

COMPLEX PROJECTIVE STRUCTURES WITH SCHOTTKY HOLONOMY

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ABSTRACT. Let S be a closed orientable surface of genus at least two. Let Γ be a Schottky group whose rank is equal to the genus of S , and Ω be the domain of discontinuity of Γ . Pick an arbitrary epimorphism $\rho: \pi_1(S) \rightarrow \Gamma$. Then Ω/Γ is a surface homeomorphic to S carrying a (complex) projective structure with holonomy ρ . We show that every projective structure with holonomy ρ is obtained by $(2\pi-)$ grafting Ω/Γ along a multiloop on S .

1. INTRODUCTION

Throughout this paper, let S denote a closed orientable surface of genus $g > 1$. Then Gallo, Kapovich and Marden ([3]) proved that a homomorphism $\rho: \pi_1(S) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ is a holonomy representation of some projective structure on S if and only if ρ satisfies:

- (i) the image of ρ is non-elementary and
- (ii) ρ lifts to a homomorphism from $\pi_1(S)$ to $\mathrm{SL}(2, \mathbb{C})$.

Their proof of this theorem implies that there are infinitely many distinct projective structures with fixed holonomy ρ satisfying (i) and (ii) (see also [1]). Then it is a natural question to ask for a characterization of projective structures with fixed holonomy ρ . In literature, this question goes back to a paper by Hubbard ([6]; see also [3, §12]). In this paper, we give a characterization of projective structures with a certain type of discrete nonfaithful holonomy representation, using a certain surgery operation.

Basic examples of projective structures arise from Kleinian groups with nonempty domain of discontinuity. Let Γ be a (not necessarily classical) *Schottky group* of rank $g > 1$, which can be defined as a subgroup of $\mathrm{PSL}(2, \mathbb{C})$ isomorphic to a free group of rank g consisting only of loxodromic elements (see [8, pp 75] [15] about Schottky groups). Let $\Omega \subset \hat{\mathbb{C}}$ denote the domain of discontinuity of Γ . Let $\rho: \pi_1(S) \rightarrow \Gamma$ be an epimorphism, which we call a *Schottky holonomy (representation)*. Then Ω/Γ enjoys a projective structure on S with holonomy

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ρ (*uniformizable structure*), when Ω/Γ and S are homeomorphically identified in an appropriate way.

A (2π) -grafting is an operation that transforms a projective structure into another projective structure on the same surface without changing holonomy representation (§3.3). It is a surgery operation that inserts a projective cylinder along a certain type of loop on a projective surface, called an admissible loop. If there is a multiloop consisting of disjoint admissible loops on a projective surface, a grafting operation can be done simultaneously along all loops.

A *Schottky structure* is a projective structure on S with Schottky holonomy. Our goal of this paper is to show:

Theorem 8.1. *Every projective structure on S with Schottky holonomy ρ is obtained by grafting Ω/Γ along a multiloop on S .*

Remark: By a quasiconformal map of $\hat{\mathbb{C}}$, the proof of Theorem 8.1 is easily reduced to the case that Γ is a real Schottky group, i.e. the limit set of Γ lies in the equator $\mathbb{R} \cup \{\infty\}$ of $\hat{\mathbb{C}}$ (as in the proof of Theorem 1.1 below from [4]).

A projective structure is called *minimal* if it can *not* be obtained by grafting another projective structure. Theorem 8.1 implies that Ω/Γ is the unique minimal structure among all projective structures on S with holonomy ρ , up to a change of marking on S . There is an incorrect theorem in the literature, implying that there are other minimal structures with holonomy ρ , i.e. they are not Ω/Γ (Theorem 3.7.3, Example 3.7.6 in [19])

Recently Thompson [20] showed that if two uniformizable projective structures with holonomy ρ differ by a Dehn twist, then they can be grafted to a common projective structure. Thus, since Dehn twists generate the subgroup of the mapping class group of S that preserves the holonomy ρ and the orientation of the surface ([13]), combining with Theorem 8.1, the set of projective structures with Schottky holonomy ρ is generated by graftings. This answers a special case of Problem 12.1.2 in [3] in the affirmative.

Theorem 8.1 is an analog to the case of a quasifuchsian holonomy: Let Γ' be a quasifuchsian group and let Ω^+ and Ω^- be the connected components of the domain of discontinuity of Γ' . Then Ω^+/Γ' and Ω^-/Γ' are projective surfaces on S with the same holonomy ρ' , where ρ' is an isomorphism from $\pi_1(S)$ onto Γ' .

Theorem 1.1 (Goldman [4]). *Every projective structure with quasifuchsian holonomy ρ' is obtained by grafting either Ω^+/Γ' or Ω^-/Γ' along a multiloop.*

In Theorem 1.1, for a given projective structure, the choice of the multiloop, up to an isotopy, and the choice of Ω^+/Γ' or Ω^-/Γ' are unique. On the other hand, in Theorem 8.1, there are typically infinitely many pairs of a multiloop and a marking on Ω/Γ for a single Schottky structure (see the remark preceding Lemma 8.2; see also [20]).

Thus we discuss an approach to describe Schottky structures in a unique manner. Fix an appropriate marking (and, therefore, an orientation) on Ω/Γ so that it has the Schottky holonomy ρ . Let \mathcal{P}_ρ be the collection of all projective structures on S of the fixed orientation with holonomy ρ . Let $\phi: S \rightarrow S$ be a homeomorphism. Then the *support* of ϕ is the minimal π_1 -injective subsurface R of S , defined up to an isotopy, such that the restriction of ϕ to $S \setminus R$ is isotopic to the identity. The homeomorphism ϕ induces an isomorphism $\phi^*: \pi_1(S) \rightarrow \pi_1(S)$. Let $Stab_\rho$ denote the subgroup of the mapping class group of S consisting of orientation-preserving mapping classes $[\phi]$ such that $\rho \circ \phi^* = \rho$. As mentioned above, $Stab_\rho$ is generated by Dehn twists along simple loops on S representing elements in $\ker(\rho)$. Let $\mathcal{AML}_\rho(S)$ denote the set of isotopy classes of multiloops on S consisting of disjoint admissible loops on Ω/Γ .

Conjecture 1.2. *Every $C \in \mathcal{P}_\rho$ can be obtained by changing the marking of Ω/Γ by a unique $\phi \in Stab_\rho$ and grafting Ω/Γ along a unique $L \in \mathcal{AML}_\rho(S)$ such that the multiloop L and the support of ϕ are disjoint:*

$$\mathcal{P}_\rho(S) \cong \{(\phi, L) \in Stab_\rho \times \mathcal{AML}_\rho(S) \mid Supp(\phi) \cap L = \emptyset\}.$$

Theorem 8.1 ensures that, for every $C \in \mathcal{P}_\rho$, there is a corresponding pair $(\phi, L) \in Stab_\rho \times \mathcal{AML}_\rho(S)$; however $Supp(\phi) \cap L$ may be nonempty. The conjecture asserts that this intersection can be uniquely “resolved”.

Outline of the proof of Theorem 8.1. Let C be a projective structure on S with Schottky holonomy ρ . The main step is to reduce the theorem to the case of “genus zero”: We decompose C , by cutting it along a multiloop M , into certain (very simple) projective structures on holed-spheres, called *good holed spheres* (§3.2): Namely each component F of $C \setminus M$ has trivial holonomy and the boundary components of F , via its developing map, cover disjoint loops that bound disjoint disks on $\hat{\mathbb{C}}$. Then we show that the the good holed sphere F is obtained by grafting a holed sphere embedded in $\hat{\mathbb{C}}$ along a multiarc (Proposition 7.5).

The multiloop on S in Theorem 8.1 is realized as the union of such multiarcs over all components of $C \setminus M$, and the uniformizable structure Ω/Γ the union of the corresponding embedded holes spheres.

It is straightforward to find a multiloop realizing the above decomposition on C except that different boundary components of F may cover the same loops on $\hat{\mathbb{C}}$ (see §4). Then the main work is to isotope this imperfect multiloop on C to a desired one. To realize this isotopy, we construct a certain embedding a handlebody bounded by S to itself associated with the projective structure C (§5).

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2. CONVENTIONS AND TERMINOLOGY

In this paper, unless otherwise stated,

- a surface is connected and orientable;
- a loop and arc are simple;
- a component is a connected component.

As usual,

- a loop on a surface F may be regarded as an element of $\pi_1(F)$.

Here is some terminology often used in this paper:

- A *multiloop* is the union of isolated and disjoint simple closed curves embedded in a surface; similarly a *multiarc* is the union of isolated and disjoint simple curves property embedded in a surface.
- Let F be a surface and let F_1 and F_2 be subsurfaces of F . Then we say that F_1 and F_2 are *adjacent* if F_1 and F_2 have disjoint interiors and share a boundary component.

3. PRELIMINARIES

3.1. Projective structures. Let F be a connected orientable surface possibly with boundary. A (*complex*) *projective structure* is a $(\hat{\mathbb{C}}, \text{PSL}(2, \mathbb{C}))$ -structure, i.e. an atlas modeled on the Riemann sphere $\hat{\mathbb{C}}$ with its transition maps lying in $\text{PSL}(2, \mathbb{C})$. Let \bar{F} denote the universal cover of F . It is well-known that a projective structure is equivalently defined as a pair (f, ρ) consisting of

- a topological immersion $\bar{f}: \bar{F} \rightarrow \hat{\mathbb{C}}$ (i.e. a locally injective continuous map) and
- a homomorphism $\rho: \pi_1(F) \rightarrow \text{PSL}(2, \mathbb{C})$

such that \bar{f} is ρ -equivariant, i.e. $\bar{f} \circ \alpha = \rho(\alpha) \circ \bar{f}$ for all $\alpha \in \pi_1(F)$ (see [21, §3.4]). The immersion \bar{f} is called the *(maximal) developing map* and the homomorphism ρ is called the *holonomy (representation)* of the projective structure. A projective structure is defined up to an isotopy of F and an element of $\mathrm{PSL}(2, \mathbb{C})$, that is, $(\bar{f}, \rho) \sim (\gamma \circ \bar{f}, \gamma \circ \rho \circ \gamma^{-1})$ for all $\gamma \in \mathrm{PSL}(2, \mathbb{C})$.

If C is a projective structure on F , the pair (F, C) is called a *projective surface*. As usual, we will often conflate the projective structure C and the projective surface (F, C) .

3.1.1. *Minimal developing maps.* Let $C = (\bar{f}, \rho)$ be a projective structure on F . Then the short exact sequence

$$1 \rightarrow \ker(\rho) \rightarrow \pi_1(F) \xrightarrow{\rho} \mathrm{Im}(\rho) \rightarrow 1$$

induces an isomorphism $\tilde{\rho}: \pi_1(F)/\ker(\rho) \rightarrow \mathrm{Im}(\rho)$. Let $\tilde{F} = \bar{F}/\ker(\rho)$ and let $\phi: \bar{F} \rightarrow \tilde{F}$ denote the corresponding covering map; then \tilde{F} is the cover of F whose deck transformation group is $\pi_1(F)/\ker(\rho)$. Then the maximal developing map $\bar{f}: \bar{F} \rightarrow \hat{\mathbb{C}}$ canonically descends to a locally injective map $f: \tilde{F} \rightarrow \hat{\mathbb{C}}$ that is $\tilde{\rho}$ -equivariant, so that $\bar{f} = \phi \circ f$. We call this map f the *minimal developing map* of C and the cover \tilde{F} the *developable cover* of F associated with ρ (c.f. [10]).

Conversely the pair (f, ρ) of the minimal developing map and the holonomy representation of C , recovers the maximal developing map \bar{f} .

Since Schottky representations have a large kernel, it is much easier to work with minimal developing maps. Thus, for the remainder of the paper, unless otherwise stated, developing maps are always minimal developing maps; in particular, we use a pair (f, ρ) of a minimal developing map and a holonomy representation to represent a projective structure. We denote the minimal developing map of a projective structure C by $dev(C)$.

3.1.2. *Restriction of projective structures to subsurfaces.* Let $C = (f, \rho)$ be a projective structure on a surface F . Let E be a subsurface of F . The *restriction* of C to E is the projective structure on E given by restricting the atlas of C on F to E , which we denote by $C|_E$. We can equivalently define the restriction $C|_E$ as a pair of the (minimal) developing map and a holonomy representation as follows: The inclusion $E \subset F$ induces a homomorphism $i^*: \pi_1(E) \rightarrow \pi_1(F)$. Let \tilde{F} be the developable cover of F associated with ρ and let \tilde{E} be a lift of E to \tilde{F} invariant under $\pi_1(E)$. Then the restriction of C to E is the pair of $f|_{\tilde{E}}: \tilde{E} \rightarrow \hat{\mathbb{C}}$ and $\rho \circ i^*: \pi_1(E) \rightarrow \mathrm{PSL}(2, \mathbb{C})$.

3.1.3. *Embedded projective structures.* A projective surface C is *uniformizable* (by Γ) if C is isomorphic to a component of the ideal boundary of a hyperbolic 3-manifold \mathbb{H}^3/Γ , where Γ is a Kleinian group. More generally a projective surface C is *embedded* if its (minimal) developing map is an embedding onto a subset of $\hat{\mathbb{C}}$. Then it is easy to show:

Lemma 3.1. *Let C be a projective structure on S with Schottky holonomy $\rho: \pi_1(S) \rightarrow \Gamma \subset \mathrm{PSL}(2, \mathbb{C})$, where Γ is a Schottky group. Then C is embedded if and only if it is uniformizable by Γ , i.e. $\mathrm{dev}(C)$ is an embedding onto the domain of discontinuity of Γ .*

Proof. Suppose that $C = (f, \rho)$ is embedded. Then we first show that the developing map $f: \tilde{S} \rightarrow \hat{\mathbb{C}}$ is an embedding into the complement of the limit set Λ of Γ . Suppose that there is $x \in \tilde{S}$ such that $f(x) \in \Lambda$. Then there is a compact neighborhood of x in \tilde{S} that f takes homeomorphically onto its image. On the other hand, since f is equivariant under the isomorphism $\tilde{\rho}: \pi_1(S)/\pi_1(\tilde{S}) \rightarrow \Gamma$, for any $y \in \tilde{S}$, then $f(\Gamma \cdot y)$ accumulates to the limit point $f(x)$. Then f is not injective, which is a contradiction.

Then f is injective, and thus C embeds into Ω/Γ . Since C and Ω/Γ are homeomorphic to S , thus f must be an embedding onto Ω . \square

3.2. Good and almost good projective structures. Let F be a surface obtained from \mathbb{S}^2 by removing disjoint points and closed (topological) disks, i.e F is a genus-zero surface. In addition we assume that the Euler characteristic of F is non-positive. Then let M be the union of the boundary circles of the removed disks, and let P be the union of the removed points.

A projective structure $C = (f, \rho)$ on the surface F is *almost good* if:

- (i) $\rho: \pi_1(F) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ is the trivial representation (so that the domain of f is F),

and there is a genus-zero surface R embedded in $\hat{\mathbb{C}}$ such that

- (ii) f continuously extends to P , taking P into the set of punctures of R , and
- (iii) each component of M , via f , covers a boundary component of R .

The surface R is called a *support* of the almost good projective structure C . Then there is a unique component of $\hat{\mathbb{C}} \setminus f(P \cup M)$ in $\hat{\mathbb{C}}$ that is also a support of C , and it is contained in R . Thus we call the component the *full support* of C and denote it by $\mathrm{Supp}(C)$ or, alternatively, $\mathrm{Supp}_f(F)$ to indicate the associated developing map. Clearly

the developing map $f: F \rightarrow \hat{\mathbb{C}}$ has the lifting property along every path disjoint from the boundary of $Supp(C)$.

Moreover a projective structure $C = (f, \rho)$ on F is *good* if it is almost good and in addition

- (vi) f yields bijections between the punctures of F and of $Supp(C)$ and the boundary components of F and of $Supp(C)$.

Then, if C is a good structure on F , there is a homeomorphism between F to $Supp(C)$ realizing the bijections in (vi). Obviously the full support enjoys a good embedded projective structure on F whose developing map is such a homeomorphism; this is called an *embedded structure associated with* the good structure C . Then this structure associated structure is unique up to an element of the pure mapping class group of F .

Let us return to the case that $C = (f, \rho)$ is an almost good projective structure on F supported on R . Then we can naturally extend C to an almost good structure on a punctured sphere homotopy equivalent to F , so that the developing map of this extension is a branched covering of \mathbb{S}^2 to $\hat{\mathbb{C}}$ as follows: Along each boundary component l of F , attach a disk D minus a point p . The loop $f(l)$ bounds a unique disk D' in $\hat{\mathbb{C}}$ whose interior is disjoint from the support R . Pick a point p' in the interior D' . Then continuously extend the developing map f to D so that $f|_D$ is a branched covering map of D onto D' such that p is the unique ramification point and p' is the unique branched point.

3.3. Grafting. Grafting was initially developed as an operation that transforms a hyperbolic structure to a projective structure on a surface by inserting a flat affine cylinder along a circular loop (Maskit [14], Hejhal [5], Sullivan-Thurston [18], Kamishima-Tan [7]). Goldman used a variation of this grafting operation, which is often called *2π -grafting*: His grafting is, to an extent, more special since it preserves the holonomy representation of a projective structure, but, to another extent, more general since it can be done along a non-circular loop ([4]).

In this paper, we basically follow the definition given by Goldman. Yet, following out convention, we define grafting with respect to minimal developing maps. In addition, we define grafting also along an arc properly embedded in a certain kind of projective surface so that grafting is compatible with identifying boundary components of projective surfaces. For simplicity, we graft only embedded projective structures.

Let R be the complement of a union of (possibly infinitely many) disjoint points and closed (topological) disks in $\hat{\mathbb{C}}$. Then R enjoys

an embedded projective structure with trivial holonomy. Let F be the topological surface homeomorphic to R . Let α be a simple arc property embedded in R that does not bound a disk in C ; then α connects different components X, Y of $\hat{\mathbb{C}} \setminus R$, which are a disk or a point. Since X and Y are disjoint, $\hat{\mathbb{C}} \setminus (X \cup Y) =: A$ is homeomorphic to a cylinder. Then A carries an embedded projective structure with trivial holonomy and α is also canonically embedded in A . Since α is property embedded in both projective surfaces R and A , we can cut and past them along α to make a new projective structure on F . Namely we can pair up the boundary segments of $R \setminus \alpha$ and $A \setminus \alpha$ coming from α in an alternating fashion (as in Figure 1).

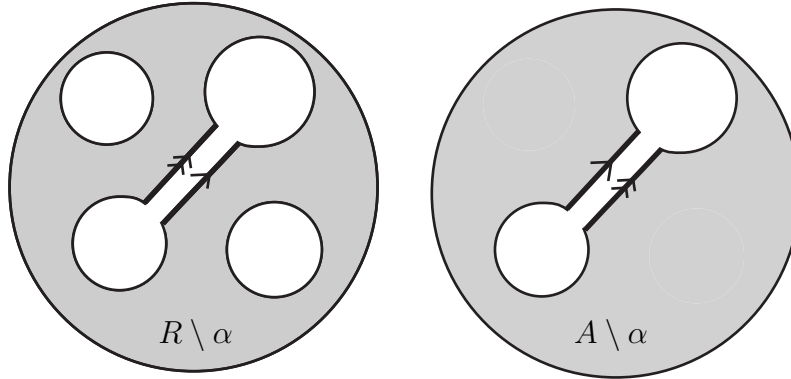


FIGURE 1. Cut-and-past for grafting of a four-holed sphere.

This operation is called the *grafting (operation)* on R along α , and the resulting structure is denoted by $Gr_\alpha(R)$. Then $Gr_\alpha(R)$ is also a good structure fully supported R : If an endpoint of α is a puncture, near this point, the new developing map is a branched covering of degree 2; similarly if an endpoint of α is on a boundary component of R , this component is covered by the corresponding boundary component of F via the new developing map and its covering degree is 2. Clearly $Gr_\alpha(R)$ is non-embedded. In addition when we isotopy α on R properly, the resulting structure $Gr_\alpha(R)$ does *not* change.

If there is a multiarc M on R consisting of arcs that, as above, connect different punctures or boundary components of R , we can simultaneously graft along all arcs of M and obtain a good projective structure with full support R . Similarly denote this resulting structure by $Gr_M(R)$.

Next we define grafting along a loop on an embedded projective structure $C = (f, \rho)$ on a surface F . We set $f: \tilde{F} \rightarrow \hat{\mathbb{C}}$, where \tilde{F} is the

developable cover of F associated with holonomy ρ . Accordingly, lifting C , we obtain a projective structure \tilde{C} on \tilde{F} . Then the holonomy of \tilde{C} is trivial and $dev(\tilde{C})$ is an embedding of \tilde{F} into $\hat{\mathbb{C}}$ (see §3.1.1). A loop ℓ on the embedded projective surface (F, C) is called *admissible* if $\rho(\ell)$ is loxodromic. If ℓ is admissible, letting $\tilde{\ell}$ be a lift of ℓ to \tilde{F} invariant under $\ell \in \pi_1(F)$, then $f(\tilde{\ell})$ is a simple arc on $\hat{\mathbb{C}}$ invariant by $\rho(\ell)$. Thus the end points of $f(\tilde{\ell})$ are the fixed points of the loxodromic element $\rho(\ell)$; denote this fixed point set by $Fix(\rho(\ell))$. If ℓ is an admissible loop, $\hat{\mathbb{C}} \setminus Fix(\rho(\ell))$ is a twice-punctured sphere and $f(\tilde{\ell})$ is a properly embedded arc in $\hat{\mathbb{C}} \setminus Fix(\rho(\ell))$ connecting the punctures. The infinite cyclic group $\langle \rho(\ell) \rangle$ acts on $\hat{\mathbb{C}} \setminus Fix(\rho(\ell))$, preserving $f(\tilde{\ell})$, and its quotient is thus a (Hopf) torus T containing a copy of ℓ . Since ℓ is embedded in both C and T , we can similarly obtain a new projective structure on F by a canonical cut-and-past operation along ℓ : Insert the cylinder $T \setminus \ell$ between the boundary components of $C \setminus \ell$. This operation is called a *grafting* on C along ℓ , and we denote the resulting structure by $Gr_\ell(C)$.

The *total lift* \mathbf{l} of ℓ on F to \tilde{F} is the union of all lifts of ℓ to \tilde{F} . Then the grafting of C along ℓ lifts the grafting of \tilde{C} along \mathbf{l} . Thus we see that $\tilde{C} \setminus \mathbf{l}$ is isomorphically embedded in $Gr_1(\tilde{C})$ and, with respect to this embedding, the developing map of $\tilde{C} \setminus \mathbf{l}$ does not change under the grafting. Thus the maximal developing map of $Gr_\ell(C)$ is also ρ -equivariant, and the grafting preserves the holonomy ρ .

If there is a multiloop on (F, C) consisting of disjoint admissible loops, similarly we can graft C simultaneously along all loops of the multiloop.

3.4. Hurwitz spaces. Let d_i ($i = 1, 2, \dots, k$) be integers greater than 1. Consider a pair consisting of a set of k distinct ordered points P_i ($i = 1, 2, \dots, k$) on $\hat{\mathbb{C}}$ and a rational function $\tau: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ such that P_i are the ramification points of τ with ramification index d_i (i.e. τ has zero of order d_i at P_i) and that $\tau(P_i)$ are distinct points on $\hat{\mathbb{C}}$. Let $\mathcal{R} = \mathcal{R}(d_1, d_2, \dots, d_k)$ denote the space of all such pairs (P_i, τ) . Then $\mathrm{PSL}(2, \mathbb{C})$ acts on \mathcal{R} by postcomposition. The quotient space $\mathcal{R}/\mathrm{PSL}(2, \mathbb{C})$ is called a *Hurwitz space*, which we denote by $\mathcal{H} = \mathcal{H}(d_1, d_2, \dots, d_k)$. It is well known that the map $\tau \mapsto (\tau(P_1), \tau(P_2), \dots, \tau(P_k))$ is a covering map from \mathcal{R} onto $\hat{\mathbb{C}}^k \setminus (\text{diagonals})$, and therefore \mathcal{R} is a complex manifold. We see that $\mathrm{PSL}(2, \mathbb{C})$ acts on \mathcal{R} freely and holomorphically. Therefore \mathcal{H} is also a complex manifold and \mathcal{R} is a $\mathrm{PSL}(2, \mathbb{C})$ -bundle over \mathcal{H} . (See [2, 22].) Moreover,

Theorem 3.2 (Liu-Osserman [12]). *\mathcal{H} is a connected manifold; hence, \mathcal{R} is also a connected manifold.*

4. DECOMPOSITION OF A SCHOTTKY STRUCTURE INTO ALMOST GOOD HOLED SPHERES

Definition 4.1. Let $C = (f, \rho)$ be a projective structure on a surface F . A loop ℓ on F is a meridian if ℓ is an essential loop on F and $\rho(\ell) = id$.

Definition 4.2. Let F be a surface and M be a multiloop on F . The essential part of M is the union of all essential loops of M , which we denote by $\lfloor M \rfloor$.

Lemma 4.3. Let F be a surface of genus zero and $C = (f, \rho_{id})$ be an almost good projective structure on F supported on a surface $R \subset \hat{\mathbb{C}}$. Let ℓ be an inessential loop in the complement of ∂R in $\hat{\mathbb{C}}$. Then $\lfloor f^{-1}(\ell) \rfloor = \emptyset$.

Proof. Suppose that ℓ is an inessential loop in $\hat{\mathbb{C}} \setminus \partial R$. Then ℓ bounds a disk E contained in $\hat{\mathbb{C}} \setminus \partial R$. Since C is almost good, f has the lifting property along every path on $\hat{\mathbb{C}} \setminus \partial R$. Thus each component of $f^{-1}(\ell)$ is a loop bounding a disk in F that f homeomorphically maps onto E . Therefore $\lfloor f^{-1}(\ell) \rfloor = \emptyset$. \square

4.1. Pullback of a multiloop via a developing map. Let Γ be a real Schottky group of rank g . (However this assumption that Γ is real is not essential since our arguments will be all topological.) Let Ω denote the domain of discontinuity of Γ . Let $C' = \Omega/\Gamma$, which is a uniformizable projective structure with Schottky holonomy $\rho: \pi_1(S) \rightarrow \Gamma$. Let H' denote the handlebody $(\Omega \cup \mathbb{H}^3)/\Gamma$ bounded by C' . Let \tilde{S} be the developable cover S with respect to ρ , so that Γ is its covering transformation group (see §3.1.1), and let $\Psi_\Gamma: \tilde{S} \rightarrow S$ denote the covering map. Then we see that \tilde{S} is naturally homeomorphic to $\hat{\mathbb{C}} \setminus \Lambda$, where Λ is the limit set of Γ , and in particular \tilde{S} is planar.

Let C be a projective structure on S with holonomy ρ . Set $C = (f, \rho)$, where $f: \tilde{S} \rightarrow \hat{\mathbb{C}}$ is the developing map of C . Let $\tilde{\rho}: \pi_1(S)/\ker(\rho) \rightarrow \Gamma$ be the isomorphism induced from ρ . Then f is $\tilde{\rho}$ -equivariant.

Let ℓ be a meridian loop on C' . Let $\tilde{\ell}$ be a lift of ℓ to Ω , which is also a loop. Then

Lemma 4.4. Then $f^{-1}(\tilde{\ell})$ is a multiloop¹ on \tilde{S} .

Proof. Since f has the path lifting property in the domain of discontinuity Ω (see [9]), $f^{-1}(\tilde{\ell})$ is a one-manifold properly embedded in \tilde{S} . Moreover, since f is $\tilde{\rho}$ -equivariant, each $x \in \tilde{S}$ has a neighborhood U ,

¹There may be locally infinitely many loops.

such that, if $\gamma \in \Gamma$, then $\gamma U \cap f^{-1}(\tilde{\ell})$ is either a single arc properly embedded in U or the empty set. Therefore $f^{-1}(\tilde{\ell})/\Gamma$ is a multiloop on (S, C) .

Suppose that $f^{-1}(\tilde{\ell})$ contains a biinfinite simple curve α . Then α covers the loop $\tilde{\ell}$ via f . On the other hand, since α/Γ is a component of the multiloop $f^{-1}(\tilde{\ell})/\Gamma$, there is a non-identity element of Γ that preserves α . Then, since f is $\tilde{\rho}$ -equivariant, $f(\alpha)$ is an unbounded curve in Ω . This is a contradiction, since $f(\alpha)$ is contained in the loop $\tilde{\ell}$. \square

A loop on the boundary of a handlebody is called *meridian* if it bounds a disk properly embedded in the handlebody. Let N' be a multiloop on $C' = \partial H'$ consisting of meridian loops. Let \tilde{N}' be the total lift of N' to Ω . Then \tilde{N}' is a Γ -invariant multiloop on Ω . Since f is $\tilde{\rho}$ -equivariant, by Lemma 4.4, $f^{-1}(\tilde{N}')$ is a Γ -invariant multiloop on \tilde{S} . (A compact subset of \tilde{S} contain infinitely many components of $f^{-1}(\tilde{N}')$.) Let $\tilde{N} = \lfloor f^{-1}(\tilde{N}') \rfloor$. Let N denote the multiloop on (S, C) obtained by quotienting \tilde{N} by Γ , which we call the *pullback* of N' (via f).

Proposition 4.5. *Assume that N' is a multiloop on C' such that*

- (I) N' consists of finitely many meridian loops and
- (II) each component of $C' \setminus N'$ is a holed sphere.

Then the pullback N of N' via f satisfies

- (i) N is nonempty,
- (ii) N consists of finitely many meridian loops on S (with respect to ρ), and
- (iii) if Q is a component of $S \setminus N$, then the restriction $C|Q'$ is an almost good holed sphere supported on a component of $\Omega \setminus \tilde{N}'$, and the full support of $C|Q'$ has at least two boundary components.

Let $\tilde{C} = (f, \rho_{id})$ denote the projective structure on \tilde{S} obtained by lifting C , where $\rho_{id}: \pi_1(\tilde{S}) \rightarrow \text{PSL}(2, \mathbb{C})$ is the trivial holonomy. Then Proposition 4.5 immediately implies

Corollary 4.6. *(i) Let \tilde{Q} be a component of $\tilde{S} \setminus \tilde{N}$. Then there is a unique component R of $\Omega \setminus \tilde{N}'$ such that $\tilde{C}|_{\tilde{Q}}$ is an almost good holed sphere supported on R (via f). (ii) Let ℓ be a loop of \tilde{N} and let \tilde{Q}_1 and \tilde{Q}_2 be the adjacent components of $\tilde{S} \setminus \tilde{N}$ along ℓ . Then the support of the almost good structures $\tilde{C}|_{\tilde{Q}_1}$ and $\tilde{C}|_{\tilde{Q}_2}$ are the adjacent components of $\Omega \setminus \tilde{N}'$ along the loop $f(\ell)$.*

The rest of the section is the proof of the proposition.

Proof of Proposition 4.5 (ii). First we show that, for each $x \in \tilde{S}$, there is a neighborhood $U(= U_x)$ of x such that $U \cap \tilde{N}$ is either the empty set or a single arc properly embedded in U . Since f is a local homeomorphism, x has a neighborhood U that f homeomorphically takes onto its image $f(U) =: U'$. The codomain $\hat{\mathbb{C}}$ of f is decomposed into Λ , \tilde{N}' and $\Omega \setminus \tilde{N}'$.

Case 1. Suppose that $f(x) \in \Lambda$. Then, by Assumption (II), we can take U so that U' is a disk bounded by a loop of \tilde{N}' . Then, since $f|_U$ is a homeomorphism onto its image, $\tilde{N}' \cap U'$ is a union of infinitely many disjoint loops. Since U' is a disk, those loops are all inessential; thus $\tilde{N} \cap U = \emptyset$.

Case 2. Suppose that $f(x) \in \tilde{N}'$. Then, since each loop of \tilde{N} is isolated, we can take U so that $U' \cap \tilde{N}'$ is a single arc properly embedded in U' . Accordingly $f^{-1}(\tilde{N}') \cap U$ is a single arc properly embedded in U . Therefore $[f^{-1}(\tilde{N}')] \cap U$ is either the arc or the empty set.

Case 3. Last suppose that $f(x) \in \Omega \setminus \tilde{N}'$. Then we can take U so that $f(U)$ is disjoint from \tilde{N}' . Then $f^{-1}(\tilde{N}') \cap U = \emptyset$ and therefore $[f^{-1}(\tilde{N}')] \cap U = \emptyset$.

The set of all U_x for all $x \in \tilde{S}$ is a cover of \tilde{S} . Via the covering map, this cover descends to a cover of S . Since S is compact, there is a finite subcover of S . Since the Γ -action preserves \tilde{N} and each U_x contains at most an single arc, N consists of at most finitely many loops.

Since \tilde{N} is a union of essential loops and the holonomy of \tilde{C} is trivial, each loop of \tilde{N} is a meridian loop on \tilde{C} . Since \tilde{N} covers N , each loop of N is a meridian loop on C . \square

Lemma 4.7. *Under the assumptions in Proposition 4.5, for every component Q of $S \setminus N$, the restriction of ρ to $\pi_1(Q)$ is the trivial representation.*

Proof. For $\alpha \in \pi_1(Q)$, let $\gamma = \rho(\alpha) \in \Gamma$. We regard α also as an oriented closed curve on Q representing it. Suppose that $\gamma \neq id$. Then γ is a loxodromic element. Therefore α lifts a bi-infinite simple curve $\tilde{\alpha}$ on \tilde{S} invariant under the infinite cyclic group $\langle \alpha \rangle$ generated by α . Let $p_1, p_2 \in \hat{\mathbb{C}}$ be the fixed points of the loxodromic element γ . Then $f|_{\tilde{\alpha}}$ is a $\rho|_{\langle \alpha \rangle}$ -equivariant curve connecting p_1 and p_2 . By Assumption (II), there is a loop μ' of \tilde{N}' separating p_1 and p_2 . By a small homotopy of α on Q if necessary, we can assume that the curve $f|_{\tilde{\alpha}}$ is disjoint from the points p_1 and p_2 and it transversally intersects the loop μ' . Since $f|_{\tilde{\alpha}}$ is $\rho|_{\langle \alpha \rangle}$ -equivariant and p_1, p_2 are contained in the different components of $\hat{\mathbb{C}} \setminus \mu'$, $f|_{\tilde{\alpha}}$ intersects μ' an odd number of times. Therefore $\tilde{\alpha}$ transversally intersects $f^{-1}(\mu')$ an odd number of times.

By Lemma 4.4, $f^{-1}(\mu')$ is a multiloop on \tilde{S} . Furthermore, by Proposition 4.5 (ii), each component of $f^{-1}(\mu')$ is either a loop of \tilde{N} or an inessential loop on \tilde{S} . The curve $\tilde{\alpha}$ is $\langle \alpha \rangle$ -invariant and properly immersed in \tilde{S} . Thus $\tilde{\alpha}$ is unbounded. Therefore, each inessential loop of $f^{-1}(\mu')$ intersects $\tilde{\alpha}$ an even number of times. Since $\tilde{\alpha}$ intersects $f^{-1}(\mu')$ an odd number of times, $\tilde{\alpha}$ must intersect at least one essential loop of $f^{-1}(\mu')$. Thus $\tilde{\alpha}$ intersects \tilde{N} , and therefore α transversally intersects N . This contradicts the assumption that α is contained in Q . Hence $\rho(\alpha) = id$ for all $\alpha \in \pi_1(Q)$. \square

Proposition 4.5 (i) immediately follows from

Lemma 4.8. *Every component of $S \setminus N$ has genus zero.*

Proof. Let Q be a component of $S \setminus N$. By Lemma 4.7, $\pi_1(Q) \subset \ker \rho$. Thus, by the definition of \tilde{S} , Q homeomorphically lifts to a component \tilde{Q} of $\tilde{S} \setminus \tilde{N}$. Since \tilde{S} is planar, Q has genus zero. \square

What is left is:

Proof Proposition 4.5 (iii). Let Q be a component of $S \setminus N$. By Lemma 4.8, Q is a holed sphere. Since Q is bounded by essential loops on S , Q has at least two boundary components. By Lemma 4.7, the holonomy of $C|Q$ is trivial. Thus we can isomorphically lift Q to a component \tilde{Q} of $\tilde{C} \setminus \tilde{N}$ (unique up to an element of Γ). Then the developing map of $C|Q$ is the restriction of f to \tilde{Q} .

For the first assertion of (iii), it suffices to show that there is a component R of $\Omega \setminus \tilde{N}'$ such that, via f , each boundary component ℓ of \tilde{Q} covers a boundary component of R . Clearly $f(\ell)$ is a loop of \tilde{N}' . Since f is a local homeomorphism, there is a small neighborhood of ℓ in \tilde{Q} so that its f -image is contained in a component R of $\Omega \setminus \tilde{N}'$. We show that, if m is another boundary component of \tilde{Q} , then the loop $f(m)$ is also a boundary component of R . Let $\alpha: [0, 1] \rightarrow \tilde{Q}$ be a properly embedded arc connecting ℓ to m , so that $\alpha(0) \in \ell$ and $\alpha(1) \in m$. We can assume that $f \circ \alpha$ is transversal to \tilde{N} , if necessary by a small isotopy of α . Suppose that $f(m)$ is *not* a boundary component of R . Then $f(m)$ is contained in the interior of a component D of $\hat{C} \setminus R$. Then D is a topological disk bounded by a boundary component of R .

Case 1. Suppose that $\partial D =: n'$ is different from $f(\ell)$. Then $f(\ell)$ and $f(m)$ are contained in the distinct components of $\hat{C} \setminus n'$. Then the curve $f \circ \alpha$ is transversal to n' . Since $f \circ \alpha$ is a curve on \hat{C} connecting $f(\ell)$ to $f(m)$, it intersects n' an odd number of times. Therefore α transversally intersects $f^{-1}(n')$ an odd number of times. Then, similarly to the proof

of Lemma 4.7, we can deduce that α transversally intersects \tilde{N} . This contradicts the assumption that α is in \tilde{Q} .

Case 2. Suppose that $f(\ell)$ bounds D . Since α is property embedded in \tilde{Q} , the curve $f \circ \alpha: [0, 1] \rightarrow \hat{\mathbb{C}}$ takes $(0, \epsilon]$ into R for sufficiently small $\epsilon > 0$. Then the point $f \circ \alpha(\epsilon)$ and the loop $f(m)$ are contained in the distinct components of $\hat{\mathbb{C}} \setminus f(\ell)$. Then the restriction of $f \circ \alpha$ to $[\epsilon, 1]$ transversally intersects $f(\ell)$ an odd number of times. Then we can similarly deduce a contradiction that α must transversally intersects \tilde{N} .

For the second assertion of (iii), we show that $f(\partial\tilde{Q})$ consists of at least two boundary components of R . Suppose that $f(\partial\tilde{Q})$ is a single boundary component ℓ' of R . Then $f|_{\tilde{Q}}: \tilde{Q} \rightarrow \hat{\mathbb{C}}$ has the path lifting property on $\hat{\mathbb{C}} \setminus \ell'$. Since $f^{-1}(\ell')$ is a union of disjoint loops on \tilde{S} , $f^{-1}(\ell') \cap \tilde{Q}$ is a union of disjoint loops on \tilde{Q} . Let P be a component of $\tilde{Q} \setminus f^{-1}(\ell')$ such that P and \tilde{Q} share a boundary component. Then this common boundary component is an essential loop on \tilde{S} . On the other hand $f|_P$ is a covering map from P onto a component of $\hat{\mathbb{C}} \setminus \ell'$, which is a disk. Thus P is also a disk, and therefore the boundary component of P is inessential on \tilde{S} , which is a contradiction \square

5. SCHOTTKY STRUCTURES AND HANDLEBODIES WITH CELLULAR STRUCTURES.

(The techniques used in this section has great influence from a paper by Oertel [16].) We carry over our notation from §4. Let Ω_0 be a standard fundamental domain of the action of the real Schottky group Γ on the domain of discontinuity Ω , i.e. Ω_0 is the complement of disjoint $2g$ round disks. We in addition assume that Ω_0 is a closed subset of $\hat{\mathbb{C}}$. The boundary components of Ω_0 are paired up by generators $\gamma_1, \gamma_2, \dots, \gamma_g$ of Γ . In addition $\partial\Omega_0$ is a multiloop in Ω , and $\Gamma(\partial\Omega_0) =: \tilde{L}'$ is a Γ -invariant multiloop splitting Ω into the fundamental domains $\gamma\Omega_0$ with $\gamma \in \Gamma$.

Recall that the projective structure $C' = \Omega/\Gamma$ is the boundary surface of the genus- g hyperbolic handlebody $H' = (\mathbb{H}^3 \cup \Omega)/\Gamma$. Let $L' = \tilde{L}'/\Gamma$. Then L' is a multiloop on C' that is the union of g disjoint non-parallel loops bounding meridian disks in H' , so that $C' \setminus L'$ is a $2g$ -holed sphere. Let L be the pullback of L' via f , which is a multiloop on S (see §4.1). Thus L satisfies the assertions of Proposition 4.5.

Let $\tilde{L} := \lfloor f^{-1}(\tilde{L}') \rfloor$. Then \tilde{L} is the total lift of L to \tilde{S} , by the definition of L ,

5.1. Cellular structures on handlebodies. For a subset X of $\mathbb{H}^3 \cup \Omega$, let $\text{Conv}(X)$ denote the convex hull of X in $\mathbb{H}^3 \cup \Omega$. Then, for each loop ℓ of \tilde{L}' , $\text{Conv}(\ell)$ is a copy of $\overline{\mathbb{H}^2}$, the union of \mathbb{H}^2 with its ideal boundary $\partial_\infty \mathbb{H}^2$. Let $\tilde{\Delta}'$ denote the union of $\text{Conv}(\ell)$ over all loops ℓ of \tilde{L}' . Then $\tilde{\Delta}'$ is a multidisk properly embedded in $\mathbb{H}^3 \cup \Omega$, and $\tilde{\Delta}'$ divides $\mathbb{H}^3 \cup \Omega$ into connected fundamental domains of the action $\Gamma \curvearrowright \mathbb{H}^3 \cup \Omega$. Let $\Delta' = \tilde{\Delta}'/\Gamma$. Then Δ' is the disjoint union of g copies of $\overline{\mathbb{H}^2}$ in H' bounded by L' , and thus Δ' cuts H' into a topological 3-disk. Thus, we can regard the pair (H', Δ') as a handlebody with a cellular structure consisting of g 2-cells, the disks of Δ' , and one 3-cell, the closure of $H' \setminus \Delta'$.

By Proposition 4.5 (iii), each component of $S \setminus L$ is a sphere with at least 2 holes. Thus we let H be a handlebody of genus g and identify S and ∂H by a homeomorphism so that each loop of L on S is a meridian loop of H . Let Δ be the multidisk bounded by L and embedded in H properly. Then Δ divides H into finitely many 3-disks. Thus, similarly, we regard (H, Δ) as the handlebody H with the natural cellular structure: its 2-cells are the disks of Δ and 3-cells are the closures of components of $H \setminus \Delta$. Let \tilde{H} denote the universal cover of H , so that $\partial \tilde{H} = \tilde{S}$. Let $\tilde{\Delta}$ denote the total lift of Δ to \tilde{H} , which is a Γ -invariant multidisk bounded by \tilde{L} .

In §5, we prove:

Proposition 5.1. *There exists an embedding $\epsilon: (H, \Delta) \rightarrow (H', \Delta')$ such that*

- (i) ϵ embeds each 2-cell of (H, Δ) into a 2-cell of (H', Δ') and each 3-cell of (H, Δ) into a 3-cell of (H', Δ') ,
- (ii) H' minus the interior of $\text{Im}(\epsilon)$ is homeomorphic to the product $S \times [0, 1]$, and
- (iii) if ℓ is a loop of \tilde{L} , then $\tilde{\epsilon}(\ell)$ is contained in $\text{Conv}(f(\ell)) \cong \overline{\mathbb{H}^2}$, where $\tilde{\epsilon}: (\tilde{H}, \tilde{\Delta}) \rightarrow (\mathbb{H}^3 \cup \Omega, \tilde{\Delta}')$ is a lift of ϵ .

Remarks: In (ii), $S \times \{0, 1\}$ corresponds to $\partial H' \sqcup \epsilon(\partial H)$. In (iii), the lift $\tilde{\epsilon}$ exists since ϵ is π_1 -injective by (ii). It is straightforward to see that (iii) can be equivalently replaced by: (iii') ρ is equal to the induced homomorphism $(\epsilon|_S)_*: \pi_1(S) \rightarrow \pi_1(H') = \Gamma$.

By Proposition 5.1 (i), (iii), and Proposition 4.5 (iii), we immediately obtain

Corollary 5.2. *If R is the closure of a component of $\tilde{S} \setminus \tilde{L}$, then $\tilde{\epsilon}$ properly embeds R into the convex hull of $\text{Supp}_f(R)$ in \mathbb{H}^3 .*

5.2. Dual graphs of cellular handlebodies. Let (M, Δ_M) be a pair of a 3-manifold M with boundary and a locally finite multidisk $\Delta_M =$

$\sqcup_{i \in I} D_i$ properly embedded in M . Then there is a graph dual to this pair: Pick pairwise disjoint regular neighborhoods N_i of the disks D_i ($i \in I$) so that N_i are homeomorphic to $\mathbb{D}^2 \times [-1, 1]$ and $N_i \cap \partial M$ are homeomorphic to $\mathbb{S}^1 \times [-1, 1]$. Collapse each $\mathbb{D}^2 \times \{x\}$ to a single point for each $i \in I$ and $x \in [-1, 1]$ and each component of $M \setminus \sqcup_i N_i$ also to a single point. Then the resulting quotient space is a graph whose edges bijectively correspond to the regular neighborhoods N_i ($i \in I$) and vertices to the components of $M \setminus \sqcup_i N_i$. This graph is called the *dual graph* of (M, Δ_M) and denoted by $(M, \Delta_M)^*$. Accordingly, if X is an edge or a vertex of the dual graph, let X^* denote its corresponding disk D_i or a component of $M \setminus \Delta_M$, and if X is a disk of Δ_M or a component of $M \setminus \Delta_M$, then let X^* denote its corresponding edge or vertex of the dual graph.

The dual graph $(M, \Delta_M)^* =: G_M$ can be embedded in (M, Δ_M) , realizing the duality: Each vertex of G_M is in the corresponding component of $M \setminus \sqcup_i N_i$ and each edge of G_M transversally intersects Δ_M in a single point contained in the disk D_i dual to it (see Figure 2).

We in addition assume that M is a handlebody of genus g and Δ_M is a union of finitely many disjoint meridian disks in M such that Δ_M splits M into 3-disks. Particularly, we have contracted such pairs (H, Δ) and (H', Δ') in §5.1. Under this assumption, G_M is a finite connected graph and every vertex of G_M has degree at least two. Since Δ_M splits M into 3-disks, we can choose the dual embedding of G_M into (M, Δ_M) so that M is a regular neighborhood of G_M . Then $\pi_1(G_M)$ is isomorphic to the free group of rank g .

Let \tilde{M} be the universal cover of the handlebody M , and let $\tilde{\Delta}_M$ be the total lift of the multidisk Δ_M to \tilde{M} . Let \tilde{G}_M be the universal cover of the dual graph G_M . Then \tilde{G}_M is the dual graph of $(\tilde{M}, \tilde{\Delta}_M)$. The action of $\pi_1(G_M) = \pi_1(M)$ commutes with the dual embedding \tilde{G}_M into $(\tilde{M}, \tilde{\Delta}_M)$. Thus, for every $\gamma \in \pi_1(M)$ and every cell X of \tilde{G}_M and $(\tilde{M}, \tilde{\Delta}_M)$, we have $(\gamma X)^* = \gamma \cdot X^*$.

Conversely, given a finite graph K , we can easily construct a pair of a handlebody H_K and a union Δ_K of disjoint meridian disks in H_K splitting H_K into 3-disks such that K is the dual graph of (H_K, Δ_K) .

5.3. Graph map. A *graph map* is a simplicial map between graphs that maps each edge onto an edge (and each vertex to a vertex). Let G and G' be the dual graphs of the cellular handlebodies (H, Δ) and (H', Δ') , respectively, constructed in §5.1. Then G' is a bouquet of g circles consisting of g edges and one vertex (e.g. Figure 2).

We will see that the developing map f induces a graph map $\kappa: G \rightarrow G'$. Observe that the identification $S = \partial H$ induces an isomorphism

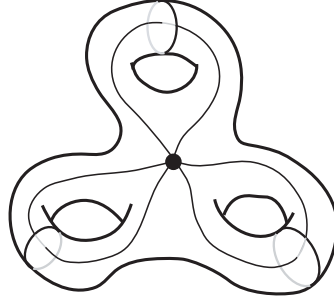


FIGURE 2. The dual graph embedded in (H', D') in the case of $g = 3$.

between $\pi_1(H)$ and $\pi_1(S)/\ker \rho$. Recall that the Schottky representation $\rho: \pi_1(S) \rightarrow \Gamma$ induces an isomorphism $\tilde{\rho}: \pi_1(S)/\ker(\rho) \rightarrow \Gamma$. Then this isomorphism can also be regarded as an isomorphism from $\pi_1(H)$ to $\pi_1(H')$ and, by the duality, from $\pi_1(G)$ to $\pi_1(G')$.

Let \tilde{G} and \tilde{G}' be the universal covers of G and G' , respectively. Then we construct a $\tilde{\rho}$ -equivariant $\tilde{\kappa}: \tilde{G} \rightarrow \tilde{G}'$. For each vertex v of \tilde{G} , its dual v^* is a component of $\tilde{H} \setminus \tilde{\Delta}$, and $v^* \cap \tilde{S}$ is a component of $\tilde{S} \setminus \tilde{L}$. Thus, by Corollary 4.6 (i), the restriction of the projective structure \tilde{C} to $v^* \cap \tilde{S}$ is an almost good holed-sphere supported on the fundamental domain $\gamma\Omega_0$ for a unique $\gamma \in \Gamma$. Then $Conv(\gamma\Omega_0)$ is a component of $(\mathbb{H}^3 \cup \Omega) \setminus \tilde{\Delta}'$. Thus we define $\tilde{\kappa}(v)$ to be the vertex of \tilde{G}' dual to $Conv(\gamma\Omega_0)$. Then

Lemma 5.3. (i) $\tilde{\kappa}$ is ρ -equivariant. (ii) Let v_1 and v_2 be the adjacent vertices of \tilde{G} . Then $\tilde{\kappa}(v_1)$ and $\tilde{\kappa}(v_2)$ are also adjacent vertices of \tilde{G}' .

Proof. (i). Let v be a vertex of \tilde{G} . Then, as above, $Supp_f(v^* \cap \tilde{S}) = \gamma\Omega_0$ for a unique $\gamma \in \Gamma$. For all $\omega \in \Gamma$, we have $(\omega \cdot v)^* = \omega \cdot v^*$. Since f is ρ -equivariant,

$$Supp_f((\omega \cdot v)^* \cap \tilde{S}) = Supp_f(\omega \cdot (v^* \cap \tilde{S})) = \omega \cdot Supp_f(v^* \cap \tilde{S}) = \omega \cdot \gamma\Omega_0.$$

Also

$$\begin{aligned} \omega \cdot Conv(\gamma\Omega_0) &= Conv(\omega \cdot \gamma\Omega_0) \text{ and} \\ \omega \cdot (Conv(\gamma\Omega_0))^* &= (Conv(\omega \cdot \gamma\Omega_0))^*. \end{aligned}$$

Thus

$$\omega \cdot \tilde{\kappa}(v) = \omega \cdot (Conv(\gamma\Omega_0))^* = (\omega \cdot Conv(\gamma\Omega_0))^* = \tilde{\kappa}(\omega \cdot v).$$

(ii). For all adjacent vertices v_1 and v_2 of \tilde{G} , there is an edge e of \tilde{G} connecting v_1 and v_2 . Since e^* is a disk of $\tilde{\Delta}$, then $e^* \cap \tilde{S}$ is a loop of \tilde{L} . Thus $v_1^* \cap \tilde{S}$ and $v_2^* \cap \tilde{S}$ are adjacent components of $\tilde{S} \setminus \tilde{L}$ along the loop $e^* \cap \tilde{S}$. By Corollary 4.6 (ii), $Supp_f(v_1^* \cap \tilde{S})$ and $Supp_f(v_2^* \cap \tilde{S})$ are adjacent components of $\Omega \setminus \tilde{L}'$, having the common boundary component $f(e^* \cap \tilde{S})$. Therefore $\tilde{\kappa}(v_1)$ and $\tilde{\kappa}(v_2)$ are adjacent vertices of \tilde{G}' . \square

By Lemma 5.3 (ii), $\tilde{\kappa}$ from the vertices of \tilde{G} to of \tilde{G}' uniquely extends to the edges of \tilde{G} : For each edge $[v_1, v_2]$ of \tilde{G} connecting a vertex v_1 to its adjacent vertex v_2 , its image $\tilde{\kappa}([v_1, v_2])$ is the edge of \tilde{G}' connecting $\tilde{\kappa}(v_1)$ to $\tilde{\kappa}(v_2)$. Then, by Lemma 5.3 (i), $\tilde{\kappa}: \tilde{G} \rightarrow \tilde{G}'$ is ρ -equivariant. Therefore, quotienting $\tilde{\kappa}$ by Γ , we obtain a graph map $\kappa: G \rightarrow G'$.

5.4. Labeling. Recall that the free groups $\pi_1(H')$, $\pi_1(G')$ and Γ are canonically identified. Let e be an oriented edge of the graph G' . Since G is a bouquet of g circles, we can regard e as a simple closed curve on G' and its homotopy class as a unique element of $\{\gamma_1^\pm, \gamma_2^\pm, \dots, \gamma_g^\pm\}$. We call this homotopy class the *label* of e and denote it by $label(e)$.

Let Ψ_Γ denote the covering map, induced by the Γ -action, from \tilde{S} to S , from Ω to $\partial H'$, from \tilde{G} to G and from \tilde{G}' to G' . We will see that the labels on the oriented edges of G' induce the unique labels on the oriented edges of G, \tilde{G}, \tilde{G}' by $\{\gamma_1^\pm, \gamma_2^\pm, \dots, \gamma_g^\pm\}$ so that the labels are preserved under the graph maps $\kappa: G \rightarrow G'$, $\tilde{\kappa}: \tilde{G} \rightarrow \tilde{G}'$, $\Psi_\Gamma: \tilde{G} \rightarrow G$ and $\Psi_\Gamma: \tilde{G}' \rightarrow G'$. First, for each oriented edge e of \tilde{G}' , define $label(e)$ to be $label(\Psi_\Gamma(e))$. Then the labels on the oriented edges of \tilde{G}' are Γ -invariant. Next, for each oriented edge e of \tilde{G} , define $label(e)$ to be $label(\tilde{\kappa}(e))$. Since $\tilde{\kappa}$ is ρ -equivariant, the labels on the oriented edges of \tilde{G} are also Γ -invariant. Finally, for each oriented edge e of G , define $label(e)$ to be $label(\tilde{e})$, where \tilde{e} is a lift of e to \tilde{G} . Since the labeling on the edges of \tilde{G} is Γ -invariant, $label(e)$ does *not* depend on the choice of the lift \tilde{e} . Then $label(e) = label(\tilde{e}) = label(\tilde{\kappa}(\tilde{e})) = label(\Psi_\Gamma \circ \tilde{\kappa}(\tilde{e})) = label(\kappa(e))$. Therefore κ also preserves the labels.

5.5. Folding maps. (See [17].) Let K be a graph. Label each oriented edge of K with an element of $\{\gamma_1^\pm, \gamma_2^\pm, \dots, \gamma_g^\pm\}$. Suppose that there are two different oriented edges $e_1 = [u, v_1]$ and $e_2 = [u, v_2]$ of K with a common vertex u and distinct vertices v_1, v_2 . In addition assume that $label(e_1) = label(e_2)$. Then we can naturally identify the edges e_1 and e_2 , yielding a new labeled graph K' (see Figure 3). This operation is called a *folding* and the graph map $\mu: K \rightarrow K'$ realizing this folding is called a *folding map*. Note that a folding decreases the number of

edges in K by 1, if K is a finite graph. Clearly K and K' are homotopy equivalent, and in particular μ induces an isomorphism $\mu^*: \pi_1(K) \rightarrow \pi_1(K')$. Using the covering theory for graphs ([17, §3.3]), we see that μ lifts to a μ^* -equivariant graph map $\tilde{\mu}$ from the universal cover of K to that of K' .

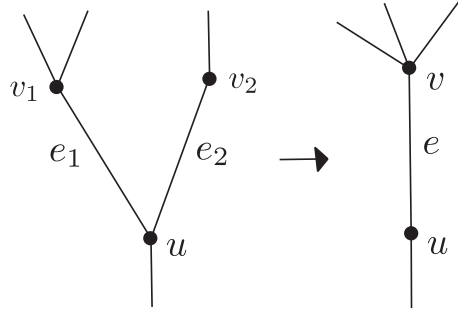


FIGURE 3. Folding.

In §5.3, we constructed the $\kappa: G \rightarrow G'$ and, in §5.4, labeled the oriented edges of G and G' so that κ preserves the labels. Then

Lemma 5.4. *There is a sequence of folding maps*

$$G = G_0 \xrightarrow{\mu_1} G_1 \xrightarrow{\mu_2} \dots \xrightarrow{\mu_n} G_n = G'$$

such that $\kappa = \mu_n \circ \mu_{n-1} \circ \dots \circ \mu_1$.

Proof. By [17, §3.3], $\kappa: G \rightarrow G'$ is the composition of a sequence of folding maps

$$G_0 \xrightarrow{\mu_1} G_1 \xrightarrow{\mu_2} G_2 \xrightarrow{\mu_3} \dots \xrightarrow{\mu_n} G_n$$

and an *immersion* $\iota: G_n \rightarrow G'$ (i.e. locally injective graph map). Since $\kappa, \mu_1, \mu_2, \dots, \mu_n$ are homotopy equivalences, $\iota^*: \pi_1(G_n) \rightarrow \pi_1(G')$ is an isomorphism; therefore ι is also a homotopy equivalence. Thus the lift of ι to a graph map between the universal covers of G_n and G' is an isomorphism that commutes with the action of Γ ; hence ι is the identity map. \square

5.6. Proof of Proposition 5.1. Set $\kappa = \mu_n \circ \mu_{n-1} \circ \dots \circ \mu_1: G = G_0 \rightarrow G_n = G'$ as in Lemma 5.4. Then, for each $i \in \{1, 2, \dots, n\}$, the folding map $\mu_i: G_{i-1} \rightarrow G_i$ lifts to a graph map $\tilde{\mu}_i: \tilde{G}_{i-1} \rightarrow \tilde{G}_i$ between their universal covers, so that $\tilde{\mu}_i$ is equivariant under the isomorphism

$\mu_i^*: \pi_1(G_{i-1}) \rightarrow \pi_1(G_i)$. Let $\kappa_0 = \kappa: G \rightarrow G'$ and $\tilde{\kappa}_0 = \tilde{\kappa}: \tilde{G} \rightarrow \tilde{G}'$. For each $i \in \{1, 2, \dots, n-1\}$, let

$$\kappa_i = \mu_n \circ \mu_{n-1} \circ \dots \circ \mu_{i+1}: G_i \rightarrow G'$$

and let $\kappa_n: G_n \rightarrow G'$ be the identity map. Then let

$$\tilde{\kappa}_i = \tilde{\mu}_n \circ \tilde{\mu}_{n-1} \circ \dots \circ \tilde{\mu}_{i+1}: \tilde{G}_i \rightarrow \tilde{G}',$$

which is a lift of κ_i . Then $\tilde{\kappa}_i$ is equivariant under $\kappa_i^*: \pi_1(G_i) \rightarrow \pi_1(G')$ and $\tilde{\kappa}_i \circ \tilde{\mu}_i = \tilde{\kappa}_{i-1}: G_{i-1} \rightarrow G'$ for each $i \in \{1, \dots, n\}$.

Let $H_0 = H$ and $\Delta_0 = \Delta$. Similarly, for each $i \in \{1, 2, \dots, n\}$, let (H_i, Δ_i) denote the cellular handlebody dual to the graph G_i , where H_i is a genus g -handlebody and Δ_i is a union of disjoint meridian disks in H_i (see §5.2). For each $i \in \{0, 1, \dots, n\}$, let L_i be the multiloop on ∂H_i bounding Δ_i . Let \tilde{H}_i denote the universal cover of H_i . Let $\tilde{\Delta}_i$ and \tilde{L}_i denote the total lifts of Δ_i and L_i to \tilde{H}_i , respectively, so that $\partial \tilde{\Delta}_i = \tilde{L}_i$.

For a multiloop M , which is a subset of a surface, let $[M]$ denote the set of all loops of M . We define a κ_i^* -equivariant map $f_i: [\tilde{L}_i] \rightarrow [\tilde{L}']$ for each $i \in \{0, 1, \dots, n\}$. For each loop ℓ of \tilde{L}_i , let D_ℓ be the disk of $\tilde{\Delta}_i$ bounded by ℓ . Then D_ℓ^* is an edge of \tilde{G}_i . Then $\tilde{\kappa}_i(D_\ell^*)$ is an edge of \tilde{G}' , and $(\tilde{\kappa}_i(D_\ell^*))^*$ is a disk of $\tilde{\Delta}'$. Define $f_i(\ell)$ to be the loop of \tilde{L}' bounding the disk $(\tilde{\kappa}_i(D_\ell^*))^*$. Then by the definition of κ we see that $f_0: [\tilde{L}] \rightarrow [\tilde{L}']$ is exactly the correspondence given by the covering map $f|_{\tilde{L}}: \tilde{L} \rightarrow \tilde{L}'$. Since $\tilde{\kappa}_i$ is κ_i^* -equivariant, f_i is also κ_i^* -equivariant for all $i \in \{0, 1, \dots, n\}$. Since $\tilde{\mu}_i$ is $\tilde{\mu}_i^*$ -equivariant, $\tilde{\mu}_i$ similarly induces a μ_i^* -equivariant map $h_i: [\tilde{L}_{i-1}] \rightarrow [\tilde{L}_i]$. Then, since $\tilde{\kappa}_{i-1} = \tilde{\kappa}_i \circ \tilde{\mu}_i$, we have $f_{i-1} = f_i \circ h_i$.

The following proposition, when $i = 0$, implies Proposition 5.1.

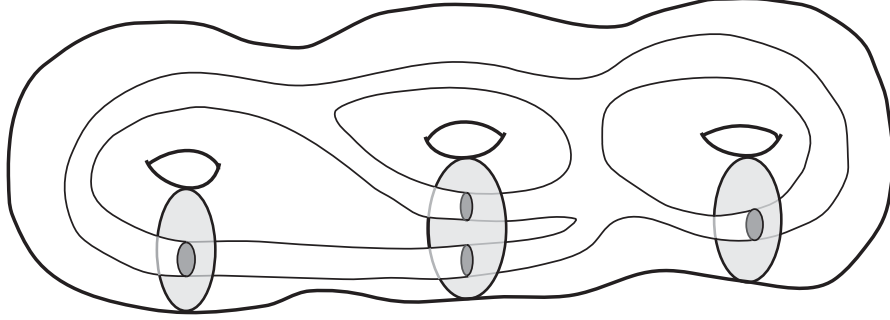
Proposition 5.5. *For each $i \in \{0, 1, 2, \dots, n\}$, there exists an embedding $\epsilon_i: (H_i, \Delta_i) \rightarrow (H', \Delta')$ such that*

(i) ϵ_i takes each 2- and 3-cell of (H_i, Δ_i) into a 2- and 3-cell of (H', Δ') , respectively,

(ii) the complement of $\text{Im}(\epsilon_i)$ is a product so that $H_i \setminus \text{int}(\text{Im}(\epsilon_i))$ is homeomorphic to $S \times [0, 1]$, and

(iii) if ℓ is a loop of \tilde{L}_i , then $\tilde{\epsilon}_i(\ell)$ is contained in $\text{Conv}(f_i(\ell)) \cong \overline{\mathbb{H}^2}$, where $\tilde{\epsilon}_i: (\tilde{H}_i, \tilde{\Delta}_i) \rightarrow (\mathbb{H}^3 \cup \Omega, \tilde{\Delta}')$ is a lift of ϵ_i .

Proof. When $i = n$, the graph map $\kappa_n: G_n \rightarrow G'$ is an isomorphism. Then κ_n induces an isomorphism from (H_n, Δ_n) to (H', Δ') . The graph G' is embedded in (H', Δ') , realizing the duality between G' and (H', Δ') , so that H' is a regular neighborhood of G' . Thus we can

FIGURE 4. An example of ϵ_i .

easily construct an embedding $\epsilon_n: (H_n, \Delta_n) \rightarrow (H', \Delta')$, in an obvious way, onto a smaller regular neighborhood of G' that satisfies (i), (ii) and (iii).

Next, assuming that there is ϵ_i satisfying (i) - (iii), we construct ϵ_{i-1} satisfying (i) - (iii). (Figure 5 illustrates this induction.) Let $e_1 = [u, v_1]$, $e_2 = [u, v_2]$ denote the edges of G_{i-1} that the holding map μ_i identifies, and let $e = [u, v]$ denote the edge of G_i obtained by this identification.

Let P and Q be the components of $H_i \setminus \Delta_i$ that are dual to the vertices u and v , respectively. Let c_1, c_2, \dots, c_p be the edges of G_{i-1} , other than e_1 , that end at v_1 , and let d_1, d_2, \dots, d_q be the edges of G_{i-1} , other than e_2 , that end at v_2 .

Let $D_{c_1}, D_{c_2}, \dots, D_{c_p}$, $D_{d_1}, D_{d_2}, \dots, D_{d_q}, D_e$ denote the disks of Δ_i that are dual to $c_1, c_2, \dots, c_p, d_1, d_2, \dots, d_q, e$, respectively. Then Q is bounded by the meridian disks $D_{c_1}, D_{c_2}, \dots, D_{c_p}, D_{d_1}, D_{d_2}, \dots, D_{d_q}, D_e$. Pick two disjoint meridian disks D_1 and D_2 of H_i parallel to D_e such that D_1 and D_2 are contained in P and that D_1 and D_e bound a solid cylinder in H_i containing D_2 .

Case 1. Suppose that u and v are different vertices of G_i . Then u, v_1, v_2 are different vertices of G_{i-1} , and P and Q are different components of $H_i \setminus \Delta_i$. For each $j \in \{1, 2\}$, let B_j be the union of Q with the solid cylinder, in P , bounded by D_j and D_e . Then B_j is topologically the 3-disk and it is bounded by the meridian disks $D_j, D_{c_1}, D_{c_2}, \dots, D_{c_p}, D_{d_1}, D_{d_2}, \dots, D_{d_q}$. Then ∂B_2 is a 2-sphere and those meridian disks are disjointly embedded in ∂B_2 , so that the complement is a $(1+p+q)$ -holed sphere. Choose an arc α properly embedded in this $(1+p+q)$ -holed sphere, such that: (I) the end points of α are distant points on ∂D_2 , (II) α separates $D_{c_1}, D_{c_2}, \dots, D_{c_p}$ and $D_{d_1}, D_{d_2}, \dots, D_{d_q}$ in the 2-disk $\partial B_2 \setminus D_2$, and (III) α transversally intersects the circle ∂D_e in exactly

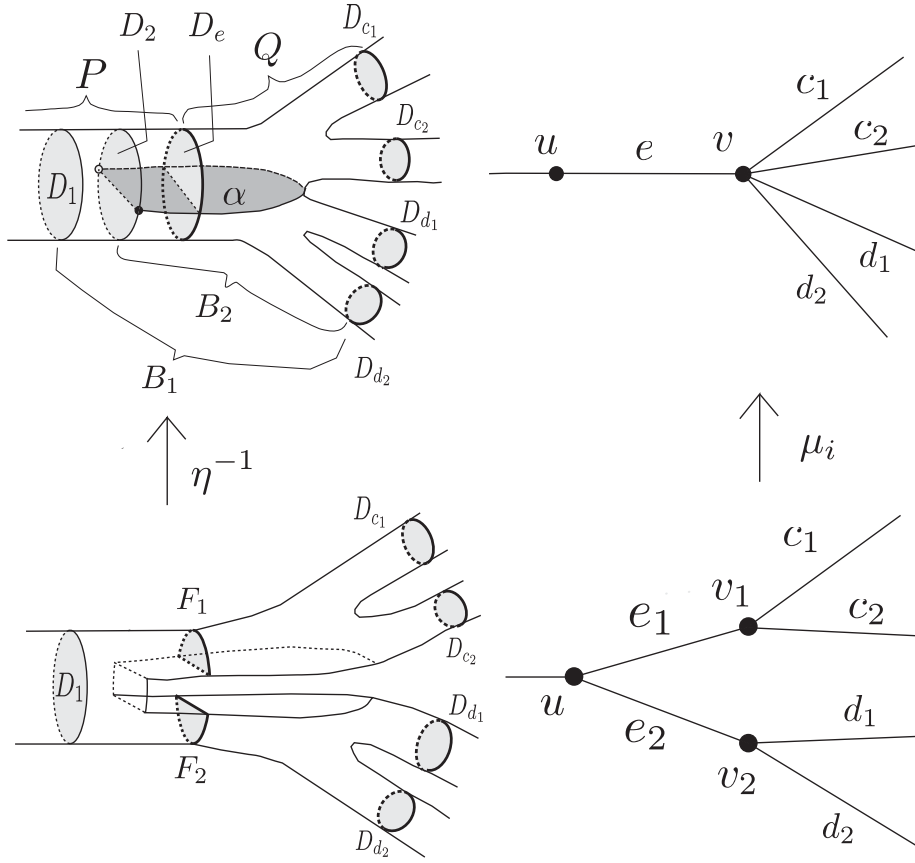


FIGURE 5.

two points. Pick an arc β properly embedded in the disk D_2 connecting the end points of α . Then $\alpha \cup \beta$ is a loop on ∂B_2 and it bounds a 2-disk E properly embedded in B_2 . In addition, we can assume that E transversally intersects D_e in a single arc.

We will see that the compression along disk E naturally transforms (H_i, Δ_i) to (H_{i-1}, Δ_{i-1}) and also ϵ_i to ϵ_{i-1} (Figure 5). Choose a small regular neighborhood N of E in B_2 , so that $D_e \setminus N$ is a union of two disjoint 2-disks, which we denote by F_1 and F_2 . Then N is a 3-disk such that $\partial N \cap \partial H_i$ is a 2-disk contained in the sphere ∂B_1 . Thus there is an isotopy η from H_i to $H_i \setminus N$ supported on B_1 . Then $Q \setminus N$ has two components; if necessarily by swapping the names F_1 and F_2 of the disks, a component of $Q \setminus N$ is bounded by $F_1, D_{c_1}, D_{c_2}, \dots, D_{c_p}$ and the other by $F_2, D_{d_1}, D_{d_2}, \dots, D_{d_q}$. Then $H_i \setminus N$ is still a genus- g handlebody. In addition $\Delta_i \cap (H_i \setminus N)$ is obtained from Δ_i by replacing the disk D_e by the disks F_1 and F_2 ; thus $\Delta_i \cap (H_i \setminus N)$ is a union of

meridian disks in the handlebody $H_i \setminus N$. Thus we see that the new cellular handlebody $(H_i \setminus N, \Delta_i \cap (H_i \setminus N))$ is isomorphic to (H_{i-1}, Δ_{i-1}) .

We see moreover that the inverse isotopy η^{-1} of $H_i \setminus N$ to H_i induces the folding map $\mu_i: G_{i-1} \rightarrow G_i$ via the duality: The dual of v_1 is the component of $H_{i-1} \setminus \Delta_{i-1}$ bounded by $F_1, D_{c_1}, D_{c_2}, \dots, D_{c_p}$, and the dual of v_2 is the component of $H_{i-1} \setminus \Delta_{i-1}$ bounded by $F_2, D_{d_1}, D_{d_2}, \dots, D_{d_q}$; the isotopy η^{-1} combines F_1 and F_2 into D_e , and accordingly μ_i folds the edges $F_1^* = e_1$ and $F_2^* = e_2$ into $D_e^* = e$; the isotopy η^{-1} is supported on B_i and in particular it preserves all the other disks of Δ_{i-1} , and accordingly μ_i is an isomorphism onto its image in the complement of e_1 and e_2 .

We define $\epsilon_{i-1}: (H_{i-1}, \Delta_{i-1}) \rightarrow (H', \Delta')$ to be the restriction of $\epsilon_i: (H_i, \Delta_i) \rightarrow (H', \Delta')$ to $(H_i \setminus N, \Delta_i \cap (H_i \setminus N))$. We show that ϵ_{i-1} satisfies (i), (ii) and (iii). Each 2- and 3-cell of (H_{i-1}, Δ_{i-1}) is contained in a 2- and 3-cell of (H_i, Δ_i) , respectively. Since ϵ_i satisfies (i), thus ϵ_{i-1} also satisfies (i). Recall that η isotopes $\text{Im}(\epsilon_i)$ to $\text{Im}(\epsilon_{i-1})$. Since ϵ_i satisfies (ii), so does ϵ_{i-1} .

We show (iii). Since the isotopy η is supported on the 3-disk B_1 embedded in H_i , it lifts to a $(\Gamma$ -equivariant) isotopy $\tilde{\eta}$ from \tilde{H}_i to \tilde{H}_{i-1} supported on the total lift \tilde{B}_1 of B_1 to \tilde{H}_i . Since each component R of \tilde{B}_1 is homeomorphic to B_1 , we can canonically identify $\tilde{\eta}|_R$ with $\eta|_{B_1}$. For each loop ℓ of \tilde{L}_{i-1} , let D_ℓ denote the disk of $\tilde{\Delta}_{i-1}$ bounded by ℓ . Let $m = h_i(\ell)$, which is a loop of \tilde{L}_i , and let D_m be the disk of $\tilde{\Delta}_i$ bounded by m . Since $f_{i-1} = f_i \circ h_i$, we have $f_{i-1}(\ell) = f_i(h_i(\ell)) = f_i(m)$.

First, suppose that ℓ does *not* bound a lift of F_1 or F_2 in \tilde{H}_{i-1} . Since η is the identity on $\Delta_i \setminus D_\ell$, the isotopy $\tilde{\eta}^{-1}$ fixes D_m . Therefore $D_\ell = D_m$ by the inclusion $(\tilde{H}_{i-1}, \tilde{\Delta}_{i-1}) \subset (\tilde{H}_i, \tilde{\Delta}_i)$. Then we have $\tilde{\epsilon}_{i-1}(D_\ell) = \tilde{\epsilon}_i(D_m)$. Since $\tilde{\epsilon}_i$ satisfies (iii), we have $\tilde{\epsilon}_i(D_m) \subset \text{Conv}(f_i(m)) = \text{Conv}(f_{i-1}(\ell))$. Therefore $\tilde{\epsilon}_{i-1}(D_\ell) \subset \text{Conv}(f_{i-1}(\ell))$.

Next, suppose that ℓ bounds a lift of F_1 or F_2 . Then, accordingly, D_ℓ is a lift of F_1 or F_2 to \tilde{H}_{i-1} . Therefore D_m is a lift of D_e to \tilde{H}_i , and it is contained in a component R of \tilde{B}_1 by the inclusion $(\tilde{H}_{i-1}, \tilde{\Delta}_{i-1}) \subset (\tilde{H}_i, \tilde{\Delta}_i)$. Since $\tilde{\eta}|_R = \eta|_{B_1}$, we have $D_\ell \subset D_m$ and thus $\tilde{\epsilon}_{i-1}(D_\ell) \subset \tilde{\epsilon}_i(D_m)$. Similarly to the previous case, we have $\tilde{\epsilon}_i(D_m) \subset \text{Conv}(f_i(m)) = \text{Conv}(f_{i-1}(\ell))$. Thus $\tilde{\epsilon}_{i-1}(D_\ell) \subset \text{Conv}(f_{i-1}(\ell))$, and therefore ϵ_{i-1} satisfies (iii).

Case 2. (For the following discussion, see Figure 6.) Suppose that u and v are the same vertices of G_i . Then, without loss of generality, we can assume that $u = v_1$ and $u \neq v_2$ (if $u = v_1 = v_2$, there is a contradiction to the definition of folding maps). Then the components

P and Q of $H_i \setminus \Delta_i$ are the same component. Since $u = v_1$, we can assume that $e = c_1$. Then $D_e = D_{c_1}$.

We define B_1 and B_2 analogously: Let B_1 be the 3-disk in H_i bounded by $D_1, D_{c_2}, \dots, D_{c_p}, D_{d_1}, D_{d_2}, \dots, D_{d_q}$. Let B_2 be the 3-disk in H_i bounded by $D_1, D_2, D_{c_2}, \dots, D_{c_p}, D_{d_1}, D_{d_2}, \dots, D_{d_q}$. Similarly let α be an arc properly embedded in the $(1 + p + q)$ -holed sphere

$$\partial B_2 \setminus (D_1 \sqcup D_2 \sqcup D_{c_2} \sqcup \dots \sqcup D_{c_p} \sqcup D_{d_1} \sqcup D_{d_2} \sqcup \dots \sqcup D_{d_q}),$$

such that: (I) the end points of α are contained in ∂D_2 ; (II) α separates $D_1, D_{c_2}, D_{c_3}, \dots, D_{c_p}$ and $D_{d_1}, D_{d_2}, \dots, D_{d_q}$ in the 2-disk $\partial B_2 \setminus D_2$, and (III) α transversally intersects $\partial D_e = \partial D_{c_1}$ in exactly two points. The rest of the proof is also similar to *Case 1*. \square

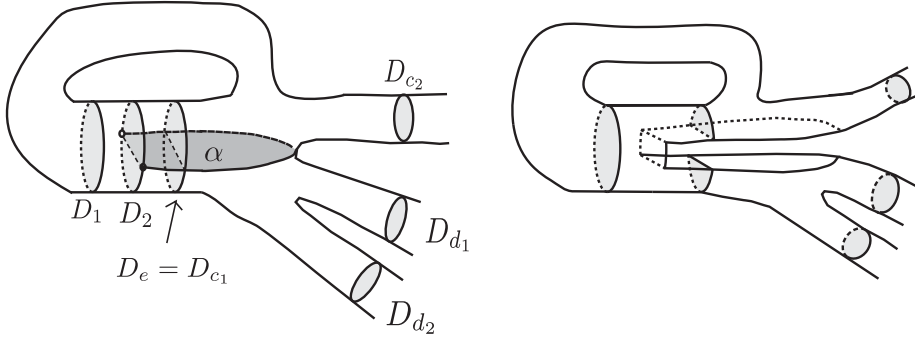


FIGURE 6.

6. DECOMPOSITION OF A SCHOTTKY STRUCTURE INTO GOOD HOLED SPHERES

We started with a projective structure C on S with real Schottky holonomy $\rho: \pi_1(S) \rightarrow \Gamma$. Then, we obtained a multiloop L on S by Proposition 4.5 (see §5), and regarded S as the boundary of the handlebody H so that L bounds a multidisk Δ in H (§5.1). There we also constructed a pair (H', Δ') of the hyperbolic handlebody H' and the multidisk Δ' consisting of copies of $\overline{\mathbb{H}^2}$ such that ∂H is the uniformizable structure $C' = \Omega/\Gamma$. Let $\epsilon: (H, \Delta) \rightarrow (H', \Delta')$ be the embedding obtained by Proposition 5.1. Then by Proposition 5.1 (ii) there is a homeomorphism $\eta: S \times [0, 1] \rightarrow H' \setminus \text{int}(\text{Im}(\epsilon))$. We set $\eta_t(s) = \eta(s, t)$ so that η_0 is a homeomorphism from S onto $\epsilon(\partial H)$ and η_1 onto $\partial H'$. By Proposition 5.1 (i), (iii), the multiloop $\eta_0(L)$ embedded in H' is the intersection of the boundary of the embedded handlebody $\epsilon(H)$ and the

multidisk Δ' . Let $M' = \eta_1(L)$. Then M' is a multiloop on $\partial H' = C'$. (This multiloop M' depends on the choice of η , however it is unique it is up to an isotopy.)

We show that M' satisfies Assumptions (I), (II) of Proposition 4.5. Since each component of $S \setminus L$ is a holed sphere by Proposition 4.5 (iii) and η_1 is a homeomorphism, each component of $C' \setminus M'$ is also a holed sphere; thus M' satisfies (II). For an arbitrary loop m' of M' , let ℓ be the corresponding loop of L , i.e. $\eta_1(\ell) = m'$. Then $\eta(\ell \times [0, 1])$ is an annulus properly embedded in the product region $H' \setminus \text{int}(\text{Im}(\epsilon))$ bounded by $\eta_1(\ell) = m'$ and $\eta_0(\ell) = \epsilon(\ell)$. Since ℓ bounds a meridian disk in H , then $\epsilon(\ell)$ also bounds a meridian disk in the embedded handlebody $\text{Im}(\epsilon)$. Thus the union of this meridian disk in $\text{Im}(\epsilon)$ and the annulus $\eta(\ell \times [0, 1])$ is a meridian disk in H' bounded by m' . Hence M' satisfies (I).

Let M be the pullback of M' via the developing map $f: \tilde{S} \rightarrow \hat{C}$ of C . Then M is a multiloop on S (§4.1), and it decomposes (S, C) into almost good holed spheres (Proposition 4.5 (iii)). In this section, we show that this decomposition is moreover into good holed spheres. Let \tilde{C} be the projective structure on \tilde{S} obtained by lifting C , and let \tilde{M} and \tilde{M}' denote the total lifts of M and M' to \tilde{S} and Ω , respectively. Then

Theorem 6.1. *There exists a $\tilde{\rho}$ -equivariant homeomorphism $\zeta: \tilde{S} \rightarrow \Omega$ such that, if \tilde{P} is a component of $\tilde{S} \setminus \tilde{M}$, then the restriction $\tilde{C}|_{\tilde{P}}$ is a good holed sphere whose full support is $\zeta(\tilde{P})$ and in particular each boundary component ℓ of \tilde{P} covers the loop $\zeta(\ell)$ via f .*

Thus, if P is a component of $S \setminus M$, then the restriction of C to P is a good holed sphere. Theorem 6.1 follows from:

Proposition 6.2. *If μ' is a loop of \tilde{M}' , then $[f^{-1}(\mu')]$ is a single loop on \tilde{S} .*

Proof of Theorem 6.1, with Proposition 6.2 assumed. The developing map f , by Proposition 6.2, yields a $\tilde{\rho}$ -equivariant bijection $f_*: [\tilde{M}] \rightarrow [\tilde{M}']$ from the set of the loops of \tilde{M} onto that of \tilde{M}' . Therefore, we can choose a $\tilde{\rho}$ -equivariant homeomorphism $\zeta: \tilde{M} \rightarrow \tilde{M}'$ such that, if m is a loop of \tilde{M} , then m covers $\zeta(m)$ via f . We have seen that, if P is a component of $\tilde{S} \setminus \tilde{M}$, then $\tilde{C}|_P$ is an almost good holed sphere whose support is a unique component R of $\Omega \setminus \tilde{M}'$ via f . Since f_* is bijection, if different loops a, b of \tilde{M} are boundary components of P , then $f(a), f(b)$ are also different boundary components of R . Since f_* is a bijection, Corollary 4.6 (ii) implies that $\zeta|_{\partial P}$ is a homeomorphism onto ∂R . Therefore $\tilde{C}|_P$ is a good holed sphere fully supported on

R . Thus $\zeta: \tilde{M} \rightarrow \tilde{M}'$ continuously extends to P so that $\zeta|_P$ is a homeomorphism onto R . Therefore we can extend ζ to a $\tilde{\rho}$ -equivariant homeomorphism from \tilde{S} to Ω with the desired property. \square

6.1. An outline of the proof of Proposition 6.2. Proposition 6.2 is the main proposition of this paper, and we here outline its proof. Let μ' be a loop of \tilde{M}' and let λ be the loop of \tilde{L} with $\tilde{\eta}_1(\lambda) = \mu'$. Let $\lambda' = f(\lambda)$, which is a loop of \tilde{L}' .

Step 1. We first reduce the proposition to a similar assertion about a good holed sphere obtained from (\tilde{S}, \tilde{C}) as follows. Find a compact subsurface F of \tilde{S} with boundary, such that the interior of F contains the loop λ and the multiloop $\lfloor f^{-1}(\mu') \rfloor$ and that the restriction $\tilde{C}|_F$ is an almost good holed sphere (Proposition 6.4 and Corollary 6.5). Let \check{F} be the puncture sphere obtained by attaching a once-punctured disk long every boundary component of F ; then F is a subsurface of \check{F} . We naturally extend $\tilde{C}|_F$ to a good structure $C_{\check{F}} = (f_{\check{F}}, \rho_{id})$ on \check{F} , so that $\lfloor f^{-1}(\mu') \rfloor = \lfloor f_{\check{F}}^{-1}(\mu') \rfloor \subset F$. Thus it suffices to show that $\lfloor f_{\check{F}}^{-1}(\mu') \rfloor$ is a single loop (Proposition 6.8). We also see, in particular, that $\lfloor f_{\check{F}}^{-1}(\lambda') \rfloor$ is a multiloop on \check{F} that decomposes $C_{\check{F}}$ into almost good holed spheres.

Step 2. Proposition 5.1 yields an embedding $\tilde{\epsilon}: \tilde{H} \rightarrow \overline{\mathbb{H}^3}$ associated with the projective structure C . We construct an analogous embedding $\epsilon_{\check{F}}$ of a 3-disk into $\overline{\mathbb{H}^3}$ associated with $C_{\check{F}}$, where the punctured sphere \check{F} sits in the boundary of the 3-disk. (See the left picture in Figure 10.) In this analogy, the punctures of \check{F} correspond to the limit set of the Schottky group.

Step 3. An isotopy of \hat{C} induces a deformation of the good projective structure $C_{\check{F}}$, by postcomposing its developing map $f_{\check{F}}$ with the isotopy. Under this deformation the projective structure remains a good structure on \check{F} . In addition the loop μ' on \hat{C} is also isotoped, and accordingly the pullback of μ' , a multiloop, is also isotoped on \check{F} . We then isotope \hat{C} so that, yet with the loop λ' fixed on \hat{C} , the pullback of λ' is a single loop homotopic to λ on \check{F} . This isotopy is realized as a composition of isotopies of \hat{C} that inductively reduce the number of the components of the pullback of λ' .

We then extend this isotopy of \hat{C} to $\overline{\mathbb{H}^3}$ (Figure 10) so that the induced good structure on \check{F} still satisfies all properties analogues to Proposition 5.1 (i) -(iii) at each induction step. Then a certain product structure analogues to Proposition 5.1 (iii) implies that, after the isotopy, the loop μ' becomes isotopic to the fixed loop λ' on the punctured

sphere fully supporting the induced structure on \check{F} . We thus conclude that the pullback of μ' is also a single loop isotopic to λ .

6.2. The proof of Proposition 6.2. *Step 1.* Recall that Ω_0 is the compact fundamental domain for the Γ -action on Ω bounded by $2g$ round loops of \tilde{L}' . Accordingly $Conv(\Omega_0)$ is a fundamental domain for the Γ -action on $\mathbb{H}^3 \cup \Omega$. Then $Conv(\Omega_0)$ is a compact subset of $\mathbb{H}^3 \cup \Omega$ bounded by $2g$ disks of $\tilde{\Delta}'$. Also $\tilde{\epsilon}: (\tilde{H}, \tilde{\Delta}) \rightarrow (\mathbb{H}^3 \cup \Omega, \tilde{\Delta}')$ is the $\tilde{\rho}$ -equivariant lift of ϵ . Similarly the homeomorphism $\eta: S \times [0, 1] \rightarrow H' \setminus \text{int}(\text{Im}(\epsilon))$ lifts to a unique $\tilde{\rho}$ -equivariant homeomorphism $\tilde{\eta}: \tilde{S} \times [0, 1] \rightarrow (\mathbb{H}^3 \cup \Omega) \setminus \text{int}(\text{Im}(\tilde{\epsilon}))$. Set $\tilde{\eta}_t(s) = \tilde{\eta}(s, t)$.

Let μ' be a loop of \tilde{M}' and λ be its corresponding loop of \tilde{L} , i.e. $\tilde{\eta}_1(\lambda) = \mu'$. Then $f(\lambda)$ is the loop of \tilde{L}' with $Conv(f(\lambda)) \supset \tilde{\epsilon}(\lambda)$ (Proposition 5.1 (iii)). Then $\tilde{\eta}(\lambda \times [0, 1])$ is a compact annulus embedded properly in the product region $(\mathbb{H}^3 \cup \Omega) \setminus \text{int}(\text{Im}(\tilde{\epsilon}))$ and bounded by $\tilde{\eta}(\lambda \times \{0\}) = \tilde{\epsilon}(\lambda)$ and $\tilde{\eta}(\lambda \times \{1\}) = \mu'$. Let

$$F'_0 = \bigcup \{ \gamma \Omega_0 \mid \gamma \in \Gamma, Conv(\gamma \Omega_0) \cap \tilde{\eta}(\lambda \times [0, 1]) \neq \emptyset \}.$$

Then

Lemma 6.3. *F'_0 is a compact connected subsurface of Ω bounded by (finitely many) loops of \tilde{L}' , and the interior of F'_0 contains the loops μ' and $f(\lambda)$.*

Proof. The multidisk $\tilde{\Delta}'$ decomposes $\mathbb{H}^3 \cup \Omega$ into the compact fundamental domains $Conv(\gamma \Omega_0)$ with $\gamma \in \Gamma$. Since $\tilde{\eta}(\lambda \times [0, 1])$ is compact, it intersects $Conv(\gamma \Omega_0)$ for only finitely many $\gamma \in \Gamma$. Therefore, letting

$$E = \bigcup \{ Conv(\gamma \Omega_0) \mid \gamma \in \Gamma, Conv(\gamma \Omega_0) \cap \tilde{\eta}(\lambda \times [0, 1]) \neq \emptyset \},$$

E is compact. Besides, since $\tilde{\eta}(\lambda \times [0, 1])$ is connected, E is also connected. Thus E is a connected compact convex subset of $\mathbb{H}^3 \cup \Omega$ bounded by (finitely many) disks of $\tilde{\Delta}'$. Since $Conv(F'_0) = E$, thus F'_0 is a connected compact subsurface of Ω bounded by finitely many loops of \tilde{L}' . Since $\mu' = \tilde{\eta}(\lambda \times \{1\}) \subset \tilde{\eta}(\lambda \times [0, 1])$, the interior of F'_0 contains μ' by the definition of F'_0 . Since the loop $\epsilon(\lambda)$ is contained in the annulus $\tilde{\eta}(\lambda \times [0, 1])$ and in the disk $Conv(f(\lambda))$ of $\tilde{\Delta}'$, therefore F'_0 contains the adjacent components of $\Omega \setminus \tilde{L}'$ along the loop $f(\lambda)$. Therefore the interior of F'_0 contains $f(\lambda)$. \square

Proposition 6.4. *There exist a compact connected subsurface F of \tilde{S} bounded by finitely many loops of \tilde{L} and a compact connected subsurface F' of Ω bounded by finitely many loops of \tilde{L}' such that:*

(i) F' contains F'_0 ,

- (ii) $\tilde{C}|F$ is an almost good holed sphere supported on F' , and
 (iii) F contains $\tilde{\epsilon}^{-1}(\text{Conv}(F'_0)) \cap \tilde{S}$.

Proof. For a component R of $\tilde{S} \setminus \tilde{L}$, we have either $\text{Supp}(R) \subset F'_0$ or $\text{Supp}(R) \subset \tilde{S} \setminus F'_0$ (here we mean $\tilde{C}|R$ by R , abusing notation). By Corollary 5.2, if $\text{Supp}(R) \subset F'_0$, then $\tilde{\epsilon}(R) \subset \text{Conv}(F'_0)$ and, if $R \subset \tilde{S} \setminus F'_0$, then $\tilde{\epsilon}(R) \cap \text{Conv}(F'_0) = \emptyset$. Therefore $\tilde{\epsilon}^{-1}(\text{Conv}(F'_0)) \cap \tilde{S} (=: X_0)$ is equal to

$$\bigcup \{cl(R) \mid R \text{ is a component of } \tilde{S} \setminus \tilde{L}, \text{Supp}_f(R) \subset F'_0\},$$

where $cl(R)$ denote the closure of R .

Since f is $\tilde{\rho}$ -equivariant, for each $\gamma \in \Gamma$, there is at least one but at most finitely many components of $\tilde{S} \setminus \tilde{L}$ supported on $\gamma\Omega_0$ (by Corollary 4.6 (ii)). Therefore X_0 is a compact subsurface of \tilde{S} bounded by finitely many loops of \tilde{L} , but X_0 is *not* necessarily connected. Thus we choose a compact connected subsurface F_0 of \tilde{S} bounded by finitely many loops of \tilde{L} such that $F_0 \supset X_0$. Then each component Q of $F_0 \setminus \tilde{L}$ is a component of $\tilde{S} \setminus \tilde{L}$, and Q is supported on a unique component of $\Omega \setminus \tilde{L}'$. Let

$$F' = \bigcup cl(\text{Supp}(Q)),$$

where Q varies over all components of $F_0 \setminus \tilde{L}$. By the definition of F_0 and F' , we have $F'_0 \subset F'$; thus (i) holds.

Since F_0 is compact, by Corollary 4.6, we see that F' is also a compact connected subsurface of Ω bounded by finitely many loops of \tilde{L}' . We see that $\tilde{\epsilon}^{-1}(\text{Conv}(F')) \cap \tilde{S}$ is the union of the closures of finitely many components R of $\tilde{S} \setminus \tilde{L}$ such that $\text{Supp}(R) \subset F'$. Then $\tilde{\epsilon}^{-1}(\text{Conv}(F')) \cap \tilde{S}$ is a compact subsurface of \tilde{S} bounded by finitely many loops of \tilde{L} , but again it is *not* necessarily connected. Since F_0 is connected, $\tilde{\epsilon}^{-1}(\text{Conv}(F')) \cap \tilde{S}$ contains a connected component containing F_0 by the definition of F' . Let F be this component of $\tilde{\epsilon}^{-1}(\text{Conv}(F')) \cap \tilde{S}$. Since F_0 contains X_0 , then F also contains X_0 ; thus (iii) holds.

Since F is a compact subsurface of \tilde{S} , its genus is zero and it has at least two boundary components. By Corollary 4.6, ∂F covers $\partial F'$ via f . We see that $f(\partial F)$ has at least two components, similarly to Proposition 4.5 (iii). Therefore $\tilde{C}|F$ is an almost good holed sphere supported on F' ; thus (ii) holds. \square

Corollary 6.5. *The multiloop $\lfloor f^{-1}(\mu') \rfloor$ is contained in the interior of the compact subsurface F of \tilde{S} given by in Proposition 6.4.*

Proof. For an arbitrary component of $\tilde{S} \setminus \tilde{L}$ that is disjoint from F , let R denote its closure. Then it suffices to show that $R \cap \lfloor f^{-1}(\mu') \rfloor = \emptyset$.

By Proposition 6.3 (iii), $Supp_f(R)$ disjoint from $int(F'_0)$. Therefore, by Lemma 6.3, μ' and $Supp_f(R)$ are disjoint. Hence, by Lemma 4.3, $R \cap [f^{-1}(\mu')] = \emptyset$. \square

Step2. Set $\tilde{C}|F = (f_F, \rho_{id})$, where $f_F: F \rightarrow \hat{\mathbb{C}}$ is its developing map and $\rho_{id}: \pi_1(S) \rightarrow \text{PSL}(2, \mathbb{C})$ is the trivial representation. Then, in order to prove Proposition 6.2, it suffices to show that, by Corollary 6.5, $[f_F^{-1}(\mu')] = [f^{-1}(\mu')] \cap F$ is a single loop on F .

Every boundary components of F bound a disk of $\tilde{\Delta}$. The union of such disks bound a 3-disk H_F in \tilde{H} . Then $H_F \cap \tilde{S} = F$. Let $\epsilon_F: H_F \rightarrow \mathbb{H}^3$ be the restriction of $\tilde{\epsilon}: \tilde{H} \rightarrow \mathbb{H}^3$ to H_F . Accordingly, restricting the embedding $\tilde{\eta}: \tilde{S} \times [0, 1] \rightarrow \overline{\mathbb{H}^3}$ to $F \times [0, 1]$, we obtain an embedding $\eta_F: F \times [0, 1] \rightarrow H'$. Let \check{F} be the punctured sphere obtained by attaching a once-punctured disk along each boundary component of F , and let p_1, p_2, \dots, p_n be the punctures of \check{F} , where n is the number of the boundary components of F . Then $\check{F} \cup p_1 \cup p_2 \dots \cup p_n =: \hat{F}$ is a 2-sphere. Let $H_{\hat{F}}$ be a closed 3-disk and identify its boundary with \hat{F} . Then similarly the multiloop $\partial F (\subset \hat{F})$ bounds a multidisk property embedded in $H_{\hat{F}}$. Then the multidisk bounds a 3-disk in $H_{\hat{F}}$; we naturally identify this 3-disk with H_F , so that $\partial H_F \cap \hat{F} = F$.

With respect to the inclusion $H_F \subset H_{\hat{F}}$, we will extend $\epsilon_F: H_F \rightarrow \overline{\mathbb{H}^3}$ to an embedding $\epsilon_{\hat{F}}: H_{\hat{F}} \rightarrow \overline{\mathbb{H}^3}$ such that

- $\epsilon_{\hat{F}}$ takes p_1, p_2, \dots, p_n to $\hat{\mathbb{C}}$, and the rest to \mathbb{H}^3 , so that $\hat{\mathbb{C}} \setminus \text{Im}(\epsilon_{\hat{F}})$ is homeomorphic to \check{F} , and
- $\overline{\mathbb{H}^3} \setminus \text{Im}(\epsilon_{\hat{F}})$ is homeomorphic to $\check{F} \times (0, 1]$, and the product extends to $\hat{F} \times \{0\}$, inducing a homeomorphism from $\epsilon_{\hat{F}}(\check{F})$ to $\hat{F} \times \{0\}$.

Since F is embedded in both \tilde{S} and \hat{F} , each boundary component ℓ of F bounds a disk of $\tilde{\Delta}$ in \tilde{H} and also a disk in $H_{\hat{F}}$ that is a component of $\partial H_{\hat{F}} \setminus F$; we denote both disks by D_ℓ . Let F_ℓ be the component of $\tilde{S} \setminus F$ bounded by ℓ . Then $F_\ell \setminus \tilde{L}$ is a union of infinitely many components of $\tilde{S} \setminus \tilde{L}$. Since $H_F \subset H_{\hat{F}}$, we let H_ℓ be the component of $H_{\hat{F}} \setminus H_F$ bounded by D_ℓ . Then H_ℓ is (topologically) a 3-disk whose boundary sphere is the union of the disk D_ℓ and the component of $\hat{F} \setminus F$ bounded by ℓ , whose interior contains a unique puncture point $p(\ell) \in \{p_1, p_2, \dots, p_n\}$.

In $\overline{\mathbb{H}^3}$, $Conv(f(\ell)) \cong \overline{\mathbb{H}^2}$ is a boundary component of $Conv(F')$. Then, let X_ℓ be the component of $\overline{\mathbb{H}^3} \setminus Conv(F')$ bounded by $Conv(f(\ell))$. Let Y_ℓ be the component of $\hat{\mathbb{C}} \setminus F'$ bounded by $f(\ell)$, so that $X_\ell \cap \hat{\mathbb{C}} = Y_\ell$.

Proposition 6.6. *There is a sequence $(R_i)_{i=1}^\infty$ of distinct connected components of $F_\ell \setminus \tilde{L}$ such that*

- (i) ℓ is a boundary component of R_1 ,
- (ii) R_i and R_{i+1} are adjacent subsurfaces of F_ℓ for all $i = 1, 2, 3, \dots$,
- (iii) $\tilde{\epsilon}(R_i) \subset X_\ell$ for all $i = 1, 2, 3, \dots$, and
- (iv) $(\tilde{\epsilon}(R_i))_{i=1}^\infty$ converges to a limit point of Γ (contained in Y_ℓ). (See Figure 8.)

Proof. Let R_1 be the component of $F_\ell \setminus \tilde{L}$ bounded by $\ell =: \ell_0$ ((i)). Then $\tilde{C}|R_1$ is a good holed sphere supported on a unique component Ω_1 of $\Omega \setminus \tilde{L}'$ (via f). By Corollary 4.6, Ω_1 and F' are adjacent subsurfaces of Ω sharing the boundary component $f(\ell)$. Thus $\Omega_1 \subset Y_\ell$, and by Corollary 5.2, $\tilde{\epsilon}(R_1) \subset \text{Conv}(\Omega_1) \subset X_\ell$.

We inductively define R_i for $i \geq 2$. For $i \geq 1$, suppose that we have components R_1, R_2, \dots, R_i satisfying (ii) with R_1 as above. Then, let Ω_i be the component of $\Omega \setminus \tilde{L}'$ that is a support of $\tilde{C}|R_i$. Let $\ell_{i-1} (\subset \tilde{L})$ denote the common boundary component of R_{i-1} and R_i . Then, by the definition of an almost good holed sphere, $f(\partial R_i)$ is a union of at least 2 boundary components of Ω_i . Thus we can pick a boundary component ℓ_i of R_i so that $f(\ell_i)$ and $f(\ell_{i-1})$ are different boundary components of Ω_i . Let R_{i+1} be the component of $F_\ell \setminus \tilde{L}$ adjacent to R_i along the boundary component ℓ_i .

Lemma 6.7. *For each $k \geq 1$, $\Omega_1, \Omega_2, \dots, \Omega_k$ are distinct components of $Y_\ell \setminus \tilde{L}'$, and $cl(\sqcup_{i=1}^k \Omega_i)$ is a holed sphere in Y_ℓ bounded by finitely many loops of \tilde{L}' such that there is a 2-disk component of $Y_\ell \setminus cl(\sqcup_{i=1}^k \Omega_i)$ bounded by $f(\ell_k)$.*

Proof. (See Figure 7.) Clearly this claim holds for $k = 1$. Suppose that this claim holds for some $k \geq 1$. Let B_k be the 2-disk component of $Y_\ell \setminus cl(\sqcup_{i=1}^k \Omega_i)$ bounded by $f(\ell_k)$. From the construction of $(R_i)_{i=1}^\infty$, we see that Ω_k and Ω_{k+1} are adjacent along the loop $f(\ell_k)$. Then Ω_{k+1} is a sphere with $2g$ holes contained in B_k . Therefore $\Omega_1, \Omega_2, \dots, \Omega_k, \Omega_{k+1}$ are distinct components of $Y_\ell \setminus \tilde{L}'$, and $cl(\sqcup_{i=1}^{k+1} \Omega_i)$ is again a holed sphere in Y_ℓ bounded by finitely many loops of \tilde{L}' . Since $f(\ell_k)$ and $f(\ell_{k+1})$ are different boundary components of Ω_{k+1} , there is a component of $B_k \setminus \Omega_{k+1}$ bounded by $f(\ell_{k+1})$, which is a disk; then this component is also a component of $Y_\ell \setminus cl(\sqcup_{i=1}^{k+1} \Omega_i)$ bounded by $f(\ell_k)$. \square

By Lemma 6.7, $\Omega_i \subset Y_\ell$ for all $i \geq 1$. Therefore $\tilde{\epsilon}(R_i) \subset \text{Conv}(\Omega_i) \subset X_\ell$ ((iii)). Lemma 6.7 also implies that $(f(\ell_i))_{i=1}^\infty$ is a sequence of nested loops of \tilde{L}' contained in Y_ℓ , that is, $Y_\ell \setminus \sqcup_{i=1}^\infty f(\ell_i)$ is a union of disjoint cylinders bounded by $f(\ell_{i-1})$ and $f(\ell_i)$ with $i = 1, 2, \dots$

(see Figure 7) and a point to which $f(\ell_i)$ converges as $i \rightarrow \infty$. Since $\tilde{\epsilon}(R_i) \subset \text{Conv}(\Omega_i)$ and $f(\ell_{i-1})$ and $f(\ell_i)$ are boundary components of Ω_i , thus $\tilde{\epsilon}(R_i)$ converges to the same limit point as $i \rightarrow \infty$ ((iv)). \square

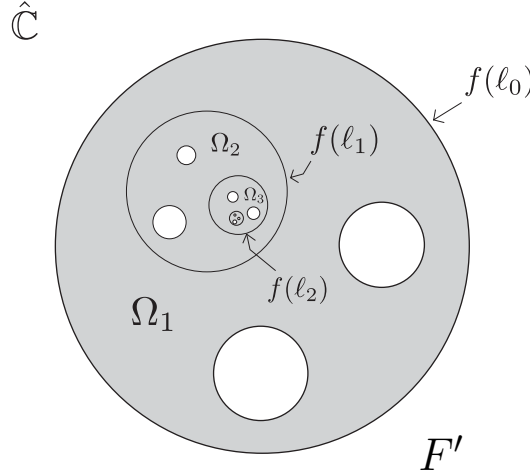


FIGURE 7. The shaded region is $\sqcup_{i=1}^k \Omega_i$.

Let G_ℓ be the union of $cl(R_i)$ with $i = 1, 2, \dots$, obtained by Proposition 6.6. Then G_ℓ is an unbounded connected subsurface of \tilde{S} contained in F_ℓ and bounded by infinitely many loops of \tilde{L} . Then ∂G_ℓ is a multiloop on \tilde{S} bounding infinitely many disks of $\tilde{\Delta}$ in \tilde{H} . Let H_{G_ℓ} be the closed (noncompact) subset of \tilde{H} bounded by these disks of $\tilde{\Delta}$, so that $G_\ell = H_{G_\ell} \cap \tilde{S}$. Then H_{G_ℓ} is homeomorphic to the closed 3-disk \mathbb{D}^3 minus a point in $\partial \mathbb{D}^3$, which corresponds to the limit point in Proposition 6.6 (iv). Thus we can naturally identify H_{G_ℓ} with H_ℓ minus the puncture point $p(\ell)$.

The set of the ends of \tilde{H} is homeomorphically identified with the limit set of Γ , which is a cantor set. Then, $\tilde{\epsilon}: \tilde{H} \rightarrow \overline{\mathbb{H}^3}$ continuously extends to the ends of \tilde{H} , realizing the homeomorphism, since $\tilde{\epsilon}$ is $\tilde{\rho}$ -equivariant. In particular, $\tilde{\epsilon}$ takes the limit of $(R_i)_i^\infty$ to the limit point of Γ in Proposition 6.6 (vi). Therefore, by identifying H_ℓ with the union of H_{G_ℓ} and its endpoint, the embedding $\epsilon_F: H_F \rightarrow \overline{\mathbb{H}^3}$ extends to an embedding of $H_F \cup H_\ell$ into $\overline{\mathbb{H}^3}$, taking $p(\ell)$ to the limit point.

By Proposition 6.6 (iii), for different boundary components ℓ of F , corresponding H_{G_ℓ} are contained in different components of $\overline{\mathbb{H}^3} \setminus \text{Conv}(F')$. Therefore, by applying such extension for all boundary components ℓ of F , we obtain an embedding $\epsilon_{\hat{F}}: H_{\hat{F}} \rightarrow \overline{\mathbb{H}^3}$. Then $\epsilon_{\hat{F}}$

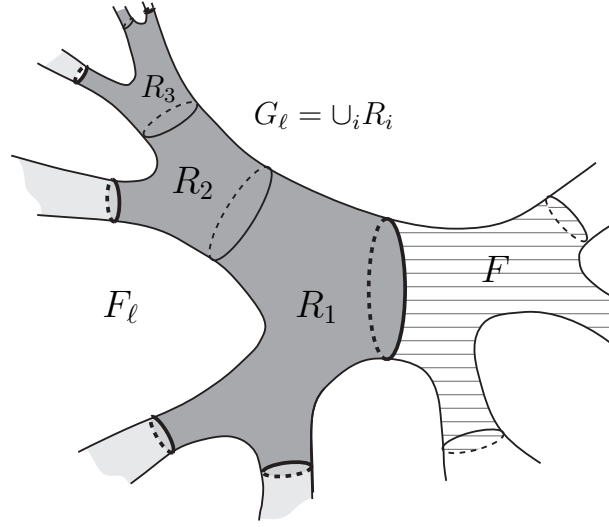


FIGURE 8.

takes $H_{\hat{F}} \setminus \{p_1, p_2, \dots, p_n\}$ to \mathbb{H}^3 and p_1, p_2, \dots, p_n to different limit points of Γ on $\hat{\mathbb{C}}$.

Next we show that $\overline{\mathbb{H}^3} \setminus \text{Im}(\epsilon_{\hat{F}})$ has a natural product structure. Let $\bar{F} = F \cup (\cup_\ell G_\ell)$, where the second union runs over all boundary components ℓ of F . Then \bar{F} is a connected subsurface of \tilde{S} bounded by infinitely many loops of \tilde{L} . By the identification of H_{G_ℓ} and $H_\ell \setminus p(\ell)$, the inclusion $F \subset \tilde{F}$ extends to the inclusion $\bar{F} \subset \tilde{F}$. Then $\tilde{F} \setminus \bar{F}$ is a union of infinitely many disjoint 2-disks bounded by boundary components of \bar{F} .

On the other hand, when \bar{F} is regarded as a subsurface of \tilde{S} , its boundary bounds infinitely many disks of $\tilde{\Delta}$ in \tilde{H} . For each boundary component m of \bar{F} , let D_m denote the disk of $\tilde{\Delta}$ bounded by m . (For the following discussion, see Figure 9.) Then $\tilde{\epsilon}(D_m)$ is a 2-disk and $\tilde{\eta}(m \times [0, 1])$ is an annulus embedded in $\overline{\mathbb{H}^3}$. Since $\tilde{\epsilon}(D_m)$ and $\tilde{\eta}(m \times [0, 1])$ share a boundary component, their union, denoted by E'_m , is a 2-disk properly embedded in $\overline{\mathbb{H}^3}$. Then the multidisk $\sqcup_m E'_m$ is properly embedded in $\overline{\mathbb{H}^3}$, where the union runs over all boundary components m of \bar{F} . We can see that $\sqcup E'_m$ bounds the union of $\tilde{\eta}(\bar{F} \times [0, 1])$ and $\text{Im}(\epsilon_{\hat{F}})$, which is homeomorphic to a closed 3-disk.

For each boundary component m of \bar{F} , let D'_m be the (disk) component of $\hat{\mathbb{C}} \setminus \tilde{\eta}(\bar{F} \times \{1\})$, bounded by $\tilde{\eta}(m \times \{1\})$. Then the 2-disks D'_m and E'_m has disjoint interior and share the boundary component $\tilde{\eta}(m \times \{1\})$. Then $D'_m \cup E'_m$ is a 2-sphere, and it bounds a 3-disk Q'_m in $\overline{\mathbb{H}^3}$. Then Q'_m is the component of $\overline{\mathbb{H}^3} \setminus (\tilde{\eta}(\bar{F} \times [0, 1]) \cup \text{Im}(\epsilon_{\hat{F}}))$ bounded

by E'_m . To give a natural product structure on Q'_m , choose a homeomorphism $\eta_m: D_m \times [0, 1] \rightarrow Q'_m$ such that $\eta_m(D_m \times \{0\}) = \tilde{\epsilon}(D_m)$, $\eta_m(D_m \times \{1\}) = D'_m$ and $\eta_m|_{\partial D_m \times [0,1]} = \tilde{\eta}|_{m \times [0,1]}$. Let $\eta_F: \bar{F} \times [0, 1] \rightarrow \overline{\mathbb{H}^3}$ denote the restriction of $\tilde{\eta}: \tilde{S} \times [0, 1] \rightarrow \overline{\mathbb{H}^3}$ to $\bar{F} \times [0, 1]$. Then, since $\tilde{F} \setminus \bar{F}$ is the union of the disks bounded by boundary components m of \bar{F} , the product structures given by $\eta_{\tilde{F}}$ and η_m match up along $\partial \bar{F} \times [0, 1]$ and they yield a homeomorphism

$$\eta_{\tilde{F}}: \tilde{F} \times [0, 1] \rightarrow \overline{\mathbb{H}^3} \setminus [\text{int}(\text{Im}(\epsilon_{\tilde{F}})) \cup (\sqcup_{i=1}^n \epsilon_{\tilde{F}}(p_i))]$$

such that $\eta_{\tilde{F}}(\tilde{F} \times \{0\}) = \epsilon_{\tilde{F}}(\tilde{F})$ and $\eta_{\tilde{F}}(\tilde{F} \times \{1\}) = \hat{\mathbb{C}} \setminus \sqcup_{i=1}^n \epsilon_{\tilde{F}}(p_i)$.

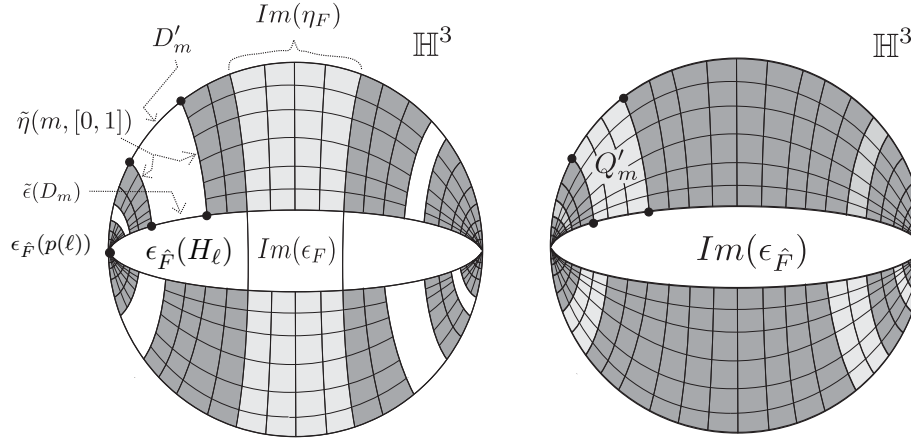


FIGURE 9. A schematic for the product structure given by $\eta_{\tilde{F}}$. On the left, the darkest region corresponds to $\tilde{\eta}(G_\ell \times [0, 1])$'s, and, On the right, to $\tilde{\eta}(\bar{F} \times [0, 1])$.

Next, since F is a subsurface of \tilde{F} , we extend the almost good structure $\tilde{C}|_F = (f|_F, \rho_{id})$ on the holed sphere F supported on F' to a good structure on the punctured sphere \tilde{F} . Each boundary component ℓ of F bounds a component of $\tilde{F} \setminus F$, which is a 2-disk with the puncture point $p(\ell)$. Accordingly $\epsilon_{\tilde{F}}(p(\ell))$ is contained in the component of $\hat{\mathbb{C}} \setminus F'$ bounded by the loop $f(\ell)$ that is also homeomorphic to a disk. Since $\epsilon_{\tilde{F}}$ takes different punctures points of \tilde{F} to different points on $\hat{\mathbb{C}}$. Therefore, as in §3.2, we can uniquely extend the almost good projective structure $\tilde{C}|_F$ on F to a good projective structure $C_{\tilde{F}} = (f_{\tilde{F}}, \rho_{id})$ on \tilde{F} such that $f_{\tilde{F}}(p(\ell)) = \epsilon_{\tilde{F}}(p(\ell))$ for all boundary components ℓ of F . Then $\text{Supp}(C_{\tilde{F}})$ is $\hat{\mathbb{C}} \setminus \sqcup_{i=1}^n \epsilon_{\tilde{F}}(p_i)$.

We have seen that, in order to prove Proposition 6.2, it suffices to show that $[f_{\tilde{F}}^{-1}(\mu')] = [f^{-1}(\mu')] \cap F$ is a single loop on F . If ℓ is a

boundary component of F , let R_ℓ be a component of $\check{F} \setminus F$ bounded by ℓ . Then R_ℓ covers, via $f_{\check{F}}$, a component of $\hat{\mathbb{C}} \setminus F'$ minus the image of the corresponding puncture point. Since μ' is contained in the interior of F' , we have $f_{\check{F}}^{-1}(\mu') \cap R_\ell = \emptyset$. Thus the proof of Proposition 6.2 is reduced to show:

Proposition 6.8. $[f_{\check{F}}^{-1}(\mu')]$ is a single loop on \check{F} .

Recall that λ and μ' are the loops on \check{F} and $\hat{\mathbb{C}}$, respectively, such that $\mu' = \tilde{\eta}(\lambda \times \{1\})$, which is equal to $\eta_{\check{F}}(\lambda \times \{1\})$. Also $\lambda' = f(\lambda)$ is the loop of \tilde{L} satisfying $\epsilon_{\check{F}}(\lambda) \subset \text{Conv}(\lambda') \cong \overline{\mathbb{H}^2}$. Let $D'_{\lambda'} = \text{Conv}(\lambda')$.

Lemma 6.9. $\epsilon_{\check{F}}^{-1}(D'_{\lambda'})$ is a multidisk properly embedded in $H_{\check{F}}$ bounded by the multiloop $[f_{\check{F}}^{-1}(\lambda')]$.

Proof. Recall that H_F and all H_ℓ have disjoint interiors, where ℓ varies over all boundary components of F and that $H_{\check{F}}$ is the union of H_F and all H_ℓ . By Lemma 6.3, λ' is contained in $\text{int}(F'_0)$ and thus in $\text{int}(F')$. Then $D'_{\lambda'}$ is contained in $\text{int}(\text{Conv}(F'))$. If ℓ is a boundary component of F , then X_ℓ is a component of $\overline{\mathbb{H}^3} \setminus \text{Conv}(F')$ and its closure contains $\epsilon_{\check{F}}(H_\ell)$, therefore $\epsilon_{\check{F}}^{-1}(D'_{\lambda'}) \cap H_\ell = \emptyset$. Hence $\epsilon_{\check{F}}^{-1}(D'_{\lambda'}) \subset H_F$.

For each boundary component ℓ of F , the once-punctured disk R_ℓ (bounded by ℓ) covers the once-punctured disk $Y_\ell \setminus \epsilon_{\check{F}}(p(\ell))$ in $\hat{\mathbb{C}}$ via $f_{\check{F}}$. Since $Y_\ell \cap \text{int}(F') = \emptyset$, thus $R_\ell \cap f_{\check{F}}^{-1}(\lambda') = \emptyset$. Therefore $[f_{\check{F}}^{-1}(\lambda')] \subset F$.

Therefore it suffices to show that $\epsilon_{\check{F}}^{-1}(D'_{\lambda'})$ is a multidisk properly embedded in H_F bounded by $[f_{\check{F}}^{-1}(\lambda')]$. By Proposition 5.1 (i), $\epsilon_{\check{F}}^{-1}(D'_{\lambda'})$ is a union of finitely many (disjoint) disks of $\tilde{\Delta} \cap H_F$. By Proposition 5.1 (iii), a disk D of $\tilde{\Delta} \cap H_F$ embeds into $D'_{\lambda'}$ if and only if $f_{\check{F}}(\partial D) = f(\partial D) = \lambda'$. This completes the proof. \square

For a surface Σ , we let P_Σ denote the set of all punctures of Σ . Let $L_{\lambda'} = [f_{\check{F}}^{-1}(\lambda')]$, which is a multiloop on \check{F} .

Lemma 6.10. Let X be a component of $\check{F} \setminus L_{\lambda'}$. Then $C_{\check{F}}|_X$ is an almost good genus-zero surface fully supported on a 2-disk with finitely many punctures $f_{\check{F}}(P_X)$, where the 2-disk is the component of $\hat{\mathbb{C}} \setminus \lambda'$ containing $f_{\check{F}}(P_X)$.

Proof. Since, by Lemma 6.9, $L_{\lambda'}$ is the boundary of the multidisk $\epsilon_{\check{F}}^{-1}(D'_{\lambda'})$ in $H_{\check{F}}$, $\epsilon_{\check{F}}: H_{\check{F}} \rightarrow \overline{\mathbb{H}^3}$ embeds X into a single component H of $\overline{\mathbb{H}^3} \setminus D'_{\lambda'}$. Then $H \cap \hat{\mathbb{C}}$ is a component of $\hat{\mathbb{C}} \setminus \lambda'$, which is a round 2-disk. If $p \in P_{\check{F}}$ (in particular if $p \in P_X$), then $f_{\check{F}}(p) = \epsilon_{\check{F}}(p)$. Since $\epsilon_{\check{F}}$ is an embedding, different points of P_X map to different points in the interior of $H \cap \hat{\mathbb{C}}$. In addition, all boundary components of X cover

λ' via $f_{\check{F}}$. Therefore $C_{\check{F}}|X$ is an almost good genus-zero surface fully supported on the punctured disk $(H \cap \hat{\mathbb{C}}) \setminus f_{\check{F}}(P_X)$. \square

Step 3. We have constructed a good projective structure $C_{\check{F}} = (f_{\check{F}}, \rho_{id})$ on a puncture sphere \check{F} ; an embedding $\epsilon_{\hat{F}}: H_{\hat{F}} \rightarrow \overline{\mathbb{H}^3}$, where the sphere \hat{F} that is the union of \check{F} with its punctures and $H_{\hat{F}}$ is a 3-disk bounded by \hat{F} ; an embedding $\eta_{\check{F}}: \check{F} \times [0, 1] \rightarrow \overline{\mathbb{H}^3}$ so that $\eta_{\check{F}}|(\check{F} \times \{0\})$ is the restriction of $\epsilon_{\hat{F}}$ to $\check{F} \subset \partial H_{\hat{F}}$ and $\eta|(\check{F} \times \{1\})$ is a homeomorphism onto the support of $C_{\check{F}}$.

Let $\phi: \overline{\mathbb{H}^3} \rightarrow \overline{\mathbb{H}^3}$ be a homeomorphism. Then we can transform $C_{\check{F}}, \epsilon_{\hat{F}}, \eta_{\check{F}}$ by postcomposing with ϕ , and, since ϕ is a homeomorphism, this transformation preserves essential topological properties of $C_{\check{F}}, \epsilon_{\hat{F}}, \eta_{\check{F}}$ and correspondences between them: Namely, we let

$$\begin{aligned} f_\phi &= \phi \circ f_{\check{F}}: \check{F} \rightarrow \hat{\mathbb{C}} \\ C_\phi &= (f_\phi, \rho_{id}) \\ \epsilon_\phi &= \phi \circ \epsilon_{\hat{F}}: H_{\hat{F}} \rightarrow \overline{\mathbb{H}^3} \\ \eta_\phi &= \phi \circ \eta_{\check{F}}: \check{F} \times [0, 1] \rightarrow \overline{\mathbb{H}^3}. \end{aligned}$$

Then

Proposition 6.11. *There exists a homeomorphism $\phi: \overline{\mathbb{H}^3} \rightarrow \overline{\mathbb{H}^3}$ such that*

- (i) $\lfloor f_\phi^{-1}(\lambda') \rfloor$ is a single loop isotopic to λ on \check{F} , and
- (ii) λ' is isotopic to $\epsilon_\phi(\lambda)$ in the product region $\text{Im}(\eta_\phi)$.

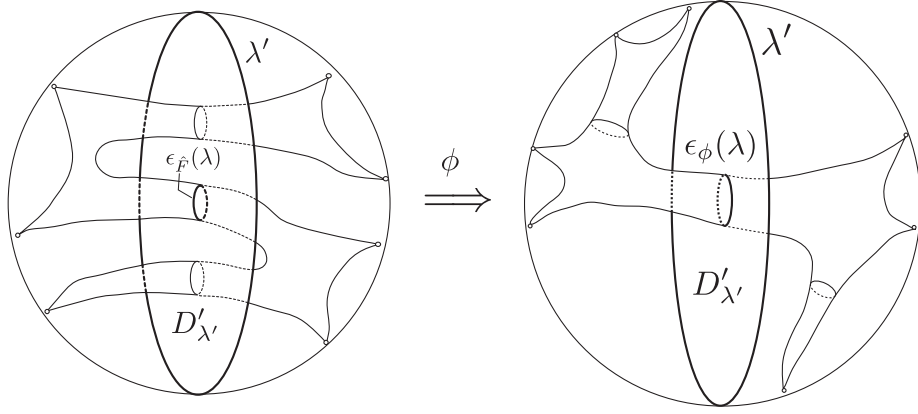
(An example of such a homeomorphism ϕ is illustrated in Figure 10.) First we show that it suffices to construct ϕ satisfying (i) and, instead of (ii),

- (II) $\epsilon_\phi^{-1}(D'_{\lambda'})$ is a multidisk properly embedded in $H_{\hat{F}} \setminus P_{\check{F}}$ and bounded by $\lfloor f_\phi^{-1}(\lambda') \rfloor$.

Assume that there is a homeomorphism $\phi: \overline{\mathbb{H}^3} \rightarrow \overline{\mathbb{H}^3}$ satisfying (i) and (II). Then, by (i), $\lfloor f_\phi^{-1}(\lambda') \rfloor$ is a loop on \check{F} isotopic to λ . Thus the loop $\epsilon_\phi(\lambda)$ is isotopic to the loop $\epsilon_\phi(\lfloor f_\phi^{-1}(\lambda') \rfloor)$ on the surface $\epsilon_\phi(\check{F})$. By (II), $D'_{\lambda'} \cap \text{Im}(\epsilon_\phi)$ is a single disk bounded by $\epsilon_\phi(\lfloor f_\phi^{-1}(\lambda') \rfloor)$. Therefore $\epsilon_\phi(\lfloor f_\phi^{-1}(\lambda') \rfloor)$ and λ' bound an annulus properly embedded in $\text{Im}(\eta_\phi)$; in particular, they are isotopic in $\text{Im}(\eta_\phi)$. Thus (ii) holds.

We will further reduce the proof of Proposition 6.11 to an induction. For this, suppose that there is a homeomorphism $\phi_1: \overline{\mathbb{H}^3} \rightarrow \overline{\mathbb{H}^3}$ satisfying:

- (I) $\lfloor f_{\phi_1}^{-1}(\lambda') \rfloor =: L_{\phi_1}$ is a multiloop on \check{F} containing a loop λ_{ϕ_1} isotopic

FIGURE 10. An example of ϕ in Proposition 6.11.

to λ ,

(II) $\epsilon_{\phi_1}^{-1}(D'_{\lambda'}) =: \Delta_{\phi_1}$ is a multidisk properly embedded in $H_{\hat{F}} \setminus P_{\hat{F}}$ and bounded by L_{ϕ_1} , and

(III) if X is a component of $\check{F} \setminus L_{\phi_1}$, then $C_{\phi_1}|_X$ is an almost good surface fully supported on the punctured disk $B_X \setminus f_{\phi_1}(P_X)$, where B_X is the component of $\hat{C} \setminus \lambda'$ containing $f_{\phi_1}(P_X)$.

In particular, if $\phi_1 = id$, then, (I) holds since $f(\lambda) = \lambda'$, (II) by Lemma 6.9, and (III) by Lemma 6.10.

A loop ℓ of a multiloop N on a surface Σ is *outermost* if ℓ is a separating loop and a component of $\Sigma \setminus \ell$ contains no loops of N ; this component is called an *outermost component*. Every loop of L_{ϕ_1} on \check{F} is separating since \check{F} is a planar surface.

Lemma 6.12. *Let ℓ be an outermost loop of L_{ϕ_1} on \check{F} . Then there is a homeomorphism $\phi_2: \mathbb{H}^3 \rightarrow \mathbb{H}^3$ satisfying the following properties:*

(I') $[f_{\phi_2}^{-1}(\lambda')] =: L_{\phi_2}$ is isotopic to $L_{\phi_1} \setminus \ell$ on \check{F} ,

(II') $\epsilon_{\phi_2}^{-1}(D'_{\lambda'}) =: \Delta_{\phi_2}$ is a multidisk properly embedded in $H_{\hat{F}} \setminus P_{\hat{F}}$ and bounded by L_{ϕ_2} , and

(III') if X is a component of $\check{F} \setminus L_{\phi_2}$, then $C_{\phi_2}|_X$ is an almost good surface fully supported on the punctured disk $B_X \setminus f_{\phi_2}(P_X)$, where B_X is the component of $\hat{C} \setminus \lambda'$ containing $f_{\phi_2}(P_X)$.

This lemma implies Proposition 6.11:

Proof of Proposition 6.11 with Lemma 6.12 assumed. Clearly Conclusions (I'), (II'), (III') on ϕ_2 correspond to Assumptions (I), (II), (III) on ϕ_1 , respectively. Therefore, starting from the base case that $\phi_1 = id$,

we repeatedly apply Lemma 6.12 and inductively reduce the number of the loops of L_{ϕ_1} . Since L_{ϕ_1} always contains at least two outer most loops, we can proceed out induction, preserving the loop isotopic to λ_{ϕ_1} . Thus, when there is exactly one loop isotopic to λ left, we obtain ϕ_2 satisfying (i), (II). Hence this ϕ_2 realizes Proposition 6.11 (as discussed above). \square

Proof (Lemma 6.12). First, we construct a homeomorphism $\psi : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ such that $\psi \circ \phi_1|_{\hat{\mathbb{C}}}$ is a homeomorphism from $\hat{\mathbb{C}}$ to itself satisfying (I') and (III'). Later, we extend ψ to a homeomorphism from $\overline{\mathbb{H}^3}$ to itself, such that $\psi \circ \phi_1 : \overline{\mathbb{H}^3} \rightarrow \overline{\mathbb{H}^3}$ satisfies (II') (with $\psi \circ \phi_1 = \phi_2$).

Let D_ℓ be the disk of Δ_{ϕ_1} bounded by ℓ . Let Q be the outermost component of $\check{F} \setminus L_{\phi_1}$ bounded by ℓ . Let H_Q be the closure of the component of $H_{\check{F}} \setminus \check{\Delta}$ bounded by Q . Then $H_Q \cap \check{F} = Q \cup \ell$, and, by (II), $\epsilon_{\phi_1}(H_Q)$ is contained in the closure of a component of $\overline{\mathbb{H}^3} \setminus D'_{\lambda'}$. Let R be the component of $\check{F} \setminus L_{\phi_1}$ adjacent to Q along ℓ . Then ∂R bounds a unique multidisk consisting of disks of Δ_{ϕ_1} . Similarly let H_R be the closure of the component of $H_{\check{F}} \setminus \Delta_{\phi_1}$ bounded by this multidisk; then $H_R \cap \check{F} = R$.

Since $\epsilon_{\phi_1} : H_{\check{F}} \rightarrow \overline{\mathbb{H}^3}$ is an embedding, we regard subsets of $H_{\check{F}}$ also as their images in $\overline{\mathbb{H}^3}$ under ϵ_{ϕ_1} . By this convention, the disk D_λ separating H_Q and H_R is contained in the disk $D'_{\lambda'}$, and the interiors of H_Q and H_R are contained in the different components of $\overline{\mathbb{H}^3} \setminus D'_{\lambda'}$. Our goal is to isotope $H_{\check{F}}$ in $\overline{\mathbb{H}^3}$ keeping the puncture of \check{F} on $\hat{\mathbb{C}}$ so that $H_Q \cup H_R$ moves to the single component of $\overline{\mathbb{H}^3} \setminus D'_{\lambda'}$ containing H_R while $H_{\check{F}} \setminus (H_Q \cup H_R)$ is fixed. Such an isotopy induces a desired homeomorphism $\psi : \overline{\mathbb{H}^3} \rightarrow \overline{\mathbb{H}^3}$.

Let $B(\lambda', +)$ and $B(\lambda', -)$ denote the components of $\hat{\mathbb{C}} \setminus \lambda'$. Since Q and R are adjacent, we can assume that $Supp(C_{\phi_1}|_Q) = B(\lambda', +) \setminus P_Q$ and $Supp(C_{\phi_1}|_R) = B(\lambda', -) \setminus P_R$. The full supports $Supp(C_{\phi_1}|_Q)$ and $Supp(C_{\phi_1}|_R)$ are punctured disks, and thus Q and R have at least one puncture by the definition of almost good projective structures. Let q_1, q_2, \dots, q_h be the punctures of Q . Let λ'' be a round circle in the punctured disk $B(\lambda', -) \setminus P_{\check{F}}$ parallel to $\lambda' = \partial B(\lambda', -)$ so that λ'' and λ' bounds a round annulus A in $B(\lambda', -) \setminus P_{\check{F}}$. Similarly, let $B(\lambda'', +)$ and $B(\lambda'', -)$ be the components of $\hat{\mathbb{C}} \setminus \lambda''$ so that $B(\lambda'', +) = B(\lambda', +) \cup A'$ and $B(\lambda'', -) = B(\lambda', -) \setminus A'$.

Let a_1, a_2, \dots, a_h be disjoint paths on the punctured disk $B(\lambda'', +) \setminus P_{\check{F}}$ such that a_i connects the point q_i to a point r_i in $int(A')$ for each $i \in \{1, 2, \dots, h\}$. We can in addition assume that each a_i transversally intersects λ' in a single point. Pick a 2-disk neighborhood U_i of

a_i in $B(\lambda'', +)$ such that U_i contains no points of $P_{\check{F}}$ except q_i and U_i intersects λ' in a single arc (Figure 11). In addition assume that U_1, U_2, \dots, U_h are disjoint. For each $i \in \{1, 2, \dots, h\}$, choose a homeomorphism $\psi_i: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ supported on U_i such that $\psi_i(q_i) = r_i$ (i.e. ψ_i is the identity map on $\hat{\mathbb{C}} \setminus U_i$). Let $\psi = \psi_1 \circ \psi_2 \circ \dots \circ \psi_h: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$. Then ψ is a homeomorphism supported on $\sqcup_{i=1}^h U_i =: U$. Then there is an isotopy $\xi_\psi: [0, 1] \times \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ of $\hat{\mathbb{C}}$ supported on U between ψ and the identity map.

The restriction of f_{ϕ_1} to $\check{F} \setminus f_{\phi_1}^{-1}(P_{\check{F}})$ is a covering map onto $\hat{\mathbb{C}} \setminus P_{\check{F}}$. Since A' is disjoint from $P_{\check{F}}$, via this covering map, A' lifts to finitely many copies A_1, A_2, \dots, A_J of A in \check{F} . Therefore, letting $L_{\phi_1}^- = [f_{\phi_1}^{-1}(\lambda'')]$, each A_j ($1 < j < J$) is bounded by a loop of L_{ϕ_1} and a loop of $L_{\phi_1}^-$. Then the components of $\check{F} \setminus L_{\phi_1}$ bijectively correspond to the components of $\check{F} \setminus L_{\phi_1}^-$ by adding or subtracting A_1, A_2, \dots, A_J . By Assumption (III), if X is a component of $\check{F} \setminus L_{\phi_1}$, then $C_{\phi_1}|X$ is an almost good surface fully supported on either $B(\lambda', +) \setminus P_X$ or $B(\lambda', -) \setminus P_X$. Accordingly, if X is a component of $\check{F} \setminus L_{\phi_1}^-$, then $C_{\phi_1}|X$ is a good genus-zero surface supported on either $B(\lambda'', +) \setminus P_X$ (Type One) or $B(\lambda'', -) \setminus P_X$ (Type Two). Since Q is outermost, Q has exactly one boundary component, which is ℓ . Thus there exists a unique $j \in \{1, 2, \dots, J\}$ such that A_j adjacent to Q along ℓ . Then $Q \cup A_j$ is a component of $\check{F} \setminus L_{\phi_1}^-$ of Type 1.

Let X be a component of $\check{F} \setminus L_{\phi_1}^-$ of Type Two. since λ' and $Supp(C_{\phi_1}|X)$ are disjoint, by Lemma 4.3, $L_{\phi_1} \cap X = \emptyset$. Since ψ is supposed on the complement of $Supp(C_{\phi_1}|X)$, we have $L_{\phi_2} \cap X = \emptyset$.

Next let X be a component of $\check{F} \setminus L_{\phi_1}^-$ of Type One. Then $f_{\check{F}}^{-1}(A') \cap X$ is a regular neighborhood of ∂X , and it is a union of some A_j 's. Thus $X \cap L_{\phi_1}$ is a multiloop on X isotopic to ∂X . Suppose that X does not contain Q . Since U is disjoint from $P_{\check{F}} \setminus P_Q$, then P_X is disjoint from U . Thus $Supp(C_{\phi_1}|X)$ contains U . Since the isotopy ξ_ψ of $\hat{\mathbb{C}}$ is supported on U , lifting ξ_ψ via $dev(C_{\phi_1}|X)$, we obtain an isotopy of $X \cap L_{\phi_1}$ to $X \cap L_{\phi_2}$ on X .

Last suppose that X contains Q . Then $X = Q \cup A_j$ for some $j \in \{1, 2, \dots, J\}$ and $X \cap L_{\phi_1} = \ell$. The homeomorphism ψ moves $P_X \subset B(\lambda', +)$ to $int(A') \subset B(\lambda', -)$. Then $C_{\phi_2}|X$ is a good surface fully supported on $B(\lambda'', +) \setminus \psi(P_X)$. Since the support $B(\lambda'', +) \setminus \psi(P_X)$ contains the disk $B(\lambda', +)$ bounded by λ' , therefore, by Lemma 4.3, $X \cap L_{\phi_2} = \emptyset$. We have seen that on all other components X of $\check{F} \setminus L_{\phi_1}^-$, the multiloop $L_{\phi_1} \cap X$ is isotopic to $L_{\phi_2} \cap X$. Hence L_{ϕ_2} is isotopic to $L_{\phi_1} \setminus \ell$ on \check{F} ((I')).

If Z is a component of $\check{F} \setminus L_{\phi_2}$, then each boundary component of Z covers λ' via $dev(C_{\phi_2}|Z) = f_{\phi_2}|Z$. In addition, under the isotopy of \check{F} moving L_{ϕ_2} to $L_{\phi_1} \setminus \ell$, the component Z is isotoped to either the union $Q \cup R$ or a component of $\check{F} \setminus L_{\phi_1}$ that is *not* Q or R . First suppose that Z is isotopic to $Q \cup R$. We have $\psi(P_Q) \subset A'$ and $\psi(P_R) = P_R \subset B(\lambda', -)$. Thus, since ψ is a homeomorphism, $dev(C_{\phi_2}|Z): Z \rightarrow \overline{\mathbb{H}^3}$ takes all punctures of Z to distinct points in $B(\lambda', -)$. Therefore, since all boundary components of Z cover λ' , thus $C_{\phi_2}|Z$ is an almost good surface whose support is $B(\lambda', -) \setminus \psi(P_Z)$. Next suppose that Z is isotopic to a component Y of $\check{F} \setminus L_{\phi_1}$. Then, since ψ fixes P_Z , obviously $\psi(P_Z)$ is contained in either $B(\lambda', +)$ or $B(\lambda', -)$. Therefore $C_{\phi_2}|Z$ is an almost good surface whose support is, accordingly, $B(\lambda', +) \setminus \psi(P_Z)$ or $B(\lambda', -) \setminus \psi(P_Z)$. (Moreover it is easy to see that $C_{\phi_2}|Z = C_{\phi_1}|Y$ since a projective structure is defined up isotopy of its base surface.) Thus (III') holds.

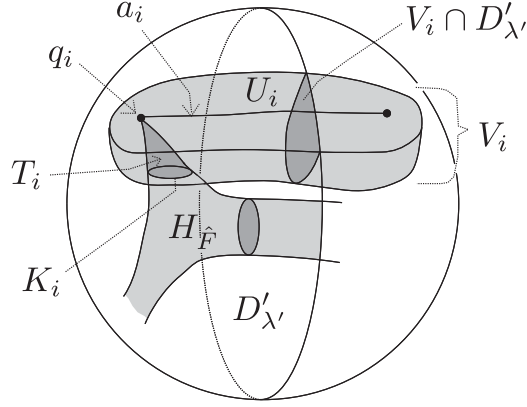
We now extend the homeomorphism $\psi: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ to a homeomorphism $\psi: \overline{\mathbb{H}^3} \rightarrow \overline{\mathbb{H}^3}$. Let $H'(\lambda', +) = Conv(B(\lambda', +))$ and $H'(\lambda', -) = Conv(B(\lambda', -))$, so that $\overline{\mathbb{H}^3} \setminus D'_{\lambda'} = H'(\lambda', +) \sqcup H'(\lambda', -)$. Similarly, let $H'(\lambda'', +) = Conv(B(\lambda'', +))$ and $H'(\lambda'', -) = Conv(B(\lambda'', -))$, so that $\overline{\mathbb{H}^3} \setminus Conv(\lambda'') = H'(\lambda'', +) \sqcup H'(\lambda'', -)$.

For each $i \in \{1, 2, \dots, h\}$, let V_i be a neighborhood of a_i in $H'(\lambda'', +)$ homeomorphic to a closed 3-disk that is a “natural extension” of U_i : Namely we require

- (i) $V_i \cap \hat{\mathbb{C}} = U_i$,
- (ii) $V_i \cap D'_{\lambda'}$ is a 2-disk,
- (iii) $\partial V_i \cap H_{\hat{F}}$ is contained in $H'(\lambda'', +)$, and it is the union of a closed 2-disk K_i in \mathbb{H}^3 and the point q_i in $\hat{\mathbb{C}}$,
- (iv) V_1, V_2, \dots, V_n are disjoint, and
- (v) $\partial V_i, D_{\lambda'}, \partial H_{\hat{F}}$ intersects transversally

(see Figure 11). By (i) and (ii), $\partial(V_i \cap D'_{\lambda'})$ is a circle that is the union of an arc properly embedded in $D'_{\lambda'}$ and the arc $U_i \cap \lambda'$. By (iii), $V_i \cap H_{\hat{F}}$ is a closed 3-disk T_i .

Then, for each $i = 1, 2, \dots, h$, we can extend the homeomorphism $\psi_i: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ supported on U_i to a homeomorphism $\sigma_i: \overline{\mathbb{H}^3} \rightarrow \overline{\mathbb{H}^3}$ supported on V_i so that the boundary of $\sigma_i(T_i)$ transversal intersects $D'_{\lambda'}$ in a single circle; let D_{T_i} be the 2-disk $\sigma_i(T_i) \cap D'_{\lambda'}$ bounded by the circle (see (i) and (ii) in Figure 12). Accordingly, $\sigma = \sigma_1 \circ \sigma_2 \circ \dots \circ \sigma_h$ has extended to a homeomorphism from $\overline{\mathbb{H}^3}$ to itself. In addition the isotopy ξ_σ of $\hat{\mathbb{C}}$ extends to an isotopy of $\overline{\mathbb{H}^3}$ supported on $V = \sqcup_{i=1}^h V_i$ between σ and the identity map.

FIGURE 11. A picture of V_i .

Since we have isotoped $H_{\hat{F}} (= \epsilon_{\phi_1}(H_{\hat{F}}))$ by ξ_σ , we now regard subsets of $H_{\hat{F}}$ as their images under the embedding $\sigma \circ \epsilon_{\phi_1}: H_{\hat{F}} \rightarrow \overline{\mathbb{H}^3}$. (In particular, the puncture q_i of \check{F} is regarded as the point $r_i \in \hat{\mathbb{C}}$.) Since σ is a homeomorphism, $\eta_\sigma := \sigma \circ \eta_{\check{F}}: \check{F} \times [0, 1] \rightarrow \sigma \circ \text{Im}(\eta_{\check{F}}) \subset \overline{\mathbb{H}^3}$ is a homeomorphism that takes $\check{F} \times \{0\}$ to $\check{F} (= \sigma \circ \epsilon_{\check{F}}(\check{F}))$ and $\check{F} \times \{1\}$ to $\hat{\mathbb{C}} \setminus P_{\check{F}} (= \hat{\mathbb{C}} \setminus \sigma \circ \epsilon_{\check{F}}(P_{\check{F}}))$.

Choose a puncture r of R . Let $\beta_1, \beta_2, \dots, \beta_h$ be disjoint paths on the punctured sphere $\check{F} (\subset \overline{\mathbb{H}^3})$ such that β_i connects $r_i (= q_i)$ to r for each $i = 1, 2, \dots, h$ (see Figure 12 (ii)). In addition, we can assume that β_i and T_j are disjoint if $i \neq j$ and that β_i transversally intersects ∂K_i and $D'_{\lambda'}$ minimally (i.e. it intersects ∂K_i in a single point and $D'_{\lambda'}$ in two points). Then, since L_{ϕ_1} is contained in $D'_{\lambda'}$, then β_i is contained in the connected subsurface $Q \cup R$ of \check{F} and it intersects ℓ in a single point.

The union of the product region $\text{Im}(\eta_\sigma)$ and the puncture points $P_{\check{F}}$ is naturally homeomorphic to the product $F_{\hat{F}} \times [0, 1]$ with the interval $[0, 1]$ above each puncture point collapsed to a single point. Then for each $i \in \{1, 2, \dots, h\}$, since β_i is a path on \check{F} connecting the puncture points r_i and r , then $\eta_\sigma(\beta_i \times [0, 1]) \cup \{r_i, r\} =: E_i$ is a 2-disk properly embedded in $\text{Im}(\eta_\sigma) \cup P_{\check{F}}$. Then the disks E_1, E_2, \dots, E_h share the point r , and $E_1 \setminus \{r\}, E_2 \setminus \{r\}, \dots, E_h \setminus \{r\}$ are disjoint. In addition, we can assume that each E_i transversally intersects $D'_{\lambda'}$, if necessarily, by a small isotopy of E_i in $\text{Im}(\eta_\sigma)$. Then there is a unique arc component e_i of $E_i \cap D'_{\lambda'}$ connecting the points of $\beta_i \cap D'_{\lambda'}$. Then e_i connects the disks D_{T_i} and D_ℓ . Take disjoint (small) regular neighborhoods $N(E_1), N(E_2), \dots, N(E_h)$ of $E_1 \setminus \{r, r_1\}, E_2 \setminus \{r, r_2\}, \dots, E_h \setminus \{r, r_h\}$, respectively, in $\text{Im}(\eta_\sigma)$, such that $N(E_i) \cap V_j = \emptyset$ for all $i, j \in \{1, 2, \dots, h\}$

with $i \neq j$. In addition, we can assume that $N(E_i) \cap D'_{\lambda'}$ is a regular neighborhood of the one-dimensional manifold $E_i \cap D'_{\lambda'}$ properly embedded in $D'_{\lambda'}$ and that $N(E_i) \cap \partial K_i$ is a single arc, which is a regular neighborhood of the single point $E_i \cap \partial K_i$ in the circle ∂K_i . In particular, there is a component of $N(E_i) \cap D'_{\lambda'}$ that is a regular neighborhood $N(e_i)$ of the arc e_i in $D'_{\lambda'}$. Then $N(e_i)$ is a rectangular strip connecting D_ℓ and D_{T_i} , and $N(e_i) \cup D_{T_i}$ is a (topological) 2-disk properly embedded in the 3-disk $T_i \cup N(E_i) \cup \{r, r_i\}$.

Take a small regular neighborhood of $N(e_i) \cup D_{T_i}$ in $T_i \cup N(E_i)$. Then, since $N(e_i) \cup D_{T_i}$ is contained in $D'_{\lambda'}$, we parametrize this regular neighborhood as $(N(e_i) \cup D_{T_i}) \times [-1, 1]$ so that $(N(e_i) \cup D_{T_i}) \times [-1, 0] \subset H'(\lambda', -)$ and $(N(e_i) \cup D_{T_i}) \times (0, 1] \subset H'(\lambda', +)$. Then $N(e_i) \times \{-1\}$ is a 2-disk properly embedded in the 3-disk $N(E_i)$, splitting $N(E_i)$ into two 3-disks. Let $N^+(E_i)$ denote the one of these two 3-disks containing $N(e_i) \times [-1, 1]$ (see Figure 12 (ii)). Then $\text{int}(N^+(E_i))$ is disjoint from $\text{int}(H_{\hat{F}})$, and the intersection of $\partial N^+(E_i)$ and $\partial H_{\hat{F}}$ is a 2-disk contained in the subsurface $Q \cup R$ of \tilde{F} . Therefore we can isotope $H_{\hat{F}}$ to $H_{\hat{F}} \cup N^+(E_i)$ in $\overline{\mathbb{H}^3}$, fixing $H_{\hat{F}} \setminus (H_Q \cup H_R)$ and $\partial \mathbb{H}^3$. Accordingly we now regard subsets of $H_{\hat{F}}$ as their images in $\overline{\mathbb{H}^3}$ under the composition of $\sigma \circ \epsilon_{\hat{F}}: H_{\hat{F}} \rightarrow \overline{\mathbb{H}^3}$ and this isotopy.

Since $N^+(E_i)$ and T_i have disjoint interiors and their boundaries intersect in a 2-disk, therefore $N^+(E_i) \cup T_i$ is a 3-disk. Then $(N(e_i) \cup D_{T_i}) \times \{-\frac{1}{2}\}$ is a 2-disk properly embedded in the 3-disk $N^+(E_i) \cup T_i$, and therefore it separates $N^+(E_i) \cup T_i$ into two closed 3-disks. Let D_i^+ and D_i^- denote these two 3-disks so that D_i^+ contains $(N(e_i) \cup D_{T_i}) \times [-\frac{1}{2}, 1]$ and D_i^- contains $(N(e_i) \cup D_{T_i}) \times [-1, -\frac{1}{2}]$ (see Figure 12 (iii)). Then D_i^+ is contained in \mathbb{H}^3 . On the other hand D_i^- is contained in $H'(\lambda', -)$ and it intersects $\hat{\mathbb{C}}$ at the point r_i . Then D_i^+ and $cl(H_{\hat{F}} \setminus D_i^+)$ are 3-disks with disjoint interiors, and their boundaries share a single 2-disk. Thus, similarly, there is an isotopy of $\overline{\mathbb{H}^3}$ supported on a small neighborhood of D_i^+ that moves $H_{\hat{F}}$ to $cl(H_{\hat{F}} \setminus D_i^+)$; in particular it fixes $H_{\hat{F}} \setminus (H_Q \cup H_R)$ and $\hat{\mathbb{C}}$ (see Figure 12 (iv)).

For each $i \in \{1, 2, \dots, h\}$, modify $\sigma_i: \mathbb{H}^3 \rightarrow \mathbb{H}^3$ by postcomposing it with be the homeomorphism induced by the composition of the isotopies of $\overline{\mathbb{H}^3}$ applied after σ_i (as in Figure 12), which transforms the initial image $\epsilon_{\phi_1}(H_{\hat{F}})$ to the 3-disk $cl(H_{\hat{F}} \setminus D_i^+)$ in the preceding paragraph. Then we still have $\sigma_i|_{\hat{\mathbb{C}}} = \psi_i$. To compare the difference between them, below we regard subsets of $H_{\hat{F}}$ as subsets of $\epsilon_{\hat{F}}(H_{\hat{F}})$, returning to the initial identification. Therefore H_Q of $H_{\hat{F}}$ was transformed to $H_Q \setminus T_i$ and H_R to $H_R \cup D_i^-$; the rest of $H_{\hat{F}}$ reminded the

same. Thus, topologically, σ_i has just moved the puncture point q_i on $\hat{\mathbb{C}}$ across the loop λ' and accordingly q_i has moved from on the boundary of H_Q to to the boundary of H_R across ℓ , while σ_i fixes $H_{\hat{F}} \setminus (H_Q \cup H_R)$.

For different $i, j \in \{1, 2, \dots, h\}$, since V_i and $N(E_j)$ are disjoint, thus T_i and $N(E_j) \setminus r$ are disjoint; then the homeomorphisms $\sigma_i: \overline{\mathbb{H}^3} \rightarrow \overline{\mathbb{H}^3}$ ($i \in \{1, 2, \dots, h\}$) have disjoint support. Then let $\sigma: \overline{\mathbb{H}^3} \rightarrow \overline{\mathbb{H}^3}$ be their composition $\sigma_1 \circ \sigma_2 \circ \dots \circ \sigma_h$, which is obviously homeomorphism. Then σ transforms H_Q to $H_Q \setminus (T_1 \cup T_2 \cup \dots \cup T_h)$, which contains no puncture points and H_R to $H_R \cup (D_1^- \cup D_2^- \cup \dots \cup D_h^-)$, which contains $P_Q \cup P_R$. Then $H_Q \setminus (T_1 \cup T_2 \cup \dots \cup T_n)$ is topologically a 3-disk in $H'(\lambda', +) \cap \mathbb{H}^3$ and its boundary intersects $D'_{\lambda'}$ in a single 2-disk as the boundary of H_Q does. Therefore, there is an isotopy of $\overline{\mathbb{H}^3}$ supported in small neighborhood of the 3-disk $H_Q \setminus (T_1 \cup T_2 \cup \dots \cup T_n)$ that moves its neighborhood $H_Q \setminus (T_1 \cup T_2 \cup \dots \cup T_n)$ into $H'(\lambda', -)$. By this isotopy the entire union $H_R \cup H_Q$ has moved into $H'(\lambda', -)$ while its complement $H_{\hat{F}} \setminus (H_R \cup H_Q)$ remains fixed. Modify the homeomorphism $\sigma: \overline{\mathbb{H}^3} \rightarrow \overline{\mathbb{H}^3}$ by postcomposing with this isotopy. Thus we have $(\sigma \circ \epsilon_{\phi_1})^{-1}(D'_{\lambda'})$ is $\Delta_{\phi_1} \setminus D_\ell$.

Combining with (I'), since $\Delta_{\phi_1} \setminus D_\ell$ in $H_{\hat{F}}$ is bounded by $L_{\phi_1} \setminus \ell$ in $H_{\hat{F}}$, thus the multiloop $L_{\phi_1} \setminus \ell$ is isotopic to $L_{\psi \circ \phi_1}$ on \check{F} . There is an isotopy of $\overline{\mathbb{H}^3}$ that fixes $\hat{\mathbb{C}}$, preserves $\sigma(H_{\hat{F}})$, in particular the surface $\sigma(\check{F}) \subset \mathbb{H}^3$, and moves $\sigma(L_{\phi_1} \setminus \ell)$ into $D'_{\lambda'}$. Finally extend $\psi: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ to the homeomorphism of \mathbb{H}^3 obtained by post-composing σ with this isotopy. Then we have $\partial \Delta_{\psi \circ \phi_1} = L_{\psi \circ \phi_1}$ ((II')). \square

Proof (Proposition 6.8). Let $\phi: \overline{\mathbb{H}^3} \rightarrow \overline{\mathbb{H}^3}$ be the homeomorphism obtained by Proposition 6.11. By Proposition 6.11 (ii), there is an isotopy between λ' and $\epsilon_\phi(\lambda)$ in $\text{Im}(\eta_\phi)$. By the product structure of η_ϕ , there is also an isotopy between the loops $\eta_\phi(\lambda \times \{0\}) = \epsilon_\phi(\lambda)$ and $\eta_\phi(\lambda \times \{1\})$ in $\text{Im}(\eta_\phi)$. Then there is an isotopy between the loops λ' and $\eta_\phi(\lambda \times \{1\})$ in $\text{Im}(\eta_\phi)$. By postcomposing with ϕ^{-1} , we have an isotopy between the loops $\phi^{-1}(\lambda')$ and $\phi^{-1} \circ \eta_\phi(\lambda \times \{1\}) = \eta_{\check{F}}(\lambda \times \{1\}) = \mu'$ in $\phi^{-1}(\text{Im}(\eta_\phi)) = \text{Im}(\eta_{\check{F}})$. Those loops $\phi^{-1}(\lambda')$ and μ' are on the punctured sphere $\hat{\mathbb{C}} \setminus \epsilon_{\check{F}}(P_{\check{F}}) = \eta_{\check{F}}(\check{F} \times \{1\})$. By the obvious projection from $\text{Im}(\eta_{\check{F}}) \cong \check{F} \times [0, 1]$ to $\check{F} \times \{1\}$, the isotopy between $\phi^{-1}(\lambda')$ and μ' in $\text{Im}(\eta_{\check{F}})$ induces to a homotopy between $\phi^{-1}(\lambda')$ and μ' on $\hat{\mathbb{C}} \setminus \epsilon_{\check{F}}(P_{\check{F}})$; Thus there is moreover an isotopy between them. Thus, via $f_{\check{F}}$, this isotopy between $\phi^{-1}(\lambda')$ and μ' lifts to an isotopy between the multiloops $f_{\check{F}}^{-1}(\phi^{-1}(\lambda')) = f_\phi^{-1}(\lambda')$ and $f_{\check{F}}^{-1}(\mu')$ on \check{F} . In particular, their essential parts $\lfloor f_\phi^{-1}(\lambda') \rfloor$ and $\lfloor f_{\check{F}}^{-1}(\mu') \rfloor$ are also isotopic on \check{F} . By

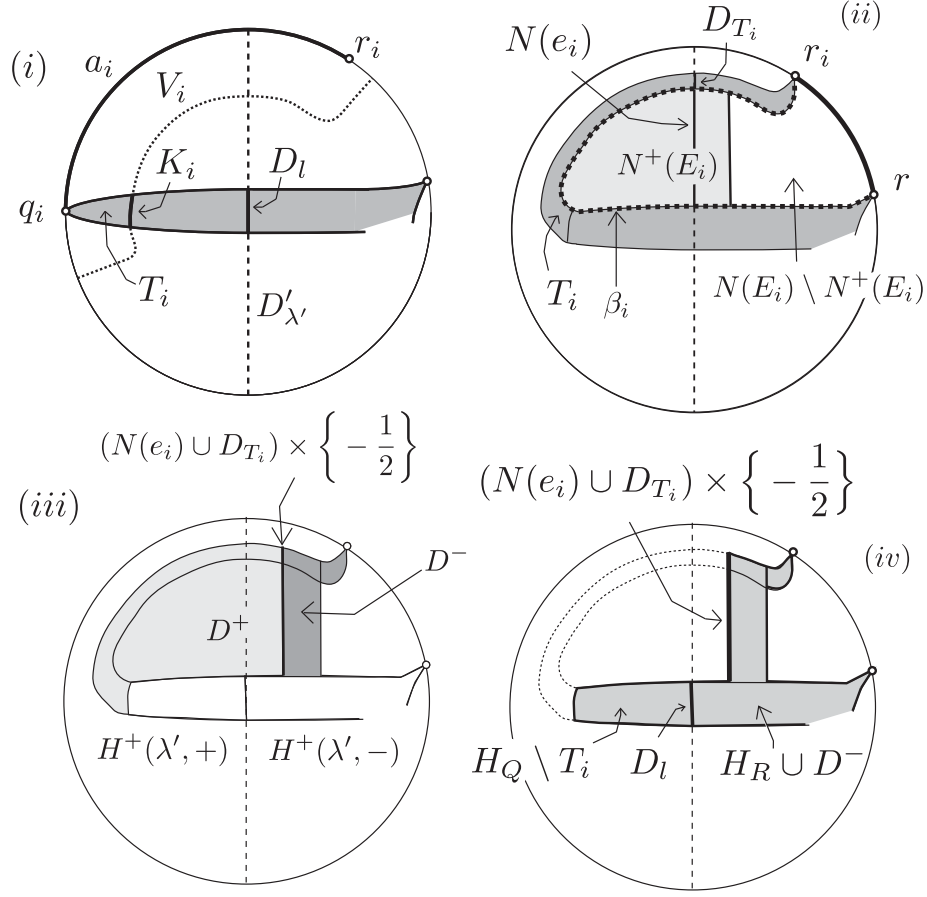


FIGURE 12. The series of isotopes of $H_{\hat{F}}$ in $\overline{\mathbb{H}^3}$.

Proposition 6.11 (i), there is an isotopy between $[f_{\phi}^{-1}(\lambda)]$ and the loop λ on \check{F} . Hence $[f_{\check{F}}^{-1}(\mu')]$ is a single loop. \square

7. A CHARACTERIZATION OF GOOD STRUCTURES BY GRAFTING

Let F be a sphere \hat{F} with n punctures p_1, p_2, \dots, p_n . Let $C = (f, \rho_{id})$ be a good projective structure on F , where $\rho_{id}: \pi_1(F) \rightarrow \text{PSL}(2, \mathbb{C})$ is the trivial representation. Then the developing map $f: F \rightarrow \hat{\mathbb{C}}$ continuously extends to a branched covering map $f: \hat{F} \rightarrow \hat{\mathbb{C}}$. Since C is a good structure, $f(p_1) =: q_1, f(p_2) =: q_2, \dots, f(p_n) =: q_n$ are distinct points on $\hat{\mathbb{C}}$, and $\text{Supp}(C)$ is the n -punctured sphere $\hat{\mathbb{C}} \setminus \{q_1, q_2, \dots, q_n\}$. Let $f_0: F \rightarrow \text{Supp}(C)$ be a homeomorphism such that (its extension

satisfies) $f_0(p_i) = q_i$ for all $i \in \{1, 2, \dots, n\}$. Then the pair (f_0, ρ_{id}) represents an embedded projective structure on F associated with C . On the other hand, every embedded projective structure on F associated with C can be obtained in such a way. We prove

Proposition 7.1. *Every good projective structure $C = (f, \rho_{id})$ on a punctured sphere F can be obtained by grafting a basic structure associated with C along a multiarc (each arc of which connects different punctures of F).*

For each $i \in \{1, 2, \dots, n\}$, let d_i be the ramification index of f at the ramified point p_i . If $d_i > 1$, then p_i is called a *proper ramification point*; if $d_i = 1$, then p_i is called a *trivial ramification point*, i.e. f is a local homeomorphism at p_i . In the latter case, we may regard p_i and q_i merely as marked points. Let d be the degree of f , i.e. the cardinality of $f^{-1}(x)$ for $x \in \text{Supp}(C)$. Let $\delta = d - 1$ and $\delta_i = d_i - 1$ for each $i \in \{1, 2, \dots, n\}$. Clearly we have $\delta \geq \delta_i (\geq 0)$ and, by the Riemann-Hurwitz formula, $2\delta = \sum_{i=1}^n \delta_i (\in 2\mathbb{N})$. Therefore we have

$$(1) \quad 2 \max_{1 \leq i \leq n} \delta_i \leq \sum_{i=1}^n \delta_i.$$

Lemma 7.2. *There is a multiarc A on F such that*

- (i) *each arc of A connects distinct punctures of F , and*
- (ii) *for each $i \in \{1, 2, \dots, n\}$, there are exactly δ_i arcs of A ending at p_i .*

The following claim implies Lemma 7.2 (c.f. [11]):

Claim 7.3. *Let X be an n -gon, and let e_1, e_2, \dots, e_n denote its edges in a cyclic order. For each edge e_i , choose δ_i distinct marked points in its interior. Then, there exists a multiarc A' properly embedded in X such that each arc of A' connects marked points on different edges of X and each marked point is an end of exactly one arc of A' .*

Proof of Lemma 7.2 with Claim 7.3 assumed. (See Figure 13.) For each $i \in \{1, 2, \dots, n\}$, let e'_i be the closed arc contained in the interior of the edge e_i so that e'_i contains all marked points on e_i . Consider the quotient space X/\sim obtained by collapsing each e'_i to a single point r_i so that the end points of the multiarc A' on each e_i are identified. Embed $X/\sim (\cong \mathbb{D}^2)$ into the sphere \hat{F} so that r_i maps to p_i for each $i \in \{1, 2, \dots, n\}$. Via this embedding, A'/\sim realizes the desired multiarc A on F . \square

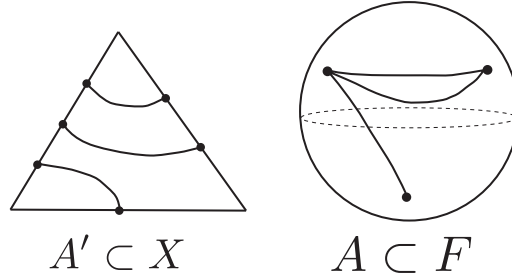


FIGURE 13. The multiarcs A and A' for $(\delta_1, \delta_2, \delta_3) = (1, 2, 3)$.

Proof of Claim 7.2. We prove this claim by induction on the even non-negative integer $\Sigma\delta_i$. Suppose that the lemma holds if $(\delta_1, \delta_2, \dots, \delta_n)$ satisfies $\Sigma\delta_i = 2(k - 1)$ for some $k \in \mathbb{Z}_{>0}$. If $(\delta_1, \delta_2, \dots, \delta_n)$ satisfies $\Sigma_{i=1}^n \delta_i = 2k$, then without loss of generality, we can in addition assume that $\delta_1 = \max_{1 \leq i \leq n} \delta_i$. Let m be the minimal integer in $\{2, 3, \dots, n\}$ with $\delta_m \neq 0$. Then there are adjacent marked points on e_1 and e_m (along the circle ∂X). Thus pick an arc α properly embedded in X connecting these marked points; then that a component of $X \setminus \alpha$ contains no marked points. Therefore it suffices to find a multiarc for the reduced n -tuple

$$(\delta_1 - 1, 0, 0, \dots, 0, \delta_m - 1, \delta_{m+1}, \delta_{m+2}, \dots, \delta_n).$$

For this reduces n -tuple, we have

$$(\delta_1 - 1) + 0 + \dots + 0 + (\delta_m - 1) + \delta_{m+1} + \delta_{m+2} + \dots + \delta_n = 2(k - 1).$$

In addition, using Inequality (1) and also from the way the inequality is obtained, we see that

$$2 \max\{\delta_1 - 1, 0, 0, \dots, 0, \delta_m - 1, \delta_{m+1}, \delta_{m+2}, \dots, \delta_n\} \leq 2(k - 1)$$

(if more than one $i \in \{1, 2, \dots, n\}$ realizes $\max_{1 \leq i \leq n} \delta_i$, then consider if there are more than two non-zero δ_i or not). Therefore, by the induction hypothesis, there is a multiarc A on X for the reduces n -tuple. Then we can in addition assume that A is contained in $X \setminus \alpha$. Then $\alpha \sqcup A$ is indeed the desired multiarc on X for the original n -tuple $(\delta_1, \delta_2, \dots, \delta_n)$. \square

Proposition 7.4. *Let $f_1, f_2: \hat{F} \rightarrow \hat{\mathbb{C}}$ be branched covering maps, such that, for each $i \in \{1, 2, \dots, n\}$, p_i is a ramification point of both f_1 and f_2 over q_i with the ramification index d_i and that f_1 and f_2 have no other proper ramification points. Then f_1 and f_2 are topologically equivalent, i.e. there are homeomorphisms $\phi: \hat{F} \rightarrow \hat{F}$ and $\phi': \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ such that*

- (i) $\phi(p_i) = p_i$ and $\phi'(q_i) = q_i$ for all $i \in \{1, 2, \dots, n\}$ and
(ii) $\phi' \circ f_1 = f_2 \circ \phi$.

Proof. Without loss of generality, we can assume that $d_1, d_2, \dots, d_k > 1$ and $d_{k+1} = d_{k+2} = \dots = d_n = 1$ for some integer $k \in \{1, 2, \dots, n\}$. For each $j = 1, 2$, let \mathcal{C}_j be the complex structure on $\hat{F} \cong \mathbb{S}^2$ obtained by pulling back the complex structure on $\hat{\mathbb{C}}$ via f_j . Then $f_j: (\hat{F}, \mathcal{C}_j) \rightarrow \hat{\mathbb{C}}$ is a meromorphic function. By the uniformization theorem, f_j is conformally equivalent to a rational function, i.e. there exist a rational function $\tau_j: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ and a conformal map $\psi_j: (\hat{F}, \mathcal{C}_j) \rightarrow \hat{\mathbb{C}}$ such that $f_j = \tau_j \circ \psi_j$. Then, for each $i \in \{1, 2, \dots, k\}$, $\psi_j(p_i)$ is the ramification point of τ_j over q_i with the ramification index d_i , and for each $i \in \{k+1, k+2, \dots, n\}$, $\psi_j(p_i)$ is the trivial ramification point of τ_i .

By Theorem 3.2, there is a path τ_t ($t \in [1, 2]$) connecting τ_1 to τ_2 in the space $\mathcal{R}(d_1, d_2, \dots, d_k)$ of rational functions defined in §3.4. Along τ_t , the (proper) ramification points of $\tau_t: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ continuously move on the source sphere $\hat{\mathbb{C}}$ without hitting each other; similarly the branched points of τ_t continuously move on the target sphere $\hat{\mathbb{C}}$ without hitting each other. Thus, for each $i \in \{1, 2, \dots, k\}$, there is a unique closed curve $Q_i(t)$ ($t \in [1, 2]$) on the target sphere such that $Q_i(1) = Q_i(2) = q_i$ and $Q_i(t)$ is a branched point of τ_t for all $t \in [1, 2]$. Then $Q_1(t), Q_2(t), \dots, Q_n(t)$ are the branched points of τ_t for all $t \in [1, 2]$. Similarly, for each $i \in \{1, 2, \dots, k\}$, we have a unique (not necessarily closed) curve $P_i(t)$ ($t \in [1, 2]$) on the source sphere, such that $P_i(t)$ is the ramification point of τ_t over $Q_i(t)$ with the ramification index d_i for each $t \in [1, 2]$. Then $P_1(t), P_2(t), \dots, P_k(t)$ are the branched points of τ_t for each $t \in [1, 2]$. In particular $P_i(1) = \psi_1(p_i)$ and $P_i(2) = \psi_2(p_i)$ for each $i \in \{1, 2, \dots, k\}$.

For each $i \in \{k+1, k+2, \dots, n\}$, pick a path $P_i(t)$ on the source sphere $\hat{\mathbb{C}}$ connecting $\psi_1(p_i)$ to $\psi_2(p_i)$ such that, for each $t \in [1, 2]$, $P_1(t), P_2(t), \dots, P_n(t)$ are different points on the source sphere. For each $i \in \{k+1, k+2, \dots, n\}$, let $Q_i(t) = \tau_t(P_i(t))$ ($t \in [1, 2]$), a path on the target sphere $\hat{\mathbb{C}}$. Then $Q_i(t)$ is a closed path based at q_i . We can in addition assume that $Q_1(t), Q_2(t), \dots, Q_n(t)$ are different points on $\hat{\mathbb{C}}$, by perturbing the paths $P_i(t)$ with $k+1 \leq i \leq n$ if necessarily.

Thus the paths $Q_i(t)$ obviously induce an isotopy between the branched points of τ_1 and of τ_2 on $\hat{\mathbb{C}}$. This isotopy of the branched points extends to an isotopy of the target sphere, denoted by $\xi'_t: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ ($t \in [1, 2]$). Since $\tau_t: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ is continuous in t , the isotopy ξ'_t on the target sphere uniquely lifts to an isotopy of the source sphere, $\xi_t: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ ($t \in [1, 2]$), via τ_t . Then $\tau_t \circ \xi_t = \xi'_t \circ \tau_1$ for all $t \in [1, 2]$, and $\xi_t(p_i) = P_i(t)$ for each

$i \in \{1, 2, \dots, n\}$. In particular $\tau_2 \circ \xi_2 = \xi'_2 \circ \tau_1$. Therefore we have

$$f_2 \circ (\psi_2^{-1} \circ \xi_2 \circ \psi_1) = \tau_2 \circ \xi_2 \circ \psi_1 = \xi'_2 \circ \tau_1 \circ \psi_1 = \xi'_2 \circ f_1.$$

Here ξ'_2 is a homeomorphism of $\hat{\mathbb{C}}$ fixing q_i , and $\psi_2^{-1} \circ \xi_2 \circ \psi_1: \hat{F} \rightarrow \hat{F}$ is a homeomorphism fixing p_i for all $i \in \{1, 2, \dots, n\}$. Thus f_1 and f_2 are topologically equivalent. \square

Proof (Proposition 7.1). Let $C_0 = (f_0, \rho_{id})$ be an embedded projective structure on F associated with C . Let A be the multiarc on F obtained by Lemma 7.2. By Lemma 7.2 (i), we can graft C_0 along A ; set $C_1 = (f_1, \phi_{id})$ to denote $Gr_A(C_0)$. Then, for all $i \in \{1, 2, \dots, n\}$, we have $f(p_i) = f_1(p_i) = q_i$ and, by Lemma 7.2 (ii), the ramification index of f and f_1 are both d_i at p_i . Therefore, by Proposition 7.4, there are homeomorphisms $\phi: \hat{F} \rightarrow \hat{F}$ fixing all p_i and a homeomorphism $\phi': \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ fixing all q_i , such that $\phi' \circ f = f_1 \circ \phi$. Therefore $f = \phi'^{-1} \circ f_1 \circ \phi$. For a homeomorphism $e: F \rightarrow Supp(C)$ and an appropriate multiarc N on F , let $Gr_N(\psi)$ denote the developing map of $Gr_N((e, \rho_{id}))$, where (e, ρ_{id}) represents an embedded projective structure on F . In particular $f_1 = Gr_A(f_0)$. Then

$$\begin{aligned} f &= \phi'^{-1} \circ Gr_A(f_0) \circ \phi = \phi'^{-1} \circ Gr_A(f_0 \circ \phi) \\ &= Gr_{\phi'^{-1}(A)}(\phi'^{-1} \circ f_0 \circ \phi). \end{aligned}$$

Thus $(\phi'^{-1} \circ f_0 \circ \phi, \rho_{id})$ is an embedded projective structure on F , and $C = (f, \rho_{id})$ is obtained by grafting this embedded projective structure along the multiarc $\phi'^{-1}(A)$. \square

As an immediate corollary of Proposition 7.1, we obtain:

Proposition 7.5. *Let C be a good projective structure on a holed sphere F . Then C can be obtained by grafting an embedded structure associated with C along a multiarc (each arc of which connects different boundary components of F).*

8. THE PROOF OF THE MAIN THEOREM

Recall that S is a closed orientable surface of genus g , Γ is a real Schottky group of rank $g > 1$, and $\rho: \pi_1(S) \rightarrow \Gamma \subset \text{PSL}(2, \mathbb{C})$ is an epimorphism.

Theorem 8.1. *Every Schottky structure $C = (f, \rho)$ on S can be obtained by grafting a uniformizable Schottky structure with holonomy ρ (along a multiloop on S).*

Remark: By Lemma 3.1, a uniformizable projective structure with holonomy ρ is Ω/Γ with some marking.

Proof. In §6, we constructed the multiloops M and M' on (S, C) and Ω/Γ , respectively; the multiloops \tilde{M} and \tilde{M}' are the total lifts of M and M' to \tilde{S} and Ω , respectively. Then M decomposes (S, C) into projective structures $(F_i, C_i)_{i=1}^n$, where F_i are components of $S \setminus M$ and $C_i = C|_{F_i}$. Then, for each $i \in \{1, 2, \dots, n\}$, by Theorem 6.1, (F_i, C_i) is a good holed sphere fully supported on a component of $\Omega \setminus \tilde{M}'$ that corresponds to C_i via the developing map f . By Proposition 7.5, $C_i = Gr_{A_i}(C_{0,i})$, where $C_{0,i}$ is an embedded structure on the holed sphere F_i with $Supp(C_{0,i}) = Supp(C_i)$ and A_i is a multiarc on F_i such that each arc of A_i connects distinct boundary components of F_i . For each loop ℓ of M , every lift $\tilde{\ell}$ of ℓ to \tilde{S} is a loop covering a loop of \tilde{M}' via f . Let d_ℓ be the degree of this covering map $f|_{\tilde{\ell}}$. The loop ℓ corresponds to exactly two boundary components of $F_1 \sqcup F_2 \sqcup \dots \sqcup F_n$. Then, on each of these two boundary components, there are exactly $d_\ell - 1$ arcs of $A_1 \sqcup A_2 \sqcup \dots \sqcup A_n$ ending. Therefore we can isotope A_i on F_i for all $i \in \{1, 2, \dots, n\}$, keeping the endpoints of A_i of ∂F_i , so that the endpoints of A_1, A_2, \dots, A_n pair up and $\cup A_i$ is a multiloop A on S . (There are infinitely many non-isotopic choices for A : We can Dehn twist A along a boundary component of F_i .)

Lemma 8.2. (i) *With respect to $S = \cup_{i=1}^n F_i$, the union of the embedded projective structures $C_{0,i}$ on F_i ($i = 1, 2, \dots, n$) is a uniformizable Schottky structure on S with holonomy ρ .*

(ii) *If α is a component of the multiloop A , then $\rho(\alpha)$ is loxodromic, i.e. $\rho(\alpha) \neq 1$.*

Proof. (i). Assume that C_i and C_j are adjacent components of $C \setminus M$, sharing a boundary component ℓ . Then $Supp(C_i)$ and $Supp(C_j)$ are adjacent components of $\Omega \setminus \tilde{M}$ (up to an element of Γ), sharing a boundary component $f(\tilde{\ell})$, where $\tilde{\ell}$ is a lift of ℓ to \tilde{S} . Since $C_{0,i}$ and $C_{0,j}$ are the canonical projective structures embedded in \tilde{C} , respectively, we can identify the boundary components of $C_{0,i}$ and $C_{0,j}$ corresponding to ℓ . Similarly identify all corresponding boundary components of $C_{0,i}$ ($i = 1, 2, \dots, n$) and obtain a projective structure C_0 on S . Let $\tilde{C}_0 = (f_0, \rho_{id})$ be the projective structure on \tilde{S} obtained by lifting C_0 to \tilde{S} , where f_0 is a $\tilde{\rho}$ -equivariant immersion from \tilde{S} to \tilde{C} . Since f_0 embeds each components of $\tilde{S} \setminus \tilde{M}$ onto a component of $\Omega \setminus \tilde{M}'$, thus it is a $\tilde{\rho}$ -equivariant embedding onto Ω . Therefore (S, C_0) is a uniformizable Schottky structure with holonomy ρ .

(ii). Let α be a loop of A . Then M decomposes α into subarcs a_1, a_2, \dots, a_m , so that $\alpha = a_1 \cup a_2 \cup \dots \cup a_m$. Let $\tilde{\alpha}$ be a lift of α to \tilde{S} . Then each a_j ($j = 1, 2, \dots, m$) is an arc properly embedded in F_i for

some $i \in \{1, 2, \dots, n\}$, connecting different boundary components of F_i . Therefore, for each component P of $\tilde{S} \setminus \tilde{M}$, either $\tilde{\alpha}$ is disjoint from P or $\tilde{\alpha}$ intersects P in a single arc connecting different boundary components of P . Thus $\tilde{\alpha}$ is a biinfinite simple curve properly embedded in \tilde{S} , and then α , as an element in $\pi_1(S)$, translates \tilde{S} along $\tilde{\alpha}$. Therefore $\rho(\alpha)$ is loxodromic. \square

Let C_0 be the uniformizable projective structure $\cup_{i=1}^n C_{0,i}$ on S given by Lemma 8.2 (i). We will show that C is obtained by grafting C_0 along A . To complete the proof, we will see that the grafting Gr_A on C_0 “commutes” with the decomposition $C_0 = \cup_{i=1}^n G_{0,i}$ as

$$\cup_{i=1}^n Gr_{A_i}(C_{0,i}) = Gr_A(C_0).$$

For each $j \in \{1, 2, \dots, m\}$, the arc a_j is property embedded in $C_{0,i}$ with some $i \in \{1, 2, \dots, n\}$; let b_j, c_j denote the boundary components of $C_{0,i} = (F_i, C_{0,i})$ connected by a_j .

Clearly the developing map $dev(C_{0,i})$ embeds $C_{0,i}$ isomorphically onto $Supp(C_{0,i})$, which is a component of $\Omega \setminus \tilde{M}'$. Via this isomorphism, (the images of) b_j and c_j bound a projective cylinder Y_j in $\hat{\mathbb{C}}$, and Y_j contains the arc a_i connecting connecting b_j and c_j . By the definition of graftings, $Gr_{a_j}(C_{0,i})$ is obtained by appropriately identifying the boundary arcs of $Y_j \setminus a_j$ and $C_{0,i} \setminus a_j$ corresponding to a_j .

Suppose that a_{j_1} and a_{j_2} ($j_1, j_2 \in \{1, 2, \dots, m\}$) are adjacent arcs in α , sharing an endpoint v . Similarly, for each $k = 1, 2$, we have $a_{j_k} \subset C_{0,i_k}$ for some $i_k \in \{1, 2, \dots, n\}$; let Y_{j_k} be its corresponding projective cylinder, as above, containing a_{j_k} . Then C_{0,i_1} and C_{0,i_2} are isomorphic to some adjacent components of $C_0 \setminus M$, sharing the boundary component containing v . Accordingly, Y_{j_1} and Y_{j_2} are also adjacent cylinders embedded in $\hat{\mathbb{C}}$ (up to an element of Γ). Thus we can identify the corresponding boundary components of Y_{j_1} and Y_{j_2} . Similarly identify all corresponding boundary components of Y_j ($j = 1, 2, \dots, m$) and obtain a projective torus $T = \cup_j Y_j$. Then the loop $\alpha = \cup_j a_j$ is naturally embedded in T .

We show that T is a hopf torus. Let N be the union of boundary components of Y_j in T . Then N is the multiloop splitting T into Y_1, Y_2, \dots, Y_m . The homotopy class of α generates an infinite cyclic subgroup $\langle \alpha \rangle$ of $\pi_1(S)$. Let \tilde{T} be the projective cylinder coving T with the covering transformation group $\langle \alpha \rangle$. Then $\tilde{\alpha} \subset \tilde{S}$ is also embedded in \tilde{T} . Let \tilde{N} denote the total lift of N to \tilde{T} . Then $\tilde{\alpha}$ transversally intersects each loop of \tilde{N} in a single point. Then $\tilde{\alpha}$ is embedded into Ω under the developing map of C_0 and, therefore, of T . Thus $dev(T)$ embeds \tilde{N} onto the multiloop on Ω consisting of the loops of \tilde{M}' interesting $\tilde{\alpha}$,

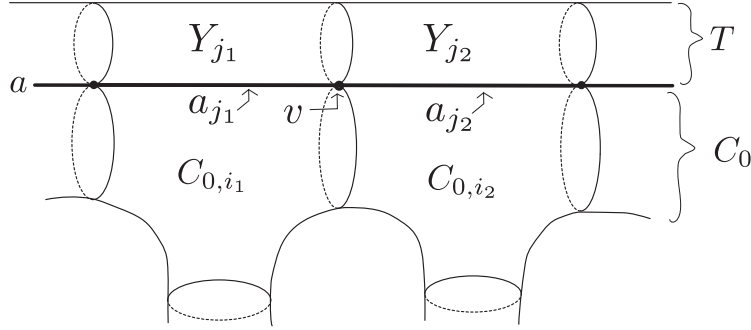


FIGURE 14.

and therefore $dev(T)$ is an embedding onto $\hat{\mathbb{C}}$ minus the fixed points of $\rho(\alpha)$. Hence T is a Hopf torus.

We adapt this observation for a single loop to the entire multiloop A . Set $A = \alpha_1 \sqcup \alpha_2 \sqcup \dots \sqcup \alpha_r$, where $\alpha_1, \alpha_2, \dots, \alpha_r$ are the loops of A . Further decomposing those loops into the components of A_1, A_2, \dots, A_n similarly to the proof Lemma 8.2 (ii), we set $A = a_1 \cup a_2 \cup \dots \cup a_m$. Then, for each $j \in \{1, 2, \dots, m\}$, a_j is an arc properly embedded in $F_{i(j)}$ with some $i(j) \in \{1, 2, \dots, n\}$. Similarly, let Y_j be the projective cylinder for the grafting of $C_{0,i(j)}$ along the arc a_j , so that $Gr_{a_j}(C_{0,i(j)}) = (C_{0,i(j)} \setminus a_j) \cup (Y_j \setminus a_j)$. Then, for each $i \in \{1, 2, \dots, n\}$, we have

$$Gr_{A_i}(C_{0,i}) = (C_{0,i} \setminus A_i) \cup (\sqcup_j \{Y_j \setminus a_j\}),$$

where the second union runs over all $j \in \{1, 2, \dots, m\}$ with $a_j \subset A_i$. For each $k \in \{1, 2, \dots, r\}$, let T_k denote the Hopf torus associated with α_k ; then $T_k = \cup Y_j$, where the union runs over all $j \in \{1, 2, \dots, m\}$ with $a_j \subset \alpha_k$. Therefore

$$\begin{aligned} C &= \cup_{i=1}^n C_i \\ &= \cup_{i=1}^n Gr_{A_i}(C_{0,i}) \\ &= \cup_{i=1}^n [(C_{0,i} \setminus A_i) \cup (\sqcup \{Y_j \setminus a_j \mid a_j \subset A_i\})] \\ &= [\cup_{i=1}^n (C_{0,i} \setminus A_i)] \sqcup [\cup_{j=1}^m (Y_j \setminus a_j)] \\ &= (C_0 \setminus A) \cup [(T_1 \setminus \alpha_1) \cup (T_2 \setminus \alpha_2) \cup \dots \cup (T_r \setminus \alpha_r)] \\ &= Gr_A(C_0). \end{aligned}$$

□

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