Q)$ is elementary, use Lemma 13.6 and Proposition 13.7). By the tangential convergence of $\rho_i(m) \to I$, the G_i-invariant foliation of $\mathbb{H}^3$ by $\mathcal{F}_i$ converges to a foliation of $\mathcal{F}_\infty$ of $\mathbb{H}^3$. If $f_{P,\infty}(\ell)$ is not the ideal point of a_∞, $\mathrm{Axis}(\rho_i(\iota\gamma))$ be eventually disjoint from every compact subset in the space of the leaves of $\mathcal{F}_\infty$. Therefore $\mathrm{Axis}(\rho_i(\gamma))$

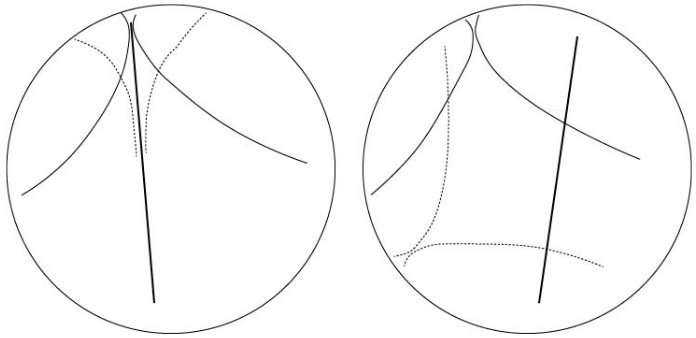

Figure 17.

leaves every compact subset of the leaf space of $\mathcal{F}_\infty$, and thus $\rho_i(\gamma)$ must diverge to ∞ in $\mathrm{PSL}_2\mathbb{C}$, which is a contradiction. (Figure 17.) □

This claim completes the proof. 13.12

It remains only to prove the surjectivity in Theorem 13.1 (3c):

Lemma 13.14. *Suppose that a_∞ is a geodesic in $\mathbb{H}^3$. Then $f_\infty(\ell)$ and $f_\infty(\ell')$ are the different endpoints of a_∞.*

Proof. By Claim 13.13. $f_i|\ell$ converges to the constant map to an endpoint of a_∞.

Let $n_i \in \mathbb{Z}$ be the power of the Dehn twist along m which gives $\iota_i \in \mathrm{MCG}(S)$. Thus $\rho_i(\gamma^{n_i})$ is a hyperbolic element whose axis a_i converges to a_∞, and its translation length diverges to infinity as $i \to \infty$. Then the attracting fixed point of $\rho_i(\gamma^{n_i})$ converges to the endpoint of a_∞ which is not $f_\infty(\ell)$. Thus $f_\infty(\ell')$ must be at the other endpoint. □

13.12

13.0.1. *Convergence of pleated surfaces when $\rho_\infty(m) = I$.* First we compare developing maps of $\mathbb{C}\mathrm{P}^1$-structures and the exponential map $\exp\colon \mathbb{C} \to \mathbb{C}^*$. Let ℓ be the geodesic in $\mathbb{H}^3$ connecting 0 to ∞ of $\partial\mathbb{H}^3 = \mathbb{C} \cup \{\infty\}$. Let $\Psi\colon \mathbb{C}^* \to \ell$ be the continuous extension of the nearest point projection $\mathbb{H}^3 \to \ell$. Then, the composition is $\Psi \circ \exp\colon \mathbb{C} \to \mathbb{H}^3$ is the Epstein map of the $\mathbb{C}\mathrm{P}^1$-structure on $\mathbb{C}$ given by exp.

Recall that, given a $\mathbb{C}\mathrm{P}^1$-structure $C = (X, q)$, for $x \in \mathbb{C}$, $d(x)$ is the Euclidean distance from x to the set of the zeros of the holomorphic differential q. Note that, if $d(x)$ is large, then we can naturally embed a large neighborhood of x into $\mathbb{C}(\cong \mathbb{E}^2)$ by an isometric map onto its image, so that vertical leaves map into horizontal lines, and horizontal leaves map into vertical lines.

Proposition 13.15. *For every $\epsilon > 0$, there is $R > 0$, such that, if $x \in C$ satisfies $d(x) > R$, then the Epstein map $\Sigma\colon \tilde{C} \cong \tilde{S} \to \mathbb{H}^3$ is ϵ-close, in the C^1-topology, to the composition of the collapsing map $\tilde{\kappa}\colon \tilde{S} \to \mathbb{H}^2$ and the bending map $\beta\colon \mathbb{H}^2 \to \mathbb{H}^3$ at every lift $\tilde{x}$ of x.*

Proof of Proposition 13.15.

Lemma 13.16. *For every $\epsilon > 0$, there is $R > 0$, such that, if $z \in \tilde{C}$ satisfies $d(z) > R$, then the maximal ball centered at z is ϵ-close to the maximal ball of the corresponding exponential map.*

Proof. As the Epstein map of C and exp are close, their developing maps are also close. This implies the closeness of their maximal balls centered at z and their ideal points. □

The proposition follows from the above lemma, and Proposition 4.9. 13.15

Recall that we have already proved Theorem 13.1 (1), (2), (3) regarding the asymptotic behavior of C_t using the decomposition of C_t into the restriction of C_t to the thin part A and its complement. We prove additional compatibility of the corresponding bending map.

Proposition 13.17. *Suppose that $\rho_\infty(m) = I$. Then for every diverging sequence $t_1 < t_2 < \dots$, up to taking a subsequence, there are a sequence of diffeomorphisms $\iota_i \colon S \to \tau_{t_i}$ representing the marking of C_{t_i} and a path of cylinders A_i in C_{t_i} homotopy equivalent to m, such that in addition to Theorem 13.1 (1), (2), (3), the following holds:*

(1) A maps to A_i by ι_i;
(2) $\beta_{t_i} \circ \tilde\kappa_{t_i} \colon \tilde S \to \mathbb{H}^3$ converges to a ρ_∞-equivariant continuous map $\tilde S \to \mathbb{H}^3 \cup \mathbb{CP}^1$ uniformly on compact subsets;
(3) for each connected component F of $\tilde S \setminus \phi^{-1}(A)$, the restriction of $\beta_{t_i} \circ \kappa_{t_i}$ to F converges to the pleated surface of the corresponding component of C_∞ or the constant map to an ideal point of a_∞ (in Proposition 13.12);
(4) letting $\tilde A$ be a connected component of $\phi^{-1}(A)$ in $\tilde S$, then $\beta_{t_i} \circ \tilde\kappa_{t_i} | \tilde A$ converges to a map onto a_∞ uniformly on compacts in $\tilde A$ with respect to a fixed closed disk metric on $\mathbb{H}^3 \cup \mathbb{CP}^1$.

Proof. For $t \gg 0$, there is a one-parameter family of loops homotopic to m such that their developments are invariant under a unique one-dimensional subgroup G_t of $\mathrm{PSL}_2\mathbb{C}$ which contains $\rho_t(m)$ (as in the proof of Proposition 13.11). Then we can pick a cylinder A_t in C_t homotopy equivalent to m, such that

- A_t is foliated by loops whose developments are invariant under G_t for each $t \gg 0$,
- $C_t \setminus A_t$ converges to C_∞ as $t \to \infty$, and
- $\mathrm{Mod}\, A_t \to \infty$ as $t \to \infty$.

By the second property, A_t is contained in a thinner and thinner part of C_t as $t \to \infty$. Then, the developing map of A_t is the restriction of $\exp \colon \mathbb{C} \to \mathbb{C}^*$ to a *bi-infinite strip* T_t, i.e. a region in $\mathbb{C}$ bounded by a pair of parallel lines. Then its deck transformation group ($\cong \mathbb{Z}$) is generated by a translation of T_t. Then A_t has a natural Euclidean metric by identifying $\mathbb{C}$ with $\mathbb{E}^2$.

Recall that A is a cylinder in S homotopic to m, and fix a finite volume Euclidean structure on A with geodesic boundary (by picking a homeomorphism $A \to \mathbb{S}^1 \times [-1, 1]$). We can easily pick a marking $\iota_t \colon S \to C_t$ such that

- ι_t takes A to A_t ((1));
- the restriction of C_t to $\iota_t(S \setminus A)$ converges to C_∞;
- $\iota_t | A$ is linear with respect to the Euclidean structures on A and A_t.

Given a component F of $\tilde S \setminus \phi^{-1}(A)$, suppose that $f_{t_i}|F$ converges to a developing map of the component of C_∞. Then, clearly $\beta_i \circ \tilde\kappa_i | F$ converges to a pleated surface for the corresponding component of C_∞. By Proposition 13.11, if $f_{t_i}|F$ does not converge to a developing map, then $\beta_i \circ \tilde\kappa_i|F$ converges to the constant map to an ideal point of the axis limit a_∞. Thus we have (3).

Last we prove (4). As we have already shown the convergence of the developing map in the thick part, we need to show that the convergence extends to the convergence on the neck. As the developing map of some components of $S \setminus m$ may degenerate as described in Proposition 13.11, accordingly one needs to be careful about the behavior of $\beta_i \circ \kappa_i$ on the neck.

By Theorem 13.1(2), the Fenchel-Nielsen twisting parameter of C_t along m diverges to either ∞ or $-\infty$ as $t \to \infty$. We can assume that the twisting of C_t along m occurs in A_t by isotopy of S.

(Case One) Suppose that $a_\infty := \lim_{i\to\infty} \operatorname{Axis} \rho_{t_i}(m)$ is a bi-infinite geodesic. Then $\rho_{t_i}(m)$ is hyperbolic if i is large enough, and the translation length of $\rho_{t_i}(m)$ time the number of twist goes to infinity as $i \to \infty$. For $r > 0$, let $U_i(r)$ be the r-neighborhood of a_i in $\mathbb{H}^3$. Clearly $U_i(r)$ is invariant under $\rho_i(m)$. Let $(\tau_i, L_i) \in \mathsf{T} \times \mathsf{ML}$ be the Thurston parameters of C_i for each i. Pick $\epsilon > 0$ less than the Bers' constant, and let $N_i = N_i^\epsilon$ be the ϵ-thin part of τ_i. Let $\tilde{N}_i$ be the lift of N_i to the universal cover $\mathbb{H}^2$ of τ_i invariant under the fixed representative the loop m in $\pi_1(S)$. Let $\ell_{i,1}, \ell_{i,2}$ denote the boundary components of $\tilde{N}_i$, which connect the endpoints of the geodesic a_i

Lemma 13.18. *If $r > 0$ is sufficiently large, then $\beta_i(\tilde{N}_i)$ is contained in $U_i(r)$ for sufficiently large i.*

Proof. Let $\tilde{A}$ be the lift of A to $\tilde{S}$ which is invariant under $m \in \pi_1(S)$. Let P_1 and P_2 be the components of $\tilde{S} \setminus \phi^{-1}(A)$ adjacent across a lift $\tilde{A}$. Suppose, to the contrary, that for every $r > 0$, the image $\beta_i(\tilde{N}_i)$ is *not* eventually contained in U_i as $i \to \infty$. Then, either

(i) for every $r > 0$, if i is sufficiently large, then $\beta_i(\ell_{i,1})$ and $\beta_i(\ell_{i,2})$ are both not contained in U_i, or
(ii) for every large $r > 0$, if i is sufficiently large, then one of $\beta_i(\ell_{i,1})$ and $\beta_i(\ell_{i,2})$ is contained in U_i but the other is not.

First, suppose (i). Then, let $\phi_i\colon \mathbb{H}^2 \to \tau_i$ be the universal covering map. Let $P'_{i,1}$ and $P'_{i,2}$ be the component of $\mathbb{H}^2 \setminus \phi_i^{-1}(N_i)$. For each $i = 1, 2, \dots$ and $j = 1, 2$, pick compact fundamental domains $D_{i,j}$ of $\operatorname{Stab} P_j \curvearrowright P'_{i,j}$, such that $D_{i,j}$ converges to a fundamental domain of the ϵ-thick components of τ_∞. Recall that U_i is invariant under $\rho_i(m)$. Then, for every $r > 0$, if i is sufficiently large, both fundamental domains of $P'_{i,1}$ or $P'_{i,2}$ map to outside U_i by β_i. Therefore, it follows from Proposition 13.3 and Proposition 13.7 that $\rho_i | \operatorname{Stab} P_1$ or $\rho_i | \operatorname{Stab} P_2$ must diverge to ∞ up to a subsequence, against to the convergence of ρ_i.

Next we suppose (ii). Without loss of generality, we can assume that $\beta_i(\ell_{i,1})$, not contained in U_i but $\beta_i(\ell_{i,2})$ is contained in U_i for sufficiently large i. Then, for every $r > 0$, similarly, the fundamental domain $P_{i,1}$ of P'_1 maps to outside U_i by β_i if i is sufficiently large. Then, by the assumption of $\beta_i(\ell_{i,2})$ being contained in U_i, one can similarly show $\rho_i | \operatorname{Stab} P_1$ diverges to ∞, up to a subsequence. □

If follows from Lemma 13.18 that, for every $\epsilon' > 0$, by taking $\delta > 0$ sufficiently smaller than $\epsilon > 0$ above, similarly letting $\tilde{N}_i^\delta$ be the $\rho_i(m)$-invariant lift of N_i^δ to the universal cover $\mathbb{H}^2$, the image $\beta_i(\tilde{N}_i^\delta)$ is ϵ'-close to the axis a_i for sufficiently large i.

Recall that we have a convergence of $\beta_i \circ \kappa_i$ on P_1, P_2 so that, in the limit, the boundary components of $\tilde{A}$ map to the endpoints of a_∞. Therefore, by taking an appropriate isotopy of S, $\beta_i \circ \kappa_i$ converges to a continuous map, up to a subsequence, such that $\tilde{N}$ maps to a_∞.

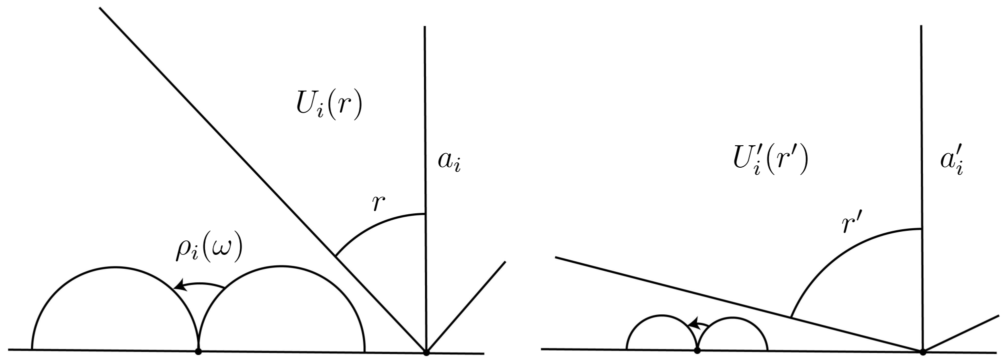

FIGURE 18. This figure illustrates the divergence of $\rho_i|\operatorname{Stab} P_j$ in the upper half space model of $\mathbb{H}^3$. The arrows indicate how the action by an element ω in $\operatorname{Stab} P_j$ changes, and it diverges as i increases in $\mathrm{PSL}_2\mathbb{C}$, where $r' > r$ and $i' > i$.

(Case Two) Suppose that a_∞ is a single point on $\mathbb{CP}^1$. Pick any horoball B in $\mathbb{H}^3$ tangent at a_∞. For each i, pick a subset $U_i \subset \mathbb{H}^3$ converging to B uniformly on compacts as $i \to \infty$, such that, if $\rho_i(\gamma)$ is either hyperbolic or elliptic, then U_i is an r_i-neighborhood of a_i for some $r_i > 0$, and if $\rho_i(\gamma)$ is parabolic, then U_i is a horoball centered at the parabolic fixed point of $\rho_i(\gamma)$.

For sufficiently large i, Let N_i be the ϵ-thin part of τ_i homotopy equivalent to m. Let $\tilde{N}_i$ be a component of $\psi_i^{-1}(N_i)$.

Lemma 13.19. *If $\epsilon > 0$ is sufficiently small, then $\beta_i(\tilde{N}_i)$ is eventually contained in U_i as $i \to \infty$. Therefore, $\beta_i \circ \kappa_i|\tilde{N}$ converges to the constant map to the point a_∞.*

Proof. Let P_1 and P_2 be the components of $\tilde{S} \setminus \phi^{-1}(A)$ adjacent across the lift $\tilde{A}$ of A invariant by $\rho_i(m)$. Suppose, to the contrary, for every $\epsilon > 0$, the image $\beta_i(\tilde{N}_i^\epsilon)$ is not eventually contained in U_i. Then, at least one of $\beta_i(\ell_{i,1})$ or $\beta_i(\ell_{i,2})$ is not contained in U_i for sufficiently large i. Therefore, it follows from using Proposition 13.3 and Proposition 13.7 that either $\rho_i|\operatorname{Stab} P_1$ or $\rho_i|\operatorname{Stab} P_2$ diverges to ∞, up to a subsequence. □

13.17

13.0.2. *Convergence of holomorphic quadratic differentials when $\rho_\infty(m) = I$.* We next describe the limit quadratic differential. In the case that $\rho_\infty(m) = I$, the singular Euclidean structure E_{t_i} contains a flat cylinder A_t homotopic to m, such that $\operatorname{Mod} A_t \to \infty$ and the complex length of its circumference converges to a positive multiple of $\pi/\sqrt{2}$, by Proposition 5.2. Therefore

Proposition 13.20. *Let C_∞ be the limit of C_t in Theorem 13.1 (3b). Then, the Schwarzian parameters of C_∞ consist of a Riemann surface with two punctures homeomorphic to $S \setminus m$ and a holomorphic quadratic differential q_∞, such that both punctures are a pole of order two and their residues are the same non-zero integer multiple of $\sqrt{2}\pi$.*

13.1. **Non-discreteness of holonomy.** We in addition show the non-discreteness of the holonomy representation ρ_t for large t.

Theorem 13.21. *Suppose that $\rho_\infty(m) = I$. Then $\operatorname{Im} \rho_t \subset \mathrm{PSL}_2\mathbb{C}$ is a non-discrete subgroup for sufficiently large $t > 0$.*

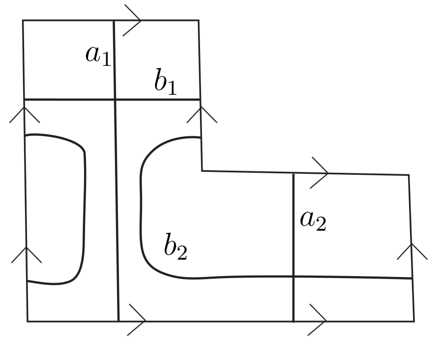

Figure 19.

Proof. Recall that $\rho_t(m) \to I$ but $\rho_t(m) \neq I$ (Theorem 13.1(1)). For each component F of $S \setminus m$, $\rho_t(\pi_1(F))$ is nontrivial for sufficiently large $t \gg 0$ (Proposition 13.3). Recall from Proposition 13.12 that, if C_{t_i} converges to a $\mathbb{C}\mathrm{P}^1$-structure of a punctured surface homeomorphic to F for a diverging sequence $t_1 < t_2 < \dots$, then, in the limit, its cusp point develops to an endpoint of the limit of the axis of ρ_{t_i}. Therefore the subgroup of $\operatorname{Im}\rho_t$ generated by $\{\, \rho_t(m)\gamma\rho_t(m)^{-1} \mid \gamma \in \rho_t(F) \,\}$ is non-elementary since the endpoint in $\mathbb{C}\mathrm{P}^1$ is not preserve by some non-trivial element in $\rho_t(\pi_1(F))$ by (Lemma 13.6 and Proposition 13.7). As $\rho_t(m) \to I$, by the Margulis lemma, $\operatorname{Im}\rho_t$ cannot be discrete. □

14. Examples of exotic degeneration

We construct examples of a path $C_t = (f_t, \rho_t)$ of $\mathbb{C}\mathrm{P}^1$-structures on S asymptotically pinched along a loop m as $t \to \infty$ such that $\rho_\infty(m) = I$ and $[\rho_t]$ converges in χ as $t \to \infty$, as in the second case of Theorem C. We construct two examples: one with $\rho_t(m)$ hyperbolic and one with $\rho_t(m)$ elliptic for all sufficiently large $t > 0$.

14.1. **Hyperbolic $\rho_t(m)$ converging to I.** Let E be the singular Euclidean surface obtained from an L-shaped polygon by identifying the opposite edges (Figure 19). Then E has exactly one cone point, and its cone angle is 6π. Let F be the underlying topological surface of E, which is a closed surface of genus two. Let E' denote E minus the cone point, and let F' denote the underlying topological surface of E'. Let ℓ_p be the (oriented) peripheral loop around the removed cone point. Let $\xi\colon \pi_1(F') \to \mathrm{PSL}_2\mathbb{C}$ be the holonomy of E'. Then, as F' has a Euclidean structure, the image of ξ consists of parabolic elements, and we can assume that its image consists of upper triangular matrices with 1's on the diagonal. In particular $\xi(\ell_p) = I$ (as before, by abuse of notation, we regard ℓ_p also as a fixed element of $\pi_1(S)$ by picking a basepoint of $\pi_1(S)$ on ℓ.) Notice that there is a point in the universal cover $\tilde{E}$ of E corresponding to $\ell_p \in \pi_1(S)$. (Namely, by lifting ℓ_p to a loop in the universal cover E starting from the base point, there is a unique cone point of $\tilde{E}$ in the disk region bounded by the lift.)

Proposition 14.1. *There is a path of $\mathbb{C}\mathrm{P}^1$-structures, $D_t = (h_t, \xi_t)$, on F' converging to $E' = (h, \xi)$ as $t \to \infty$, such that $\xi_t(\ell_p)$ is a hyperbolic translation whose axis converging to a geodesic connecting the global (parabolic) fixed point of ξ and the h-image of the corresponding singular point of $\tilde{E}$.*

Proof. Note that elements of $\operatorname{Im}\xi$ are translations of $\mathbb{C}$. Pick non-separating simple closed curves a_1, b_1, a_2, b_2 on E as in Figure 19 forming a standard generating set of $\pi_1(F)$ so that

- for each $i = 1, 2$, a_i and b_i intersect in a single point, and $[a_1, b_1][a_2, b_2] = I$,

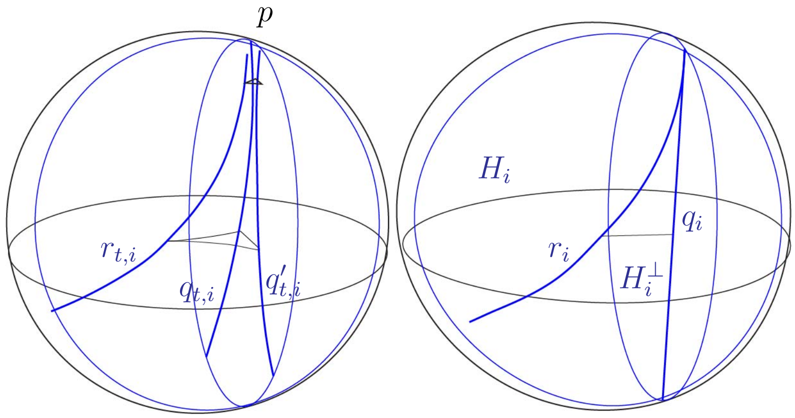

Figure 20.

- the translation directions of a_1 and a_2 are the same and the translation direction of b_1 and b_2 are the same, and
- the translation directions of a_i and b_i are orthogonal for each $i = 1, 2$.

Let c be a separating loop on E which separates $\{a_1, b_1\}$ and $\{a_2, b_2\}$. Then, let F_1 and F_2 be the components of $F \setminus c$ which are homeomorphic to a torus minus a disk.

Lemma 14.2. *Let q_i be any geodesic in $\mathbb{H}^3$ starting from the global fixed point $p \in \mathbb{C}\mathrm{P}^1$ of* Hol E*, and let H_i be the hyperbolic plane, in $\mathbb{H}^3$, containing an $\langle a_i \rangle$-orbit of q_i. For each $i = 1, 2$, given any path $h_{i,t}$ $(t \geq 0)$ of hyperbolic elements in* $\mathrm{PSL}_2\mathbb{C}$ *such that*

(1) the axis of $h_{i,t}$ is orthogonal to H_i at a point in q_i for all $t \geq 0$, and
(2) $h_{i,t} \to I$ as $t \to \infty$.

Then, there is a path $\zeta_{i,t} \colon \pi_1(F_i) \to \mathrm{PSL}_2\mathbb{C}$ of homomorphisms which converges to the restriction of Hol(E) *to $\pi_1(F_i)$ as $t \to \infty$ such that $\zeta_{i,t}(c) = h_{i,t}$.*

Proof. The point p is contained in the ideal boundary of H_i. Let r_i be a geodesic in H_i, such that $R(r_i)R(q_i) = \xi(a_i)$, where $R(r_i)$ and $R(q_i)$ are the π-rotations of $\mathbb{H}^3$ about r_i and q_i, respectively (Figure 20, Right).

Let $H_i^\perp$ be the hyperbolic plane in $\mathbb{H}^3$ orthogonal to H_i in the geodesic q_i. As $\mathrm{Axis}(h_{i,t})$ is in $H_i^\perp$ and orthogonal to q_i, we let $q_{i,t}$ and $q'_{i,t}$ be continuous paths of geodesics in $H_i^\perp$ such that $R(q_{i,t})R(q'_{i,t}) = h_{i,t}$, the geodesics $q_{i,t}$ and $q'_{i,t}$ converge to q_i as $t \to \infty$ uniformly on compact subsets, and the π-rotation $R(q_i)$ exchanges $q_{i,t}$ and $q'_{i,t}$. By this symmetry, there is a path of geodesics $r_{i,t}$ in H_i such that, for all $t \gg 0$,

- there is a hyperbolic plane intersecting $r_{t,i}, q_{t,i}, q'_{i,t}$ orthogonally, and
- $d_{\mathbb{H}^3}(r_{i,t}, q_{i,t}) = d_{\mathbb{H}^3}(r_{i,t}, q'_{i,t})$.

Thus by the symmetry, $\mathrm{tr}\, R(q_{i,t})R(r_i) = \mathrm{tr}\, R(q'_{i,t})R(r_i) \in \mathbb{R} \setminus [-2, 2]$.

The surface $F_i \setminus a_i$ is a pair of pants, and two of its boundary components correspond to a_i. Consider the path of homomorphisms $\zeta_{i,t} \colon \pi_1(F_i \setminus a_i) \to \mathrm{PSL}_2\mathbb{C}$ for $t > 0$, such that the two boundary components corresponding to a_i map to $R(q_{i,t})R(r_i)$ and $R(r_i)R(q'_{i,t})$— thus the other boundary corresponding to ∂F_i maps to $R(q_{i,t})R(q'_{i,t}) = h_{i,t}$ (see [Gol09]). Then, by Theorem 5.6, there is a path of $\mathbb{C}\mathrm{P}^1$-structures on $F_i \setminus a_i$ with holonomy $\zeta_{i,t}$ which converges to the component of $E \setminus (c \cup a_i)$ as $t \to \infty$ corresponding to $F_i \setminus a_i$. As the holonomies along the two boundary components are conjugate, for large enough $t > 0$, there is a path of $\mathbb{C}\mathrm{P}^1$-structures $\Sigma_{i,t}$ on F_i which converges to the

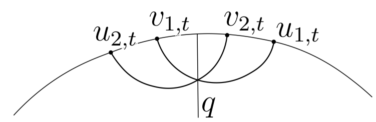

FIGURE 21.

component of $E \setminus c$, so that $\mathrm{Hol}\,\Sigma_{i,t}|\pi_1 F_i = \zeta_{i,t}$. In particular, the holonomy of $\Sigma_{i,t}$ around the puncture is the hyperbolic element $R(q_{i,t})R(q'_{i,t}) = h_{i,t}$. □

Notice that H_1 and H_2 are totally geodesic hyperbolic planes in $\mathbb{H}^3$ tangent at p. Therefore we can, in addition, assume that H_1 and H_2 are different and $H_1^\perp = H_2^\perp =: H$. Pick a geodesic q in H initiating from p contained in the region bounded by the geodesics $q_1 = H \cap H_1$ and $q_2 = H \cap H_2$.

Proposition 14.3. *We can choose, the path of the hyperbolic isometries $h_{1,t}, h_{2,t}$ (given by Lemma 14.2) so that their composition $h_{1,t}h_{2,t}$ is eventually a hyperbolic element whose axis converges to q as $t \to \infty$.*

Proof. Pick $h_{1,t}$ and $h_{2,t}$ such that their axes converge to the parabolic fixed point p. Since $h_{1,t}$ and $h_{2,t}$ converge to I, their product $h_{1,t}h_{2,t}$ also converges to I in $\mathrm{PSL}_2\mathbb{C}$.

For each $i = 1, 2$, let $u_{i,t}$ be the attracting fixed point, and let $v_{i,t}$ be the repelling fixed point of $h_{i,t}$. We may first assume that the endpoints of $\mathrm{Axis}\,h_{1,t}, \mathrm{Axis}\,h_{2,t}$ lie on ∂H in this cyclic order $u_{2,t}, v_{1,t}, v_{2,t}, u_{1,t}$ (Figure 21). The composition $h_{1,t}h_{2,t}$ fixes a point on the arc in ∂H between $v_{1,t}$ and $v_{2,t}$ for each $t > 0$. Note that the segment contains p. Then as $\mathrm{Axis}(h_{1,t}), \mathrm{Axis}(h_{2,t})$ converge to the parabolic fixed point p, there is a fixed point of $h_{1,t}h_{2,t}$ converging to p. Moreover, as $h_{1,t} \to I$, one can continuously adjust the translation length of $h_{2,t}$ so that $h_{1,t}h_{2,t}$ also fixes the other endpoint of q for sufficiently large $t > 0$. Let s be the endpoint of the geodesic q which is *not* p. Then, after this adjustment, clearly $h_{2,t}(h_{1,t}(s)) = s$ holds for all large $t > 0$ and $h_{1,t}(s) \to s$ as $t \to \infty$. Since the axis of the hyperbolic element $h_{2,t}$ converges to the ideal point $p (\neq s)$, the translation length of $h_{2,t}$ must converge to zero; thus $h_{2,t}$ converges to the identity.

Clearly the composition $h_{1,t}h_{2,t}$ does *not* fix the endpoints of the axes of the hyperbolic elements $h_{1,t}$ and $h_{2,t}$ for all large $t > 0$. Therefore $h_{1,t}h_{2,t}$ is a hyperbolic element with the axis q for sufficiently large $t > 0$, which is not the identity. □

Let $h_{1,t}, h_{2,t} \in \mathrm{PSL}_2\mathbb{C}$ be the paths given by Proposition 14.3. Then, by Lemma 14.2, for each $i = 1, 2$, we have a path of homomorphisms $\zeta_{i,t}\colon \pi_1(F_i) \to \mathrm{PSL}_2\mathbb{C}$ such that $\zeta_{i,t}(\ell) = h_{i,t}$ for $t \gg 0$. Then there is a unique path $\zeta_t\colon \pi(F') \to \mathrm{PSL}_2\mathbb{C}$ so that $\zeta_t|\pi_1(F_i) = \zeta_{i,t}$ for $i = 1, 2$; thus $\zeta_t(\ell_p) = h_{1,t}h_{2,t}$. Then, by the holonomy theorem (Theorem 5.6), there is a path D_t of $\mathbb{C}\mathrm{P}^1$-structures on F' with holonomy ζ_t for $t \gg 0$ such that D_t converges to E' as $t \to \infty$. 14.1

Remark 14.4. *Since ξ_t converges to the parabolic representation ξ and the axis of the hyperbolic element $\rho_t(\ell_p)$ converges to a geodesic starting from the parabolic fixed point of ξ as $t \to \infty$, by normalizing by an appropriate power r_t of isometries $\xi_t(\ell_p)$, the conjugation $\xi_t(\ell_p)^{r_t} \cdot \xi_t \cdot \xi_t(\ell_p)^{-r_t}$ converges to the trivial representation, and the developing map $\xi_t(\ell_p)^{r_t} h_t$ converges to the constant map to the endpoint of q which is not p.*

14.1.1. *Constructing a closed surface from punctured surfaces.* To make a desired example of exotic degeneration, we take two copies D_t of $\mathbb{C}P^1$-surfaces with a single puncture from Proposition 14.1, and glue them together with many twists.

Theorem 14.5. *There is a path of $\mathbb{C}P^1$-structures $C_t = (f_t, \rho_t)$ on a closed surface S of genus four with following properties:*

- *The conformal structure X_t is pinched along a separating loop m as $t \to \infty$; let F_1 and F_2 be the connected components of $S \setminus m$.*
- *$\rho_t\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ converges in the representation variety as $t \to \infty$.*
- *Pick an element $\gamma \in \pi_1(S)$ whose free homotopy class is m. Then $\rho_\infty(\gamma) = I$, and, for all $t > 0$, the holonomy $\rho_t(\gamma)$ is a hyperbolic element such that its axis a_t converges to a geodesic a_∞ in $\mathbb{H}^3$ as $t \to \infty$.*
- *Let $\tilde{F}_1, \tilde{F}_2$ be the connected components of $\tilde{S} \setminus \phi^{-1}(m)$ which are adjacent across the lift $\tilde{m}$ of m preserved by $\gamma \in \pi_1(S)$. Then $C_t|\tilde{F}_1$ converges to the developing map of a $\mathbb{C}P^1$-structure on a genus two surface minus a point such that the cusp maps to an endpoint of a_∞ as $t \to \infty$.*
- *$f_t|\tilde{F}_2$ converges to the constant map to the other endpoint of a_∞ uniformly on compacts, and $\rho_\infty|\pi_1(F_2)$ is the trivial representation.*

Remark 14.6. *In fact, $\mathrm{Im}\,\rho_\infty$ consists of parabolic elements with a global fixed point on $\mathbb{C}P^1$, and therefore the limit representation ρ_∞ is identified with the trivial representation in the character variety χ. In other words, the frontier of $\mathrm{PSL}_2\mathbb{C}$-orbit of ρ_∞ contains the trivial representation. Thus, there is a path α_t $(t > 0)$ in $\mathrm{PSL}_2\mathbb{C}$ such that $\alpha_t \rho_t \alpha_t^{-1}$ converges to the trivial representation.*

Proof. For sufficiently large $t > 0$, the $\mathbb{C}P^1$-structure D_t with a single puncture from Proposition 14.1 has a cusp neighborhood N_t foliated by admissible loops whose developments are invariant under the one-dimensional subgroup G_t of $\mathrm{PSL}_2\mathbb{C}$ containing $\xi_t(\ell_p)$. We can assume that N_t changes continuously in t and is asymptotically the empty set on E' as $t \to \infty$. Note that as G_t is a one-dimensional subgroup $\rho_t(m)$ of $\mathrm{PSL}_2\mathbb{C}$, integer powers $\rho_t(m)^n$ for $n \in \mathbb{Z}$ continuously extends to real powers.

First take two copies $\Sigma_{1,t}, \Sigma_{2,t}$ of $D_t \setminus N_t$ and, since the boundary of N_t are invariant by the one-parameter subgroup G_t, glue them together along their boundary components without adding a twist. Let $C'_t = (f'_t, \rho'_t)$ be the resulting developing pair. Then we can normalize by $\mathrm{PSL}_2\mathbb{C}$ so that the axis of the hyperbolic element $\rho'_t(m)$ is the geodesic q for all t. In addition, we can renormalize the developing pair by $\mathrm{PSL}_2\mathbb{C}$ so that the restriction of f'_t to $\tilde{F}_1$ and the restriction of ρ'_t to the stabilizer $\mathrm{Stab}\,\tilde{F}_1$ of $\tilde{F}_1$ in $\pi_1(S)$ converges to a developing pair of E' as $t \to \infty$. Then, as N_t converges to the empty set, the restriction of ρ'_t to $\mathrm{Stab}\,\tilde{F}_2$ leaves every compact in the representation variety, and the restriction of f'_t to $\tilde{F}_2$ does not converge to a continuous map as $t \to \infty$.

Recall that the holonomy ρ'_t along m is a hyperbolic element with axis q, and the translation length of $\rho'_t(m)$ goes to zero as $t \to \infty$. Therefore, when we glue $\Sigma_{1,t}, \Sigma_{2,t}$ of $D_t \setminus N_t$, we can continuously add more and more twists along m, which conjugates the structure on F_2 by $\rho'_t(m)$ raised to the power of the amount of twist along q, so that

- the restriction of f'_t to $\tilde{F}_1$ and the restriction of ρ'_t to $\mathrm{Stab}\,\tilde{F}_1$ still converges to a developing pair for E', and
- the restriction of ρ'_t to $\mathrm{Stab}\,\tilde{F}_2$ converges to the trivial representation, and the restriction of f'_t to $\tilde{F}_2$ converges to the constant map to the other endpoint of q (by Remark 14.4) as $t \to \infty$.

We obtained a desired path C'_t. □

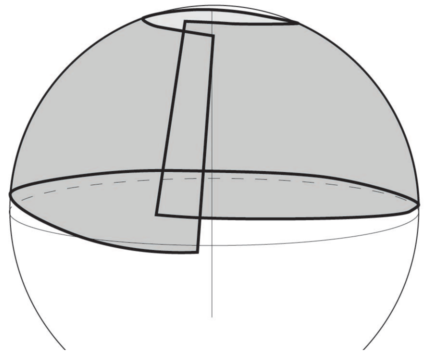

FIGURE 22.

14.2. **Elliptic $\rho_t(m)$ converging to the identity.** In this section, we construct an example of $C_t = (f, \rho_t)$ in Theorem C (ii) such that $\rho_t(m)$ is an elliptic element for all sufficiently large $t > 0$ and it converges to I as $t \to \infty$.

Given an elliptic element $e \in \mathrm{PSL}_2\mathbb{C}$, normalize the unit disk model $\mathbb{D}^3 \subset \mathbb{R}^3$ of $\mathbb{H}^3$ centered at the origin, so that $\mathrm{Axis}(e)$ is contained in the axis of the third coordinate. Let $\zeta \in (0, 2\pi)$ be the rotation angle of e. Then, define $b_e\colon \mathbb{R} \to \partial\mathbb{H}^3$ by $x \mapsto (\cos(\zeta x)\sin x, \sin(\zeta x)\sin x, \cos x)$ which is equivariant under $\mathbb{Z} \to \langle e \rangle$.

Lemma 14.7. *Let r be a geodesic in $\mathbb{H}^3$. Pick a parallel vector field $V \subset T\mathbb{H}^3$ along r such that V is orthogonal to r. Then, there are a path of (nontrivial) elliptic elements $e_t \in \mathrm{PSL}_2\mathbb{C}$ and a continuous function $\theta_t \in \mathbb{R}_{\geq 0}$ in $t > 0$ which satisfies the following:*

- *$e_t \to I$ as $t \to \infty$.*
- *$\mathrm{Axis}(e_t)$ orthogonally intersects r, and $\mathrm{Axis}(e_t)$ converges to an endpoint of r on $\mathbb{C}\mathrm{P}^1$ as $t \to \infty$.*
- *Letting $\theta_t \in \mathbb{R}$ be a continuous function such that the angle between $\mathrm{Axis}\, e_t$ and V is $\theta_t \mod 2\pi$, when an orientation of Axis(e_t) is fixed continuously in t.*
- *Let $u_t = 2\theta_t$. Then the rotation angle of $e_t^{u_t}$ is π for all $t \geq 0$, so that $e_t^{u_t}$ takes r to itself, reversing the orientation.*

Proof. It is easy to construct an example satisfying the first three conditions. Then adjust the rotation angle of e_t so that it also satisfies the last condition. □

Lemma 14.8. *Let e_t be as in Lemma 14.7. Let p be the endpoint of r to which $\mathrm{Axis}(e_t)$ converges. Pick a round disk D in $\mathbb{C}\mathrm{P}^1$ containing p such that the hyperbolic plane in bounded by the boundary of D is orthogonal to the geodesic r. Then, there is a path A_t of $\mathbb{C}\mathrm{P}^1$-structures on an annulus A with smooth boundary for sufficiently large $t \gg 0$, such that*

- *A_t converges to the once-punctured disk $D \setminus \{p\}$ as $t \to \infty$ as a $\mathbb{C}\mathrm{P}^1$-structure, and*
- *the developments of the both boundary components of A_t are curves equivalent to b_{e_t} by elements of $\mathrm{PSL}_2\mathbb{C}$.*

Proof. For sufficiently large $t > 0$, one can easily construct the fundamental membrane for A_t for sufficiently large $t > 0$ (Figure 22). □

Proposition 14.9. *Let P be a pair of pants, and pick a boundary component ℓ of P. Let $\tilde{\ell}$ be a lift of ℓ to the universal cover of P. Consider a (flat) Euclidean cylinder with geodesic boundary, and let P_∞ be the surface obtained by removing an interior point p of P_∞; regard P_∞ as a $\mathbb{C}\mathrm{P}^1$-structure on P, and let (h, ξ) be its developing pair, so that h takes $\tilde{\ell}$ to a single point v on $\mathbb{C}\mathrm{P}^1$.*

Let r be the geodesic in $\mathbb{H}^3$ connecting v and the parabolic fixed point of h, and let $e_t \in \mathrm{PSL}_2\mathbb{C}$ be a path of (non-trivial) elliptic elements given by Lemma 14.7 for r.

Then, there is a path of $\mathbb{C}\mathrm{P}^1$-structures $P_t = (h_t, \xi_t)$ on P satisfying the following:

(1) For all $t > 0$, $\xi_t(\ell) = e_t$.
(2) P_t converges to P_∞ as $t \to \infty$. Let $\gamma_t \in \mathrm{PSL}_2\mathbb{C}$ be a path of hyperbolic elements with the axis r, such that $\gamma_t \,\mathrm{Axis}(e_t)$ converges to a geodesic g_∞ in $\mathbb{H}^3$ orthogonal to r as $t \to \infty$ (so that γ_t is a large hyperbolic translation towards v for $t \gg 0$). Let $H \subset \mathbb{H}^3$ be the totally geodesic hyperbolic plane orthogonal to r and containing g_∞. Then, the developing pair $\gamma_t(h_t, \xi_t)$ normalized by γ_t converges to a developing pair for a round disk minus a point, where the removed point is v and the disk is the component of $\mathbb{C}\mathrm{P}^1 \setminus \partial H$ containing v.
(3) Let ℓ_t be the boundary component of P_t corresponding to ℓ. Then $\mathrm{dev}\, P_t$ along a lift of ℓ_t is b_{ℓ_t} (up to $\mathrm{PSL}_2\mathbb{C}$).
(4) Let α be a boundary component of P not equal to ℓ. Then $\xi_t(\alpha)$ is a hyperbolic element for all $t \gg 0$ (converging to a parabolic element as $t \to \infty$).

Proof. First we construct an appropriate path of representations $\xi_t \colon \pi_1(P) \to \mathrm{PSL}_2\mathbb{C}$. Let a_t denote $\mathrm{Axis}(e_t)$. Pick a pair of geodesics q_t, q'_t in $\mathbb{H}^3$ for each $t > 0$ such that

- $R(q_t)R(q'_t) = e_t$, where $R(q_t), R(q'_t) \in \mathrm{PSL}_2\mathbb{C}$ are the π-rotations of $\mathbb{H}^3$ about q_t, q'_t, respectively;
- q_t and q'_t change continuously in $t > 0$;
- q_t and q'_t intersect at the intersection $a_t \cap r$;
- q_t and q'_t are symmetric about r;
- q_t and q'_t are orthogonal to a_t;
- q_t and q'_t converge to r as $t \to \infty$ (see Figure 23).

There is a path of geodesics h_t $(t \geq 0)$ in $\mathbb{H}^3$ such that

- h_t is disjoint from q_t and q'_t for all $t \geq 0$, and
- h_t converges to a geodesic h in $\mathbb{H}^3$ sharing an endpoint with r as $t \to \infty$, such that the composition $R(r)R(h)$ is the parabolic holonomy along a boundary geodesic of P_∞.

Indeed one can first find the limit geodesic h which satisfies the second condition, then as q_t, q'_t converges to r, one can take a desired path h_t.

Let $\xi_t \colon \pi_1(P) \to \mathrm{PSL}_2\mathbb{C}$ be such that the holonomy along boundary components are $R(h_t)R(q_t), R(q_t)R(q'_t), R(q'_t)R(h_t)$. Note that $R(h_t)R(q_t), R(q'_t)R(h_t)$ are hyperbolic elements, as the rotation axes are disjoint, and they converge to the parabolic holonomy along the boundary geodesics of P_∞.

Pick a round disk D on P_∞ containing p such that ∂D on $\mathbb{C}\mathrm{P}^1$ bounds a hyperbolic plane in $\mathbb{H}^3$ orthogonal to r. Then, apply Lemma 14.8 to D, let D_t be a path of $\mathbb{C}\mathrm{P}^1$-structures on an annulus converging to $D \setminus \{p\}$, so that it gives the desired path only near the punctured of P_∞.

Pick a smaller closed regular neighborhood D' of the puncture p of P_∞ such that $\partial D'$ bounds a hyperbolic plane orthogonal to r and that D' is contained in the interior of D. Clearly its complement K in P_∞ and the interior of $D \setminus \{p\}$ form an open cover of P_∞. Then K is topologically a pair of pants. By the Thurston Ehresmann principle, there is a path of $\mathbb{C}\mathrm{P}^1$-structures on a pair of pants K_t for sufficiently large $t > 0$ such that K_t converges to K and e_t is the holonomy of K_t around the boundary component corresponding to $\partial D'$. Moreover, by deformation nearly the boundary, we can in addition assume that the boundary of K_t is equivalent to b_{ℓ_t}.

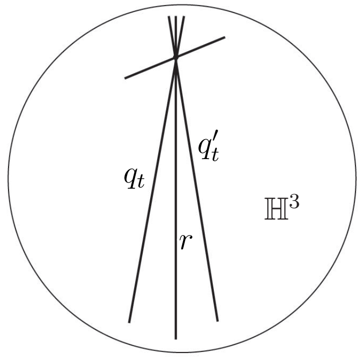

FIGURE 23. Realize e_t as the compositions of the π-rotations about q_t and q'_t.

Then, since K and $D \setminus \{p\}$ form an open cover of P_∞, for sufficiently large t. by gluing K_t and A_t in the overlapping region, we obtained a desired path of $\mathbb{C}\mathrm{P}^1$-structures P_t. □

Proposition 14.10. *Let $P_t = (h_t, \xi_t)$ be a path of $\mathbb{C}\mathrm{P}^1$-structures on a pair of pants from Proposition 14.9. Then, there is a path Σ_t of $\mathbb{C}\mathrm{P}^1$-structures on a closed surface F minus a point which satisfies the following:*

- *There is a subsurface A of F whose interior contains p, such that A is homeomorphic to a pair of pants, and $\Sigma_t | A = P_t$ for all large enough $t > 0$.*
- *Σ_t converges to a $\mathbb{C}\mathrm{P}^1$-structure Σ_∞ on F as $t \to \infty$.*

Proof. First we construct the limit structure Σ_∞. Take any complete hyperbolic surface τ with a single cusp, such that τ is homeomorphic to a closed surface minus a point, denoted by F'. Pick a cusp neighborhood N of τ, a horodisk quotient. The pair of pants P_∞ has two boundary components and one puncture. As the two boundary components of P_∞ lift to horocycles, we can glue a copy of $\tau \setminus N$ along each boundary component of P_∞. We thus obtained a $\mathbb{C}\mathrm{P}^1$-structure on a closed surface with a single puncture so that P_∞ is its subsurface.

There are paths $\zeta_{1,t}$ and $\zeta_{2,t}$ of representations $\pi_1(\tau) \to \mathrm{PSL}_2\mathbb{C}$ which converge to the holonomy of τ as $t \to \infty$, such that their images of the peripheral loop are $R(r_t)R(q'_t)$ and $R(q_t)R(r_t)$, respectively, which are hyperbolic elements (c.f. [Gol09]). Let $\tau_{1,t}, \tau_{2,t}$ be paths of $\mathbb{C}\mathrm{P}^1$-structures homeomorphic to $\tau \setminus N$ for $t \gg 0$ such that $\mathrm{Hol}(\tau_{1,t}) = \zeta_{1,t}$, and $\mathrm{Hol}(\tau_{2,t}) = \zeta_{2,t}$ and $\tau_{1,t}, \tau_{2,t}$ converge to $\tau \setminus N$. We may in addition assume that the boundary components of $\tau_{1,t}, \tau_{2,t}$ are invariant under one-dimensional subgroups of $\mathrm{PSL}_2\mathbb{C}$ containing $R(r)R(q'_t)$ and $R(q_t)R(r)$, respectively.

Then by gluing $\tau_{t,1}, \tau_{t,2}, P_t$ along their boundary, we obtain a desired path Σ_t of $\mathbb{C}\mathrm{P}^1$-structures. □

Let Σ_t be the path of $\mathbb{C}\mathrm{P}^1$ -structures, obtained from Proposition 14.10, on a compact surface with one boundary component. Let R_t be the π-rotation of $\mathbb{H}^3$ around the axis a_t of the elliptic e_t. By Proposition 14.9(2, 3), we can glue two copies of Σ_t by the involution R_t, and we obtain a path of $\mathbb{C}\mathrm{P}^1$-structures C_t on a closed surface, so that two copies of Σ_t are embedded in C_t disjointly up to an isotopy. Let m be the loop along which the two copies are glued. Then, to obtain a marked projective structure, we need to specify the twisting along m. We glue then so that the Fenchel-Nielson twisting parameter matches to be u_t so that, by the π-rotation along a_t, the developing maps of adjacent components of $\tilde{S} \setminus \tilde{m}$ are identical. Let $\Sigma_t^1 = (h_t^1, \rho_t^1), \Sigma_t^2 = (h_t^2, \rho_t^2)$ are the subsurfaces of C_t corresponding to Σ_t.

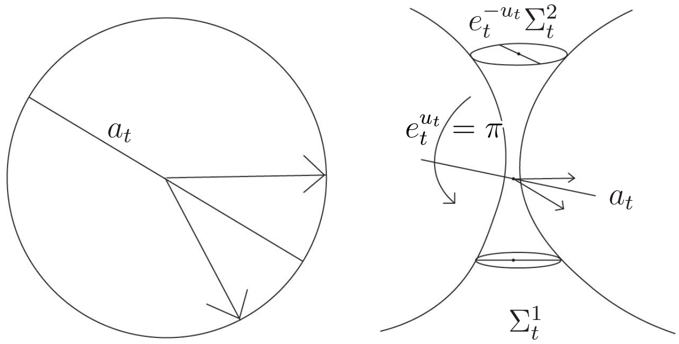

FIGURE 24. The left figure is a section of the right figure by a horizontal plane containing a_t. It illustrates the rotation about a_t by π, and it makes the restriction of f_t on F_1 coincide with that to F_2 coincide.

Theorem 14.11. *Let $C_t = (f_t, \rho_t)$ be the path of $\mathbb{C}\mathrm{P}^1$-structures as above, and let m be the loop on C_t corresponding to the boundary components of Σ_t^1 and Σ_t^2. Let N be the regular neighborhood of m. Then, by taking an appropriate isotopy of S, C_t satisfies the following.*

(1) $\rho_t(m)$ converges to I as $t \to \infty$, and $\rho_t(m)$ is an elliptic element for all $t > 0$;
(2) the axis of $\rho_t(m)$ converges to the point p of $\mathbb{C}\mathrm{P}^1$;
(3) $f_t \colon \tilde{S} \setminus \phi^{-1}(N) \to \mathbb{C}\mathrm{P}^1$ converges to a ρ_∞-equivariant continuous map $f_\infty \colon \tilde{S} \setminus \phi^{-1}(N) \to \mathbb{C}\mathrm{P}^1$, such that f_∞ is a local homeomorphism in the interior;
(4) for each connected component $\tilde{N}$ of $\phi^{-1}(N)$, the boundary components of $\tilde{N}$ map to its corresponding limit given by (2).

Proof. Let F_1, F_2 be the connected components of $S \setminus N$. We normalize the developing pair of C_t by a path of $\mathrm{PSL}_2\mathbb{C}$ so that the restriction to $\tilde{F}_1$ converges to a developing pair for Σ_∞. Then (1) and (2) clearly hold. Moreover, we can take an appropriate isotopy of S so that each boundary component of $\tilde{F}_1$ converges to the corresponding limit point of its corresponding axis. Since the rotation angle of $e_t^{u_t}$ is π by Lemma 14.7, the restriction of f_t to $\tilde{F}_2$ is the same as that to to $\tilde{F}_1$ (Figure 24). Therefore, the restriction of f_t to $\tilde{F}_2$ converges to a developing map of Σ_∞ as well. Thus we have (3). Then, by the equivariant property, we also have (4). □

14.11

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