BERS' SIMULTANEOUS UNIFORMIZATION AND THE INTERSECTION OF POINCARÉ HOLONOMY VARIETIES

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Abstract. We introduce the space of ordered pairs of distinct $\mathbb{C}\mathrm{P}^1$-structures on Riemann surfaces (of any orientations) which have identical holonomy, so that the quasi-Fuchsian space is identified with a connected component of this space. This space holomorphically maps to the product of the Teichmüller spaces minus its diagonal.

In this paper, we prove that this map is a complete local branched covering map. As a corollary, we reprove Bers' simultaneous uniformization theorem without the measurable Riemann mapping theorem. Along the way, we show in particular that the intersection of arbitrary two Poincare holonomy varieties ($\mathrm{SL}_2\,\mathbb{C}$-opers) is a non-empty discrete set.

Contents

1. Introduction

In 1960, Bers established a bijection between pairs of Riemann surface structures of opposite orientations and typical discrete and faithful representations of a surface group into $\mathrm{PSL}(2,\mathbb{C})$

up to conjugacy ([Ber60]). It is called *Bers' simultaneous uniformization theorem*, and it gave a foundation for the later evolutional development of the hyperbolic three-manifold theory by Thurston ([Thu81]) and many others. In this paper, we partially generalize Bers' theorem, in a certain sense, to generic surface representations into $\mathrm{PSL}(2,\mathbb{C})$, which are not necessarily discrete.

Throughout this paper, let S be a closed orientable surface of genus $g > 1$. Given a *quasi-Fuchsin representation* $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$, the domain of discontinuity is the union of disjoint topological open disks Ω^+, Ω^- in $\mathbb{CP}^1$. Then, their quotients $\Omega^+/\mathrm{Im}\rho, \Omega^-/\mathrm{Im}\rho$ have marked Riemann surface structures with opposite orientations.

Let S^+, S^- be S with opposite orientations. Then Bers' simultaneous uniformization theorem asserts that this correspondence gives a biholomorphism

$$\mathsf{QF} \;\to\; \mathsf{T} \times \mathsf{T}^* (= \mathbb{R}^{6g-6} \times \mathbb{R}^{6g-6}) \tag{1}$$

where QF is space of the quasi-Fuchsian representations $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ up to conjugation, T is the Teichmüller space of S^+ and T^* is the Teichmüller space of S^-; see [Hub06] [EK06] for the analyticity. (Note that T^* is indeed anti-holomorphic to T; see [Wol10].)

The $\mathrm{PSL}(2,\mathbb{C})$-character variety of S is the space of homomorphisms $\pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$, roughly, up to conjugation, and it has two connected components ([Gol88]). Let χ denote the component consisting of representations $\pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ which lift to $\pi_1(S) \to \mathrm{SL}(2,\mathbb{C})$; then χ strictly contains the (Euclidean) closure of QF.

A $\mathbb{CP}^1$-structure on S is a locally homogeneous structure modeled on $\mathbb{CP}^1$, and its holonomy is in χ. The quotients $\Omega^+/\mathrm{Im}\rho$ and $\Omega^-/\mathrm{Im}\rho$ discussed above have not only Riemann surfaces structures but also $\mathbb{CP}^1$-structures on S^+ and S^-, respectively. In fact, almost every representation in χ is the holonomy of some $\mathbb{CP}^1$-structure on S [GKM00]; see §2.1 for details.

In fact, each $\mathbb{CP}^1$-structure on S corresponds to a holomorphic quadratic differential on a Riemann surface structure on S (§2.1.2). Let P be the space all (marked) $\mathbb{CP}^1$-structures on S^+ with the fixed orientation, which is identified with the cotangent bundle of T. Similarly, let P^* be the space of all marked $\mathbb{CP}^1$ on S^-, identified with the cotangent bundle of T^*.

By sending each quasi-Fuchsian representation $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ to the $\mathbb{CP}^1$-structures $\Omega^+/\mathrm{Im}\rho$ and $\Omega^-/\mathrm{Im}\rho$, the quasi-Fuchsian space QF holomorphically embeds into $\mathsf{P} \times \mathsf{P}^*$ as a closed half-dimensional submanifold. The holonomy map

$$\mathrm{Hol}\colon \mathsf{P} \sqcup \mathsf{P}^* \to \chi$$

takes each $\mathbb{CP}^1$-structure to its holonomy representation. Now we introduce the space B of all ordered pairs of distinct $\mathbb{CP}^1$-structures sharing holonomy

$$\left\{(C, D) \in (\mathsf{P} \sqcup \mathsf{P}^*)^2 \,\middle|\, \mathrm{Hol}(C) = \mathrm{Hol}(D), C \neq D\right\}.$$

Since Hol is locally biholomorphic, B is also a half-dimensional closed holomorphic submanifold. The map switching the order of C and D is a fixed-point-free biholomorphic involution of B. Then, the quasi-Fuchsian space QF is biholomorphically identified with two connected components of B, which are identified by this involution (Lemma 2.5). Every connected component of $(\mathsf{P} \sqcup \mathsf{P}^*)^2$ contains at least one component of B which does not corresponds to QF (see Lemma 2.4).

Let $\psi \colon \mathsf{P} \sqcup \mathsf{P}^* \to \mathsf{T} \sqcup \mathsf{T}^*$ be the projection from the space of all $\mathbb{C}\mathrm{P}^1$-structures on S^+ and S^- to the space of all Riemann surface structures on S^+ and S^-. Define $\Psi \colon \mathsf{B} \to (\mathsf{T} \sqcup \mathsf{T}^*)^2 \setminus \Delta$ by $\Psi(C, D) = (\psi(C), \psi(D))$, where Δ is the diagonal $\{(X, X) \mid X \in \mathsf{T} \sqcup \mathsf{T}^*\}$ (which can not intersect $\Psi(B)$).

It is a natural question to ask to what extent connected components of B resemble the quasi-Fuchsian space QF. In this paper, we prove a local and a global property of the holomorphic map Ψ:

Theorem A. *The map Ψ is a complete local branched covering map.*

(For the definition of complete local branched covering maps, see §2.5.) In particular, Ψ is open, and its fibers are discrete subsets of B. Thus its ramification locus is a no-where dense analytic subset, which may possibly be the empty set. (The completeness of Theorem A is given by Theorem 12.2, and the local property by Theorem B below.)

Note that, by the completeness in Theorem A, for every connected component Q of B, the restriction $\Psi|Q$ is surjective onto its corresponding component of $(\mathsf{T} \sqcup \mathsf{T}^*)^2 \setminus \Delta$. We also show that, towards the diagonal Δ, the holonomy of $\mathbb{C}\mathrm{P}^1$-structures leaves every compact in χ (see Proposition 12.6).

The deformation theory of hyperbolic cone manifolds is developed, especially, by Hodgson, Kerckhoff and Bromberg [HK98, HK05, HK08, Bro04a, Bro04b]). If cone angles exceed 2π, their deformation theory is established only under the assumption that the cone singularity is short and, thus, the tube radius is large. More generally, a conjecture of McMullen ([McM98, Conjecture 8.1]) asserts that the deformation space of geometrically-finite hyperbolic cone-manifolds are parametrized by using the cone angles and the conformal structures on the ideal boundary. Theorem A provides some additional evidence for the conjecture, when the cone angles are 2π-multiples (c.f. [Bro07]).

Bers' simultaneous uniformization theorem is a consequence of the measurable Riemann mapping theorem, and it is important that the domain $\Omega^+ \sqcup \Omega^-$ is a (full measure) subset of $\mathbb{C}\mathrm{P}^1$. However, in general, developing maps of $\mathbb{C}\mathrm{P}^1$-structures are not embeddings, and Bers' proof does not apply to the other components of B. In fact, Theorem A implies the simultaneous uniformization theorem, without any aid of the measurable Riemann mapping theorem (§13).

Next we describe the local property in Theorem A. Since Hol is locally biholomorphic, for every $(C, D) \in \mathsf{B}$, if an open neighborhood V of (C, D) in B is sufficiently small, then Hol embeds V onto a neighborhood U of $\mathrm{Hol}(C) = \mathrm{Hol}(D)$ in χ. Let T_C and T_D be T or T^* so that $\psi(C) \in \mathsf{T}_C$ and $\psi(D) \in \mathsf{T}_D$, and define a holomorphic map $\Psi_{C,D} \colon U \to \mathsf{T}_C \times \mathsf{T}_D$ by the restriction of Ψ to V and the identification $V \cong U$. The following gives a finite-to-one "parametrization" of U by pairs of Riemann surface structures associated with V.

Theorem B. *Let $(C, D) \in \mathsf{B}$. Then, there is a neighborhood V of (C, D) in B, such that* Hol *embeds V into χ, and the restriction of Ψ to V is a branched covering map onto its image in $T_C \times T_D$ (Theorem 10.3.)*

By the simultaneous uniformization theorem, for every $X \in \mathsf{T}^*$ and $Y \in \mathsf{T}$, the slices $\mathsf{T} \times \{Y\}$ and $\{X\} \times \mathsf{T}^*$, called the Bers' slices, intersect transversally in the point in QF corresponding to (X, Y) by (1). The Teichmüller spaces T and T^* are, as complex manifolds, open bounded pseudo-convex domains in $\mathbb{C}^{3g-3}$, where g is the genus of S. In order to prove Theorem A and Theorem B, we consider the analytic extensions of $\mathsf{T} \times \{Y\}$ and $\{X\} \times \mathsf{T}^*$ in the character variety χ and analyze their intersection.

For each $X \in \mathsf{T} \sqcup \mathsf{T}^*$, let P_X be the space of all $\mathbb{C}\mathrm{P}^1$-structures on X. Then P_X is an affine space of holomorphic quadratic differentials on X, and thus $\mathsf{P}_X \cong \mathbb{C}^{3g-3}$. Although the restrictions of the holonomy map Hol to P and P^* are non-proper and non-injective, the restriction of Hol to P_X is a proper embedding ([Poi84, GKM00], see also [Tan99, Kap95, Dum17]). Let $\chi_X = \mathrm{Hol}(\mathsf{P}_X)$, which we shall call the Poincaré holonomy variety of X as its injectivity is due to Poincaré. Note that, if $X \in \mathsf{T}$, then χ_X contains $\{X\} \times \mathsf{T}^*$ as a bounded pseudo-convex subset, and similarly, if $Y \in \mathsf{T}^*$, then χ_Y contains $\mathsf{T} \times \{Y\}$ as a bounded open subset. Since $\dim \chi_X$ is half of $\dim \chi$, it is a basic question to ask what the intersection of such smooth subvarieties look like.

Theorem C. *For all distinct X, Y in $\mathsf{T} \sqcup \mathsf{T}^*$, the intersection of χ_X and χ_Y is a non-empty discrete set.*

Other than Bers' theorem, this theorem is completely new. In particular, we will show that $\chi_X \cap \chi_Y$ contains at least one point if the orientations of X and Y are the same, and at least two points if the orientations are opposite (Corollary 12.7).

The deformation spaces, P and P^*, of $\mathbb{C}\mathrm{P}^1$-structures have two distinguished parametrizations: namely, *Schwarzian parametrization* (§2.1.2) and *Thurston parametrization* (§2.1.4). In order to understand points in $\chi_X \cap \chi_Y$, we give a comparison theorem between those two parametrizations.

Let C be a $\mathbb{C}\mathrm{P}^1$-structure on a Riemann surface X. Then the quadratic differential of its Schwarzian parameters gives a vertical measured (singular) foliation V on X. The Thurston parametrization of C gives the measured geodesic lamination L on the hyperbolic surface. Dumas showed that V and L *projectively* coincide in the limit as C leaves every compact in P_X ([Dum06, Dum07]), see also [OSWW].)

The measured geodesic lamination L of the Thurston parameter is also realized as a circular measured lamination $\mathcal{L}$ on C, so that $\mathcal{L}$ and L are the same measured lamination on S (§2.1.5). In this paper, we prove more explicit asymptotic companion between the Thurston lamination $\mathcal{L}$ and the vertical foliation V, without projectivization. For a quadratic differential $q = \phi\, dz^2$ on a Riemann surface X, let $\|q\| = \int_X |\phi|\, dx\, dy$, the L^1-norm. Then we have the following.

Theorem D. *Let $X \in \mathsf{T} \sqcup \mathsf{T}^*$. For every $\epsilon > 0$, there is $r > 0$, such that, if the holomoprhic quadratic differential q on X satisfies $\|q\| > r$, then, letting C be the $\mathbb{C}\mathrm{P}^1$-structure on X given by*

q, the vertical foliation V of q is $(1+\epsilon,\epsilon)$-quasi-isometric to $\sqrt{2}$ times the Thurston lamination $\mathcal{L}$ on C, up to an isotopy of X supported the ϵ-neighborhood of the zero set of q in the uniformizing hyperbolic metric on X. (Theorem 4.1.)

(See 4 for the definition of being quasi-isometric, and see §2.1.5 for the Thurston lamination on a $\mathbb{C}\mathrm{P}^1$-surface.)

Last we address the following question aiming to strengthen Theorem A.

Question 1.1. *For every (or even some) non-quasi-Fuchsian component Q of B, is the restriction of Φ to Q biholomorphic map onto its corresponding component of $(\mathsf{T} \sqcup \mathsf{T}^*)^2$?*

1.1. Outline of this paper.

In §3, we analyze the geometry of Epstein-Schwarz surfaces corresponding to $\mathbb{C}\mathrm{P}^1$-structures, using [Dum17] and [Bab]. In §4, we analyze the horizontal foliations of $\mathbb{C}\mathrm{P}^1$-structures on X and Y corresponding to the intersection points of $\chi_X \cap \chi_Y$ in Theorem C. In fact, we show that such horizontal projectivized measured foliations projectively coincide towards infinity of $\chi_X \cap \chi_Y$ (Theorem 4.9).

A (fat) train-track is a surface obtained by identifying edges of rectangles in a certain manner. In §5, we introduce more general train-tracks whose branches are not necessarily rectangles but more general polygons, cylinders, and even surfaces with staircase boundary (*surface train tracks*). In §6, given a certain pair of flat surfaces, we decompose them into the surface train tracks in a compatible manner.

In §7, we prove Theorem D. In §8, for every holonomy ρ in $\chi_X \cap \chi_Y$ outside a large compact subset K of χ, we construct certain surface train-track decompositions of $\mathbb{C}\mathrm{P}^1$-structures on X and Y corresponding to ρ in a compatible manner, using the decomposition of flat surfaces. In §9, from the compatible decompositions of the $\mathbb{C}\mathrm{P}^1$-structures, we construct an integer-valued cocycle which changes continuously in $\rho \in \chi_X \cap \chi_Y \setminus K$. In §10, by this cocycle and some complex geometry, we prove the discreetness in Theorem C. In §12, the completeness of Theorem C is proven. In §11, we discuss the case when the orientations of X and Y are opposite. In §13, we give a new proof of Bers' theorem.

1.2. Acknowledgements.

I thank Ken Bromberg, David Dumas, Misha Kapovich, and Shinnosuke Okawa for helpful conversations.

The author is partially supported by the Grant-in-Aid for Research Activity Start-up (19K21023) and the Grant-in-Aid for Scientific Research C (20K03610).

2. Preliminaries

2.1. $\mathbb{C}\mathrm{P}^1$-structures.

(General references are [Dum09], [Kap01, §7].) Let F be a connected orientable surface. A $\mathbb{C}\mathrm{P}^1$-structure on F is a $(\mathbb{C}\mathrm{P}^1, \mathrm{PSL}(2,\mathbb{C}))$-structure. That is, a maximal atlas of charts embedding open sets of F into $\mathbb{C}\mathrm{P}^1$ with transition maps in $\mathrm{PSL}(2,\mathbb{C})$. Let $\tilde{F}$ be the universal cover of F. Then, equivalently, a $\mathbb{C}\mathrm{P}^1$-structure is a pair of

- a local homeomorphism $f\colon \tilde{F} \to \mathbb{C}\mathrm{P}^1$ and
- a homomorphism $\rho\colon \pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$

such that f is ρ-equivariant ([Thu97]). It is defined up to an isotopy of the surface and an element α of $\mathrm{PSL}(2, \mathbb{C})$, i.e. $(f, \rho) \sim (\alpha f, \alpha^{-1}\rho\alpha)$. The local homeomorphism f is called the developing map and the homomorphism ρ is called the holonomy representation of a $\mathbb{C}\mathrm{P}^1$-structure. We also write the developing map of C by $\operatorname{dev} C$.

2.1.1. *The holonomy map.* The $\mathrm{PSL}(2, \mathbb{C})$-character variety of S is the space of the equivalence classes homomorphisms

$$\{\pi_1(S) \to \mathrm{PSL}(2, \mathbb{C}))\} /\!\!/ \mathrm{PSL}(2, \mathbb{C}),$$

where the quotient is the GIT-quotient (see [New] for example). For the holonomy representations of $\mathbb{C}\mathrm{P}^1$-structures on S, the quotient is exactly given by the conjugation by $\mathrm{PSL}(2, \mathbb{C})$. Then, the character variety has exactly two connected components, distinguished by the lifting property to $\mathrm{SL}(2, \mathbb{C})$; see [Gol88]. Let χ be the component consisting of representations which lift to $\pi_1(S) \to \mathrm{SL}(2, \mathbb{C})$, and let P be the space of marked $\mathbb{C}\mathrm{P}^1$-structures on S. Then the holonomy map

$$\mathrm{Hol}\colon \mathsf{P} \to \chi$$

takes each $\mathbb{C}\mathrm{P}^1$-structure to its holonomy representation. Then Hol is a locally biholomorphic map, but not a covering map onto its image ([Hej75, Hub81, Ear81]). By Gallo, Kapovich, and Marden ([GKM00]), $\rho \in \operatorname{Im} \mathrm{Hol}$ if and only if ρ is non-elementary and ρ has a lift to $\pi_1(S) \to \mathrm{SL}(2, \mathbb{C})$. In particular, Hol is almost onto χ.

2.1.2. *The Schwarzian parametrization.* (See [Dum09] [Leh87].) Let X be a Riemann surface structure on S. Then, the hyperbolic structure τ_X uniformizing X is, in particular, a $\mathbb{C}\mathrm{P}^1$-structure on X. For an arbitrary $\mathbb{C}\mathrm{P}^1$-structure C on X, the Schwarzian derivative gives a holomorphic quadratic differential on X by comparing with τ_X, so that τ_X corresponds to the zero differential. Then (X, q) is the Schwarzian parameters of C. Let $\mathsf{QD}(X)$ be the space of the holomorphic quadratic differentials on X, which is a complex vector space of dimension $3g - 3$. Thus, the space P_X of all $\mathbb{C}\mathrm{P}^1$ structures on X is identified with $\mathsf{QD}(X)$.

Theorem 2.1 ([Poi84, Kap95], see also [Tan99, Dum17])**.** *For every Riemann surface structure X on S, the set P_X of projective structures on X is property embedded in χ by* Hol.

For $X \in \mathsf{T} \sqcup \mathsf{T}^*$, let χ_X denote the smooth analytic subvariety $\mathrm{Hol}(\chi_X)$. Pick any metric d on T and T^* compatible with their topology (for example, the Teichmüller metric or the Weil-Peterson metric).

Lemma 2.2. *Let B be an arbitrary bounded subset of either T or T^*. For every compact subset K in χ, there is $\epsilon > 0$, such that, if distinct $X, Y \in B$ satisfy $d(X, Y) < \epsilon$, then $\chi_X \cap \chi_Y \cap K = \emptyset$.*

Proof. For each $X \in \mathsf{T} \sqcup \mathsf{T}^*$, by Theorem 2.1, P_X is properly embedded in χ. For a neighborhood U of X, let $D_r(U)$ denote the set of all holomorphic quadratic differentials q on Riemann surfaces

Y in U such that the L^1-norm $\|q\|$ is less than r. Since Hol is a local biholomorphism, for every $X \in \mathsf{T} \sqcup \mathsf{T}^*$ and $r \in \mathbb{R}_{>0}$, there is a neighborhood U of X, Hol embeds $D_r(U)$ into χ. Let P_U be the space of all $\mathbb{C}\mathrm{P}^1$-structures whose complex structures are in U. Then, if $r > 0$ is sufficiently large, we can, in addition, assume that $K \cap \mathrm{Hol}(\mathsf{P}_U) = K \cap \mathrm{Hol}(D_r(U))$. Therefore, for all $Y, W \in U$, we have $\chi_Y \cap \chi_W \cap K = \emptyset$. □

2.1.3. *Singular Euclidean structures.* (See [Str84], [FM12].) Let $q = \phi\, dz^2$ be a quadratic differential on a Riemann surface X. Then q induces a singular Euclidean structure E on S from the Euclidean structure on $\mathbb{C}$: Namely, for each non-singular point $z \in X$, we can identify a neighborhood U_z of z with an open subset of $\mathbb{C} \cong \mathbb{E}^2$ by the integral

$$\eta(w) = \int_z^w \sqrt{\phi}\, dz$$

along a path connecting z and w, where $w \in U_z$ is a fixed base point (for details, see [Str84]). Then the zeros of q correspond to the singular points of E. Note that, for $r > 0$, if the differential q is scaled by r, then the Euclidean metric E is scaled by $\sqrt{r}$. Let E^1 denote the normalization $\frac{E}{\mathrm{Area} E}$ of E by the area.

The complex plane $\mathbb{C}$ is foliated by horizontal lines and, by the identification $\mathbb{C} = \mathbb{E}^2$, the vertical length dy gives a canonical transversal measure to the foliation. Similarly, $\mathbb{C}$ is also foliated by the vertical lines, and the horizontal length dx gives a canonical transversal measure to the foliation. Then, those vertical and horizontal foliations on $\mathbb{C}$ induce vertical and horizontal singular foliations on E which meet orthogonally.

In this paper, a flat surface is the singular Euclidean structure obtained by a quadratic differential on a Riemann surface, which has vertical and horizontal foliations.

2.1.4. *Thurston's parameterization.* By the uniformization theorem of Riemann surfaces, the space of all marked hyperbolic structures on S is identified with the space T of all marked Riemann surface structures. Let ML be the space of measured laminations on S. Note that $\mathbb{C}\mathrm{P}^1$ is the ideal boundary of $\mathbb{H}^3$, so that $\mathrm{Aut}\, \mathbb{C}\mathrm{P}^1 = \mathrm{Isom}^+ \mathbb{H}^3$. In fact, Thurston gave a parameterization of P using the three-dimensional hyperbolic geometry.

Theorem 2.3 (Thurston, see [KP94, KT92]). *There is a natural (tangential) homeomorphism*

$$\mathsf{P} \to \mathsf{T} \times \mathsf{ML}.$$

Suppose that, by this homeomorphism, $C = (f, \rho) \in \mathsf{P}$ corresponds to a pair $(\sigma, L) \in \mathsf{T} \times \mathsf{ML}$. Let $\tilde{L}$ be the $\pi_1(S)$-invariant measured lamination on $\mathbb{H}^2$ obtained by lifting L. Then (σ, L) yields a ρ-equivariant pleated surface $\beta\colon \mathbb{H}^2 \to \mathbb{H}^3$, obtained by bending $\mathbb{H}^2$ along $\tilde{L}$ by the angles given by its transversal measure. The map β is called a **bending map**, and it is unique up to post-composing with $\mathrm{PSL}(2, \mathbb{C})$.

2.1.5. *Collapsing maps.* ([KP94]; see also [Bab20].) Let $C \cong (\tau, L)$ be a $\mathbb{C}\mathrm{P}^1$-structure expressed in Thurston parameters. Let $\tilde{C}$ be the universal cover of C. Then $\tilde{C}$ can be regarded as the domain of f, so that $\tilde{C}$ is holomorphically immersed in $\mathbb{C}\mathrm{P}^1$. A round disk is a topological open disk whose development is a round disk in $\mathbb{C}\mathrm{P}^1$, and a maximal disk is a round disk which is not contained in contained in a strictly bigger round disk. In fact, for all $z \in \tilde{C}$, there is a unique maximal disk D_z whose core contains z. Then there is a measured lamination $\mathcal{L}$ on C obtained from the cores of maximal disks in the universal cover $\tilde{C}$, such that $\mathcal{L}$ is equivalent to L in ML. This lamination is the Thurston lamination on C. In addition, there is an associated continuous map $\kappa\colon C \to \tau$ which takes $\mathcal{L}$ to L, called the collapsing map.

Then, the bending map and the developing of C are related by the collapsing map κ and appropriate nearest point projections in $\mathbb{H}^3$: Let $\tilde{\kappa}\colon \tilde{C} \to \mathbb{H}^2$ be the lift of κ to a map between the universal covers. Let H_z be the hyperbolic plane in $\mathbb{H}^3$ bounded by the boundary circle of D_z. There is a unique nearest point projection from D_z to H_z. Then $\beta \circ \tilde{\kappa}(z)$ is the nearest point projection of $f(z)$ to H_z.

2.2. **Bers' space.** Recall, from §1, that B is the space of ordered pairs of $\mathbb{C}\mathrm{P}^1$-structures on S with identical holonomy, which may have different orientations.

Lemma 2.4. *every component of* $(\mathsf{P} \sqcup \mathsf{P}^*)^2$ *contains, at least, one connected component of* B *which is not identified with the quasi-Fuchsian space.*

Proof. By [GKM00], every non-elementary representation $\rho\colon \pi_1(S) \to \mathrm{SL}(2, \mathbb{C})$ is the holonomy representation of infinitely many $\mathbb{C}\mathrm{P}^1$-structures on S^+ whose developing maps are not embedding, and also of infinitely many $\mathbb{C}\mathrm{P}^1$-structures of S^- whose developing maps are not embedding. Therefore, since a quasi-Fuchsian component of B consists of pairs of $\mathbb{C}\mathrm{P}^1$-structures whose developing maps are embedding, every component of $(\mathsf{P} \sqcup \mathsf{P}^*)^2$ contains at least one connected component of B, which is not a quasi-Fuchsian component. ☐

Lemma 2.5. B *is a closed analytic submanifold of* $\mathsf{P} \sqcup \mathsf{P}^*$ *of complex dimension* $6g - 6$.

Proof. It is a holomorphic submanifold, since $\mathrm{Hol}\colon \mathsf{P} \sqcup \mathsf{P}^* \to \mathcal{X}$ is a local biholomorphism. As $\dim_{\mathbb{C}} \mathcal{X} = 6g - 6$, the complex dimension of B is also $6g - 6$. Let (C_i, D_i) be a sequence in B converging to (C, D) in $(\mathsf{P} \sqcup \mathsf{P}^*)^2$. Then, since $\mathrm{Hol}\, C_i = \mathrm{Hol}\, D_i$, by the continuity of Hol, $\mathrm{Hol}(C) = \mathrm{Hol}(D)$. Therefore B is closed. ☐

2.3. **Angles between laminations.** Let F be a surface with a hyperbolic or singular Euclidean metric. Let ℓ_1, ℓ_2 be (non-oriented) geodesics on F with non-empty intersection. Then, for $p \in \ell_1 \cap \ell_2$, let $\angle_p(\ell_1, \ell_2) \in [0, \pi/2]$ denote the angle between ℓ_1 and ℓ_2 at p.

Let L_1 L_2 be geodesic laminations or foliations on F. Then $\angle(L_1, L_2)$ be the infimum of $\angle_p(\ell_1, \ell_2) \in [0, \pi/2]$ over all $p \in L_1 \cap L_2$ where ℓ_1 and ℓ_2 are leaves of L_1 and L_2, respectively, containing p. By convention, if $L_1 \cap L_2 = \emptyset$, then $\angle(L_1, L_2) = 0$. We say that L_1 and L_2 are ϵ-parallel, if $\angle(L_1, L_2) < \epsilon$.

2.4. **The Morgan-Shalen compactification.** (See [CS83, MS84], see also [Kap01, §10.3].) The Morgan-Shalen compactification is a compactification of $\mathrm{PSL}(2,\mathbb{C})$-character variety, introduced in [CS83, MS84]. For our $\mathcal{X}$, each boundary point corresponds to a minimal action of $\pi_1(S)$ on a $\mathbb{R}$-tree, $\pi_1(S) \curvearrowright T$.

Every holonomy $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ induces a translation length function $\rho^*\colon \pi_1(S) \to \mathbb{R}_{\geq 0}$, and a minimal action $\pi_1(S)$ on a $\mathbb{R}$-tree also induces a translation length function. Then $\rho_i \in \mathcal{X}$ converges to a boundary point $\pi_1(S) \curvearrowright T$ if the length function ρ_i^* projectively converges to the projective class of the translation function of $\pi_1(S) \curvearrowright T$ as $i \to \infty$.

2.5. **Complex geometry.** We recall some basic complex geometry used in this paper. Let U, W be complex manifolds of the same dimension. A holomorphic map $\phi\colon U \to W$ is a (finite) branched covering map if

- there are closed analytic subsets U', W' of dimensions strictly smaller than $\dim U = \dim W$, such that the restriction of ϕ to $U \setminus U'$ is a covering map onto $W \setminus W'$, and
- its covering degree is finite. (See [FG02, p227].)

A holomorphic map $\phi\colon U \to W$ is a local branched covering map if, for every $z \in U$, there is a neighborhood V of z in U such that the restriction $\phi|V$ is a branched covering map onto its image. A holomorphic map $U \to W$ is complete if it has the (not necessarily unique) path lifting property ([AS60]).

Let U be an open subset of $\mathbb{C}^n$. Then a subset V of U is *analytic* if it is locally an intersection of zeros of finitely many holomorphic functions.

Proposition 2.6 (Proposition 6.1 in [FG02])**.** *Every connected bounded analytic set in $\mathbb{C}^n$ is a discrete set.*

Theorem 2.7 (p107 in [GR84], Theorem 7.9 in [HY99])**.** *Let $U \subset \mathbb{C}^n$ be a region. Suppose that $f\colon U \to \mathbb{C}^n$ is a holomorphic map with discrete fibers. Then it is an open map.*

3. Approximations of Epstein-Schwarz surfaces

3.1. **Epstein surfaces.** (See Epstein [Eps], and also Dumas [Dum17].) Let C be a $\mathbb{C}\mathrm{P}^1$-structure on S. Fix a developing pair (f, ρ) of C, where $f\colon \tilde{C} \to \mathbb{C}\mathrm{P}^1$ is the developing map and $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ is the holonomy representation, which is unique up to $\mathrm{PSL}(2,\mathbb{C})$. For $z \in \mathbb{H}^3$, by normalizing the ball model of $\mathbb{H}^3$ so that z is the center, we obtain a spherical metric $\nu_{\mathbb{S}^2}(z)$ on $\partial_\infty \mathbb{H}^3 = \mathbb{C}\mathrm{P}^1$.

Given a conformal metric μ on C, there is a unique map $\mathrm{Ep}\colon \tilde{C} \to \mathbb{H}^3$ such that, for each $x \in \tilde{C}$, the pull back of $\nu_{\mathbb{S}^2}\, \mathrm{Ep}(z)$ coincides with μ at z. This map is ρ-equivariant, and called the Epstein surface.

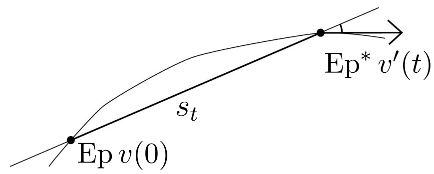

FIGURE 1.

3.2. **Approximation.** Let $C = (X, q)$ be a $\mathbb{C}P^1$-structure on S expressed in Schwarzian coordinates, where q is a holomorphic quadratic differential on a Riemann surface X. Then q yields a flat surface structure E on S. Moreover q gives a vertical measured foliation V and a horizontal measured foliation H on E.

Let $\mathrm{Ep}\colon \tilde{S} \to \mathbb{H}^3$ be the Epstein surface of C with the conformal metric given by E. Then, let $\mathrm{Ep}^*\colon T\tilde{S} \to T\mathbb{H}^3$ be the derivative of Ep, where $T\tilde{S}$ and $T\mathbb{H}^3$ denote the tangent bundles. Let $d\colon \tilde{E} \to \mathbb{R}_{\geq 0}$ be the distance function from the singular set $\tilde{Z}_q$ with respect to the singular Euclidean metric of $\tilde{E}$.

Let $v'(z)$ be the vertical unite tangent vector of $\tilde{E}$ at a smooth point z. Similarly, let $h'(z)$ be the horizontal unite tangent vector at a smooth point z of $\tilde{E}$.

Lemma 3.1 ([Eps], Lemma 2.6 and Lemma 3.4 in [Dum17])**.**

(1) $\| \mathrm{Ep}^*\, h'(z)\| < \frac{6}{d(z)^2}$;
(2) $\sqrt{2} < \| \mathrm{Ep}^*\, v'(z)\| < \sqrt{2} + \frac{6}{d(z)^2}$;
(3) $h'(z), v'(z)$ *are principal directions of* Ep *at* z;
(4) $k_v < \frac{6}{d(z)^2}$, *where* k_v *is the principal curvature of* Ep *in the vertical direction.*

Consider the Euclidean metric on $\mathbb{C} \cong \mathbb{E}^2$. By the exponential map $\exp\colon \mathbb{C} \to \mathbb{C}^*$, we push forward a complete Euclidean metric to $\mathbb{C}^*$, which is invariant under the action of $\mathbb{C}^*$. If a simply connected region Q in the flat surface E contains no singular points, then Q is immersed into $\mathbb{C}$ locally isometrically preserving horizontal and vertical directions. Using Lemma 3.1 and the definition of Epstein surfaces, one obtains the following.

Lemma 3.2. *(*[Bab, Lemma 12.15]*.) For every $\epsilon > 0$, there is $r > 0$, such that if Q is a region in E satisfying*

- *Q has E-diameter less than r, and*
- *the distance from the singular set of E is more than r.*

then $\exp\colon \mathbb{C} \to \mathbb{C}^*$ *and the developing map are ϵ-close pointwise with respect to the complete Euclidean metrics.*

We shall further analyze vertical curves on Epstein surfaces. Let $v\colon [0, \ell] \to \tilde{E}$ be a path in a vertical leaf, such that v contains no singular point and has a constant speed $\frac{1}{\sqrt{2}}$ in the Euclidean metric. Let $\mathrm{Ep}^{\perp}(z)$ be the unite normal vector of the Epstein surface Ep at each smooth point $z \in \tilde{E}$. Let s_t be the geodesic segment in $\mathbb{H}^3$ connecting $\mathrm{Ep}\, v(0)$ and $\mathrm{Ep}\, v(t)$. (See Figure 2.)

Lemma 3.3. *For every $\epsilon > 0$, if there is (large) $\omega > 0$ only depending on ϵ, such that, w.r.t. the E-metric, the distance of the vertical segment v from the zeros Z_q of q is more than ω, then the angle between $\mathrm{Ep}^* v'(t)$ and the geodesic containing s_t is less than ϵ for all t. (Figure 1.)*

Proof. By Lemma 3.1 (2) (4), the proof is similar to the proof of [CEG87, Theorem I.4.2.10]; see also [EMM04] [Bab10, Lemma 5.3]. □

Define $\theta\colon [0,\ell] \to T_{\mathrm{Ep}\, v(0)}$ by the parallel transport of $\mathrm{Ep}^\perp(t)$ along s_t to the starting point $\mathrm{Ep}(v(0))$. Let H be the (totally geodesic) hyperbolic plane in $\mathbb{H}^3$ orthogonal to the tangent vector $\mathrm{Ep}^* v'(0)$, so that H contains $\mathrm{Ep}^\perp v(0)$. Then, Lemma 3.3, implies

Corollary 3.4. *For every $\epsilon > 0$, there is (large) $\omega > 0$ only depending on ϵ such that, if the Hausdorff distance between v and the zeros Z_q of q is more than ω w.r.t. the E-metric, then $\angle_{v(0)}(\theta(t), H) < \epsilon$ for all $t \in [0,\ell]$.*

Proposition 3.5 (Total curvature bound in the vertical direction)**.** *For all $X \in \mathsf{T} \cup \mathsf{T}^*$ and all $\epsilon > 0$, there is a bounded subset $K = K(X,\epsilon)$ in χ_X, such that, for $\rho \in \chi_X \setminus K$, if a vertical segment v has normalized length less than $\frac{1}{\epsilon}$ and has normalized Euclidean distance from the zeros of $q_{X,\rho}$ at least ϵ, then the total curvature along v is less than ϵ.*

Proof. Immediately follows from Dumas' estimate in Lemma 3.1 (4) □

Consider the projection $\hat\theta(t)$ of $\theta(t) \in T^1_{v(0)}\mathbb{H}^3$ to the unite tangent vector in H at v_0. Let $\eta\colon [0,\ell] \to \mathbb{R}$ be the continuous function of the total increase of $\hat\theta(t)\colon [0,\ell] \to \mathbb{R}$, so that $\eta(0) = 0$ and $\eta'(t) = \hat\theta'(t)$.

Proposition 3.6. *Let $X \in \mathsf{T} \sqcup \mathsf{T}^*$. For every $\epsilon > 0$, there is a bounded subset $K = K(X,\epsilon) > 0$ in χ_X, such that, if*

- *$C \in \mathsf{P}_X$ has holonomy in $\chi_X \setminus K$;*
- *a vertical segment v of the normalized flat surface E^1_C has the length less than $\frac{1}{\epsilon}$;*
- *the normalized distance of v from the singular set Z_C of E^1_C is more than ϵ,*

then, $|\eta'(t)| < \epsilon$ for $t \in [0,1]$ and $\int_0^\ell |\eta'(t)| < \epsilon$. In particular, $|\eta(t)| < \epsilon$ for all $t \in [0,1]$.

Proof. As s_t changes smoothly, we consider the surface $\cup_{u\in[0,t]} s_u$ in $\mathbb{H}^3$ spanned by s_t. For each $t \in [0,\ell]$, let $A(t)$ be the area of the surface $\cup_{u\in[0,t]} s_u$. The total curvature bound in Proposition 3.5 and the angle bounds in Lemma 3.3 imply, using the Gauss-Bonnet theorem, that: for every $\epsilon > 0$, if K is sufficiently large, then $A(\ell) < \epsilon$ and $A'(t) < \epsilon/\ell$ for all $t \in [0,\ell]$. Therefore, by the Gauss-Bonnet theorem and Lemma 3.1 (4), $|\eta'(t)|$ is bounded from above the curvature. Thus Proposition 3.5 implies the assertion. □

Let α be the bi-infinite geodesic in $\mathbb{H}^3$ through $\mathrm{Ep}(v(0))$ and $\mathrm{Ep}(v(\ell))$. Let p_1, p_2 denote the endpoints of α in $\mathbb{CP}^1$. If a hyperbolic plane in $\mathbb{H}^3$ is orthogonal to α, then its ideal boundary is a round circle in $\mathbb{CP}^1 \setminus \{p_1, p_2\}$. Moreover $\mathbb{CP}^1 \setminus \{p_1, p_2\}$ is foliated by round circles which bound hyperbolic planes orthogonal to α.

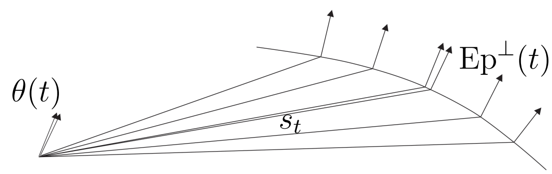

FIGURE 2.

If a hyperbolic plane in $\mathbb{H}^3$ contains the geodesic α, then its ideal boundary is a round circle containing p_1 and p_2. Then, by considering all such hyperbolic planes, we obtain another foliation $\mathcal{V}$ of $\mathbb{C}\mathrm{P}^1 \setminus \{p_1, p_2\}$ by circular arcs connecting p_1 and p_2. Then $\mathcal{V}$ is orthogonal to the foliation by round circles. Note that $\mathcal{V}$ has a natural transversal measure given by the angles between the circular arcs at p_1 (and p_2). Then the transversal measure is invariant under the rotations of $\mathbb{H}^3$ about α, and its total measure is 2π. Given a smooth curve c on $\mathbb{C}\mathrm{P}^1 \setminus \{p_1, p_2\}$ such that c decomposes into finitely many segments $c_1, c_2, \dots c_n$ which are transversal to $\mathcal{V}$, possibly, except at their endpoints. Let $\mathcal{V}(c)$ denote the "total" transversal measure of c given by $\mathcal{V}$, the sum of the transversal measures of $c_1, c_2, \dots, c_n$. Then, Proposition 3.6 implies the following.

Corollary 3.7. *For every $\epsilon > 0$, there is a bounded subset $K \subset \chi_X$, such that, if*

- *$C \in \mathsf{P}_X$ has holonomy in $\chi_X \setminus K$,*
- *a vertical segment v of E_C has the normalized length less than $\frac{1}{\epsilon}$, and*
- *the normalized distance of v from the zeros Z_C is more than ϵ,*

then, the curve $f|v\colon [0, \ell] \to \mathbb{C}\mathrm{P}^1$ intersects $\mathcal{V}$ at angles less than ϵ, and the total $\mathcal{V}$-transversal measure of the curve is less than ϵ.

Definition 3.8. *Let v be a unite tangent vector of $\mathbb{H}^3$ at $p \in \mathbb{H}^3$. Let H be a totally geodesic hyperbolic plane in $\mathbb{H}^3$. For $\epsilon > 0$, v is ϵ-almost orthogonal to H if $dist_{\mathbb{H}^3}(H, p) < \epsilon$ and the angle between the geodesic g tangent to v at p and H is ϵ-close to $\pi/2$.*

Fix a metric on the unite tangent space $T^1\mathbb{H}^3$ invariant under $\mathrm{PSL}(2, \mathbb{C})$. For $\epsilon > 0$, let $N_\epsilon Z^1_{X,\rho}$ denote the ϵ-neighborhood of the singular set $Z^1_{X,\rho}$ of the normalized flat surface $E^1_{X,\rho}$.

Theorem 3.9. *Fix arbitrary $X \in \mathsf{T} \sqcup \mathsf{T}^*$. For every $\epsilon > 0$, if a bounded subset $K_\epsilon \subset \chi_X$ is sufficiently large, then, for all $\rho \in \chi_X \setminus K_\epsilon$,*

1. *if a vertical segment v in $E^1_{X,\rho} \setminus N_\epsilon Z^1_{X,\rho}$ has length less than $\frac{1}{\epsilon}$, then the total curvature of $\mathrm{Ep}_{X,\rho}|v$ is less than ϵ, and*
2. *if a horizontal segment h in $E^1_{X,\rho} \setminus N_\epsilon Z^1_{X,\rho}$ has length less than $\frac{1}{\epsilon}$, then for the vertical tangent vectors w along the horizontal segment h, their images $\mathrm{Ep}^*_{X,\rho}(w)$ are ϵ-close in the unite tangent bundle of $\mathbb{H}^3$.*

Proof. (1) is already by Proposition 3.5. By [Bab, Proposition 4.7], we have (2). 3.9

4. Comparing measured foliations

4.1. **Thurston laminations and vertical foliations.** Let L_1, L_2 be measured laminations or foliations on a surface F. Then L_1 and L_2 each define a pseudo-metric almost everywhere on F: for all $x, y \in F$ not contained in a leaf of L_i with atomic measure, consider the minimal transversal measure of all arcs connecting x to y. We say that, for $\epsilon > 0$, L_1 is $(1+\epsilon, \epsilon)$-quasi-isometric to L_2, if for almost all $x, y \in F$,

$$(1+\epsilon)^{-1} d_{L_1}(x,y) - \epsilon < d_{L_2}(x,y) < (1+\epsilon) d_{L_1}(x,y) + \epsilon.$$

We shall compare a measured lamination of the Thurston parametrization and a measured foliation from the Schwarzian parametrization of a $\mathbb{C}\mathrm{P}^1$-surface.

Theorem 4.1. *For every $\epsilon > 0$, there is $r > 0$ with the following property: For every $C \in \mathsf{P} \sqcup \mathsf{P}^*$, then, letting (E, V) be its associated flat surface, if disk D in E has radius less than $\frac{1}{\epsilon}$ and the distance between D and the singular set Z of E is more than r, then the vertical foliation V of C is $(1+\epsilon, \epsilon)$-quasi isometric to $\sqrt{2}$ times the Thurston lamination L of C on D.*

Proof of Theorem 4.1. It suffices to show the assertion when D is a unite disk. Since D contains no singular point, we can regard D as a disk in $\mathbb{C}$ by the natural coordinates given by the quadratic differential. The scaled exponential map

$$\exp(\sqrt{2}\,*) \colon \mathbb{C} \to \mathbb{C}^* \colon z \mapsto \exp(\sqrt{2} z).$$

is a good approximation of the developing map sufficiently away from zero (Lemma 3.2), which was proved using Dumas' work [Dum17]). Let C_0 be the $\mathbb{C}\mathrm{P}^1$-structure on $\mathbb{C}$ whose developing map is $\exp(\sqrt{2}\,*)$. The next lemma immediately follows from the construction of Thurston coordinates.

Lemma 4.2. *The Thurston lamination on C_0 is the vertical foliation of $\mathbb{C}$ with a transversal measure is given by the horizontal Euclidean distance.*

Let D_x be the maximal disk in $\tilde{C}$ centered at x. Let $D_{0,x}$ be the maximal disk in C_0 centered at x by the inclusion $D \subset \mathbb{C}$. When $\mathbb{C}\mathrm{P}^1$ is identified with $\mathbb{S}^2$ so that the center O of the disk D map to the north pole and the maximal disk in $\tilde{C}$ centered at O maps to the upper hemisphere. If $r > 0$ is sufficiently large, then the $\mathrm{dev}\,|D$ is close to $\exp(\sqrt{2}\,*)$. Then, for every $x \in D$, its maximal disk D_x in $\tilde{C}$ is ϵ-close to the maximal disk $D_{0,x}$ in C_0, and the ideal point $\partial_\infty D_x$ is ϵ-Hausdorff close to the idea boundary $\partial_\infty D_{0,x}$ on $\mathbb{S}^2$.

Therefore, by [Bab17, Theorem 11.1, Proposition 3.6], the convergence of canonical neighborhoods implies the assertion. 4.1

A staircase polygon is a polygon in a flat surface whose edges are horizontal or vertical (see Definition 5.1).

Theorem 4.3. *For every $X \in \mathsf{T} \sqcup \mathsf{T}^*$ and every $\epsilon > 0$, there is a constant $r > 0$ with the following property: Suppose that C is a $\mathbb{C}\mathrm{P}^1$-structure on X and C contains a staircase polygon P w.r.t.*

its flat surface structure (E, V), such that the $(E$-$)$distance from ∂P to the singular set Z of E is more than r. Then, letting $\mathcal{L}$ denote the Thurston lamination of C, the restriction of $\mathcal{L}$ of C to P with its transversal measured scaled by $\sqrt{2}$ is $(1+\epsilon, \epsilon)$-quasi-isometric to the vertical foliation V on P up to a diffeomorphism supported on the $r/2$-neighborhood of the singular set in P.

Proof. Let $N_{r/2}Z$ denote the $r/2$-neighborhood of Z. If $r > 0$ is sufficiently large, then, for each disk D of radius $\frac{r}{4}$ centered at a point on $E \setminus N_{r/2}(Z)$, the assertion holds by Theorem 4.1.

Since $\partial P \cap N_{r/2}Z = \emptyset$, there is an upper bound for lengths of edges of such staircase polygons P with respect to the normalized Euclidean metric E^1.

Lemma 4.4. *For every $\epsilon > 0$, if $r > 0$ is sufficiently large, then for every vertical segment v of $V|P$ whose distance from the singular set Z is more than $r/2$, we have $\mathcal{L}(v) < \epsilon$.*

Proof. This follows from Corollary 3.7. □

By Theorem 4.1 and Lemma 4.4, V and $\mathcal{L}$ are $(1+\epsilon, \epsilon)$-quasi-isometric on P minus $N_{r/2}Z$. Note that V and $\mathcal{L}$ in $P \cap N_{r/2}$ are determined by V and $\mathcal{L}$ in $P \setminus N_{r/2}$ up to an isotopy, respectively. Therefore, as desired, V and $\mathcal{L}$ are $(1+\epsilon, \epsilon)$-quasi-isometric on P, up to a diffeomorphism supported on $N_{r/2}Z$. 4.3

4.2. **Horizontal foliations asymptotically coincide.** For a group G, a G-tree is a metric tree with an isometric group action. A G-tree T is minimal, if there is no proper subtree T' invariant under the G-action. A (non-trivial) G-tree is unipotent if G fixes a point of the ideal boundary $\partial_\infty T$ of T.

Lemma 4.5 (p231 in [Kap01])**.** *Let T be a non-unipotent minimal G-tree. Then, for every segment s in T, there is a hyperbolic element $\gamma \in G$ such that the axis of γ contains s.*

Let $X, Y \in \mathsf{T} \sqcup \mathsf{T}^*$ with $X \neq Y$. Let $\chi_X = \mathrm{Hol}\,\mathsf{P}_X$ and let $\chi_Y = \mathrm{Hol}\,\mathsf{P}_Y$, the holonomy varieties of X and Y, respectively. Suppose that ρ_i is a sequence in $\chi_X \cap \chi_Y$ which leaves every compact in χ. Then, let $C_{X,i}$ and $C_{Y,i}$ be the $\mathbb{C}\mathrm{P}^1$-structures on X and Y, respectively, with holonomy ρ_i. Similarly, let $H_{X,i}$ and $H_{Y,i}$ denote the horizontal measured foliations of $C_{X,i}$ and $C_{Y,i}$. Then, up to a subsequence, we may assume that ρ_i converges to a $\pi_1(S)$-tree T in the Morgan-Shalen boundary of χ, and that the projective horizontal foliations $[H_{X,i}]$ and $[H_{Y,i}]$ converge to $[H_X]$ and $[H_Y] \in \mathsf{PML}(S)$, respectively, as $i \to \infty$.

Let $\tilde{H}_X$ be the total lift of the horizontal foliation H_X to the universal cover of X, which a $\pi_1(S)$-invariant measured foliation. Then, collapsing each leaf of $\tilde{H}_X$ to a point, we obtain a $\mathbb{R}$-tree T_X, where the metric is induced by the transversal measure (dual tree of $\tilde{H}$). Let $\phi_X \colon \tilde{S} \to T_X$ be the quotient collapsing map, which commutes with the $\pi_1(S)$-action. By Dumas ([Dum17, Theorem A, §6]), there is a unique straight map $\psi_X \colon T_X \to T$ such that ψ_X is also $\pi_1(S)$-equivariant, and that every non-singular vertical leaf of $\tilde{V}|_X$ maps to a geodesic in T.

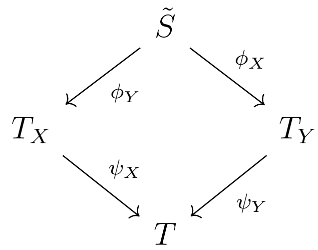

FIGURE 3.

Similarly, let $\phi_Y : \tilde{S} \to T_Y$ be the map which collapses each leaf of $\tilde{H}_Y$ to a point. Let $\psi_Y : T_Y \to T$ be the straight map. Let d_T be the induced metric on T.

Lemma 4.6. *The diagram in Figure 3* roughly commutes*. That is, there is $K > 0$ such that for every $p \in \tilde{S}$, $d_T(\psi_X \circ \phi_X(p), \psi_Y \circ \phi_Y(p)) < K$.*

Proof. Since S is a closed surface, we can pick a compact fundamental domain D in $\tilde{S}$. Then, let K be the diameter of the union of $\psi_X \circ \phi_X(K)$ and $\psi_Y \circ \phi_Y(K)$ in T. Then, the assertion follows from the $\pi_1(S)$-equivariant property. □

Next we show the unmeasured horizontal foliations coincide in the limit.

Proposition 4.7. $|H_X| = |H_Y|$.

Proof. Let h be a smooth leaf of H_X. It suffices to show that h is contained in a leaf of H_Y up to an isotopy of S (note that, by the same argument, every smooth leaf of H_Y is contained in a leaf of H_X).

Suppose, to the contrary, that h is not contained in a leaf of H_Y. Then $\phi_Y(h)$ is a quasi-geodesic in T_Y. Let h_Y be the geodesic in T_Y fellow-traveling with $\phi_Y(h)$. By Lemma 4.6, $\psi_Y \circ \phi_Y(h)$ is a bounded subset of T.

Consider the restriction of ψ_Y to the geodesic h_Y. Then, by the properties of the straight map, the set of non-locally injective points on h_Y is a discrete set. By Lemma 4.6, $\psi_Y \circ \phi_Y(h)$ is bounded. Therefore, the non-injective points decompose h_Y into segments of length bounded from above. Thus, there are geodesics ℓ_1, ℓ_2 in T_Y, such that ℓ_1 and ℓ_2 are disjoint and each of the geodesics intersects h_Y in a single point, and that ψ_Y embeds ℓ_1 and ℓ_2 onto disjoint geodesics in T.

Then, by Lemma 4.5, there are $\gamma_1, \gamma_2 \in \pi_1(S)$ acting T_Y as hyperbolic translations, such that, letting m_1, m_2 be their geodesic axes in T_Y,

- m_1 and m_2 are disjoint;
- m_1 and m_2 each intersect h_Y in a single point;
- the axis of γ_1 acting on T is disjoint from the axis of γ_2 acting on T.

Then, the axis of γ_1 and of γ_2 are transversal to h in $\tilde{S}$. Therefore, on T_X, the axis of γ_1 intersects the axis of γ_2. Hence, on T, the axis of γ_1 must intersect the axis of γ_2, which is a contradiction. 4.7

Proposition 4.7 immediately implies the following.

Corollary 4.8. *There is a $\pi_1(S)$-equivariant homeomorphism $\eta\colon T_X \to T_Y$ such that $\psi_X = \psi_Y \circ \eta$.*

Moreover, we prove that $H_{X,i}$ and $H_{Y,i}$ are asymptotic as measured laminations in the following sense.

Theorem 4.9. *There are sequences of positive real numbers k_i and k'_i, such that $\lim_{i\to\infty} \frac{k_i}{k'_i} = 1$ and $\lim_{i\to\infty} k_i H_{X,\rho_i} = \lim_{i\to\infty} k'_i H_{Y,\rho_i}$ in ML. In particular $[H_X] = [H_Y]$.*

Proof. Let E_{X,H_X} be the flat surface given by (X, H_X). Take finitely many monotone staircase loops $\ell_1, \ell_2, \dots, \ell_n$ on E_{X,H_X} (Definition 5.1) such that ℓ_i does not intersect any singular point of E_{X,H_X} for each $i = 1, 2, \dots, n$, and that the map $\mathsf{ML} \to \mathbb{R}^n_{\geq 0}$ defined by

$$L \mapsto (L(\ell_1), L(\ell_2), \dots, L(\ell_n))$$

is injective, where $L(\ell_i)$ denotes the transversal measure of ℓ_i given by L.

Proposition 4.10. *For every $\epsilon > 0$, if $i \in \mathbb{Z}_{>0}$ is sufficiently large, then*

- *$\rho_i(\ell_j)$ is hyperbolic for each $j = 1, \dots, n$, and*
- *the translation length of $\rho_i(\ell_j)$ is $(1+\epsilon)$-bilipschitz to both transversal measures $\sqrt{2}H_{X,\rho_i}(\ell_j)$ and $\sqrt{2}H_{Y,\rho_i}(\ell_j)$.*

Proof. For each $j = 1, 2, \dots, n$, if i is sufficiently large, then there is a staircase curve $\ell_{j,i}$ such that $\ell_{j,i}$ smoothly converging to ℓ_j. Set $\ell_{j,i} = v_{1,i} \sqcup h_{1,i}, \sqcup \dots \sqcup v_{n,i} \sqcup h_{n,i}$, where $v_{k,j}$ is a vertical segment and $h_{k,j}$ is a horizontal segment, so that $v_{k,j}$ and $h_{k,j}$ converge to a vertical and a horizontal segment of ℓ_j, respectively, as $i \to \infty$. If i is sufficiently large, by Lemma 3.1 (2) (4), $\mathrm{Ep}_{X,i}(v_{k,j})$ is ϵ-close to a geodesic segment of length $\sqrt{2}\,\mathrm{length}_{E_{X,i}} v_{k,j}$ in $\mathbb{H}^3$. The $\mathrm{Ep}_{X,i}(v_{k,j})$ has length less than ϵ by Lemma 3.1 (1), but the vertical tangent vectors are very close by Theorem 3.9 (2). Therefore, if i is sufficiently large, $\mathrm{Ep}_i |\ell_j$ is $(1+\epsilon, \epsilon)$-quasigeodesic with respect to the transversal measure given by H_{X,ρ_i} times $\sqrt{2}$. Thus $\rho_i(\ell_j)$ is a hyperbolic element whose translation length is $(1+\epsilon)$-bilipschitz to $\sqrt{2}H_{X,\rho_i}(\ell_j)$.

There is a sequence $r_i > 0$ converging to zero such that, for every loop $\ell \in \pi_1(S)$, the translation length of $\rho_i(\ell)$ times r_i converging to the translation length of ℓ on T as $i \to \infty$ (see for example, [Kap01, Theorem 10.24]). Then, by Corollary 4.8, for every $j = 1, 2, \dots, n$, if i is sufficiently large, then $\sqrt{2}H_{Y,\rho_i}(\ell_j)$ is $(1+\epsilon)$-bilipschitz to $\rho_i(\ell_j)$. □

By Proposition 4.10,

$$\frac{H_{Y,\rho_i}(\ell_j)}{H_{X,\rho_i}(\ell_j)} \to 1,$$

for each $i = 1, \dots, n$. 4.9

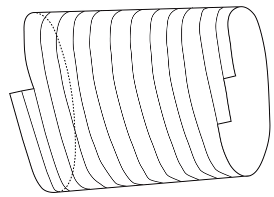

FIGURE 4. A spiral cylinder decomposed into rectangles.

5. Train tracks

5.1. **Train-track graphs.** A train track graph is a C^1-smooth graph Γ embedded in a smooth surface in the following sense:

- Each edge of Γ is C^1-smoothly embedded in the surface.
- At every vertex v of Γ, the unite vectors at v tangent to the edges starting from v are unique up to a sign, and the opposite unite tangent vectors are both realized by the edges.

A weight system w of a train-track graph is an assignment of a non-negative real number $w(e)$ to each edge e of Γ, such that at every vertex v of Γ, letting $e_1, \dots, e_n$ be the edges from one direction and $e'_1, e'_2, \dots, e'_m$ the opposite direction, the equation $w(e_1) + w(e_2) + \cdots + w(e_n) = w(e'_1) + w(e'_2) + \cdots + w(e'_n)$ holds.

5.2. **Singular Euclidean surfaces.** *A singular Euclidean structure* on a surface is given by a Euclidean metric with a discrete set of cone points. In this paper, all cone angles of singular Euclidean structures are π-multiples, as we consider singular Euclidean structures induced by holomorphic quadratic differentials. In addition, by a singular Euclidean polygon, we mean a polygon with geodesic edges and discrete set of singular points whose cone angles are π-multiples. A polygon is right-angled if the interior angles are $\pi/2$ or $\pi/3$ at all vertices. A Euclidean cylinder is a non-singular Euclidean structure on a cylinder with geodesics boundary. By a flat surface, we mean a singular Euclidean surface with (singular) vertical and horizontal foliations, which intersect orthogonally.

Definition 5.1. *Let E be a flat surface. A curve ℓ on E is a* staircase*, if ℓ contains no singular point and ℓ is piecewise vertical or horizontal. Then, a staircase curve is* monotone *if the angles at the vertices alternate between $\pi/2$ and $3\pi/2$ along the curve, so that it is a geodesic in the L^∞-metric. A staircase curve is* vertically geodesic*, if for every horizontal segment, the angle at one endpoint is $\pi/2$ and the angle at the other endpoint is $3\pi/2$.*

A staircase surface *is a flat surface whose boundary components are staircase curves. A (L^∞-)*convex staircase polygon *P is a staircase polygon, such that, if p_1, p_2 are adjacent vertices of P, then at least, one of the interior angles at p_1 and p_2 is $\pi/2$. A staircase cylinder A embedded in a flat surface E is a* spiral cylinder*, if A contains no singular point and each boundary component is a monotone staircase loop (see Figure 4).*

Clearly, we have the following decomposition.

Lemma 5.2. *Every spiral cylinder decomposes into finitely many rectangles when cut along some horizontal segments each starting from a vertex of a boundary component. (Figure 4.)*

5.3. **Surface train tracks.** Let F be a compact surface with boundary, such that each boundary component of F is either a smooth loop or a loop with even number of corner points. Then a (boundary-)marking of F is an assignment of "horizontal" or "vertical" to every smooth boundary segment, such that every smooth boundary component is horizontal and, along every non-smooth boundary component, horizontal edges and vertical edges alternate. From the second condition, every boundary component with at least one corner point has even number of conner points.

For example, a marking of a rectangle is an assignment of horizontal edges to one pair of opposite edges and vertical edges to the other pair, and a marking of a $2n$-gon is an assignment of horizontal and vertical edges, such that the horizontal and vertical edges alternate along the boundary. Clearly, there are exactly two ways to give a $2n$-gon a marking. A marking of a flat cylinder is the unique assignment of horizontal components to both boundary components.

Recall that a (fat) train track T is a surface with boundary and corners obtained by gluing marked rectangles R_i along their horizontal edges, in such a way that the identification is given by subdividing every horizontal edge into finitely many segments, pair up all edge segments, and identifying the paired segments by a diffeomorphism; see for example [Kap01, §11].

In this paper, we may allow any marked surfaces as branches.

Definition 5.3. *A* surface train track T *is a surface having boundary with corners, obtained by gluing marked surfaces F_i in such a way that the identification is given by (possibly) subdividing each horizontal edge and horizontal boundary circle of F_i into finitely segments, pairing up all segments, and identifying each pair of segments by a diffeomorphism.*

Given a surface train track $T = \cup F_i$, if all branches F_i are cylinders with smooth boundary and polygons, then we call T a polygonal train track.

Suppose that a surface F is decomposed into marked surfaces with disjoint interior so that the horizontal edges of marked surfaces overlap only with other horizontal edges, and vertical edges overlap with other vertical edges (except at corner points); we call this a surface train-track decomposition of F. Given a train-track decomposition of a surface F, the union of the boundaries of its branches is a finite graph on F, and we call it the edge graph.

Let $F = \cup F_i$ be a train-track decomposition of a surface F. Clearly the interior of a branch is embedded in F, but the boundary of a branch may intersect itself. The closure of a branch F_i in F is called the support of the branch, and denoted by $|F_i|$, which may not be homotopy equivalent to F_i on F.

Next, we consider geometric train-track decompositions of flat surfaces. Let E be a flat surface, and let V and H be it vertical and horizontal foliations, respectively. Then, when we say that a staircase surface F is on E, we always assume that horizontal edges of F are contained in leaves of H and vertical edges in leaves of V. Note that a marked rectangle R on E may self-intersect in its horizontal edges, so that it forms a spiral cylinder. Then a staircase train-track decomposition of a flat surface E is a decomposition of E into finitely staircase surfaces on E, so that their union along horizontal edges yields a polygonal train-track.

More generally, a trapezoidal train track decomposition of E is a surface train-track decomposition, such that each vertical edge is contained in a vertical leaf and each horizontal edge is a non-vertical line segment disjoint from the singular set of E.

Given a flat surface, we shall construct a canonical staircase train track. Let q be a holomorphic quadratic differential on a Riemann surface X homeomorphic to S. Let E be the flat surface given by q, which is homeomorphic to S. As above, let V, H be the vertical and horizontal foliations of E. Let E^1 be the unite-area normalization of E, so that $E^1 = \frac{E}{\operatorname{Area} E}$.

Let $z_1, z_2 \dots z_p$ be the zeros of q, which are the singular points of E. For each $i = 1, \dots, p$, let ℓ_i be the singular leaf of V containing z_i. For $r > 0$, let n_i be the closed r-neighborhood of z_i in ℓ_i with respect to the path metric of ℓ_i induced by E^1 (vertical r-neighborhood). Let N_r be their union $n_1 \cup \dots \cup n_p$ in E, which may not be a disjoint union as a singular leaf may contain multiple singular points. If $r > 0$ is sufficiently small, then each (connected) component of N_r is contractible. Let $\mathsf{QD}^1(X)$ denote the set of all unite area quadratic differentials on X. Since the set of unite area differentials on X is a sphere, by its compactness, we have the following.

Lemma 5.4. *For every X in $\mathsf{T}^+ \cup \mathsf{T}^-$, if $r > 0$ is sufficiently small, then, for all $q \in \mathsf{QD}^1(X)$, each component of N_r is a simplicial tree (i.e. contractible).*

Fix X in $\mathsf{T}^+ \cup \mathsf{T}^-$, and let $r > 0$ be the small value given by Lemma 5.4. Let p be an endpoint of a component of N_r. Then p is contained in horizontal geodesic segments, in E, of finite-length, such that their interiors intersect N only in p. Let h_p be a maximal horizontal geodesic segment or a horizontal geodesic loop, such that the interior of h_p intersects N_r only in p. If h_p is a geodesic segment, then the endpoints of h_p are also on N_r. If h_p is a geodesic loop, h_p intersects N_r only in p.

Consider the union $\cup_p h_p$ over all endpoints p of N_r. Then $N_r \cup (\cup_p h_p)$ decomposes E into staircase rectangles and, possibly, flat cylinders. Thus we obtain a staircase train track decomposition whose branches are all rectangles.

Next, we construct a polygonal train-track structure of E so that the singular points are contained in the interior of the branches. Let $b_i \in \mathbb{Z}_{\geq 3}$ be the balance of the singular leaf ℓ_i at the zero z_i.

We constructed the vertical r-neighborhood n_i of the zero z_i. Let P_i^r be the set of points on E whose horizontal distance from n_i is at most $\sqrt[4]{r}$ (horizontal neighborhood). Then, as E is fixed, if $r > 0$ is sufficiently small, then P_i^r is a convex staircase $2b_i$-gon whose interior contains z_i. We say that P_i^r is the $(r, \sqrt[4]{r})$-neighborhood of z_i.

When we vary $q \in \mathsf{QD}^1(X)$, fixing r, the convex polygons for different zeros may intersect. Nonetheless, by compactness, we have the following.

Lemma 5.5. *Let $X \in \mathsf{T}^+ \cup \mathsf{T}^-$. If $r > 0$ is sufficiently small, then, for every $q \in \mathsf{QD}^1(X)$, each connected component of $P_1^r \cup P_2^r \cup \dots \cup P_{n_q}^r$ is a staircase polygon.*

Then, let $r > 0$ and $P^r (= P_q^r)$ be $P_1^r \cup P_2^r \cup \dots \cup P_{p_q}^r$ as in Lemma 5.5. Then, similarly, for each horizontal edge h of P^r, let $\hat{h}$ be a maximal horizontal geodesic segment or a horizontal geodesic loop on E, such that the interior point of $\hat{h}$ intersects P^r exactly in h. Then, either

- $\hat{h}$ is a horizontal geodesic segment whose endpoints are on the boundary of P^r, or
- $\hat{h}$ is a horizontal geodesic loop intersecting P^r exactly in h.

Consider the union $\cup_h \hat{h}$ over all horizontal edges h of P^r. Then the union decomposes $E \setminus P^r$ into finitely many staircase rectangles and, possibly, flat cylinders. Thus we have a staircase train-track structure, whose branches are polygons and flat cylinders. Note that the singular points are all contained in the interiors of polygonal branches.

For the later use, we modify the train track to eliminate *thin* rectangular branches, i.e. they has short horizontal edges. Note that each vertical edge of a rectangle is contained in a vertical edge of a polygonal branch. Thus, if a rectangular branch R has horizontal length less than $\sqrt[4]{r}$, then naturally glue R with both adjacent polygonal branches along the vertical edges of R. After applying such gluing for all thin rectangles, we obtain a train-track structure t^r of E.

Lemma 5.6. *For every $X \in \mathsf{T}^+ \cup \mathsf{T}^-$, if $r > 0$ is sufficiently small, then, for every $q \in \mathsf{QD}^1(X)$, the branches of the train-track structure t^r on E are staircase polygons and staircase flat cylinder, and every rectangular branch of t^r has width at least $\sqrt[4]{r}$.*

Definition 5.7. *Let E be a flat surface. A train-track structure T_1 is a* refinement *of another train-track structure T_2 of E, if the T_1 is a subdivision of T_2 (which includes the case that $T_1 = T_2$).*

Let E_i be a sequence of flat surfaces converging to a flat surface E. Let T be a train-track structure on a flat surface E, and let T_i be a sequence of train-track structures on a flat surface E_i for each i. Then T_i converges *to T as $i \to \infty$ if the edge graph of T_{k_i} converging to the edge graph of T_∞ on E in the Hausdorff topology. Then T_i* semi-converges *to T as $i \to \infty$ if every subsequence T_{k_i} of T_i subconverges to a train-track structure T' on E, such that T' is a refinement of T.*

Lemma 5.8. *t^r_q is* semi-continuous *in the Riemann surface X and the quadratic differential q on X, and the (small) train-track parameter $r > 0$ given by Lemma 5.6. That is, if $r_i \to r$ and $q_i \to q$, then $t^{r_i}_{q_i}$ semi-converges to t^r_q as $i \to \infty$.*

Proof. Clearly, the flat surface E changes continuously in q. Accordingly P^r changes continuously in the Hausdorff topology in q and r. Then the semi-continuity easily follows from the construction of t^r_q. □

5.4. **Straightening foliations on flat surfaces.** Let E be the flat surface homeomorphic to S, and let V be its vertical foliation. Let V' be another measured foliation on S.

For each smooth leaf ℓ of V', consider its geodesic representative $[\ell]$ in E. If ℓ is non-periodic, the geodesic representative is unique. Suppose that ℓ is periodic. Then, if $[\ell]$ is not unique, then the set of its geodesic representatives foliates a flat cylinder in E.

Consider all geodesic representatives, in E, of smooth leaves ℓ of V', and let $[V']$ be the set of such geodesic representative and the limits of those geodesics. We still call the geodesics of $[V']$ leaves. We can regard $[V']$ as a map from a lamination $[V']$ on S to E which is a leaf-wise embedding.

6. Compatible surface train track decompositions

Let $X, Y \in \mathsf{T} \sqcup \mathsf{T}^*$ with $X \neq Y$. Clearly, for each $\rho \in \chi_X \cap \chi_Y$, there are unique $\mathbb{C}\mathrm{P}^1$-structures C_X and C_Y on X and Y, respectively, with holonomy ρ. Set $C_X = (X, q_X)$ and $C_Y = (Y, q_Y)$, in Schwarzian coordinates, where $q_X \in \mathsf{QD}(X)$ and $q_Y \in \mathsf{QD}(Y)$. Then, define $\eta\colon \chi_X \cap \chi_Y \to \mathsf{PML} \times \mathsf{PML}$ to be the map taking $\rho \in \chi_X \cap \chi_Y$ to the ordered pair of the projectivized horizontal foliations of $q_{X,\rho}$ and $q_{Y,\rho}$. Let $\Lambda_\infty \subset \mathsf{PML} \times \mathsf{PML}$ be the set of the accumulation points of η towards the infinity of χ — namely, $(H_X, H_Y) \in \Lambda_\infty$ if and only if there is a sequence ρ_i in $\chi_X \cap \chi_Y$ which leaves every compact in χ such that $\eta(\rho_i)$ converges to (H_X, H_Y) as $i \to \infty$.

Let $\Delta \subset \mathsf{PML} \times \mathsf{PML}$ be the diagonal set. Then, by Proposition 4.7, Λ_∞ is contained in Δ. Given a Riemann surface X and a projective measured foliation H, by Hubbard and Masur [HM79], there is a unique holomorphic quadratic differential on X such that its horizontal foliation coincides with the measured foliation. Let $E_{X,H} = E^1_{X,H}$ denote the unite-area flat surface induced by the differential. Given $H_X \in \mathsf{PML}$, let V_X be the vertical measured foliation realized by (X, H_X), and let V_Y be the vertical foliation of (Y, H_Y).

Noting that a smooth leaf of a (singular) foliation may be contained in a singular leaf of another foliation, we let Δ^* be the set of all $(H_X, H_Y) \in \mathsf{PML} \times \mathsf{PML}$ which satisfies either

- there is a leaf of H_X contained in a leaf of V_Y;
- there is a leaf of V_Y contained in a leaf of H_X;
- there is a leaf o H_Y contained in a leaf of V_X; or
- there is a leaf of V_X contained in a leaf of H_X.

Then Δ^* is a closed measure zero subset of $\mathsf{PML} \times \mathsf{PML}$, and disjoint from the diagonal Δ. (For the proof of our theorems, we will only consider a sufficiently small neighborhood of Δ, which is disjoint from Δ^*.)

6.1. **Straightening maps.** Fix a transversal pair $(H_X, H_Y) \in (\mathsf{PML} \times \mathsf{PML}) \setminus \Delta^*$. Let p be a smooth point in E_{Y,H_Y}, and let $\tilde{p}$ be a lift of p to the universal cover $\tilde{E}_{Y,H_Y}$.

Let v be the leaf of the vertical foliation $\tilde{V}_Y$ on $\tilde{E}_{Y,H_Y}$ containing $\tilde{p}$, and let h be the leaf of the horizontal foliation $\tilde{H}_Y$ on the universal cover containing $\tilde{p}$. Then, let $[v]_X$ denote the geodesic representative of v in $\tilde{E}_{X,H_X}$, and let $[h]_X$ denote the geodesic representative of h in $\tilde{E}_{X,H_X}$. Since $\tilde{E}_{X,H_X}$ is a non-positively curved space, $[v]_X \cap [h]_Y$ is a point or a segment of a finite length in $\tilde{E}_{X,H_X}$; let $\mathrm{st}(p)$ be the subset of E_{X,H_X} obtained by projecting the point or a finite segment.

6.2. **Non-transversal graphs.** Let E be a flat surface with horizontal foliation H. Let $\ell\colon \mathbb{R} \to E$ be a (non-constant) geodesic on E parametrized by arc length. A horizontal segment of ℓ is a maximal segment of ℓ which is tangent to the horizontal foliation H. Note that a horizontal segment is, in general, only immersed in E.

Let $X, Y \in \mathsf{T} \sqcup \mathsf{T}^*$ with $X \neq Y$, and let $(H_X, H_Y) \in (\mathsf{PML} \times \mathsf{PML}) \setminus \Delta^*$. For a smooth leaf ℓ_Y of V_Y, let $[\ell_Y]_X$ denote the geodesic representative of ℓ_Y on the flat surface E_{X,H_X}. The geodesic $[\ell_Y]_X$ is not necessarily embedding and should be regarded as an immersion $\mathbb{R} \to E_{X,H_X}$.

Lemma 6.1. *Every horizontal segment v of $[\ell_Y]_X$ is a segment (i.e. finite length) connecting singular points of E.*

Proof. If h has infinite length, then ℓ_Y must be contained in a leaf of H_Y. This contracts against $(H_X, H_Y) \in (\mathsf{PML} \times \mathsf{PML}) \setminus \Delta^*$. □

Let $[V_Y]_X$ denote the set of all geodesic representatives of smooth leaves of V_Y on $E_{X,H}$. Let $G_Y \subset E_{X,H_X}$ be the union of (the images of) all horizontal segments of $[V_Y]_X$. Then it follows that G_Y is a finite graph, such that

- every connected component of G is contained in a horizontal leaf of H_X, and
- every vertex of G is a singular point of $E_{X,H}$.

Proposition 6.2. *For all distinct $X, Y \in \mathsf{T} \sqcup \mathsf{T}^*$ and all $(H_X, H_Y) \in \mathsf{PML} \times \mathsf{PML} \setminus \Delta^*$, there is $B > 0$, such that, for all leaves ℓ_Y of V_Y, every horizontal segment of the geodesic representative $[\ell_Y]_X$ is bounded by B from above.*

Proof. By Lemma 6.1, each horizontal segment has a finite length and its endpoints are at singular points of $E_{X,H}$.

Suppose, to the contrary, that there is no upper bound. Then there is a sequence of horizontal segments s_i in $[V_Y]_X$ with $\text{length}_{i \to \infty} s_i \to \infty$ as $i \to \infty$. Then $\sqcup s_i$ is embedded in V_Y. Therefore, since $\cup s_i = G_Y$, there is a small regular neighborhood N of G_Y and a small homotopy of $\sqcup s_i \to G_Y$, such that $\sqcup s_i$ is embedded in N and the endpoints of s_i are on the boundary of N. This can not happen as G_Y is a finite graph and $\text{length}\, s_i \to \infty$ as $i \to \infty$. □

By continuity and the compactness of PML, the uniformness follows:

Corollary 6.3. *The upper bound B can be taken uniformly in $H \in \mathsf{PML}$.*

Let V_Y' denote $[V_Y]_X \setminus G_Y$, the set of geodesic representatives $[\ell_Y]_X$ minus their horizontal segments, for all leaves ℓ_Y of V_Y. Then V_Y' is transversal to H_X at every point, and the angle between them (§2.3) is uniformly bounded:

Lemma 6.4. *For every $(H_X, H_Y) \in \mathsf{PML} \times \mathsf{PML} \setminus \Delta^*$, $\angle_{E_{X,H}}(V_Y', H_X) > 0$.*

Proof. Suppose, to the contrary, that there is a sequence of distinct points x_i in V_Y' such that $(0 <) \angle_{x_i}(V_Y', H_X) \to 0$ as $i \to \infty$. We may, in addition, assume that x_i are non-singular points. Let ℓ_i be a leaf of $[V_Y]_X$ such that $\angle_{x_i}(\ell_i, H_X) \to 0$ as $i \to \infty$. Then, since E_X is compact, up to a subsequence x_i, converges to a point $x \in E$. Then, by basic Euclidean geometry, the segments of ℓ_i near x_i are contained in parallel lines, taking a subsequence if necessarily. This is a contradiction. □

6.3. Train-track decompositions for diagonal horizontal foliations.

Definition 6.5. *Let (E, V) be a flat surface. Let T be a train track decomposition of E. A curve $\mathbb{R} \to E$ is* carried by *T, if B is a branch of T, then, for every component s of $\ell \cap intB$, both endpoints of s are on different horizontal edges of B,*

A (topological) lamination on E is carried by *T if every leaf is carried by T.*

In this paper, a train-track may have "bigon regions" which correspond to vertical edges of T. Thus a measured lamination may be carried by a train track in essentially different ways. As a lamination is usually defined up to an isotopy on the entire surface, when a measured lamination is carried by a train-track, we call it a realization of the measured lamination.

Definition 6.6. *Let (E, V) be a flat surface. Let T be a train track decomposition of E. A geodesic ℓ on E is* essentially carried by *T, if, for every rectangular branch B of T and every component s of $\ell \cap intB$,*

- *both endpoints of s are (different) horizontal edges of B, or*
- *the endpoints of s are on adjacent (horizontal and vertical edges) of B.*

The measured foliation V on E is essentially carried by *T if every smooth leaf of V is essentially carried by T.*

Because of the horizontal segment, $[V_Y]$ is not necessarily carried by $t^r_{X,H}$ even if the train-track parameter $r > 0$ is very small. Let $\mathbf{t}^r_{X,H}$ be the train-track decomposition of $E_{X,H}$ obtained by, for each component of the horizontal graph G_Y, taking the union of the branches intersecting the component. A branch of $\mathbf{t}^r_{X,H}$ is transversal if it is disjoint from G_Y, and non-transversal if it contains a component of G_Y.

Lemma 6.7. *For every $(H_X, H_Y) \in (\mathsf{PML} \times \mathsf{PML}) \setminus \Delta^*$, if $r > 0$ sufficiently small, then*

(1) $[V_Y]_X$ *is essentially carried by* $\mathbf{t}^r_{X,H_X} =: \mathbf{t}_{X,H_X}$, *and*
(2) $[V_Y]_X$ *can be homotoped along leaves of* H_X *to a measured lamination* W_Y *carried by* $\mathbf{t}_{X,H_X}$ *so that, by the homotopy, every point of* $[V_Y]_X$ *either stays in the same branch or moves to the adjacent branch across a vertical edge.*

Moreover, these properties hold in a small neighborhood of $(H_X, H_Y) \in (\mathsf{PML} \times \mathsf{PML}) \setminus \Delta^*$.

We call W_Y a realization of $[V_Y]_X$ on $\mathbf{t}_{X,H_X}$.

Proof. By the construction of the train track, each vertical edge of a rectangular branch has length less than $2r$ and horizontal edge has length at least $\sqrt[4]{r}$. Then, (1) follows from Lemma 6.4.

We shall show (2). Recall from §5.3, that the construction of T_{X,H_X} started with taking an r-neighborhood of the zeros in the vertical direction and then taking points $\sqrt[4]{r}$-close to the neighborhood in the horizontal direction. Therefore, each vertical edge of $\mathbf{t}_{X,H_X}$ has length at least $\sqrt[4]{r}$ and each horizontal edge has length less than r.

Similarly to a Teichmüller mapping, we rescale the Euclidean structure of E_{X,H_X} with area one by scaling the horizontal distance by $\sqrt[4]{r}$ and the vertical distance by $\frac{1}{\sqrt[4]{r}}$, its reciprocal. Then, by this mapping, the flat surface $E_{X,H}$ is transformed to another flat surface E'_{X,H_X} and the train-track structure $\mathbf{t}_{X,H_X}$ is transformed to $\mathbf{t}'_{X,H_X}$. Then, the horizontal edges of rectangular branches of $\mathbf{t}'_{X,H_X}$ have length at least $\sqrt{r}$, and the vertical edges have length less than $2r^{\frac{3}{4}}$. Thus, as the train track parameter $r > 0$ is sufficiently small, the vertical edge is still much shorter than the horizontal edge. Note that, the foliations V_X and H_X persist by the map, except the transversal measures are scaled.

As $r >$ is sufficiently small, the geodesic representative $[V_Y]'_X$ of V_Y on E'_{X,H_X} is almost parallel to V_X. Since N is a compact subset of $(\mathsf{PML} \times \mathsf{PML}) \setminus \Delta^*$, by Lemma 6.4, $\angle_{E_X}(H_X, [V_Y)]_X$ is bounded from below by a positive number uniformly in $\mathbf{H} = (H_X, H_Y) \in N$. Then, indeed, for every $\upsilon > 0$, if $r > 0$ is sufficiently small, then $\angle_{E'_X}(V_X, [V_Y]'_X) < \upsilon$.

Then, let ℓ be a leaf of V_Y. Let ℓ_X be the geodesic representative of ℓ in E'_X. Consider the set $N^v_{\sqrt{r}}$ of points on E'_X whose horizontal distance to the set of the vertical edges of $\mathbf{t}'_{X,H_X}$ is less than $\sqrt{r}$. Let s be a maximal segment of ℓ_X, such that s is contained in $N^h_{\sqrt{r}}$ and that each endpoint of s is connected to a vertex of $\mathbf{t}'_{X,H_X}$ by a horizontal segment (which may not be contained in a horizontal edge of $\mathbf{t}'_{X,H_X}$). Clearly, if $r > 0$ is sufficiently small, s does not intersect the same vertical edge twice nor the same branch twice.

Claim 6.8. *There is a staircase curve c on E'_{X,H_X}, such that*

- *c is $r^{\frac{3}{4}}$-close to s in the horizontal direction,*
- *each vertical segment of c is a vertical edge of $\mathbf{t}'_{X,H_X}$, and*
- *each horizontal segment of c contains no vertex of $\mathbf{t}'_{X,H_X}$ in its interior.*

(See Figure 5.)

Pick finitely many segments $s_1, \dots, s_n$ in leaves of $[V_Y]'_X$ as above, such that if a vertical edge v of $\mathbf{t}'_{X,H_X}$ intersects $[V_Y]'_X$, then there is exactly one s_i which is $r^{\frac{3}{4}}$-Hausdorff close to v. Let $c_1, \dots, c_n$ be their corresponding staircase curves on E'_X.

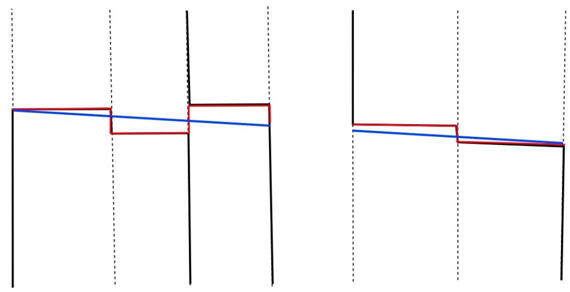

Figure 5. Examples of staircase curves given by Claim 6.8.

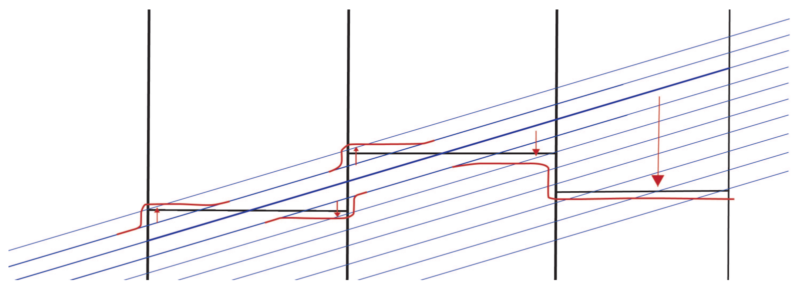

Figure 6.

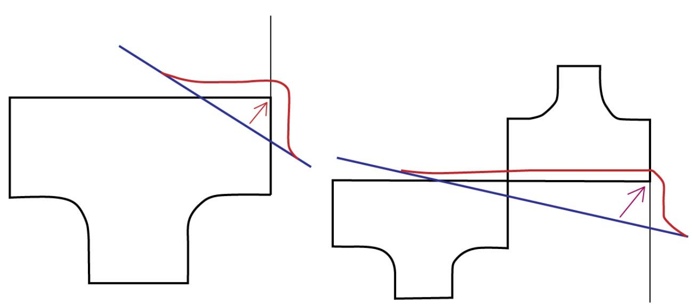

Figure 7.

Then, we can homotope $[V_Y]_X$ in a small neighborhood of the triangular region R_i bounded by s_i and c_i, such that, while homotoping, the leaves do not intersect s_i, and that the homotopy moves each point horizontally (Figure 6).

Each point on $[V_Y]'_X$ is homotoped at most to an adjacent branch. Then, after this homotopy, $[V_Y]'_X$ is carried by $\mathbf{t}'_{X,H_X}$. This homotopy induces a desired homotopy of $[V_Y]_X$. 6.7

Let $\mathbf{H} = (H_X, H_Y) \in \mathsf{PML} \times \mathsf{PML} \setminus \Delta^*$ and W_Y denote the realization of $[V_Y]_X$ on $\mathbf{t}_{X,H_X}$ given by Lemma 6.7.

A measured lamination in PML is defined up to an isotopy of the surface. The union of the vertical edges of $\mathbf{t}_{X,H_X}$ consists of disjoint vertical segments. Each vertical segment of the union

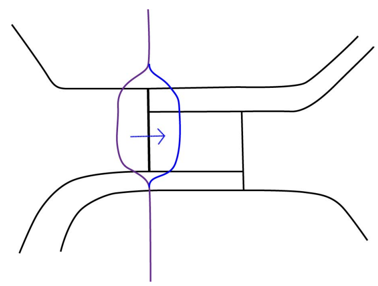

FIGURE 8. A shifting across a vertical slit.

is called a (vertical) slit. Then, a measured lamination can be carried by a train track in many different ways by homotopy across slits:

Definition 6.9 (Shifting)**.** *Suppose that T is a train-track structure of a flat surface E, and let L_1 be a realization of $L \in \mathsf{ML}$ on T. For a vertical slit v of T, consider the branches on T whose boundary intersects v in a segment. A* shifting *of L_1 across v is a homotopy of L_1 on E to another realization L_2 of L which reduces the weights of the branches on one side of v by some amount and increases the weights of the branches on the other side of v by the same amount (Figure 8). Two realizations of L on T are* related by shifting *if they are related by simultaneous shifts across some vertical slits of T.*

The homotopy of $[V_Y]_X$ in Lemma 6.7 moves points at most to adjacent branches in the horizontal direction. Thus we have the following.

Lemma 6.10. *In Lemma 6.7, the realizations given by different choices s_i are related by shifting.*

Proposition 6.11. *Let $\mathbf{H}_i = (H_{X,i}, H_{Y,i})$ be a sequence in $\mathsf{PML} \times \mathsf{PML} \setminus \Delta^*$ converging to $\mathbf{H} = (H_X, H_Y)$ in $\mathsf{PML} \times \mathsf{PML} \setminus \Delta^*$. Let W_i be a realization of $[V_{Y_i}]_{X_i}$ on $\mathbf{t}_{X,H_{X,i}}$, and let W be a realization of $[V_Y]_X$ on $\mathbf{t}_{X,H_X}$ given by Lemma 6.7. Then, a limit of the realization W_i and the realization W_∞ are related by shifting across vertical slits.*

Proof. By the semi-continuity of $\mathbf{t}_{X,H_X}$ in H_X (Proposition 6.12), the limit of the train tracks $\mathbf{t}_{X,H_{X,i}}$ is a subdivision of $\mathbf{t}_{X,H_X}$. Let $s_{i,1}, \dots, s_{i,k_i}$ be the segments from the proof of Lemma 6.7 which determine the realization W_i. The segment $s_{j,i}$ converges up to a subsequence. Then, the assertion follows from Lemma 6.10. □

In summary, we have obtained the following.

Proposition 6.12 (Staircase train tracks)**.** *For all distinct $X, Y \in \mathsf{T} \sqcup \mathsf{T}^*$ and a compact neighborhood $\mathbf{N}_\infty$ of Λ_∞ in $(\mathsf{PML} \times \mathsf{PML}) \setminus \Delta^*$, if the train-track parameter $r > 0$ is sufficiently small, then, for every $\mathbf{H} = (H_X, H_Y)$ of $\mathbf{N}_\infty$, the staircase train track $\mathbf{t}^r_{X,V_X}$ satisfies the following:*

1. *$\mathbf{t}^r_{X,H_X}$ changes semi-continuously in $\mathbf{H} \in N_\infty$.*
2. *V_Y is essentially carried by $\mathbf{t}^r_{X,H_X}$, and its realization on $\mathbf{t}^r_{X,H_X}$ changes continuously up to shifting across vertical slits.*

6.4. **An induced train-track structure for diagonal horizontal foliations.** We first consider the diagonal case when $H_X = H_Y =: H \in \mathsf{PML}$. We have constructed a staircase train track decomposition $\mathbf{t}_{X,H}$ of $E_{X,H}$. Moreover, the geodesic representative $[V_Y]_X$ is essentially carried by $\mathbf{t}_{X,H}$. Thus, we homotope $[V_Y]_X$ along leaves of H_X, so that it is carried by the train track $\mathbf{t}_{X,H}$

(Lemma 6.7). Let W_Y denote this topological lamination being carried on $\mathbf{t}_{X,H}$ which is homotopic to $[V_Y]_X$.

From the realization W_Y on $\mathbf{t}_{X,H}$, we shall construct a polygonal train-track structure on E_{Y,H_Y}. The flat surfaces $E_{X,H}$ and $E_{Y,H}$ have the same horizontal foliation, and the homotopy of $[V_Y]_X$ to W_Y is along the horizontal foliation. Therefore, for each rectangular branch R_X of $\mathbf{t}_{X,H}$, if the weight of W_Y is positive, by taking the inverse-image of the straightening map $\mathrm{st}\colon E_{Y,H} \to E_{X,H}$ in §6.1, we obtain a corresponding rectangle R_Y on $E_{Y,H}$ whose vertical length is the same as R_X and horizontal length is the weight. Note that an edge of R_Y may contain a singular point of E_{Y,H_Y}.

Next let P_X be a polygonal branch of $\mathbf{t}_{X,H}$. Similarly, let P_Y be the inverse-image of P_X by the straighten map. Note that P_Y is not necessarily homeomorphic to P_X. In particular, P_Y can be the empty set, a staircase polygon which may have a smaller number of vertices than P_X. Moreover, P_Y may be disconnected (Figure 9). Then, we have a (staircase) polygonal train-track decomposition $\mathbf{t}_{Y,H}$ of $E_{Y,H}$. By convention, non-empty P_Y, as above, is called a branch of $\mathbf{t}_{Y,H}$ corresponding to P_X (which may be disconnected). In comparison to $\mathbf{t}_{X,H}$, the one-skeleton of $\mathbf{t}_{Y,H}$ may contain some singular points of E_{Y,H_Y}. The semi-continuity of $\mathbf{t}_{X,H}$ (Proposition 6.12(1)) gives a semi-continuity of $\mathbf{t}_{Y,H}$.

Lemma 6.13. *$\mathbf{t}_{Y,H}$ changes semi-continuously in the horizontal foliation H in* PML *and the realization W_Y of $[V_Y]_X$ on $\mathbf{t}_{X,H}$.*

6.5. Filling properties.

Lemma 6.14. *Let $X \neq Y \in \mathsf{T} \sqcup \mathsf{T}^*$. For every diagonal $H_X = H_Y$, every component of $H_{X,H} \setminus [V_Y]_X$ is contractible, i.e. a tree.*

Proof. Recall that H_Y and V_Y are the horizontal and vertical foliations of the flat surface E_{Y,H_Y}. Then, since $H_X = H_Y$, the lemma follows. □

A horizontal graph is a connected graph embedded in a horizontal leaf (whose endpoints may not be at singular points). Then, Lemma 6.14 implies the following.

Corollary 6.15. *Let $X \neq Y \in \mathsf{T} \sqcup \mathsf{T}^*$. For every diagonal pair $H_X = H_Y$, let $r > 0$ be the train-track parameter given by Lemma 6.7. Then, for sufficiently small $\epsilon > 0$, if a horizontal graph h of H_X has the transversal measure less than ϵ by the realization W_Y, then h is contractible.*

By continuity,

Proposition 6.16. *There is a neighborhood N of the diagonal Δ in* PML × PML *and $\epsilon > 0$ such that, if the train-track parameter $r > 0$ is sufficiently small, then for every $(H_X, H_Y) \in N$, if a horizontal graph h of H_X has transversal measure less than δ by W_Y, then ϵ is contractible.*

6.6. Semi-diffeomorphic surface train-track decompositions.

6.6.1. *Semi-diffeomorphic train tracks for diagonal foliation pairs.*

Definition 6.17. *Let F_1 and F_2 are surfaces with staircase boundary. Then F_1 is* semi-diffeomorphic *to F_2, if there is a homotopy equivalence $\phi\colon F_1 \to F_2$ which collapses some horizontal edges of F_1 to points: To be more precise,*

- *the restriction of ϕ to the interior $\mathrm{int}F_1$ is a diffeomorphism onto the interior $\mathrm{int}F_2$;*
- *ϕ takes ∂F_1 to ∂F_2, and $\mathrm{int}F_1$ to $\mathrm{int}F_2$;*

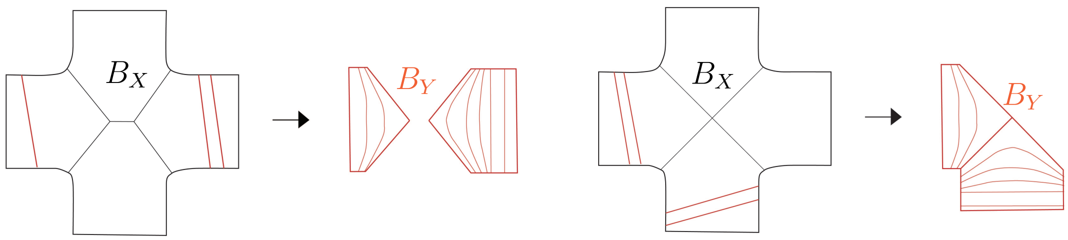

FIGURE 9. Some non-diffeomorphic correspondences of branches.

- *for every vertical edge v of F_1, the map ϕ takes v diffeomorphically onto a vertical edge or a segment of a vertical edge in F_2;*
- *for every horizontal edge h of F_1, the map ϕ takes h diffeomorphically onto a horizontal edge of F_2 or collapses h to a single point on a vertical edge of F_2.*

Let T and T' be train-track structures of flat surfaces E and E', respectively, on S. Then T is semi-diffeomorphic *to T', if there is a marking preserving continuous map $\phi\colon E \to E'$, such that,*

- *T and T' are homotopy equivalent by ϕ (i.e. their 1-skeletons are homotopy equivalent), and*
- *for each branch B of T, there is a corresponding branch B' of T' such that $\phi|B$ is a semi-diffeomorphism onto B'.*

In §6.4, for every $H \in \mathsf{PML}$, we constructed a staircase train-track structure $\mathbf{t}_{Y,H}$ of the flat surface $E_{Y,H}$ with staircase boundary from a realization W_Y of $[V_Y]_X$ on the train-track structure $\mathbf{t}_{X,H}$ of $E_{X,H}$. However, when a branch B_X of $\mathbf{t}_{X,H}$ corresponds to a branch B_Y of $\mathbf{t}_{Y,H}$, in fact, B_Y might not be connected, and in particular not semi-diffeomorphic to B_X (Figure 9, Left). In this section, we modify $\mathbf{t}_{X,H}$ and $\mathbf{t}_{Y,H}$ by gluing some branches in a corresponding manner, so that corresponding branches are semi-diffeomorphic after a small perturbation.

Let v be a (minimal) vertical edge of $\mathbf{t}_{X,H}$, i.e. a vertical edge not containing a vertex in its interior. Let B_X be a branch of $\mathbf{t}_{X,H}$ whose boundary contains v. Suppose that α is an arc in B_X connecting different horizontal edges of B_X. Then, we say that v and α are vertically parallel in B_X if

- α is homotopic in B_X to an arc α' transversal to the horizontal foliation $H|B_X$, keeping its endpoints on the horizontal edges, and
- v diffeomorphically projects into α' along the horizontal leaves $H_X|B_X$ (see Figure 10).

The W_Y-weight of v in B_X is the total weight of the leaves of $W_Y|B_X$ which are vertically parallel to v.

Let w be the W_Y-weight of v in B_X. Then, there is a staircase rectangle in B_Y such that a vertical edge corresponds to v and the horizontal length is w.

Consider a horizontal arc α_h in B connecting a point on v to a point on another vertical edge of B; clearly, the transversal measure of W_Y of α_h is a non-negative number. Then, the W_Y-weight of v in B is the minimum of the W_Y-transversal measures of all such horizontal arcs α_h starring from v.

Fix $0 < \delta < r$ be a sufficiently small positive number. We now consider both branches B_1, B_2 of $\mathbf{t}_{X,H}$ whose boundary contains v. Suppose that, the W_h-weight of v is less than δ in B_i for both

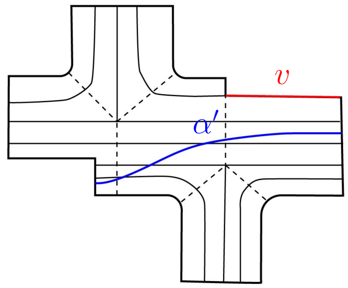

FIGURE 10. The curve α' is vertically parallel to v.

$i = 1, 2$; then, glue B_1 and B_2 along v, so that B_1 and B_2 form a single branch. Let $T^{r,\delta}_{X,H}$, or simply $T_{X,H}$, denote the train-track structure of $E_{X,H}$ obtained by applying such gluing, simultaneously, branches of $\mathbf{t}_{X,H}$ along all minimal vertical edges satisfying the condition. Then, since $\mathbf{t}_{X,H}$ is a refinement of $T_{X,H}$, the realization W_Y of $[V_Y]_X$ on $\mathbf{t}_{X,H}$ is also a realization on $T_{X,H}$. Similarly, let $T^{r,\delta}_{Y,H}$, or simply $T_{Y,H}$, be the train-track structure of $E_{Y,H}$ obtained by the realization W_Y on $T_{X,H}$; then $\mathbf{t}_{Y,H}$ is a refinement of $T_{Y,H}$. Lemma 6.14 implies the following.

Lemma 6.18. *Every transversal branch of $T_{X,H}$ has a non-negative Euler characteristic.*

Let B be a branch of $T_{Y,H}$, and let v be a minimal vertical edge of $T_{Y,H}$ contained in the boundary of B. Let B' be the branch of $T_{Y,H}$ adjacent to B across v. Suppose that the W_Y-weight of v is less than δ in B. Then, it follows from the construction of $T_{X,H}$, that there is a staircase rectangle R_v in B', such that the horizontal length of R_v is $\delta/3$ and that v is a vertical edge of R_v. Let v be a vertical edge of B. Then we enlarge B by gluing the rectangle R_v along v, and we remove R_v from B' (Figure 11)— this cut-and-paste operation transforms $T_{Y,H}$ by pushing the vertical edge v by $\delta/3$ into B' in the horizontal direction. For all minimal vertical edges v of $T_{Y,H}$ whose W-weights are less than δ as above, we apply such modifications simultaneously and obtain a train-track structure $T'_{Y,H}$ of $E_{Y,H}$ homotopic to $T_{Y,H}$.

Lemma 6.19.

- *The edge graph of $T'_{Y,H}$ is, at least, $\frac{\delta}{3}$ away from the singular set of $E^1_{Y,H}$;*
- *$T'_{Y,H}$ is δ-Hausdorff close to $T_{Y,H}$ in $E^1_{Y,H}$;*
- *$T_{X,H}$ is semi-diffeomorphic to $T'_{Y,H}$;*
- *$T_{X,H}$ changes semi-continuously in H;*
- *$T_{Y,H}$ changes semi-continuously in H, and the realization of W on $T_{X,H}$.*

Proof. First three assertions follows from the construction of $T_{X,H}$ and $T_{Y,H}$. The semi-continuity of $T_{X,H}$ is given by its construction and the semi-continuity of $\mathbf{t}_{X,H}$ (Proposition 6.12). Similarly, the semi-continuity of $T_{Y,H}$ follows from its construction and the semi-continuity of $\mathbf{t}_{Y,H}$. □

6.6.2. *Semi-diffeomorphic train-tracks for almost diagonal horizontal foliations.* In this section, we extend the construction form §6.6.1 to the neighborhood of the diagonal $(\mathsf{PML} \times \mathsf{PML}) \setminus \Delta^*$. By Lemma 6.4, for every compact neighborhood N of the diagonal Δ in $(\mathsf{PML} \times \mathsf{PML}) \setminus \Delta^*$, there is $\delta > 0$, such that

$$\angle_{E_{X,H_X}}([V_Y]_X, H_X) > \delta$$

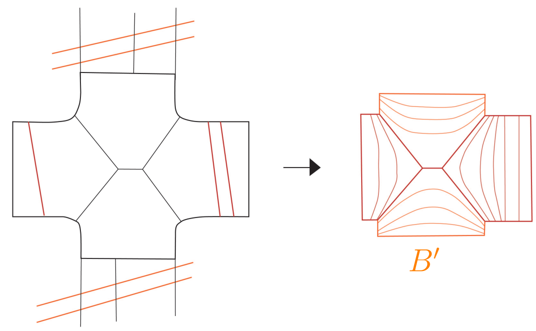

FIGURE 11.

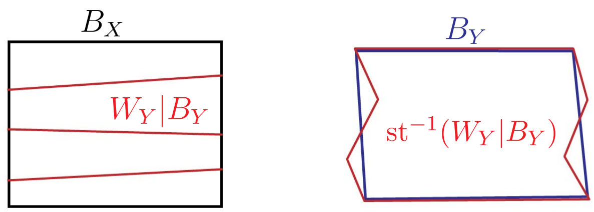

FIGURE 12.

for all $(H_X, H_Y) \in N$, where V_Y is the vertical measured foliation of the flat surface structure on Y with the horizontal foliation H_Y. Let $\mathbf{t}^r_{X,H_X}(= \mathbf{t}_{X,H_X})$ be the train-track decomposition of E_{X,H_X} obtained in §6.3. Lemma 6.7 clearly implies the following.

Proposition 6.20. *For a compact subset N in $(\mathsf{PML} \times \mathsf{PML}) \setminus \Delta^*$, if the train-track parameter $r > 0$ is sufficiently small, then for all $(H_X, H_Y) \in N$, $\mathbf{t}^r_{X,H_X}$ essentially carries $[V_Y]_X$.*

Let W_Y be a realization of $[V_Y]_X$ on $\mathbf{t}_{X,H_X}$ by a homotopy along horizontal leaf H_X (§6.3). For every branch B_X of $\mathbf{t}_{X,H_X}$, consider the subset of E_{Y,H_Y} which maps to $W_Y|B_X$ by the straightening map $\mathrm{st}\colon E_{Y,H_Y} \to E_{X,H_X}$ (§6.1) and the horizontal homotopy. Then, the boundary of the subset consists of straight segments in the vertical foliation V_Y and curves topologically transversal to V_Y (Figure 12 for the case when B_X is a rectangle). We straighten each non-vertical boundary curves of the subset keeping its endpoints. let B_Y be the region in E_{Y,H_Y} after straightening all non-vertical curves, so that the boundary of B_Y consists of the segments parallel to V_Y and segments transversal to V_Y. Then, for different branches B_X of $\mathbf{t}_{X,H_X}$, corresponding regions B_Y have disjoint interior; thus the regions B_Y yield a trapezoidal surface train-track decomposition of E_{Y,H_Y}.

Let E be a flat surface, and let H be its horizontal foliation. Then, for $\epsilon > 0$, a piecewise-smooth curve c on E is ϵ-almost horizontal, if $\angle_E(H, c) < \epsilon$, i.e. the angles between the tangent vectors along c and the foliation H are less than ϵ. More generally, c is ϵ-quasi horizontal if c is ϵ-Hausdorff close to a geodesic segment which is ϵ-almost horizontal to the horizontal foliation H. (In particular, the length of c is very short, then it is ϵ-quasi horizontal.)

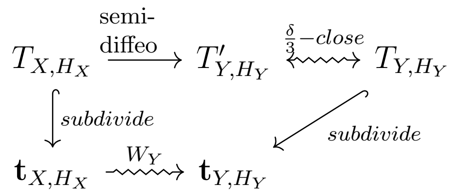

FIGURE 13. Relations between constructed train tracks.

Definition 6.21. *Let E be a flat surface. For $\epsilon > 0$, an* ϵ-quasi-staircase *train-track structure of E is a trapezoidal train-track structure of E such that its horizontal edges are all ϵ-quasi horizontal straight segments.*

If $H_X = H_Y$, then $\mathbf{t}_{Y,H}$ is a staircase train track, by continuity, we have the following.

Lemma 6.22. *Let $r > 0$ be a train-track parameter given by Corollary Proposition 6.20. Then, for every $\epsilon > 0$, if the neighborhood N of the diagonal in $\mathsf{PML} \times \mathsf{PML}$ is sufficiently small, then, for all $(H_X, H_Y) \in N$, the trapezoidal train-track decomposition $\mathbf{t}^r_{Y,H_Y}$ of E_{Y,H_Y} is ϵ-quasi staircase.*

Next, similarly to §6.6.1, we modify $\mathbf{t}_{X,H_X}$ and $\mathbf{t}_{Y,H_Y}$ by gluing some branches, so that corresponding branches have small diffeomorphic neighborhoods. Let W_Y be a realization of $[V_Y]_X$ in $\mathbf{t}_{X,H_X}$. Fix small $\delta > 0$. Let v be a vertical edge v of $\mathbf{t}_{X,H_X}$, and let B_1, B_2 be the branches of $\mathbf{t}_{X,H_X}$ whose boundary contains v. We glue B_1 and B_2 along v, if the W_Y-measure of v in B_i is less than δ for both $i = 1, 2$. By applying such gluing for all vertical edges satisfying the condition, we obtain a staircase train-track $T^{r,\delta}_{X,H_X} = T_{X,H_X}$, so that $\mathbf{t}_{X,H_X}$ is a refinement of T_{X,H_X}.

Then, W_Y is still carried by T_{X,H_X}. Therefore, let T_{Y,H_X} be the trapezoidal train-track decomposition of E_{Y,H_Y} obtained by this realization, so that $\mathbf{t}_{Y,H_Y}$ is its refinement.

Let v be a vertical edge of T_{X,H_X}. Let B_X be a branch of T_{X,H_X} whose boundary contains v. Let B'_X be the branch of T_{X,H_X} adjacent to B_X across v. Let B_Y and B'_Y be the branches of T_{Y,H_Y} corresponding to B_X and B'_X, respectively. Then, there is a vertical edge v of T_{Y,H_Y}corresponding to v, contained in the boundary of both B_Y and B'_Y.

If the W_Y-weight of v in B is less than δ, then the W_Y-weight of V in B'_X is at least δ, by the construction of T_{X,H_X}. Therefore, B'_Y contains an ϵ-quasi-staircase trapezoid R_Y, such that w is a vertical edge of R_Y and the horizontal length between the vertical edges is $\delta/3$.

Then, we can modify the train track T_{Y,H_Y} by removing R_Y from B'_Y and gluing R_Y with B_Y along w — this modified T_{Y,H_Y} by a homotopy. By simultaneously applying this modification for all vertical edges v of T_{X,H_X} satisfying the condition, we obtain a trapezoidal train-track decomposition T'_{Y,H_Y}.

Proposition 6.23. *For every compact neighborhood N of the diagonal Δ in $(\mathsf{PML} \times \mathsf{PML}) \setminus \Delta^*$, if the train-track parameter $r > 0$ and the parameter $\delta > 0$ are sufficiently small, the for every $(H_X, H_Y) \in N$,*

- *T_{X,H_X} is semi-diffeomorphic to T'_{Y,H_Y};*
- *T'_{Y,H_Y} is δ-Hausdorff close to T_{Y,H_Y} in the normalized metric E^1_{Y,H_Y};*
- *the $\delta/4$-neighborhood of the singular set is disjoint from the one-skeleton of T'_{Y,H_Y}.*

A sliding is an operation of train track moving some vertical edges in the horizontal direction without changing the homotopy type of the train-track structure. If we change the realization

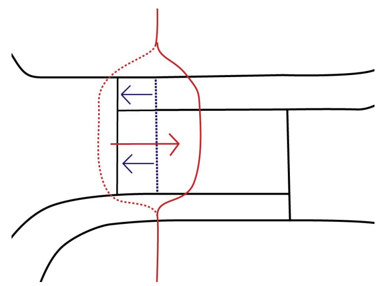

FIGURE 14. A slide corresponding a shift in Figure 8.

W_Y on $\mathbf{t}_{X,H}$ by shifting across a vertical slit, the induced train-track $\mathbf{t}_{Y,H}$ changes by sliding its corresponding vertical segment (Figure 14).

As before, a branch of T_{X,H_X} disjoint from the non-transversal graph G_Y is called a transversal branch. A branch of T_{X,H_X} containing a component of G_Y is called the non-transversal branch. By the semi-continuity of T_{Y,H_Y} in Lemma 6.19 and the construction of T'_{Y,H_Y}, we obtain a semi-continuity of T_{Y,H_Y} up to sliding.

Lemma 6.24. *Let $\mathbf{H}_i = (H_{X,i}, H_{Y,i})$ be a sequence converging to $\mathbf{H} = (H_X, H_Y)$. Then, up to a subsequence, $T_{Y,H_{Y,i}}$ semi-converges to a train track structure T''_{Y,H_Y} of E_{Y,H_Y}, such that either $T''_{Y,H_Y} = T'_{Y,H_Y}$ or T''_{Y,H_Y} can be transformed to a refinement of T'_{Y,H_Y} by sliding some vertical edges by $\delta/3$.*

6.7. **Bounded polygonal train tracks for the Riemann surface X.** The train tracks which we constructed so far may have rectangular branches with very long horizontal edges. In this section, we further modify the train-track structures T_{X,H_X} and T_{Y,H_Y} from §6.6 by reshaping those long rectangles into spiral cylinders.

Given a rectangular branch of a train track, although its interior is embedded in a flat surface, its boundary may intersect itself. Let T be a train-track structure of a flat surface E. The diameter of a branch B of T is the diameter of the interior of B with the path metric in B. The diameter of a train track T is the maximum of the diameters of the branches of T.

Recall that we have fixed a compact neighborhood $N \subset (\mathsf{PML} \times \mathsf{PML}) \setminus \Delta^*$ of the diagonal. Recall that, for $(H_X, H_Y) \in N$, E^1_{X,H_X} and E^1_{Y,H_Y} are the unit-area flat structures realizing (X, H_X) and (Y, H_Y), respectively. Pick a small $r > 0$ given by Proposition 6.12, so that, for every $(H_X, H_Y) \in N$, there are train-track structures T_{X,H_X} of E^1_{X,H_X} and T_{Y,H_Y} of E^1_{Y,H_Y} from §6.6.2.

Lemma 6.25. (1) *Let $\mathbf{H}_i = (H_{X,i}, H_{Y,i}) \in N$ be a sequence converging to $\mathbf{H} = (H_X, H_Y) \in N$. Suppose that $T_{X,H_{X,i}} =: T_{X,i}$ contains a rectangular branches R_i for each i, such that the horizontal length of R_i diverges to infinity as $i \to \infty$. Then, up to a subsequence, the support $|R_i| \subset E^1_{X,H_{X,i}} =: E_{X,i}$ converges to either*
- *a flat cylinder which is a branch of T_{X,H_X} or*
- *a close leaf of H_X which is contained in the union of the horizontal edges of T_{X,H_X}.*

(2) *Let A be the limit flat cylinder or a loop in (1). For sufficiently large $i > 0$, let $R_{i,1}, \ldots, R_{i,n_i}$ be the set of all rectangular branches of $T_{X,i}$ which converge to A as $i \to \infty$ in the Hausdorff metric. Then, the union $R_{i,1} \cup \cdots \cup R_{i,n_i} \subset E^1_{X,H_i}$ is a spiral cylinder for all sufficiently large i. (See Figure 4.)*

Proof. (1) Let R_i be a rectangular branch of $T_{X,i}$ such that the horizontal length of R_i diverges to infinity as $i \to \infty$. Then, as Area $E_i = 1$, the vertical length of R_i must limit to zero. Then, in the universal cover $\tilde{E}_i$ of E_i, we can pick a lift $\tilde{R}_i$ of R_i which converges, uniformly on compact, to a smooth horizontal leaf of $\tilde{H}_X$ or a copy of $\mathbb{R}$ contained in a singular leaf of $\tilde{H}_X$. Let $\tilde{\ell}$ denote the limit, and let ℓ be its projection into a leaf of H_X.

Claim 6.26. *ℓ is a closed leaf of H_X.*

Proof. Suppose, to the contrary, that ℓ is not periodic. Then ℓ is either a leaf of an irrational sublamination or a line embedded in a singular leaf of H_{X,H_X}. Then, the distance from ℓ to the singular set of E_{X,H_X} is zero.

Recall that the $(r, \sqrt[4]{r})$-neighborhood of the singular set of $E_{X,H_{X,i}}$ is contained in the (non-rectangular) branches of $T_{X,H_{X,i}}$. Thus, the distance from R_i to the singular set of $E_{X,H_{X,i}}$ is at least $r > 0$ for all i. This yields a contradiction. □

By Claim 6.26, as a subset of E_i, the rectangular branch R_i converges to the union of closed leaves $\{\ell_j\}_{j \in J}$ of H_{X,L_X}. Thus the Hausdorff limit A of R_i in $E_{X,H}$ must be a connected subset foliated by closed horizontal leaves. Therefore, A is either a flat cylinder or a single closed leaf.

First, suppose that the limit A is a flat cylinder. Then, the vertical edges of R_i are contained in the vertical edges of non-rectangular branches. The limit of the vertical edges of R_i are points on the different boundary components of A. Therefore, each boundary component of A must intersect a non-rectangular branch in its horizontal edge. Therefore, the cylinder is a branch of $T_{X,H}$ by the construction of $\mathbf{t}_{X,H}$.

If the limit A is a single leaf, similarly, one can show that the vertical one-skeleton of T_{X,H_X}, since a loop can be regard as a degeneration of a flat cylinder.

(2) First assume that the limit A is a flat cylinder. As the $(\epsilon, \sqrt[4]{\epsilon})$-neighborhood of the singular set is disjoint from the interior of A, we can enlarge A to a maximal flat cylinder $\hat{A}$ in $E_{X,H}$ whose interior contains (the closure of) A. Then, each boundary component of $\hat{A}$ contain at least one singular point. Since A is a cylindrical branch, each boundary component ℓ of A contains a horizontal edge of a non-rectangular branch P_ℓ of T_{X,H_X} which contains a singular point in the boundary of $\hat{A}$. Let $P_{i,1}, \dots, P_{i,k_i}$ be all non-rectangular branches of $T_{X,i}$, such that their union $P_{i,1} \cup \dots \cup P_{i,k_i}$ converges to the union of all non-rectangular branches of T_{X,H_X} which have horizontal edges contained in the boundary of A. Then, for sufficiently, large i, the vertical edges of $R_{i,1} \dots, R_{i,n_i}$ are contained in vertical edges of polygonal branches $P_{i,1}, \dots, P_{i,k_i}$. Then the assertion follows.

The similar argument holds in the case when the limit is a closed loop in a singular leaf. 6.25

By Lemma 6.25 (1), (2), there is a constant $c > 0$, such that, for $H_X \in N$, if a rectangular branch R of T_{X,H_X} has horizontal edge more than c, then R is contained in a unique spiral cylinder, which may contain other rectangular branches. The diameter of such spiral cylinders is uniformly bounded from above by a constant depending only on X. Thus, replace all rectangular branches R of T_{X,H_X} with corresponding spiral cylinders, and we obtain a staircase train track $\mathbf{T}_{X,H_X}$:

Corollary 6.27. *There is $c > 0$, such that, for all $\mathbf{H} = (H_X, H_Y) \in N$, the diameters of the branches of the staircase train track $\mathbf{T}_{X,H_X}$ are bounded by c.*

6.8. **Semi-diffeomorphic bounded almost polygonal train-track structures for Y.** For $\epsilon > 0$, we have constructed, for all $\mathbf{H} = (H_X, H_Y)$ in some neighborhood N of the diagonal Λ_∞ in $(\mathsf{PML} \times \mathsf{PML}) \setminus \Delta^*$, a staircase train-track structure T_{X,H_X} of E_{X,H_X} and an ϵ-quasi-staircase train-track structure T_{Y,H_Y} of $E_{Y,H}$, such that T_{X,H_X} is semi-diffeomorphic to T_{Y,H_Y}. In §6.7, we modify T_{X,H_X} and obtain a uniformly bounded train-track $\mathbf{T}_{X,H_X}$ creating spiral cylinders. In this section, we accordingly modify T_{Y,H_Y} to a bounded ϵ-quasi-staircase train-track structure.

Lemma 6.28. (1) *For every spiral cylinder A of $\mathbf{T}_{X,H_X}$, letting $R_{X,1}, R_{X,2}, \dots, R_{X,n}$ be the rectangular branches of $T_{X,\mathbf{H}}$ whose union is A, there are corresponding branches $R_{Y,1}, R_{Y,2}, \dots, R_{Y,n}$ of $T_{Y,\mathbf{H}}$, such that*
- *their union $R_{Y,1} \cup R_{Y,2} \cup \dots \cup R_{Y,n}$ is a spiral cylinder in E_{Y,H_Y}, and*
- *A is semi-diffeomorphic to $R_{Y,1} \cup R_{Y,2} \cup \dots \cup R_{Y,n}$.*

(2) *Moreover, there is a constant $c > 0$, such that, if a rectangular branch of T_{Y,H_Y} has horizontal length more than c, then it is contained in a spiral cylinder as above.*

Proof. As $(H_X, H_Y) \cap \Delta^* = \emptyset$, the geodesic representative $[V_Y]_X$ essentially intersects A. Thus, the realization W_Y has positive weights on $R_{X,1}, R_{X,2}, \dots, R_{X,n}$. Thus $R_{X,j}$ corresponds to a rectangular branch $R_{Y,j}$ of T_{Y,H_Y}, and their union $\cup_j R_{Y,j}$ is a spiral cylinder in T_{Y,H_Y}.

Let R_X and R_Y be corresponding rectangular branches of T_{X,H_X} and T_{Y,H_Y}, respectively. Then, the horizontal lengths of R_X and R_Y are quasi-isometric uniformly for all $(H_X, H_Y) \in N$ and such all R_X and R_Y. Then the second assertion follows. □

For every spiral cylinder A of $\mathbf{T}_{X,H_X}$, by applying Lemma 6.28, we replace the branches $R_{Y,1}, R_{Y,2}, \dots, R_{Y,n}$ of T_{Y,H_Y} with the spiral cylinder $R_{X,1} \cup R_{X,2} \cup \dots \cup R_{X,n}$ of T_{Y,H_Y}. Then, we obtain an ϵ-quasi-staircase train-track decomposition $\mathbf{T}_{Y,H_Y}$ without long rectangles:

Proposition 6.29. *For every $\epsilon > 0$, there are $c > 0$ and a neighborhood N of the diagonal in $(\mathsf{PML} \times \mathsf{PML}) \setminus \Delta^*$, such that, for every $\mathbf{H} = (H_X, H_Y) \in N \subset \mathsf{PML} \times \mathsf{PML}$, there is an ϵ-quasi-staircase train-track decomposition $\mathbf{T}_{Y,H_Y}$ of E_{Y,H_Y}, such that*

1. *$\mathbf{T}'_{Y,H_Y}$ is δ-hausdorff close to $\mathbf{T}_{Y,H_Y}$ in $E^1_{Y,H}$;*
2. *the diameters of $\mathbf{T}_{Y,H_Y}$ and $\mathbf{T}'_{Y,H_Y}$ are less than c;*
3. *$\mathbf{T}_{X,H_X}$ is semi-diffeomorphic with $\mathbf{T}'_{Y,H_Y}$;*
4. *$\mathbf{T}_{Y,H_Y}$ changes semi-continuously in (H_X, H_Y) and the realization of $[V_Y]_X$ on $\mathbf{T}_{X,H_X}$.*

Proof. Assertion (2) follows from Lemma 6.28 (2). Assertion (1) follows from Proposition 6.23. Assertion (3) follows from Proposition 6.23 and Lemma 6.28 (1). Assertion (4) holds, by Lemma 6.19, since T_{Y,H_Y} changes semi-continuously in (H_X, H_Y) and the realization W_Y of $[V_Y]_X$ on T_{X,H_X}. □

7. Thurston laminations and vertical foliations

7.0.1. *Model Euclidean Polygons and projective circular polygons.* A polygon with circular boundary is a projective structure on a polygon such that the development of each edge is contained in a round circle in $\mathbb{CP}^1$. Let σ be an ideal hyperbolic n-gon $(n \geq 3)$. Let L be a measured lamination on σ except that each boundary geodesic of σ is a leaf of weight ∞. From a view point of the Thurston parameterization, it is natural to add such weight-infinity leaves. In fact, there is a unique $\mathbb{CP}^1$-structure $\mathcal{C} = \mathcal{C}(\sigma, L)$ on the complex plane $\mathbb{C}$ whose Thurston's parametrization is the pair (σ, L); see [GM21]. Let $\mathcal{L}$ be the Thurston lamination on $\mathcal{C}$. Denote, by $\kappa\colon \mathcal{C} \to \sigma$, the collapsing map (§2.1.5).

For each boundary edge l of σ, pick a leaf ℓ of $\mathcal{L}$ which is sent diffeomorphically onto l by κ. Then, those circular leaves bound a circular projective n-gon $\mathcal{P}$ in $\mathcal{C}$, called an ideal projective polygon.

For each $i = 1, 2, \ldots, n$, let v_i be an ideal vertex of σ, and let l_i and l_{i+1} be the edges of σ starting from v_i. Consider the geodesic g starting from v_i in the middle of l_i and l_{i+1}, so that the reflection about g exchanges l_i and l_{i+1}. Embed σ isometrically into a totally geodesic plane in $\mathbb{H}^3$. Accordingly $\mathcal{P}$ is embedding in $\mathbb{C}\mathrm{P}^1$ so that the restriction of κ to $\mathcal{P}$ is the nearest point projection to σ in $\mathbb{H}^3$.

Then, pick a round circle c_i on $\mathbb{C}\mathrm{P}^1$ such that the hyperbolic plane, $\mathrm{Conv}\, c_i$, bounded by c_i is orthogonal to g, so that l_i and l_{i+1} are transversal to $\mathrm{Conv}\, c_i$.

Let ℓ_i and ℓ_i be the edges of $\mathcal{P}$ corresponding to l_i and l_{i+1}, respectively. If c_i is close to v_i enough, then there is a unique arc a_i in $\mathcal{P}$ connecting ℓ_i to ℓ_{i+1} which is immersed into c_i by the developing map. Then, the region in $\mathcal{P}$ bounded by $a_1 \ldots a_n$ is called the truncated ideal projective polygon.

Definition 7.1. *Let C be a $\mathbb{C}\mathrm{P}^1$-structure on S. Let E^1 be the normalized flat surface of the Schwarzian parametrization of C. Let P be a staircase polygon in E^1. Then P is ϵ-*close *to a truncated ideal projective polygon $\mathcal{P}$, if $\mathcal{P}$ isomorphically embeds onto a polygon in C which ϵ-Hausdorff close to P in the normalized Euclidean metric.*

For $X \in \mathsf{T} \sqcup \mathsf{T}^*$, recall that χ_X be the holonomy variety of the $\mathbb{C}\mathrm{P}^1$-structures on X. For $\rho \in \chi_X$, let $C_{X,\rho}$ be the $\mathbb{C}\mathrm{P}^1$-structure on X with holonomy ρ, and let $E_{X,\rho}$ be the flat surface given by the holomorphic quadratic differential of $C_{X,\rho}$. Similarly, for $\epsilon > 0$, let $N^1_\epsilon Z_{X,\rho}$ be the ϵ-neighborhood of the singular set in the normalized flat surface $E^1_{X,\rho}$. Let $\mathcal{L}_{X,\rho}$ be the Thurston lamination of $C_{X,\rho}$.

Then, by combining what we have proved, we obtain the following.

Theorem 7.2. *Let $X \in \mathsf{T} \sqcup \mathsf{T}^*$. Then, for every $\epsilon > 0$, there is a bounded subset $K = K(X, \epsilon)$ of χ_X satisfying the following: Suppose that ρ is in $\chi_X \setminus K$, and that the flat surface $E_{X,\rho}$ contains a staircase polygon P such that*

- *∂P disjoint from $N^1_\epsilon Z_{X,\rho}$ and*
- *the diameter of P is less than $\frac{1}{\epsilon}$.*

Then

(1) *$\mathcal{L}_{X,\rho}|P$ is $(1+\epsilon, \epsilon)$-quasi-isometric to $V_{X,\rho}|P$ up to an isotopy supported on $N^1_\epsilon Z_{X,\rho} \cap P$, such that, in the normalized Euclidean metric $E^1_{X,\rho}$,*
   (a) *on P, each leaf of V is ϵ-hausdorff-close to a leaf of $\mathcal{L}$, and*
   (b) *the transversal measure of V is ϵ-close to the transversal measure of $\mathcal{L}$ for all transversal arcs whose length are less than one.*

(2) *In the (unnormalized) Euclidean metric, P is ϵ-close to a truncated circular polygon of the hyperbolic surface in the Thurston parameters.*

Proof. The assertion (1a) follows from Lemma 4.4. The assertion (1b) is given by Proposition Theorem 4.3.

We shall prove (2). Set $C_{X,\rho} = (\tau, L) \in \mathsf{T} \times \mathsf{ML}$ be the $\mathbb{C}\mathrm{P}^1$ structure on X with holonomy $\rho \in \mathcal{X} \setminus K$ in Thurston coordinates, and let $\kappa\colon C_{X,\rho} \to \tau$ be the collapsing map. Since sufficiently away from the zero, the developing map is well-approximated by the exponential map (Lemma 3.2).

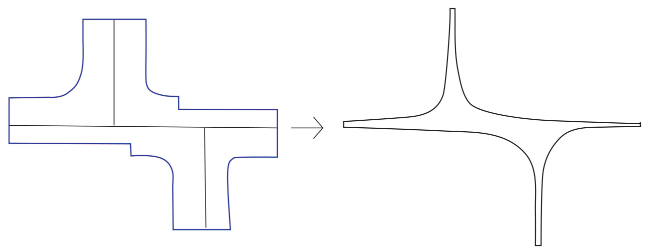

FIGURE 15.

If K is sufficiently large, then for every vertical edge v of P, the restriction $\mathrm{Ep}_{X,\rho}|v$ is a $(1+\epsilon)$-bilipschitz embedding on the Epstein surface. Therefore, by the closeness of $\mathrm{Ep}_{X,\rho}$ and $\hat{\beta}_{X,\rho}$, $\kappa(v)$ is ϵ-close to a geodesic segment s_v of length $\sqrt{2}\,\mathrm{length}\, v$. By (1b), if K is large enough, $L(s_v) < \epsilon$.

Every horizontal edge h of P is very short on the Epstein surface (Lemma 3.1). As the developing map is approximated by the Exponential map and $\mathrm{Area}\, \tau = 2\pi|\chi(S)|$, it follows that, if $\kappa(h)$ has length less than ϵ on τ. Therefore, the image of P on the hyperbolic surface is ϵ-close to a truncated ideal polygon.

7.2

7.1. **Equivariant circle systems.** For $\rho \in \mathcal{X}_X$, we shall pick a system of a ρ-equivalent round circles on $\mathbb{CP}^1$, which will be used to construct a circular train-track structure of $C_{X,\rho}$. Let $\tilde{\mathbf{T}}_{X,\rho}$ be the $\pi_1(S)$-invariant train-track structure on $\tilde{E}_{X,\rho}$ obtained by lifting the train-track structure $\mathbf{T}_{X,\rho}$ on $E_{X,\rho}$. Let $\mathrm{Ep}^*_{X,\rho} \colon T\tilde{E}_{X,\rho} \to T\mathbb{H}^3$ be the differential of $\mathrm{Ep}_{X,\rho} \colon \tilde{E}_{X,\rho} \to \mathbb{H}^3$.

Lemma 7.3. *For every $\epsilon > 0$, there is a bounded subset K_ϵ of $\mathcal{X}$ such that, if $\rho \colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ belongs to $\mathcal{X}_X \setminus K_\epsilon$, then, we can assign a round circle c_h to every minimal horizontal edge h of $\tilde{\mathbf{T}}_{X,\rho}$ with the following properties:*

(1) *The assignment $h \mapsto c_h$ is ρ-equivariant.*
(2) *The hyperbolic plane bounded by c_h is ϵ-almost orthogonal to the $\mathrm{Ep}^*_{X,\rho}$-images of the vertical tangent vectors along h.*
(3) *If h_1, h_2 are horizontal edges of $\tilde{\mathbf{T}}_{X,\rho}$ connected by a vertical edge v of length at least ϵ, then the round circles c_{h_1} and c_{h_2} are disjoint.*
(4) *If h_1, h_2, h_3 are "vertically consecutive" horizontal edges, such that*
    - *h_1 and h_2 are connected by a vertical edge v_1 of $E_{X,\rho}$-length at least ϵ;*
    - *h_2 and h_3 are connected by a vertical edge v_3 of length at least ϵ;*
    - *h_1 and h_3 are on the different sides of h_2, i.e. the normal vectors of h_2 in the direction of v_1 and v_3 are opposite,*

    then c_{h_1} and c_{h_3} are disjoint, and they bound a round cylinder whose interior contains c_{h_2}.

Proof. Without loss of generality, we can assume that $\epsilon > 0$ is sufficiently small. For each minimal horizontal edge h of $\tilde{\mathbf{T}}_{Y,\rho}$, pick a round circle c_h, such that the assignment of c_h is holonomy equivariant and that the images of vertical tangent vectors along h are ϵ^2-orthogonal to the hyperbolic plane bounded by c_h.

Then, if v is a vertical edge sharing an endpoint with h, then $\mathrm{Ep}_{X,\rho}(v)$ is ϵ^2-almost orthogonal to the hyperbolic plane bounded by c_h. For every sufficiently small $\epsilon > 0$, if $K > 0$ is sufficiently large, then the geodesic segment of length, at least, ϵ connects the hyperbolic planes bounded by c_{h_1} and c_{h_2}, and the geodesic segment is ϵ^2-almost orthogonal to both hyperbolic planes. Therefore, if $\epsilon > 0$ is sufficiently small, then, by elementary hyperbolic geometry, the hyperbolic planes are disjoint, and (3) holds. By a similar argument, (4) also holds. □

The circle system in Lemma 7.3 is not unique, but unique up to an appropriate isotopy:

Proposition 7.4. *For every $\epsilon_1 > 0$, there is $\epsilon_2 > 0$, such that, for every $\rho \in \chi_X \setminus K_{\epsilon_2}$ given two systems of round circles $\{c_h\}$ and $\{c'_h\}$ realizing Lemma 7.3 for $\epsilon_2 > 0$, there is a one-parameter family of equivalent circles systems $\{c_{t,h}\}$ $(t \in [0,1])$ realizing Lemma 7.3 for $\epsilon_1 > 0$ which continuously connects $\{c_h\}$ to $\{c'_h\}$.*

Proof. The proof is left for the reader. □

7.2. **Pleated surfaces are close.** The following gives a measure theoretic notion of almost parallel measured laminations.

Definition 7.5 (Quasi-parallel)**.** *Let L_1, L_2 be two measured geodesic laminations on a hyperbolic surface τ. Then, L_1 and L_2 are ϵ-quasi parallel, if a leaf ℓ_1 of L_1 and a leaf ℓ_2 of L_2 intersect at a point p and $\angle_p(\ell_1, \ell_2) > \epsilon$, then letting s_1 and s_2 be the unite length segments in ℓ_1 and ℓ_2 centered at p,*

$$\min(L_1(s_2), L_2(s_1)) < \epsilon.$$

Proposition 7.6. *For every $\epsilon > 0$, if a bounded subset $K \subset \chi_X \cap \chi_Y$ is sufficiently large, then $L_{Y,\rho}$ is ϵ-quasi-parallel to $L_{X,\rho}$ on $\tau_{X,\rho}$ away from the non-transversal graph G_Y.*

Proof. If K is sufficiently large, $\angle_{E_{X,\rho}}(H_{X,\rho}, V'_{Y,\rho})$ is uniformly bounded from below by a positive number by Theorem 6.4. Then, the assertion follows from Lemma 3.1 and Proposition 7.2. □

In this section, we show that the pleated surfaces for $C_{X,\rho}$ and $C_{Y,\rho}$ are close away from the non-transversal graph. Recall that $\hat\beta_{X,\rho} \colon \tilde C_{X,\rho} \to \mathbb{H}^3$ denotes the composition of the collapsing map and the bending map for $C_{X,\rho}$, and similarly $\hat\beta_{Y,\rho} \colon \tilde C_{Y,\rho} \to \mathbb{H}^3$ denotes the composition of the collapsing map and the bending map for $C_{Y,\rho}$.

Theorem 7.7. *Let $X, Y \in \mathsf{T} \sqcup \mathsf{T}^*$ with $X \neq Y$. For every $\epsilon > 0$, there is a bounded subset K_ϵ in $\chi_Y \cap \chi_X$ such that, for every $\rho \in \chi_X \cap \chi_Y \setminus K_\epsilon$, there are a homotopy equivalence map $\phi \colon E_{Y,\rho} \to E_{Y,\rho}$ and a semi-diffeomorphism $\psi \colon \mathbf{T}_{X,\rho} \to \mathbf{T}'_{Y,\rho}$ (Proposition 6.29) satisfying the following:*

1. *$d_{E^1_{Y,\rho}}(\phi(z), z) < \epsilon$;*
2. *the restriction of ϕ to $E_{X,\rho} \setminus N^1_\epsilon Z_{X,\rho}$ can be transformed to the identify by a homotopy along vertical leaves of $E_{X,\rho}$;*
3. *$\hat\beta_{X,\rho}(z)$ is ϵ-close to $\hat\beta_{Y,\rho}\tilde\psi(z)$ in $\mathbb{H}^3$ for every point $z \in \tilde E_{X,\rho}$ which are not in the interior of the non-transversal branches of $\mathbf{T}_{X,\rho}$.*

Using Lemma 3.1, one can prove the following.

Lemma 7.8. *Let $\epsilon > 0$ and let $X \in \mathsf{T} \sqcup \mathsf{T}^*$. Then, there is a compact subset K of χ_X such that, for every $\rho \in \chi_X \setminus K$, if α is a monotone staircase closed curve in $E_{X,\rho}$, such that*

- *the total vertical length of α is more than ϵ times the total horizontal length of α, and*

- *α is disjoint from the ϵ-neighborhood of the singular set in the normalized metric $E^1_{X,\rho}$,*

then $\mathrm{Ep}_{X,\rho}\,\tilde{\alpha}$ *is a $(1+\epsilon, \epsilon)$-quasi-geodesic with respect to the vertical length.*

Lemma 7.9. *Let α_X be a staircase curve carried by $t_{X,\rho}$ satisfying the conditions in Lemma 7.8. Then, there is a staircase geodesic closed curve α_Y carried by $\mathbf{T}'_{Y,\rho}$ satisfying the conditions in Lemma 7.8, such that the image of α by the semi-diffeomorphism $\mathbf{T}_{X,\rho} \to \mathbf{T}'_{Y,\rho}$ is homotopic to α_Y in the train-track $\mathbf{T}'_{Y,\rho}$.*

Proof. The proof is left for the reader. □

Let W_Y be a realization of $[V_Y]_X$ on $\mathbf{T}_{X,\rho}$ (§6.3). Let x be a point of the intersection of the realization W_Y and a horizontal edge of h_X of $\mathbf{T}_{X,\rho}$. Let y be a corresponding point of $V_{Y,\rho}$ (on $E_{Y,\rho}$). Recall that r is the train-track parameter, so that, in particular, horizontal edges are distance, at least, r away from the singular set in the normalized Euclidean metric. Let v_x be a vertical segment of length $r/2$ on $E^1_{X,\rho}$ such that x is the middle point of v_x. Similarly, let v_y be the vertical segment of length $r/2$ on $E^1_{Y,\rho}$ such that y is the middle point of v_y. We normalize the Epstein surfaces for $C_{X,\rho}$ and $C_{Y,\rho}$ so that they are ρ-equivariant for a fixed representation $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ (not a conjugacy class).

Proposition 7.10 (Corresponding vertical edges are close in $\mathbb{H}^3$)**.** *For every $\epsilon > 0$, there is a compact subset K in $\mathcal{X}$, such that, for every $\rho \in \mathcal{X}_X \cap \mathcal{X}_Y \setminus K$, if v_X and v_Y are vertical segments of $E_{X,\rho}$ and of $E_{Y,\rho}$, respectively, as above, then there is a (bi-infinite) geodesic ℓ in $\mathbb{H}^3$ satisfying the following:*

- $\mathrm{Ep}_{X,\rho}\, v_X$ *is ϵ-close to a geodesic segment α_X of ℓ in C^1-metric;*
- $\mathrm{Ep}_{Y,\rho}\, v_Y$ *is ϵ-close to a geodesic segment α_Y of ℓ in C^1-metric;*
- *if p_X and p_Y are corresponding endpoints of α_X and α_Y, then the distance between* $\mathrm{Ep}_{X,\rho}\, p_X$ *and* $\mathrm{Ep}_{Y,\rho}\, p_Y$ *is at most ϵ times the diameters of $E_{X,\rho}$ and $E_{Y,\rho}$.*

Proof. Then, pick a L^∞-geodesic staircase closed curves $\ell_{X,1}, \ell_{X,2}$ on $E_{X,\rho}$ containing v_x such that, for $i = 1, 2$, by taking appropriate lift $\tilde{\ell}_{X,1}$ and $\tilde{\ell}_{X,2}$ to $\tilde{E}_{X,\rho}$,

(1) $\ell_{X,i}$ is carried by $\mathbf{T}_{X,\rho}$;
(2) $\tilde{\ell}_{X,1} \cap \tilde{\ell}_{X,2}$ is a single staircase curve connecting singular points of $\tilde{E}_{X,\rho}$, and the projection of $\tilde{\ell}_{X,1} \cap \tilde{\ell}_{X,2}$ to $E_{X,\rho}$ does not meet a branch of $\mathbf{T}_{X,\rho}$ more than twice;
(3) if a branch B of $\tilde{T}_{X,H_X}$ intersects both $\tilde{\ell}_{X,1}$ and $\tilde{\ell}_{X,2}$, then B intersects $\tilde{\ell}_{X,1} \cap \tilde{\ell}_{X,2}$;
(4) $\tilde{\ell}_{X,1}$ and $\tilde{\ell}_{X,2}$ intersect, in the normalized metric of $\tilde{E}^1_{X,\rho}$, the ϵ-neighborhood of the singular set only in the near the endpoints of $\tilde{\ell}_{X,1} \cap \tilde{\ell}_{X,2}$.

Then, there are homotopies of $\ell_{X,1}, \ell_{X,2}$ to staircase vertically-geodesic closed curves $\ell'_{X,1}, \ell'_{X,2}$ carried by $\mathbf{T}_{X,\rho}$, such that the homotopies are supported in the 2ϵ-neighborhood of the singular set of $E^1_{X,\rho}$ and that $\ell'_{X,1}, \ell'_{X,2}$ are disjoint from the ϵ-neighborhood of the zero set. Then $\mathrm{Ep}_{X,\rho}\, \tilde{\ell}'_{X,1}$ and $\mathrm{Ep}_{X,\rho}\, \tilde{\ell}'_{X,2}$ are $(1+\epsilon, \epsilon)$-quasi-geodesics which are close only near the segment corresponding to $\tilde{\ell}_{X,1} \cap \tilde{\ell}_{X,2}$.

Pick closed geodesic staircase-curves $\ell_{Y,1}, \ell_{Y,2}$ on $E_{Y,\rho}$, such that

- $\ell_{Y,i}$ contains v_y;
- the semi-diffeomorphism $\mathbf{T}_{X,\rho} \to \mathbf{T}_{Y,\rho}$ takes $\ell'_{X,i}$ to a curve homotopic to $\ell_{Y,i}$ on $\mathbf{T}_{Y,\rho}$;
- $\ell_{Y,i}$ is carried by $\mathbf{T}_{Y,\rho}$;

- $\ell_{Y,i}$ is disjoint from $N^1_\epsilon Z_{X,\rho}$.

Let α be the geodesic such that a bounded neighborhood of α contains the quasi-geodesic $\mathrm{Ep}_{X,\rho}\, \tilde{\ell}'_{X,i}$. Let $\tilde{\ell}_{Y,i}$ be a lift of $\ell_{Y,i}$ to $\tilde{E}_{Y,\rho}$ corresponding to $\tilde{\ell}'_{X,i}$ (connecting the same pair of points in the ideal boundary of $\tilde{S}$).

Lemma 7.11. *For every $\epsilon > 0$, if a compact subset K of $\mathcal{X}$ is sufficiently large and $\upsilon > 0$ is sufficiently small, then, for all $\rho \in \mathcal{X}_X \cap \mathcal{X}_Y \setminus K$, $\mathrm{Ep}_{Y,\rho}\, \tilde{\ell}_{Y,i}$ is $(1+\epsilon, \epsilon)$-quasi-isometric with respect to the vertical length for both $i = 1, 2$.*

Then $\mathrm{Ep}_{X,\rho}\, \tilde{\ell}'_{X,1} \cup \tilde{\ell}'_{X,2}$ and $\mathrm{Ep}_{Y,\rho}\, \tilde{\ell}_{Y,1} \cup \tilde{\ell}_{Y,2}$ are both ϵ-close in the Hausdorff metric of $\mathbb{H}^3$. Therefore, corresponding endpoints of $\mathrm{Ep}_{Y,\rho}\, \tilde{\ell}'_{X,1} \cap \tilde{\ell}'_{X,2}$ and $\mathrm{Ep}_{Y,\rho}\, \tilde{\ell}'_{Y,1} \cap \tilde{\ell}'_{Y,2}$ have distance, at most, ϵ times the diameters of $E_{X,\rho}$ and $E_{Y,\rho}$. By (2), the length of $\tilde{\ell}'_{X,1} \cap \tilde{\ell}'_{X,2}$ can not be too long relative to the diameter of $E_{X,\rho}$. Letting ℓ be the geodesic in $\mathbb{H}^3$ fellow-traveling with $\mathrm{Ep}_{X,\rho}\, \tilde{\ell}'_{X,1}$ (or $\mathrm{Ep}_{X,\rho}\, \tilde{\ell}'_{X,2}$), the vertical segment v_x and v_y have the desired property. □

Finally Theorem 7.7 follows from the next proposition.

Proposition 7.12. *Suppose that a branch B'_Y of the train track $\mathbf{T}'_{Y,\rho}$ corresponds transversally to a branch B_X of $\mathbf{T}_{X,\rho}$.*

Then, there is an ϵ-small isotopy of B'_Y in the normalized surface $E^1_{Y,\rho}$ such that

- *in the complement of the $\frac{r}{2}$-neighborhood of the zero set, every point of B'_Y moves along the vertical foliation $V_{Y,\rho}$, and*
- *after the isotopy $\hat{\beta}_{X,\rho}|B_X$ and $\hat{\beta}_{Y,\rho}|B'_Y$ are ϵ-close pointwise by a diffeomorphism $\psi\colon B'_Y \to B_X$.*

Proof. By Proposition 7.10, there is an ϵ-small isotopy of the boundary of B'_Y satisfying the conditions on the boundaries of the branches. Since the branches are transversal, by Theorem 4.3, if K is sufficiently large, then the restriction of $L_{X,\rho}$ to B_X and $L_{Y,\rho}$ on B_Y are ϵ-quasi parallel on the hyperbolic surface $\tau_{X,\rho}$ (Proposition 7.6). Therefore we can extend to the interior of the branch by taking an appropriate diffeomorphism $\psi\colon B'_Y \to B_X$. □

8. Compatible circular train-tracks

In §6, for every ρ in $\mathcal{X}_X \cap \mathcal{X}_Y$ outside a large compact K, we constructed semi-diffeomorphic train-track structures $\mathbf{T}_{X,\rho}$ and $\mathbf{T}'_{Y,\rho}$ of the flat surfaces $E_{X,\rho}$ and $E_{Y,\rho}$, respectively. In this section, as $E_{X,\rho}$ and $E_{Y,\rho}$ are the flat structures on $C_{X,\rho}$ and $C_{Y,\rho}$, using Theorem 7.7, we homotope $\mathbf{T}_{X,\rho}$ and $\mathbf{T}'_{Y,\rho}$ to make them circular in a compatible manner.

8.1. **Circular rectangles.** A round cylinder is a cylinder on $\mathbb{CP}^1$ bounded by two disjoin round circles. Given a round cylinder A, the boundary components of A bound unique (totally geodesic) hyperbolic planes in $\mathbb{H}^3$, and there is a unique geodesic ℓ orthogonal to both hyperbolic planes. Moreover A is foliated by round circles which, in $\mathbb{H}^3$, bound hyperbolic planes orthogonal to ℓ — we call this foliation the horizontal foliation. In addition, A is also foliated by circular arcs which are contained in round circles bounding hyperbolic planes, in $\mathbb{H}^3$, containing ℓ — we call this foliation the vertical foliation. Clearly, the horizontal foliation is orthogonal to the vertical foliations of A.

Definition 8.1. *Let $\mathcal{R}$ be a $\mathbb{C}\mathrm{P}^1$-structure on a marked rectangle R, and let $f\colon R \to \mathbb{C}\mathrm{P}^1$ be its developing map. Then $\mathcal{R}$ is* circular *if there is a round cylinder A on $\mathbb{C}\mathrm{P}^1$ such that*

- *the image of f is contained in A;*
- *the horizontal edges of R are immersed into different boundary circles of A;*
- *for each vertical edge v of R, its development $f(v)$ is a simple arc on A transverse to the horizontal foliation.*

Given a circular rectangle $\mathcal{R}$, the support of $\mathcal{R}$ consists of the round cylinder A and the simple arcs on A which are the developments of the vertical edges of $\mathcal{R}$ in Definition 8.1. We denote the support by $\mathrm{Supp}\,\mathcal{R}$. We can pull-back the horizontal foliation on A to a foliation on $\mathcal{R}$ by the developing map, and call it the horizontal foliation of $\mathcal{R}$.

Given projective structures $\mathcal{R}$ and $\mathcal{Q}$ on a marked rectangle R, we say that $\mathcal{P}$ and $\mathcal{Q}$ are compatible if $\mathrm{Supp}\,\mathcal{R} = \mathrm{Supp}\,\mathcal{Q}$. Let $\mathcal{R}$ be a circular rectangle, such that the both vertical edges are supported on the same arc α on a circular cylinder. Then, we say that $\mathcal{R}$ is semi-compatible with α.

8.1.1. *Grafting a circular rectangle.* (See [Bab10].) Let $\mathcal{R}$ be a circular $\mathbb{C}\mathrm{P}^1$-structure on a marked rectangle R. Let A be the round cylinder in $\mathbb{C}\mathrm{P}^1$ which supports $\mathcal{R}$. Pick an arc α on $\mathcal{R}$, such that α connects the horizontals edges and it is transversal to the horizontal foliation of $\mathcal{R}$. Then α is embedded into A by $\mathrm{dev}\,\mathcal{R}$ — we call such an arc α an admissible arc. By cutting and gluing A and $\mathcal{R}$ along α in an alternating manner, we obtain a new circular $\mathbb{C}\mathrm{P}^1$-structure on R whose support still is $\mathrm{Supp}\,\mathcal{R}$. This operation is the grafting of $\mathcal{R}$ along α, and the resulting structure on R is denoted by $\mathrm{Gr}_\alpha\,\mathcal{R}$.

One can easily show that $\mathrm{Gr}_\alpha\,\mathcal{R}$ is independent on the choice of the admissible arc α, since an isotopy of α preserving its initial conditions do not change $\mathrm{Gr}_\alpha\,\mathcal{R}$.

8.2. **Circular staircase loops.** Let $C = (f, \rho)$ be a $\mathbb{C}\mathrm{P}^1$-structure on S. A topological staircase curve is a piecewise smooth curve, such that

- its smooth segments are labeled by "horizontal" or "vertical" alternatively along the curve, and
- at every singular point, the horizontal and vertical tangent directions are linearly independent in the tangent space.

Then, a topological staircase curve s on C is circular, if the following conditions are satisfied: Letting $\tilde{s}$ be a lift of s to $\tilde{S}$,

- every horizontal segment h of $\tilde{s}$ is immersed into a round circle in $\mathbb{C}\mathrm{P}^1$ by f, and
- for every vertical segment v of $\tilde{s}$, letting h_1, h_2 be the horizontal edges starting from the endpoints of v,
  - the round circles c_1, c_2 containing $f(h_1)$ and $f(h_2)$ are disjoint, and
  - $f|v$ is contained in the round cylinder bounded by c_1, c_2 and, it is transverse to the horizontal foliation of the round cylinder.

8.3. **Circular polygons.** Let P be a marked polygon with even number of edges. Then, let $e_1, e_2, \ldots, e_{2n}$ denote its edges in the cyclic order so that the edges with odd indices are vertical edges and with even indices horizontal edges. Suppose that $c_2, c_4 \ldots c_{2n}$ are round circles in $\mathbb{C}\mathrm{P}^1$ such that, for every $i \in \mathbb{Z}/n\mathbb{Z}$,

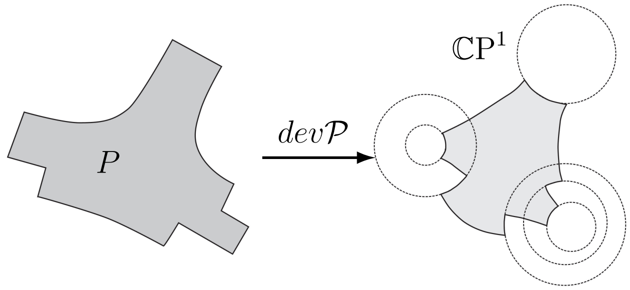

FIGURE 16. A development of a projective polygon supported on round circles (when the developing map is injective).

- c_{2i} and $c_{2(i+1)}$ are disjoint, and
- $c_{2(i-1)}$ and $c_{2(i+1)}$ are contained in the same component of $\mathbb{C}\mathrm{P}^1 \setminus c_{2i}$.

Let $\mathcal{A}_i$ denote the round cylinder bounded by c_{2i} and $c_{2(i+1)}$. A circular $\mathbb{C}\mathrm{P}^1$-structure $\mathcal{P}$ on P is supported on $\{c_{2i}\}_{i=1}^n$ if

- e_{2i} is immersed into the round circles of c_{2i} by $\operatorname{dev}\mathcal{P}$ for every $i = 1, \ldots n$, and
- e_{2i+1} is immersed into $\mathcal{A}_i$ and its image is transversal to the horizontal foliation of $\mathcal{A}_i$ (Figure 16) for every $i = 0, 1, \ldots, n-1$.

Let $\mathcal{P}$ be a circular $\mathbb{C}\mathrm{P}^1$-structure on a polygon P supported on a circle system $\{c_{2i}\}_i^n$. For $\epsilon > 0$, $\mathcal{P}$ is ϵ-circular, if

- for every vertical edge v_i is ϵ-parallel to the vertical foliation $\mathcal{V}$ of the support cylinder $\mathcal{A}_i$, and
- the total transversal measure of v given by the vertical foliation $\mathcal{V}$ is less than ϵ.

(Here, by the "total" transversal measure, we mean that if v intersects a leaf of $\mathcal{V}$ more than once, and the measure is counted with multiplicity.)

Let $\mathcal{P}_1, \mathcal{P}_2$ be circular $\mathbb{C}\mathrm{P}^1$-structures on a $2n$-gon P. Then $\mathcal{P}_1$ and $\mathcal{P}_2$ are compatible if, for each $i = 1, \ldots, n$, $\operatorname{dev}\mathcal{P}_1$ and $\operatorname{dev}\mathcal{P}_2$ take e_{2i} to the same round circle and the arcs $f_1(v_{2i-1})$ and $f_2(v_{2i-1})$ are the same.

Let A be a flat cylinder with geodesic boundary; then its universal cover $\tilde{A}$ is an infinite Euclidean strip. A projective structure (f, ρ) on A is circular, if the developing map $f\colon \tilde{A} \to \mathbb{C}\mathrm{P}^1$ is a covering map onto a round cylinder in $\mathbb{C}\mathrm{P}^1$.

Next, let A be a spiral cylinder. Then each boundary component b of A is a monotone staircase loop. Let $\tilde{b}$ be the lift of b to the universal cover universal cover $\tilde{A}$. Let $\{e_i\}_{i\in\mathbb{Z}}$ be the segments of $\tilde{b}$ linearly indexed so that e_i with an odd index is a vertical edge and with an even index is a horizontal edge; clearly $\tilde{b} = \cup_{i\in\mathbb{Z}} e_i$. Then, a $\mathbb{C}\mathrm{P}^1$-structure (f, ρ) on A is circular, if, for each boundary staircase loop b of A and each $i \in \mathbb{Z}$,

- the horizontal edge e_{2i} is immersed into a round circle c_i on $\mathbb{C}\mathrm{P}^1$;
- c_{i-1}, c_i and c_{i+1} are disjoint, and the round annulus bounded by c_{i-1} and c_{i+1} contains c_i in its interior;
- f embeds v_i in the round cylinder $\mathcal{A}_i$ bounded by c_i and c_{i+1}, and $f(v_i)$ is transverse to the circular foliation of $\mathcal{A}_i$.

Two circular $\mathbb{C}\mathrm{P}^1$-structures $\mathcal{A}_1 = (f_1, \rho_1), \mathcal{A}_2 = (f_2, \rho_2)$ on a spiral cylinder A are compatible

- ρ_1 is equal to ρ_2 up to conjugation by an element of $\mathrm{PSL}(2, \mathbb{C})$ (thus we can assume $\rho_1 = \rho_2$);
- for each boundary component h of $\tilde{A}$, f_1 and f_2 take h to the same round circle;
- for each vertical edge v of $\tilde{A}$, $f_1|v = f_2|v$.

More generally, let $\mathcal{F} = (f_1, \rho_1)$ and $\mathcal{F}' = (f_2, \rho_2)$ be two circular $\mathbb{C}\mathrm{P}^1$-structures on stair-case surfaces F and F'. First suppose that there is a diffeomorphism $\phi\colon F \to F'$, which takes the vertices of F bijectively to those of F'. Then $\mathcal{F}$ is compatible with $\mathcal{F}'$ if

- ρ_1 is conjugate to ρ_2 (thus we can assume that $\rho_1 = \rho_2$);
- for every vertex p_1 of $\mathcal{F}_1$, the development of p_1 coincides with the development of $\phi(p_2)$;
- for every a horizontal edge h of $\mathcal{F}_1$, letting h' be its corresponding horizontal edge of $\mathcal{F}'$, then the developments of h and h' are contained in the same round circle;
- for every vertical edge v of $\mathcal{F}$, letting v' be its corresponding edge v' of $\mathcal{F}'$, then the developments of v' and v coincide.

Next, in stead of a diffeomorphism, we suppose that there is a semi-diffeomorphism $\phi\colon F \to F'$. Then $\mathcal{F}$ is semi-compatible with $\mathcal{F}'$ if

- ρ_1 is conjugate to ρ_2 (thus we can assume that $\rho_1 = \rho_2$);
- for every vertex p_1 of $\mathcal{F}_1$, the development of p_1 coincides with the development of $\phi(p_2)$;
- if a horizontal edge h of $\mathcal{F}$ corresponds to a horizontal edge h' of $\mathcal{F}'$, then h and h' are supported on the same round circle on $\mathbb{C}\mathrm{P}^1$;
- for every vertical edge v of $\mathcal{F}$, letting v' be its corresponding vertical edge (segment) of $\mathcal{F}'$, then the developments of v' and v coincide.

8.4. Construction of circular train tracks $\mathcal{T}_{Y,\rho}$.

In this section, if ρ is in $\chi_X \cap \chi_Y$ minus a large compact subset, we construct a circular train-track structure of $C_{Y,\rho}$ related to the polygonal train-track decomposition $\mathbf{T}'_{Y,\rho}$.

Two train-track structures T_1, T_2 on a flat surface E is (p,q)-quasi-isometric for $p > 1$ and $q > 0$ if there is a continuous (p,q)-quasi-isometry $\phi\colon E \to E$ homotopic to the identity such that $\phi(T_1) = T_2$ and the restriction of ϕ to T_1 is a homotopy equivalence between T_1 and T_2.

Theorem 8.2. *For every $\epsilon > 0$, there is a bounded subset $K = K_\epsilon$ in $\chi_X \cap \chi_Y$, such that, for every $\rho \in \chi_X \cap \chi_Y \setminus K$, there is an ϵ-circular surface train track decomposition $\mathcal{T}_{Y,\rho}$ of $C_{Y,\rho}$ with the following properties:*

1. *$\mathcal{T}_{Y,\rho}$ is diffeomorphic to $\mathbf{T}'_{Y,\rho}$, and it is $(1+\epsilon, \epsilon)$-quasi-isometric to both $\mathbf{T}_{Y,\rho}$ and $\mathbf{T}'_{Y,\rho}$ in the normalized metric $E^1_{Y,\rho}$.*
2. *For every vertical edge v of $\mathbf{T}'_{Y,\rho}$, then its corresponding edge of $\mathcal{T}_{Y,\rho}$ is contained in the leaf of the vertical foliation $V_{Y,\rho}$.*
3. *For a branch B_X of $\mathbf{T}_{X,\rho}$, letting B_Y be its corresponding branch of $\mathbf{T}'_{Y,\rho}$ and letting $\mathcal{B}_Y$ be the branch of $\mathcal{T}_{Y,\rho}$ corresponding to B_Y, the restriction of $\hat{\beta}_{X,\rho}$ to $\partial\tilde{B}_X$ is ϵ-close to the restriction of $\hat{\beta}_{Y,\rho}$ to $\partial\tilde{\mathcal{B}}_Y$ pointwise; moreover, if B_X is a transversal branch, then $\hat{\beta}_{X,\rho}|\tilde{B}_X$ is ϵ-close to $\hat{\beta}_{Y,\rho}|\tilde{\mathcal{B}}_Y$ pointwise.*

We fix a metric on the unit tangent bundle of $\mathbb{H}^3$ which is left-invariant under $\mathrm{PSL}(2, \mathbb{C})$.

Proposition 8.3. *For every $\epsilon > 0$, if a bounded subset K_ϵ of χ_X is sufficiently large, then, for every $\rho \in \chi_X \setminus K_\epsilon$ and every horizontal edge h of $\mathbf{T}_{X,\rho}$, the $\mathrm{Ep}^*_{X,\rho}$-images of the vertical unite tangent vectors of along h are ϵ-close.*

Proof. The assertion immediately follows from Theorem 3.9 (2). □

Recall that we have constructed a system of equivariant circles for horizontal edges of $\tilde{\mathbf{T}}_{X,\rho}$ in Lemma 7.3. Let $h = [u, w]$ denote the horizontal edge of $\mathbf{T}_{X,\rho}$ where u, w are the endpoints. We shall perturb the endpoints of each horizontal edge of $\mathbf{T}_{Y,\rho}$ so that the endpoints map to the corresponding round circle.

Proposition 8.4. *For every $\epsilon > 0$, there are sufficiently small $\delta > 0$ and a (large) bounded subset K_ϵ of $\chi_X \cap \chi_Y$ satisfying the following: For every $\rho \in \chi_X \cap \chi_Y \setminus K_\epsilon$, if $\mathbf{c} = \{c_h\}$ is a circle system for horizontal edges h of $\mathbf{T}_{X,\rho}$ given by Lemma 7.3 for δ, then, for every horizontal edge $h = [u, w]$ of $\tilde{\mathbf{T}}_{Y,\rho}$, there are, with respect to the normalized metric $E^1_{Y,\rho}$, ϵ-small perturbations u' and w' of u and w along $V_{Y,\rho}$, respectively, such that $f_{Y,\rho}(u')$ and $f_{Y,\rho}(w')$ are contained in the round circle c_h.*

Proof. This follows from Theorem 7.7 and Lemma 7.3 (2). □

Proof of Theorem 8.2. By Proposition 8.4, for each horizontal edge $h = [u, w]$ of $\mathbf{T}'_{Y,\rho}$, there is an ϵ-homotopy of h to the circular segment h' the perturbations u', w' such that, letting $\tilde{h}$ be a lift of h to $\tilde{E}_{Y,\rho}$, the corresponding lift $\tilde{h}'$ of h' is immersed into the round circle $c_{\tilde{h}}$. For each vertical edge v of $\mathbf{T}'_{Y,\rho}$, at each endpoint of v, there is a horizontal edge of $\mathbf{T}'_{Y,\rho}$ starting from the point; then the round circles corresponding to the horizontal edges bound a round cylinder.

Note that a vertex u of $\mathbf{T}'_{Y,\rho}$ is often an endpoint of different horizontal edges h_1 and h_2. Thus, if the perturbations u'_1 and u'_2 of u are different for h_1 and h_2, then $\mathcal{T}_{Y,\rho}$ has a new short vertical edge connecting u'_1 and u'_2, and $\mathcal{T}_{Y,\rho}$ is non-diffeomorphic to $\mathbf{T}'_{Y,\rho}$.

Recall that the $\delta/4$-neighborhood of the singular points of $E^1_{Y,\rho}$ is disjoint from the one-skeleton of $\mathbf{T}_{Y,\rho}$ by Proposition 6.23. Thus, every vertical edge v of $\mathbf{T}'_{Y,\rho}$ is ϵ-circular with respect to the round cylinder by Corollary 3.7. Thus we have (2). Thus we obtained an ϵ-circular train-track decomposition $\mathcal{T}_{Y,\rho}$ of $E_{Y,\rho}$.

As the applies homotopies are ϵ-small, $\mathcal{T}_{Y,\rho}$ are ϵ-close to $\mathbf{T}'_{Y,\rho}$ (1). Thus we may, in addition, assume that $\mathcal{T}_{Y,\rho}$ is ϵ-close to $\mathbf{T}'_{Y,\rho}$ by Proposition 6.23. Moreover, Theorem 7.7 give (3). 8.2

8.5. **Construction of $\mathcal{T}_{X,\rho}$.** Given a train-track structure on a surface, the union of the edges of its branches is a locally finite graph embedded on the surface. An edge of a train-track decomposition is an edge of the graph, which contains no vertex in its interior (whereas an edge interior of a branch may contain a vertex of the train track).

Definition 8.5. *Let C, C' be $\mathbb{C}\mathrm{P}^1$-structures on S with the same holonomy $\rho\colon \pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$, so that $\operatorname{dev} C$ and $\operatorname{dev} C'$ are ρ-equivariant. A circular train-track decomposition $\mathcal{T} = \cup_i \mathcal{B}_i$ of C is* semi-compatible *with a circular train-track decomposition $\mathcal{T}' = \cup \mathcal{B}'_j$ of C' if there is a marking-preserving continuous map $\Theta\colon C \to C'$ such that, for each branch $\mathcal{B}$ of $\mathcal{T}$, Θ takes $\mathcal{B}$ to a branch of $\mathcal{B}'$ of $\mathcal{T}'$, and that $\mathcal{B}$ and $\mathcal{B}'$ are compatible by Θ.*

Theorem 8.6. *For every $\epsilon > 0$, if a bounded subset K_ϵ in $\chi_X \cap \chi_Y$ is sufficiently large, then, for every $\rho \in \chi_X \cap \chi_Y \setminus K_\epsilon$, there is an ϵ-circular train track decomposition $\mathcal{T}_{X,\rho}$ of $C_{X,\rho}$, such that*

(1) *$\mathcal{T}_{X,\rho}$ is semi-compatible with $\mathcal{T}_{Y,\rho}$, and*
(2) *$\mathcal{T}_{X,\rho}$ additively 2π-hausdorff-close to $\mathbf{T}_{X,\rho}$ with respect to the (unnormalized) Euclidean metric $E_{X,\rho}$: More precisely, in the vertical direction, $\mathcal{T}_{X,\rho}$ is ϵ-close to $\mathbf{T}_{X,\rho}$, and in the horizontal direction, 2π-close in the Euclidean metric of $E_{X,\rho}$ for all $\rho \in \mathcal{X}_X \setminus K_\epsilon$.*

Proof. First, we transform $\mathbf{T}_{X,\rho}$ by perturbing horizontal edges so that horizontal edges are circular. Recall that, the branches of $\mathcal{T}_{Y,\rho}$ are circular with respect to a fixed system $\mathbf{c}$ of equivariant circles given by Lemma 7.3. Thus, the $\beta_{X,\rho}$-images of vertical tangent vectors along h are ϵ-close to a single vector orthogonal to the hyperbolic plane bounded by c_h. Therefore, similarly to Theorem 8.2, we can modify the train-track structure $\mathbf{T}_{X,\rho}$ so that horizontal edges are circular and ϵ-Hausdorff close to the original train-track structure in the Euclidean metric of $E_{X,\rho}$ (this process may create new short vertical edges). Thus we obtained an ϵ-circular train track $\mathbf{T}'_{X,\rho}$ whose horizontal edges map to their corresponding round circles of $\mathbf{c}$.

Next, we make the vertical edges compatible with $\mathcal{T}_{Y,\rho}$. Recall that $\mathbf{T}_{X,H_X}$ has no rectangles with short vertical edges (Lemma 5.6). Therefore, we have the following.

Lemma 8.7. *For every $R > 0$, if the bounded subset K of $\mathcal{X}$ is sufficiently large, then, for each vertical edge of $\mathbf{T}'_{X,\rho}$, the horizontal distance to adjacent vertical edges is at least R.*

Thus, by Lemma 8.7, there is enough room to move vertical edges, less than 2π, so that the train-track is compatible with $\mathcal{T}_{Y,\rho}$ along vertical edges as well.

Since $\mathbf{T}_{X,\rho}$ is semi-diffeomorphic to $\mathbf{T}_{Y,\rho}$ (Proposition 6.29 (3)), $\mathcal{T}_{X,\rho}$ is semi-compatible with $\mathcal{T}_{Y,\rho}$. 8.6

9. Grafting cocycles and intersection of holonomy varieties

In this section, given a certain pair of $\mathbb{C}\mathrm{P}^1$-structures on S with the same holonomy, we shall construct a $\mathbb{Z}$-valued cocycle.

9.1. Relative degree of rectangular $\mathbb{C}\mathrm{P}^1$-structures.

Let $a < b$ be real numbers. Let $f, g\colon [a, b] \to \mathbb{S}^1$ be orientation preserving immersions or constant maps, such that $f(a) = g(a)$ and $f(b) = g(b)$.

Definition-Lemma 9.1. *The integer $\sharp f^{-1}(x) - \sharp g^{-1}(x)$ is independent on $x \in \mathbb{S}^1 \setminus \{f(a), f(b)\}$, where $\sharp$ denotes the cardinality. We call this integer the* degree of f relative to g*, or simply, the* relative degree*, and denote it by* $\deg(f, g)$.

Clearly, it is not important that f and g are defined on the same interval as long as corresponding endpoints map to the same point on $\mathbb{S}^1$. Moreover, the degree is additive in the following sense.

Lemma 9.2 (Subdivision of relative degree)**.** *Suppose in addition that $f(c) = g(c)$ for some $c \in (a, b)$. Then*

$$\deg(f, g) = \deg(f|_{[a,c]}, g|_{[a,c]}) + \deg(f|_{[a,c]}, g|_{[a,c]}).$$

The proofs of the lemmas above are elementary. Let R, Q be circular projective structures on a marked rectangle, and suppose that R and Q are compatible: By their developing maps, corresponding horizontal edges of R and Q are immersed into the same round circle on $\mathbb{C}\mathrm{P}^1$, and the corresponding vertices map to the same point.

Then, the degree of R relative to Q is the degree of a horizontal edge of R relative to its corresponding horizontal edge of Q — we similarly denote the degree by $\deg(R, Q) \in \mathbb{Z}$. Although R has two horizontal edges, this degree is well-defined:

Lemma 9.3 (c.f. Lemma 6.2 in [Bab15]). *The degree* $\deg(R, Q)$ *is independent on the choice of the horizontal edge.*

Proof. Let A be the round cylinder on $\mathbb{C}\mathrm{P}^1$ supporting both R and Q. Then, the horizontal foliation $\mathcal{F}_A$ of A by round circles c induces foliations $\mathcal{F}_R$ and $\mathcal{F}_Q$ on R and Q, respectively. Then, for each leaf c of $\mathcal{F}_A$, the corresponding leaves ℓ_R and ℓ_Q of $\mathcal{F}_R$ and $\mathcal{F}_Q$, respectively, are immersed into c, and the endpoints of ℓ_R and ℓ_Q on the corresponding vertical edges of R and Q map to the same point on c. The degree of ℓ_R relative to ℓ_Q is an integer, and it changes continuously in the leaves c of $\mathcal{F}_A$. Thus, the assertion follows immediately. □

Lemma 9.4 (cf. Lemma 6.2 in [Bab15]). *Let* R, Q *be* $\mathbb{C}\mathrm{P}^1$*-structures on a marked rectangle with* $\operatorname{Supp} R = \operatorname{Supp} Q$. *Then*

- *if* $\deg(R, Q) > 0$, *then* R *is obtained by grafting* Q *along an admissible arc* $\deg(R, Q)$ *times;*
- *if* $\deg(R, Q) < 0$, *then* Q *is obtained by grafting* R *along an admissible arc* $-\deg(R, Q)$ *times;*
- *if* $\deg(R, Q) = 0$, *then* R *is isomorphic to* Q *(as* $\mathbb{C}\mathrm{P}^1$*-structures).*

By Lemma 9.4, the "difference" of $\mathbb{C}\mathrm{P}^1$-rectangles R and Q can be represented by an arc α with weight $\deg(R, Q)$ such that α sits on the base rectangle connecting the horizontal edges.

Lemma 9.5. *Let* R *and* Q *be circular projective structures on a marked rectangle such that* $\operatorname{Supp} R = \operatorname{Supp} Q$. *Let* A *be the round cylinder on* $\mathbb{C}\mathrm{P}^1$ *supporting* R *and* Q. *Suppose that there are admissible arcs* α_R *on* R *and* α_Q *on* Q *which develop onto the same arc on* A *(transversal to the horizontal foliation), so that the arcs decompose* R *and* Q *into two circular rectangles* R_1, R_2 *and* Q_1, Q_2, *respectively and* $\operatorname{Supp} R_1 = \operatorname{Supp} Q_1$ *and* $\operatorname{Supp} R_2 = \operatorname{Supp} Q_2$. *Then*

$$\deg(R, Q) = \deg(R_1, Q_1) + \deg(R_2, Q_2).$$

Proof. This follows from Lemma 9.2. □

9.2. **Train-track graphs for planar polygons.** Let P be a L^∞-convex staircase polygon in $\mathbb{E}^2$, which contains no singular points. We can decompose P into finitely many rectangles $P_1, P_2, \ldots, P_n$ by cutting P along $n-1$ horizontal arcs each connecting a vertex and a point on a vertical edge. Let $\mathcal{P}, \mathcal{Q}$ be compatible circular projective structures on P such that the round circles supporting horizontal edges are all disjoint. The decomposition P into $P_1, P_2, \ldots P_n$ gives decompositions of $\mathcal{P}$ into $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ and $\mathcal{Q}$ into $\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_n$ such that $\operatorname{Supp} \mathcal{P}_i = \operatorname{Supp} \mathcal{Q}_i$ for $i = 1, 2, \ldots, n$. As in §9.1, for each i, we obtain an arc α_i connecting horizontal edges of P_i with weight $\deg(\mathcal{P}_i, \mathcal{Q}_i)$. Then, by splitting and combining $\alpha_1, \alpha_2, \ldots, \alpha_n$ appropriately, we obtain a $\mathbb{Z}$-valued train-track graph $\Gamma(\mathcal{P}, \mathcal{Q})$ on P transversal to the decomposition (Figure 17).

9.3. **Train-track graphs for cylinders.** Let A_X be a cylindrical branch of $\mathbf{T}_{X,\rho}$, and let A_Y be the corresponding cylindrical branch of $\mathbf{T}_{Y,\rho}$.

Pick a monotone staircase curve α on A_X, such that

(1) α connects different boundary components of A_X, and its endpoints are on horizontal edges (Figure 18),

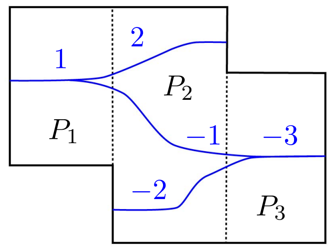

FIGURE 17. An example of a weighted train track on an L^∞-convex polygon in $\mathbb{E}^2$.

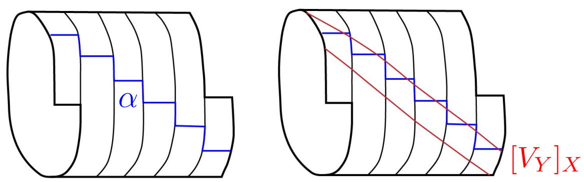

FIGURE 18.

(2) the restriction of $[V_{Y,\rho}]_X$ to A_X has a leaf disjoint from A_X.

Then $[V_{Y,\rho}]_X$ is essentially carried by A_X. Then, one can easily show that the choice of α is unique though an isotopy preserving the properties.

Lemma 9.6. *Suppose there are two staircase curves α_1, α_2 on A_X satisfying Conditions (1) and (2). Then, α_1 and α_2 are isotopic through staircase curves α_t satisfying Conditions (1) and (2).*

In Proposition 6.7, pick a realization of $[V_Y]_X$ on the decomposition (A_X, α_X) by a homotopy of $[V_Y]_X$ sweeping out triangles. This induces an ϵ-almost staircase curve α_Y. Similarly to Lemma 7.3, pick a system of round circles $\mathbf{c} = \{c_h\}$ corresponding to horizontal edges h of α_X so that the $\mathrm{Ep}_{X,\rho}$-images of vertical tangent vectors along h are ϵ-close to a single vector orthogonal to the hyperbolic plane bounded by c_h.

Then, as in §8.4 we can accordingly isotope the curve α_Y so that the horizontal edges are supported on their corresponding circle of $\mathbf{c}$ and vertical edges remain vertical— let $\alpha_Y^{\mathbf{c}}$ denote the curve after this isotopy. Then $\mathcal{A}_Y \setminus \alpha_Y^{\mathbf{c}}$ is a circular projective structure on a staircase polygon in $\mathbb{E}^2$.

Then, (similarly to Theorem 8.6), we can isotope α_X to an ϵ-almost circular staircase curve $\alpha_X^{\mathbf{c}}$ so that

- α_X is 2π-hausdorff close to $\alpha_X^{\mathbf{c}}$;
- the horizontal edge h of $\alpha_X^{\mathbf{c}}$ is supported on c_h;
- $\mathcal{A}_X \setminus \alpha_X^{\alpha}$ is an ϵ-almost circular staircase polygon compatible with $\mathcal{A}_Y \setminus \alpha_Y^{\mathbf{c}}$.

As in §9.2, $\mathcal{A}_X \setminus \alpha_X^{\mathbf{c}}$ and $\mathcal{A}_Y \setminus \alpha_Y^{\mathbf{c}}$ yield a $\mathbb{Z}$-valued weighted train track $\Gamma_{A\setminus\alpha_X}$ on the polygon $A \setminus \alpha_X$ such that $\Gamma_{A_X\setminus\alpha_X}$ is transversal to the horizontal foliation. Up to a homotopy preserving endpoints on the horizontal edges, the endpoints of $\Gamma_{A_X \setminus \alpha_X}$ match up along α_X as $\mathbb{Z}$-weighted arcs. Thus, we obtain a weighted train-track graph Γ_{A_X} on A_X.

Consider the subset of the boundary of the circular cylinder $\mathcal{A}_X$ which is the union of the vertical boundary edges and the vertices of $\mathcal{T}_{X,\rho}$ contained in $\partial \mathcal{A}_X$. Let A be the homotopy class of arcs in $\mathcal{A}_X$ connecting different points in this subset. Then $[\Gamma_{A_X}] \colon \mathsf{A} \to \mathbb{Z}$ be the map which takes an arc to its total signed intersection number with Γ_{A_X}.

Then Lemma 9.6 gives a uniqueness of $[\Gamma_{A_X}]$:

Proposition 9.7. *$[\Gamma_{A_X}] \colon \mathsf{A} \to \mathbb{Z}$ is independent on the choice of the staircase curve α and the realization of $[V_Y]_X$ on $(\mathcal{A}_X, \alpha_X)$.*

9.4. **Weighted train tracks and $\mathbb{C}\mathrm{P}^1$-structures with the same holonomy.** In this section, we suppose that Riemann surfaces X, Y have the same orientation. Let C be the set of the homotopy classes of closed curves on S (which are not necessarily simple). Given a weighted train-track graph immersed on S, it gives a cocycle taking $\gamma \in \mathsf{C}$ to its weighted intersection number with the graph.

Theorem 9.8. *For all distinct $X, Y \in \mathsf{T}$, there is a bounded subset K in $\chi_X \cap \chi_Y$, such that*

1. *for each $\rho \in \chi_X \cap \chi_Y \setminus K$, the semi-compatible train-track decompositions $\mathcal{T}_{X,\rho}$ of $C_{X,\rho}$ and $\mathcal{T}_{Y,\rho}$ of $C_{Y,\rho}$ in Theorem 8.6 yield a $\mathbb{Z}$-weighted train track graph Γ_ρ carried by $\mathcal{T}_{X,\rho}$ (immersed in S);*
2. *the intersection cocycle $[\Gamma_\rho] \colon \mathsf{C} \to \mathbb{Z}$ is independent on the choices for the construction of $\mathcal{T}_{X,\rho}$ and $\mathcal{T}_{Y,\rho}$;*
3. *$[\Gamma_\rho] \colon \mathsf{C} \to \mathbb{Z}$ is continuous in $\rho \in \chi_X \cap \chi_Y \setminus K$.*

Since $[\Gamma_\rho]$ takes values in $\mathbb{Z}$, the continuity immediately implies the following.

Corollary 9.9. *For sufficiently large boundary subset K of $\chi_X \cap \chi_Y$, $[\Gamma_\rho]$ is well-defined and constant on each connected component of $\chi_X \cap \chi_Y \setminus K$.*

We first construct a weighted train-track in (1). Let $h_{X,1} \dots h_{X,n}$ be the horizontal edges of branches of $\mathcal{T}_{X,\rho}$.

Since $\mathcal{T}_{X,\rho}$ and $\mathcal{T}_{Y,\rho}$ are semi-compatible (Proposition 8.6), for each $i = 1, 2, \dots, n$, letting $h_{Y,i}$ be its corresponding edge of a branch of $\mathcal{T}_{Y,\rho}$ or a vertex of $\mathcal{T}_{Y,\rho}$. Then, $h_{X,i}$ and $h_{Y,i}$ develop into the same round circle on $\mathbb{C}\mathrm{P}^1$ and corresponding endpoints map to the same point by the semi-compatibility. Thus we have the degree of $h_{X,i}$ relative to $h_{Y,i}$ taking a value in $\mathbb{Z}$ (Definition 9.1) for each horizontal edge:

$$\gamma_\rho \colon \{h_{X,1}, \dots, h_{X,n}\} \to \mathbb{Z}.$$

We will construct a $\mathbb{Z}$-weighted train track Γ_ρ carried by $\mathcal{T}_{X,\rho}$ so that the intersection number with $h_{X,i}$ is $\gamma_\rho(h_{X,i})$. The train-track graph Γ_ρ will be constructed on each branch of $\mathcal{T}_{X,\rho}$:

- For each rectangular branch R of $\mathcal{T}_{X,\rho}$, we will construct a $\mathbb{Z}$-train track graph embedded in R (Proposition 9.10).
- For each cylinder A of $\mathcal{T}_{X,\rho}$, we have obtained a $\mathbb{Z}$-weighted train track graph embedded in A (§9.3).
- For each transversal branch, we will construct a $\mathbb{Z}$-weighted train track graph embedded in the branch (Proposition 9.12).
- For each non-transversal branch of $\mathcal{T}_{X,\rho}$, we will construct a $\mathbb{Z}$-weighted train track immersed in the branch (Lemma 9.15).

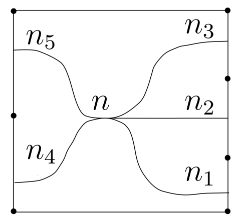

FIGURE 19. A $\mathbb{Z}$-weight train-track graph for a rectangle. Here $n = n_1 + n_2 + n_3 = n_4 + n_5$. The black dots are vertices of $\mathcal{T}_{X,\rho}$.

9.4.1. *Train tracks for rectangular branches.* Given an ordered pair of compatible $\mathbb{C}\mathrm{P}^1$-structures on a rectangle, Lemma 9.4 gives a $\mathbb{Z}$-weighted arc connecting the horizontal edges of the rectangle. Since the horizontal edges of a rectangular branch of $\mathcal{T}_{X,\rho}$ may contain a vertex, we transform the weighted arc to a weighted train-track graph so that it matches with γ_ρ.

Proposition 9.10. *For every $\epsilon > 0$, there is a bounded subset K of $\chi_X \cap \chi_Y$, such that, for every $\rho \in \chi_X \cap \chi_Y \setminus K$, for each rectangular branch $\mathcal{R}_X$ of $\mathcal{T}_{X,\rho}$, there is a $\mathbb{Z}$-weighted train track graph $\Gamma_{\rho,\mathcal{R}_X}$ embedded in $\mathcal{R}_X$ satisfying the following:*

- *$\Gamma_{\mathcal{R}_X}$ induces γ_ρ;*
- *$\Gamma_{\mathcal{R}_X}$ is transversal to the horizontal foliation on $\mathcal{R}_X$;*
- *each horizontal edge of $\mathcal{T}_{X,\rho}$ in $\partial\mathcal{R}_X$ contains, at most, one endpoint of $\Gamma_{\rho,\mathcal{R}_X}$;*
- *As cocycles, Γ_R is $(1+\epsilon,\epsilon)$-quasi-isometric to $(V_{X,\rho}|R_X) - (V_{Y,\rho}|R_Y)$, where R_X is a branch of $\mathbf{T}_{X,\rho}$ corresponding to $\mathcal{R}_X$ and R_Y is the branch of $\mathbf{T}'_{Y,\rho}$ corresponding to R_X .*

Proof. Since $\mathcal{R}_X$ and $\mathcal{R}_Y$ share their support, let $n = \deg(\mathcal{R}_X, \mathcal{R}_Y)$ as seem in Lemma 9.4. Let h_X and h_Y be corresponding horizontal edges of $\mathcal{R}_X$ and $\mathcal{R}_Y$. Then $h_X = h_{X,1} \cup \cdots \cup h_{X,m}$ be the decomposition of h_X into horizontal edges of $\mathcal{T}_X$, and let $h_Y = h_{Y,1} \cup \cdots \cup h_{Y,m}$ be the corresponding decomposition into horizontal edges and vertices of $\mathcal{T}_X$ compatible with the semi-diffeomorphism $\mathcal{T}_{X,\rho} \to \mathcal{T}_{Y,\rho}$. Let $n_i \in \mathbb{Z}$ be $\deg(h_{X,i}, h_{Y,i})$. Then, by Lemma 9.2, $n = n_1 + \cdots + n_m$. Then it is easy to construct a desired $\mathbb{Z}$-weighted train track realizing such decomposition for both pairs of corresponding horizontal edges (see Figure 19).

The last assertion follows from Lemma 4.1 and Theorem 7.7. □

9.4.2. *Train tracks for cylinders.* For each cylindrical branch $\mathcal{A}_X$ of $\mathcal{T}_{X,\rho}$, in §9.3, we have constructed a train-track graph $\Gamma_{\rho,\mathcal{A}_X}$ on $\mathcal{A}_X$, representing the difference between $\mathcal{A}_X$ and its corresponding cylindrical branch $\mathcal{A}_Y$ of $\mathcal{T}_{Y,\rho}$.

Proposition 9.11. *For every $\epsilon > 0$, if a bounded subset K of $\chi_X \cap \chi_Y$ is sufficiently large, then, for each cylindrical branch $\mathcal{A}_X$ of $\mathcal{T}_{X,\rho}$, the induced cocycle $[\Gamma_{\mathcal{A}_X}]\colon \mathsf{A} \to \mathbb{Z}$ times 2π is $(1+\epsilon,\epsilon)$-quasi-isometric to $V_{Y,\rho}|A_X - V_{X,\rho}|A_Y$, where A_X and A_Y are the corresponding cylindrical branches of $\mathbf{T}_{X,\rho}$ and $\mathbf{T}'_{Y,\rho}$.*

Proof. Recall that $\Gamma_{\mathcal{A}_X}$ is obtained from $\mathbb{Z}$-weighted train-track graphs on the rectangles. There is a uniform upper bound, which depends only on S, for the number of the rectangles used to define $[\Gamma_{\mathcal{A}_X}]$, since the decomposition was along horizontal arcs starting from singular points. Then, on each rectangle, the weighted graph is $(1+\epsilon,\epsilon)$-quasi-isometric to the difference of $V_{X,\rho}$ and $V_{Y,\rho}$ by Proposition 9.10. Thus $[\Gamma_{\mathcal{A}_X}]$ is also $(1+\epsilon,\epsilon)$-quasi-isometric to the difference of $V_{X,\rho}$ and $V_{Y,\rho}$ if K is sufficiently large. □

9.4.3. *Train-track graphs for transversal polygonal branches.* Recall that all transversal branches are polygonal or cylindrical (Lemma 6.18), i.e. their Euler characteristics are non-negative.

Proposition 9.12. *For every $\epsilon > 0$, there is a bounded subset K in $\chi_X \cap \chi_Y$ such that, for each transversal polygonal branch $\mathcal{P}_X$ of $\mathcal{T}_{X,\rho}$, there is a $\mathbb{Z}$-weighted train track $\Gamma_{\rho,\mathcal{P}_X}$ embedded in $\mathcal{P}_X$, letting P_X and P_Y be the branches of $\mathbf{T}_{X,\rho}$ and $\mathbf{T}_{Y,\rho}$, respectively, corresponding to $\mathcal{P}_X$, respectively, such that*

(1) *each horizontal edge h of $\mathcal{P}_X$ contains, at most, one endpoint of $\Gamma_{\rho,P}$;*
(2) *$[\Gamma_{\mathcal{P}_X}]$ agrees with γ_ρ on the horizontal edges of $\mathcal{T}_{X,\rho}$ contained in $\partial\mathcal{P}_X$;*
(3) *Γ_P is transversal to the horizontal foliation $H_{X,\rho}$ on P_X;*
(4) *$2\pi[\Gamma_P]$ is $(1+\epsilon,\epsilon)$-quasiisometric to $(V_{X,\rho}|P_X - V_{Y,\rho}|P_Y)$.*

For every $\epsilon > 0$, if K is sufficiently large, then, by Theorem 7.2 (2), let $\hat{\mathcal{Q}}_X$ be an ideal circular polygon whose truncation $\mathcal{Q}_X$ is ϵ-close to $\mathcal{P}_X$ in $C_{X,\rho}$. Similarly, $\hat{\mathcal{Q}}_Y$ be an ideal circular polygon whose truncation $\mathcal{Q}_Y$ is ϵ-close to $\mathcal{P}_Y$. Since $\operatorname{Supp}\mathcal{P}_X = \operatorname{Supp}\mathcal{P}_Y$ as circular polygons, we may in addition assume that $\operatorname{Supp}\mathcal{Q}_X = \operatorname{Supp}\mathcal{Q}_Y$ as truncated idea polygons.

Let $\overline{\mathcal{Q}}_X$ be the canonical polynomial $\mathbb{C}\mathrm{P}^1$-structure on $\mathbb{C}$ which contains $\hat{\mathcal{Q}}_X$ (§7.0.1). Let $\hat{\mathcal{L}}_X$ be the restriction of the Thurston lamination of $\overline{\mathcal{Q}}_X$ to $\hat{\mathcal{Q}}_X$. Similarly, let $\overline{\mathcal{Q}}_Y$ be the canonical polynomial $\mathbb{C}\mathrm{P}^1$-structure on $\mathbb{C}$ which contains $\mathcal{Q}_Y$. Let $\hat{\mathcal{L}}_Y$ be the restriction of the Thurston lamination of $\overline{\mathcal{Q}}_Y$ to $\hat{\mathcal{Q}}_Y$. As $\operatorname{Supp}\mathcal{Q}_X = \operatorname{Supp}\mathcal{Q}_Y$, thus $\overline{\mathcal{Q}}_X$ and $\overline{\mathcal{Q}}_Y$ share their ideal vertices. Then $\mathcal{Q}_X$ and $\mathcal{Q}_Y$ are ϵ-close to $\mathcal{P}_X$ and $\mathcal{P}_Y$, respectively. Thus, since γ_ρ takes values in $\mathbb{Z}$, $\hat{\mathcal{L}}_X - \hat{\mathcal{L}}_Y$ satisfies (2).

Theorem 7.2 (1) implies taht $\hat{\mathcal{L}}_X$ is $(1+\epsilon,\epsilon)$-quasi-isometric to $V_X|P_X$ and $\hat{\mathcal{L}}_Y$ is $(1+\epsilon,\epsilon)$-quasi-isometric to $V_Y|P_Y$, and therefore $\hat{\mathcal{L}}_X - \hat{\mathcal{L}}_Y$ satisfies (4).

(1) is easy to be realized by homotopy combining the edges of the train-track graph ending on the same horizontal edge. We show that there is a $\mathbb{Z}$-weighted train track Γ which is ϵ-close to $\hat{\mathcal{L}}_X - \hat{\mathcal{L}}_Y$, satisfying (3).

Let Γ^P_X be a weighted train track graph on P which represents $\hat{\mathcal{L}}_X$. Let Γ^P_Y be the weighted train-track graph which represents $\hat{\mathcal{L}}_Y$. By Theorem 7.7, the pleated surface of $\mathcal{P}_X$ is ϵ-close to the plated surface of $\mathcal{P}_Y$, because of the quasi-parallelism in Proposition 7.6. Let $\check{\Gamma}^P_X$ be the subgraph of $|\Gamma^P_X|$ obtained by eliminating the edges of weights less than a sufficiently small ϵ. Similarly, let $\check{\Gamma}^P_Y$ be the subgraph of Γ^P_Y obtained by eliminating the edges of weight less than ϵ.

Then, there is a minimal train-track graph Γ^P containing both $\check{\Gamma}^P_Y$ and $\check{\Gamma}^P_X$ and satisfying (1). Since the pleated surfaces are sufficiently close, by approximating the weights of $\Gamma^P_X - \Gamma^P_Y$ by integers, we obtain a desired $\mathbb{Z}$-weighted train-track graph supported on Γ^P.

Since $\mathcal{P}_X$ is a transversal branch, Γ^P_X and Γ^P_Y are both transversal to $H_{X,\rho}|P$, thus Γ^P is transversal to $H_{X,\rho}|P$ (3). 9.12

9.4.4. *Train-track graphs for non-transversal branches.* Let P_X and P_Y be the corresponding branches of $\mathbf{T}_{X,\rho}$ and $\mathbf{T}_{Y,\rho}$, respectively, which are non-transversal. Let $\mathcal{P}_X$ and $\mathcal{P}_Y$ be the branches of $\mathcal{T}_{X,\rho}$ and $\mathcal{T}_{Y,\rho}$ corresponding to P_X and P_Y, respectively. Then, by Theorem 8.2, $\hat{\beta}_{X,\rho}|\partial\mathcal{P}_X$ is ϵ-close to $\hat{\beta}_{Y,\rho}|\partial\mathcal{P}_Y$ in the C^0-metric and C^1-close along the vertical edges. Let σ_X be a pleated surface with crown-shaped boundary whose truncation approximates $\hat{\beta}_X|\mathcal{P}_X$, and let σ_Y be the pleated surfaced boundary with crown-shaped boundary whose truncation approximates $\hat{\beta}_Y|\mathcal{P}_Y$, such that σ_X and σ_Y share their boundary (in $\mathbb{H}^3$). In particular, the base hyperbolic surfaces for σ_X and σ_Y are diffeomorphic preserving marking and spikes.

Let ν_X and ν_Y be the bending measured laminations for σ_X and σ_Y, respectively; then ν_X and ν_Y contain only finite many leaves whose connect ideal points.

In Thurston coordinates, the developing map and the pleated surface of a $\mathbb{C}\mathrm{P}^1$-structure are related by the nearest point projection to the supporting planes of the pleated surface ([KP94, Bab20]). For small $\upsilon > 0$, let F_X, F_Y be the surfaces in $\mathbb{H}^3$ which are at distance υ from σ_X and σ_Y in the direction of the nearest point projections of P_X and P_Y ([EM87, Chapter II.2]).

Thus, if necessarily, refining ν_X to ν_Y to ideal triangulations of the hyperbolic surfaces appropriately, pick an (irreducible) sequence of flips w_i which connects ν_X to ν_Y. Clearly the sequence w_i corresponds to a sequence of triangulations.

Lemma 9.13. *If a bounded subset $K \subset \mathcal{X}$ is sufficiently large, for every $\rho \in \mathcal{X}_X \cap \mathcal{X}_Y \setminus K$ and all non-transversal branches P_X and P_Y, there is a uniform upper bound on the length of the flip sequence which depend only on $X, Y \in \mathsf{T}$, or appropriate refinements into triangulations.*

Proof. This follows from the length bound in Lemma 6.2. □

A triangulation in the sequence given by w_i is **realizable**, if there is an equivariant pleated surface homotopic to σ_X (and σ_Y) relative to the boundary such that the pleating locus agrees with the triangulation. In general, a triangulation in the sequence is not be realizable when the endpoints of edges develop to the same point on $\mathbb{C}\mathrm{P}^1$. However a generic perturbation makes the triangulation realizable:

Lemma 9.14. *For almost every perturbation of the holonomy of P_X and holonomy equivariant perturbation of the (ideal) vertices of $\tilde{\sigma}_X$ (and $\tilde{\sigma}_Y$) in $\mathbb{C}\mathrm{P}^1$, all triangulations in the flip sequence w_i are realizable. Moreover, the set of realizable perturbation is connected.*

Proof. If σ_X is an ideal polygon, the holonomy is trivial. Then, since there are only finitely many vertices and $\mathbb{C}\mathrm{P}^1$ has real dimension two, almost every perturbation are realizable.

If σ_X is not a polygon, an edge of a triangulation forms a loop if the endpoints are at the same spike of τ_X. For each loop ℓ of τ_X, the condition that the holonomy of ℓ is the identity is a complex codimension, at least, one in the character variety (and also in the representation variety). Since the flip sequence is finite, for almost all perturbations of the holonomy, if an edge of a triangulation in the sequence forms a loop, then its holonomy is non-trivial. Clearly, such a perturbation is connected. Then, for every such perturbation of the holonomy, it is easy to see that, for almost all equivariant perturbations of the ideal points, the triangulations in the sequence are realizable. □

For every perturbation of the holonomy and the ideal vertices given by Lemma 9.14, the flip sequence w_i gives the sequence of pleated surfaces $\sigma_X = \sigma_1 \xrightarrow{w_1} \sigma_2 \xrightarrow{w_2} \ldots \xrightarrow{w_{n-1}} \sigma_n = \sigma_Y$ in $\mathbb{H}^3$ connecting σ_Y to σ_X, such that σ_i's share their boundary geodesics and ideal vertices.

For each flip w_i, the pairs of triangles of the adjacent pleated surfaces σ_i and σ_{i+1} bound a tetrahedron in $\mathbb{H}^3$. To be precise, if the four vertices are contained in plane, we have the tetrahedron is collapsed into a quadrangle, but it does not affect the following argument. The edges exchanged by the flip correspond to the opposite edges of the tetrahedron. Then pick a geodesic segment

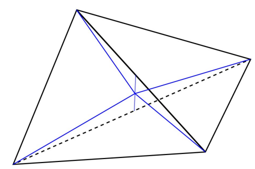

FIGURE 20. The blue lines are the pleating lamination interpolating the pleated surfaces which differ by a flip.

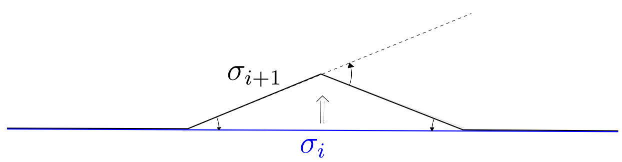

FIGURE 21. The link of a vertex of the ideal rectangle —— the sum of the indicated singed angles is zero.

connecting those opposite edges. Then there is a path σ_t $(i \leq t \leq i+1)$ of pleated surfaces with a single cone point of angle more than 2π such that

- σ_t connects σ_i to σ_{i+1};
- the pleated surfaces σ share their quadrangular boundary, which corresponds to the ideal quadrangle supporting the flip w_i;
- by the homotopy, σ_t sweeps out the tetrahedron;
- the cone point on the geodesic segment (see Figure 20).

In this manner, this sequence of pleated surfaces σ_i continuously extends to a homotopy of the pleated surfaces with, at most, one singular point of cone angle greater than 2π. This interpolation also connects a bending cocycle on σ_i to a bending cocycle on σ_{i+1} continuous, although the induced cocycle on σ_{i+1} may correspond to a measured lamination only immersed on the surface, since the edges of the triangulations transversally intersect. Thus, ν_X induces a sequence of the bending (immersed) measured laminations ν_i of σ_i supported a union of the pleating loci of $\sigma_1, \ldots, \sigma_i$.

For each i, the difference $\nu_{i+1} - \nu_i$ of the transverse cocycles is supported on the geodesics corresponding to the edges of the tetrahedron, so that, on the surface, the edges form an ideal rectangle with both diagonals. Let μ_i be the difference cocycle $\nu_{i+1} - \nu_i$. From each vertex of the ideal rectangle, there are three leaves of $\nu_{i+1} - \nu_i$ starting, and the sum of their weights is zero by Euclidean geometry (Figure 21). Note that P_X can be identified with σ_X by collapsing each horizontal edge of P_X to a point. Hence, for every i, if α is a closed curve on P_X or an arc connecting vertical edges of P_X, then $\mu_i(\alpha) = 0$. By regarding ν_j is a geodesic lamination on σ_X, their union $\cup_{j=1}^{i} \nu_j$ is a graph on σ_X whose vertices are the transversal intersection points of the triangulations. A small regular neighborhood N of $\cup_{j=1}^{i} \nu_j$ is decomposed into a small regular neighborhood N_0 of the vertices and small regular neighborhood of the edges minus N_0 in $N \setminus N_0$.

Since, after the whitehead moves, the pleated surface σ_X is transformed to a pleated surface σ_Y. Thus $\nu_n \setminus \nu_Y$ gives an integral transversal cocycle.

By the construction of the regular homotopy, we have the following.

Proposition 9.15 (Train tracks for non-transversal branches)**.** *For every non-transversally compatible branches $\mathcal{P}_X$ of $\mathcal{T}_{X,\rho}$ and $\mathcal{P}_Y$ of $\mathcal{T}_{Y,\rho}$, there is a $\mathbb{Z}$-weighted immersed train-track graph Γ_{P_X} representing the transversal cocycle supported on $\cup_{j=1}^{i} \nu_j$. Moreover, the train-track cocycle is independent on the choice of the flip sequence w_i.*

Proof. Given two flip sequences $(w_i), (w'_j)$ connecting the triangulations of σ_X to σ_Y, there are connected by a sequence of sequences (v_i^k) of triangulations connecting σ_X to σ_Y, such that (v_i^k) and (v_i^{k+1}) differ by either an involutivity, a commutativity or a pentagon relation ([Pen12, Chapter 5, Corollary 1.2]). Clearly, the difference by an involutivity and a commutativity do not affect the resulting cocycle. Also by the pentagon relation, the pleated surface does not change including the bending measure since each flip preserves the total bending along the vertices. Therefore (v_i^k) and (v_i^{k+1}) give the same train-track cocycle. □

Therefore we obtain $\mathsf{A} \to \mathbb{Z}$. By continuity and the connectedness in Lemma 9.14. we have the following.

Corollary 9.16. $[\Gamma_P]$ *is independent on the choice of the perturbation in Lemma 9.14.*

There are only finitely many combinatorial types of the train-tracks $\mathbf{T}_{X,\rho}$. We say that a branch B_X of $\mathbf{T}_{X,\rho}$ and a branch B'_X of $\mathbf{T}_{X,\rho'}$ are isotopic if they are diffeomorphic and isotopic on S. Then there are only finitely many isotopy classes of branches of $\mathbf{T}_{X,\rho}$ for all $\rho \in \mathcal{X}_X \setminus K$. Let α be an arc α on a branch B_X, such that each endpoint of α is at either on a vertical edge or a vertex of $\mathbf{T}_{X,\rho}$.

Proposition 9.17. *Let $X, Y \in \mathsf{T}$. For every $\epsilon > 0$, there a compact subset K in $\mathcal{X}$ with the following property: For every pair (B_X, α) of an isotopy class of a branch B_X and an arc α as above, there is a constant $k_\alpha > 0$ such that, if B_X is a (non-transversal) branch of $\mathbf{T}_{X,\rho}$ for some $\rho \in \mathcal{X}_X \cap \mathcal{X}_Y \setminus K$, then, $2\pi[\Gamma_{P_X}](\alpha)$ is $(1+\epsilon, k_\alpha)$-quasi-isometric to $\sqrt{2}(V_X|P_X - V_Y|P_Y)(\alpha)$.*

Proof. Since the length of the flip sequence is bounded from above, the difference between ν_X and ν_n is uniformly bounded in the space of transverse cocycles on S. Then the assertion follows. □

9.4.5. *Independency of the transverse cocycle.* From the train-track decompositions $\mathcal{T}_{X,\rho}$ and $\mathcal{T}_{Y,\rho}$, we have constructed a weighted train-track graph Γ_ρ (Theorem 9.8 (1)). Next we show its cocycle is independent on the train-track decompositions $\mathcal{T}_{X,\rho}$ and $\mathcal{T}_{Y,\rho}$ (Theorem 9.8(2)).

Recall that the train-track decompositions $\mathcal{T}_{X,\rho}$ and $\mathcal{T}_{Y,\rho}$ are determined by

(1) the holonomy equivariant circle system $\mathbf{c} = \{c_h\}$ indexed by horizontal edges h of $\tilde{\mathbf{T}}_{X,\rho}$ (given by Lemma 7.3),
(2) the realization W_Y of $[V_{Y,\rho}]_{X,\rho}$ on $\mathbf{T}_{X,\rho}$ (Lemma 6.7), and
(3) the choice of vertical edges of $\mathcal{T}_{X,\rho}$ (Theorem 8.6 (2)).

Proposition 9.18. *The cocycle $[\Gamma_\rho]\colon \mathsf{C} \to \mathbb{Z}$ constructed above is independent of the construction for $\mathcal{T}_{X,\rho}$ and $\mathcal{T}_{Y,\rho}$, i.e. (1), (2), (3).*

Proof. (1) By Proposition 7.4, given two appropriate circle systems $\{c_h\}$ and $\{c'_h\}$, there is an equivariant isotopy of circles systems $\{c_{t,h}\}$ connecting $\{c_h\}$ to $\{c'_h\}$. Then accordingly, we obtain a continuous family of cocycles $[\Gamma_{t,\rho}]\colon \mathsf{C} \to \mathbb{Z}$. As it takes discrete values, $[\Gamma_{t,\rho}]$ must remain the same.

By the different choices for (2) and (3), the $\mathbb{Z}$-weights on Γ shift across bigon regions corresponding vertical edges of $T_{Y,\rho}$ by integer values. These weight shifts clearly preserve the cocycle $[\Gamma_\rho]\colon \mathsf{C} \to \mathbb{Z}$. □

9.4.6. *Continuity of the transverse cocycle.* Next we prove the continuity of $[\Gamma_\rho]$ in ρ claimed in (3).

Definition 9.19 (Convergence and semi-convergence of train tracks)**.** *Suppose that $C_i \in \mathsf{P}$ converges to $C \in \mathsf{P}$. In addition, suppose, for each i, there are a train-track structure $\mathcal{T}_i$ of C_i and a train-track structure $\mathcal{T}$ for C. Then,*

- $\mathcal{T}_i$ converges *to $\mathcal{T}$ if*
  - *for every branch $\mathcal{P}$ of $\mathcal{T}$, there is a sequence of branches $\mathcal{P}_i$ of $\mathcal{T}_i$ converging to $\mathcal{P}$, and*
  - *for every sequence of branches $\mathcal{P}_i$ of $\mathcal{T}_i$, up to a subsequence, converges to either a branch of $\mathcal{T}$ or an edge of a branch of $\mathcal{T}$.*
- $\mathcal{T}_i$ semi-converges *to $\mathcal{T}$ if there is a subdivision of $\mathcal{T}$ into another circular train-track structure $\mathcal{T}'$ so that $\mathcal{T}_i$ converges to $\mathcal{T}'$.*

Lemma 9.20. *Let ρ_i be a sequence in $\chi_X \cap \chi_Y$ converging to $\rho \in \chi_X \cap \chi_Y \setminus K$, where K is a sufficiently large compact (as in Theorem 8.6). Pick an equivariant circle system $\mathbf{c}_i$ for $\mathbf{T}_{X,\rho_i}$ by Lemma 7.3 which converges to a circle system $\mathbf{c}$ for $\mathbf{T}_{X,\rho}$. Then, up to a subsequence,*

- *the circular train track $\mathcal{T}_{X,\rho_i}$ of C_{X,ρ_i} semi-converges to a circular train track $\mathcal{T}_{X,\rho}$ of $C_{X,\rho}$;*
- *the circular train track $\mathcal{T}_{Y,\rho_i}$ of C_{Y,ρ_i} semi-converges to a circular train track $\mathcal{T}_{Y,\rho}$ of $C_{Y,\rho}$;*
- *$\mathcal{T}_{X,\rho}$ is semi-compatible with $\mathcal{T}_{Y,\rho}$.*

Proof. By Lemma 6.19, $\mathbf{T}_{X,\rho_i}$ semi-converges to $\mathbf{T}_{X,\rho}$. Therefore $\mathbf{T}_{X,\rho_i}$ converges to a subdivision $\mathbf{T}'_{X,\rho}$ of $\mathbf{T}_{X,\rho}$ as $i \to \infty$. Then, if $\mathbf{T}_{X,\rho} \neq \mathbf{T}'_{X,\rho}$, then $\mathbf{T}_{X,\rho}$ is obtained from $\mathbf{T}'_{X,\rho}$ by gluing non-rectangular branches with rectangular branches of small width or replacing long rectangles into spiral cylinders (as in §5.3 and §6.7).

Recall that the realization of $[V_{Y,\rho_i}]_{X,\rho_i}$ in the train track $\mathbf{T}_{X,\rho_i}$ is unique up to shifting across vertical edges of non-rectangular branches (Proposition 6.12 (2)). Therefore, up to a subsequence, the realization of $[V_{Y,\rho_i}]_{X,\rho_i}$ on $\mathbf{T}_{X,\rho_i}$ converges to a realization of $[V_{Y,\rho}]_X$ on $\mathbf{T}'_{X,\rho}$, Since $\mathbf{T}'_{X,\rho}$ is a subdivision of $\mathbf{T}_{X,\rho}$, the limit can be regarded as also a realization on $\mathbf{T}_{X,\rho}$. Since the realization determines the train-track structure of $E_{Y,\rho}$, up to a subsequence, $\mathbf{T}_{Y,\rho_i}$ converges to a bounded train-track $\mathbf{T}'_{Y,\rho}$. Then $\mathbf{T}_{Y,\rho}$ is transformed to $\mathbf{T}'_{Y,\rho}$ by possibly sliding vertical edges and subdividing spiral cylinders to wide rectangles. Moreover, by Theorem 8.2 (1), $\mathcal{T}_{Y,\rho}$ is $(1+\epsilon, \epsilon)$-quasiisometric to $\mathbf{T}_{Y,\rho}$. Therefore, up to a subsequence, $\mathcal{T}_{Y,\rho_i}$ converges to a circular train-track structure $\mathcal{T}'_Y$ of $C_{Y,\rho}$. If $\mathcal{T}_{Y,\rho}$ is different from $\mathcal{T}'_{Y,\rho}$, then $\mathcal{T}_{Y,\rho}$ can be transformed to $\mathcal{T}'_{Y,\rho}$ by sliding vertical edges and subdividing spiral cylinders into rectangles.

By Theorem 8.6, $\mathcal{T}_{X,\rho_i}$ is additively 2π-close to $\mathbf{T}_{X,\rho_i}$ in the Hausdorff metric of $E^1_{X,\rho}$. Therefore, up to a subsequence $\mathcal{T}_{X,\rho_i}$ converges to a circular train track decomposition $\mathcal{T}'_{X,\rho}$ semi-diffeomorphic to $\mathcal{T}_{Y,\rho}$. Moreover $\mathcal{T}_{X,\rho}$ can be transformed to $\mathcal{T}'_{X,\rho}$ possibly by subdividing and sliding by 2π or 4π.

We have already shown that $\mathcal{T}_{X,\rho}$ is semi-diffeomorphic to $\mathcal{T}_{Y,\rho}$ (Theorem 8.6). 9.20

Finally we have the continuity (3).

Corollary 9.21. $[\Gamma_{\rho_i}]\colon \mathsf{C} \to \mathbb{Z}$ *converges to* $[\Gamma_\rho]\colon \mathsf{C} \to \mathbb{Z}$ *as* $i \to \infty$.

Proof. Since $\mathcal{T}_{X,\rho_i}$ semi-converges to $\mathcal{T}_{X,\rho}$, up to taking a subsequence, $\mathcal{T}_{X,i}$ converges to a subdivision $\mathcal{T}'_{X,\rho}$ of $\mathcal{T}_{X,\rho}$. Accordingly, there is a subdivision $\mathcal{T}'_{Y,\rho}$ of $\mathcal{T}_{Y,\rho}$, such that, to up a subsequence, $\mathcal{T}_{Y,i}$ converges to $\mathcal{T}'_{Y,\rho}$ and that $\mathcal{T}'_{X,\rho}$ is semi-diffeomorphic to $\mathcal{T}'_{Y,\rho}$.

Let Γ_{ρ_i} be the $\mathbb{Z}$-weighted train-track given by $\mathcal{T}_{X,\rho_i}$ and $\mathcal{T}_{Y,\rho_i}$. Let Γ'_ρ be the $\mathbb{Z}$-weighted train track given $\mathcal{T}'_{X,\rho}$ and $\mathcal{T}'_{Y,\rho}$. Then, by the convergence of the train tracks, Γ_{ρ_i} converges to Γ'_ρ as $i \to \infty$. Since $\mathcal{T}'_{X,\rho}$ and $\mathcal{T}'_{Y,\rho}$ are obtained by sliding and subdividing $\mathcal{T}_{X,\rho}$ and $\mathcal{T}_{Y,\rho}$ respectively, thus Γ'_ρ and Γ_ρ yield the same cocycle $\mathsf{C} \to \mathbb{Z}$. □

9.5. **Approximation of the grafting cocycle $[\Gamma_\rho]$ by vertical foliations.** Suppose that X, Y be distinct marked Riemann surfaces homeomorphic to S such that X and Y have the same orientation. For a branch B_X of $\mathbf{T}_{X,\rho}$, let $\mathsf{A}(B_X)$ be the homotopy class of arcs α on $\mathcal{R}_X$ such that every endpoint of α is either on a vertical edge or a vertex of $\mathbf{T}_{X,\rho}$.

Theorem 9.22. *Let $c_1, \dots, c_n$ be essential closed curves on S. Then, for every $\epsilon > 0$, there is a bounded subset K_ϵ of $\chi_X \cap \chi_Y$ such that, for every $\rho \in \chi_X \cap \chi_Y \setminus K$, the intersection cocycle $[\Gamma_\rho]$ times 2π is $(1+\epsilon, q)$-quasi-isometric to $\sqrt{2}(V_{X,\rho} - V_{Y,\rho})$ along $c_1, \dots, c_n$. That is,*

$$(1-\epsilon)2\pi\Gamma_\rho(c_i) - q < \sqrt{2}(V_{X,\rho}(c_i) - V_{Y,\rho}(c_i)) < (1+\epsilon)2\pi\Gamma_\rho(c_i) + q \tag{2}$$

for all $i = 1, 2, \dots, n$.

Proof. Let $H \in \mathsf{PML}$. Recall that $E^1_{X,H}$ is the flat surface conformal to X, such that the horizontal foliation is H and Area$E^1(X,H) = 1$. Let $\mathbf{T}_{X,H}$ be the bounded train-track structure of $E_{X,H}$.

Then, every closed curve c can be isotoped to a closed curve c' so that, for each branch of B of $\mathbf{T}_{X,\rho}$, $c'|B$ is an arc connecting different vertices. Let $c'_1, \dots, c'_m$ be the decomposition into sub-arcs. By the finiteness of possible train-tracks, the number m of the subarcs is bounded from above for all ρ. Then $2\pi\Gamma_\rho|c'_j$ is, if B is a transversal branch, $(1+\epsilon,\epsilon)$-quasi-isometric to $\sqrt{2}(V_{X,\rho}|B - V_{Y,\rho}|B)c'_k$ by Proposition 9.12(4), Proposition 9.11, Proposition 9.10, and, if non-transversal, $(1+\epsilon, q)$-quasi-isometric by Proposition 9.17. As the number of subarcs is bounded, the assertion follows. □

10. The discreteness

10.1. The discreteness of the intersection of holonomy varieties.

Theorem 10.1. *Suppose that X, Y are marked Riemann surfaces structures on S with the same orientation. Then every (connected) component of $\chi_X \cap \chi_Y$ is bounded.*

Proof. Let K be a component of $\chi_X \cap \chi_Y$. Suppose, to the contrary, that V is unbounded in χ. Then, there is a path ρ_t in $\chi_X \cap \chi_Y$ which leaves every compact subset. Then, by Theorem 4.9, by taking a diverging sequence $t_1 < t_2 < \dots$, there are $k_i, k'_i \in \mathbb{R}_{>0}$ such that $\frac{k_i}{k'_i} \to 1$ as $i \to \infty$ and

$$\lim_{i\to\infty} k_i H_{X,\rho_{t_i}} = \lim_{i\to\infty} k'_i H_{Y,\rho_{t_i}} \in \mathsf{ML}.$$

By taking a subsequence, we may, in addition, assume that their vertical foliations $[V_{X,\rho_{t_i}}]$ and $[V_{Y,\rho_{t_i}}]$ converge in PML. Thus let $[V_{X,\infty}]$ and $[V_{Y,\infty}]$ be their respective limits in PML. Since $X \neq Y$, $V_{X,\infty}$ and $V_{Y,\infty}$ can not be asymptotically the same, in comparison to their horizontal foliations. Then $V_{X,\rho_{t_i}} - V_{Y,\rho_{t_i}}$ "diverges to ∞". That is, there is a closed curve α on S, such that

$$|V_{X,\rho_{t_i}}(\alpha) - V_{Y,\rho_{t_i}}(\alpha)| \to \infty$$

as $i \to \infty$.

Let $[\Gamma_{\rho_t}]\colon \mathsf{C} \to \mathbb{Z}$ be the function given by Theorem 9.8. As $[\Gamma_{\rho_t}]$ is continuous (Theorem 9.8 (3)), $[\Gamma_{\rho_t}]\colon \mathsf{C} \to \mathbb{Z}$ is a constant function (for $t \gg 0$). On the other hand, by Theorem 9.22, there is $q > 0$ such that $\sqrt{2}(V_{X,\rho_{t_i}} - V_{Y,\rho_{t_i}})(\alpha)$ is $(1 + \epsilon_i, q)$-quasi-isometrically close to $2\pi[\Gamma_{\rho_{t_i}}](\alpha)$, and $\epsilon_i \to 0$ as $i \to \infty$. We thus obtain a contradiction. 10.1

Since χ_X and χ_X are complex analytic, thus their intersection is also a complex analytic set (Theorem 5.4 in [FG02]). Therefore, since every bounded connect analytic set is a singleton (Proposition 2.6), Theorem 10.1 implies the following.

Corollary 10.2. *$\chi_X \cap \chi_Y$ is a discrete set.*

We will, moreover, show that this intersection is non-empty in §12.

10.2. **A weak simultaneous uniformization theorem.** In this section, using Corollary 10.2, we prove a weak version of a simultaneous uniformization theorem for general representations. Let $\rho\colon \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ be any non-elementary representation which lifts to $\mathrm{SL}(2,\mathbb{C})$. Let C, D be $\mathbb{C}\mathrm{P}^1$-structures on S^+ with the holonomy ρ. Then, if a neighborhood U_ρ of ρ in χ is sufficiently small, then there are (unique) neighborhoods V_C and W_D of C and D in P, respectively, which are biholomorphic to U_ρ by $\mathrm{Hol}\colon \mathsf{P} \to \chi$. Then, for every $\eta \in U_\rho$, there are unique $\mathbb{C}\mathrm{P}^1$-structures C_η in V_C and D_η in W_D with holonomy η. Let $\Phi_{\rho,U} = \Phi\colon U_\rho \to \mathsf{T} \times \mathsf{T}$ be the map which takes $\eta \in U_\rho$ to the pair of the marked Riemann surface structures of C_η and D_η.

Theorem 10.3. *$\Phi_{\rho,U}$ is a finite-to-one open mapping.*

Proof. By Corollary 10.2, the fiber of Φ is discrete. In addition, Φ is holomorphic and $\dim U_\rho = 2\dim \mathsf{T}$. Therefore, by Theorem 2.7, Φ is an open map. □

11. Opposite orientations

In this section, when the orientations of the Riemann surfaces are opposite, we show the discreteness of $\chi_X \cap \chi_Y$ analogous to Theorem 11.1 and the local uniformization theorem analogous to Theorem 10.3.

Theorem 11.1. *Fix $X \in \mathsf{T}$ and $Y \in \mathsf{T}^*$. Then, $\chi_X \cap \chi_Y$ is a non-empty discrete set.*

Since the proof is similar to the case when the orientations coincide, we simply outline the proof, yet explaining how some parts are modified. We leave the details to the reader.

Recall that we have constructed compatible train track decomposition regardless of the orientation of XY (§8.4, §8.5). In summary, we have the following (in the case of opposite orientataions):

Proposition 11.2. *Fix $X \in \mathsf{T}$ and $Y \in \mathsf{T}^*$. For every $\epsilon > 0$, there is a bounded subset K_ϵ in $\chi_X \cap \chi_Y$, such that, if $\rho \in \chi_X \cap \chi_Y \setminus K_\epsilon$, then there are circular polygonal train-track decompositions $\mathcal{T}_{X,\rho}$ of $C_{X,\rho}$ and $\mathcal{T}_{Y,\rho}$ of $C_{Y,\rho}$, such that*

- *$\mathcal{T}_{X,\rho}$ and $\mathcal{T}_{Y,\rho}$ are semi-diffeomorphic, and*
- *$\mathcal{T}_{X,\rho}$ and $\mathcal{T}_{Y,\rho}$ are $(1+\epsilon, \epsilon)$-quasi-isometric to the train track decompositions $\mathbf{T}_{X,\rho}$ of the flat surface $E^1_{X,\rho}$ and $\mathbf{T}_{Y,\rho}$ of the flat surface $E^1_{Y,\rho}$, respectively, with respect to the normalized metrics.*

In the case when the orientation of X and Y are the same, in Theorem 9.8, we constructed a $\mathbb{Z}$-weighted train-track graph representing the "difference" of projective structures on X and Y with the same holonomy. As the orientations of X and Y are different, we shall construct a $\mathbb{Z}$-weighted train track graph representing, in this case, the "sum" of the $\mathbb{C}\mathrm{P}^1$-structures on X and Y with the same holonomy.

Let $\mathcal{T}_{X,\rho}$ and $\mathcal{T}_{Y,\rho}$ be circular train track decompositions of $C_{X,\rho}$ and $C_{Y,\rho}$ given by Proposition. 11.2 Let $\{h_{X,1}, h_{X_2}, \dots, h_{X_n}\}$ be the horizontal edges of $\mathcal{T}_{X,\rho}$. Similarly to §9.4, we first define the $\mathbb{Z}$-valued function on the set of horizontal edges. For each $i = 1, \dots, n$, let c_i be the round circle on $\mathbb{C}\mathrm{P}^1$ supporting the development of $h_{X,i}$. First, suppose that $h_{X,i}$ corresponds to an edge $h_{Y,i}$ of $\mathcal{T}_{Y,\rho}$ by the collapsing map $\mathcal{T}_{X,\rho} \to \mathcal{T}_{Y,\rho}$. Since $\mathcal{T}_{X,\rho}$ and $\mathcal{T}_{Y,\rho}$ are compatible, the corresponding endpoints of $h_{X,i}$ and $h_{Y,i}$ map to the same point on c_i by their developing maps. Thus, by identifying the endpoints, we obtain a covering map from a circle $h_{X,i} \cup h_{Y,i}$ onto c_i. Then, define $\gamma_{X,i}(h_{X,i}) \in \mathbb{Z}_{>0}$ to be the covering degree.

Next, suppose that $h_{X,i}$ corresponds to a vertex of $\mathcal{T}_{Y,i}$. Then the endpoints of $h_{X,i}$ develop to the same point on c_i. The circle obtained by identifying endpoints of $h_{X,i}$ covers c_i. Thus, let $\gamma_{X,i}(h_{X,i}) \in \mathbb{Z}_{>0}$ be the covering degree.

Similarly to §9.4, we shall construct a $\mathbb{Z}$-weighted train-track graph Γ_ρ immersed in $\mathcal{T}_{X,\rho}$. On each branch $\mathcal{P}_X$ of $\mathcal{T}_{X,\rho}$, we construct a $\mathbb{Z}$-weighted train-track graph $\Gamma_{\mathcal{P}_X}$ on $\mathcal{P}_X$ such that, for every $\epsilon > 0$, there is a compact subset K of χ satisfying the following:

- The endpoints of $\Gamma_{\mathcal{P}_X}$ are on horizontal edges of $\mathcal{P}_X$.
- They agree with $\gamma_{X,\rho}$ along the horizontal edges.
- If $\mathcal{P}_X$ is a transversal branch, then, for $\rho \in \chi_X \cap \chi_Y \setminus K$, then $2\pi\Gamma_{\mathcal{P}_X}(\alpha)$ is $(1+\epsilon, \epsilon)$-quasi-isometric to $\sqrt{2}(V_{X,\rho}|P_X)(\alpha) + \sqrt{2}(V_{Y,\rho}|P_Y)(\alpha)$ for all $\alpha \in \mathcal{A}(\mathcal{P}_X)$.
- For every smooth isotopy class of a staircase surface B on a flat surface homeomorphic to S and every arc α on B connecting points on horizontal edges or vertices, there is a positive constant $q(B, \alpha)$ such that, if $\mathcal{T}_{X,\rho}$ contains a non-transversal branch $\mathcal{B}_X$ smoothly isotopic to B on S, then $2\pi\Gamma_{\mathcal{B}_X}(\alpha)$ is $(1+\epsilon, q(B,\alpha))$-quasi-isometric to $\sqrt{2}\, V_{X,\rho}|B_X(\alpha) + \sqrt{2}\, V_{Y,\rho}|B_Y(\alpha)$.

Theorem 11.3. *Let $X \in \mathsf{T}$ and $Y \in \mathsf{T}^*$. For every $\epsilon > 0$, there is a bounded subset $K_\epsilon \subset \chi_X \cap \chi_Y$ such that, for every $\rho \in \chi_X \cap \chi_Y \setminus K_\epsilon$, there is a $\mathbb{Z}$-weighted graph Γ_ρ carried in $\mathcal{T}_{X,\rho}$ such that*

(1) *the induced cocycle $[\Gamma_\rho]\colon \mathsf{C} \to \mathbb{Z}$ changes continuously in $\rho \in \chi_X \cap \chi_Y \setminus K$,*
(2) *for every loop α on S, there is $q_\alpha > 0$, such that $2\pi\Gamma_\rho(\alpha)$ is $(1+\epsilon, q_\alpha)$-quasi-isometric to $\sqrt{2}(V_{X,\rho}(\alpha) + V_{Y,\rho}(\alpha))$ for all $\rho \in \chi_X \cap \chi_Y \setminus K$.*

Proof. The proof is similar to Theorem 9.8 (3) and Theorem 9.22. □

Then, Theorem 11.3 implies, similarly to Theorem 10.1, the following:

Theorem 11.4. *Each connected component of $\chi_X \cap \chi_Y$ is bounded.*

12. The completeness

In this section, we prove the completeness in Theorem A. Let Q be a connected component of the Bers' space B; then Q is a complex submanifold of $(\mathsf{P} \sqcup \mathsf{P}^*) \times (\mathsf{P} \sqcup \mathsf{P}^*)$, and $\dim_{\mathbb{C}} Q = 6g - 6$. We call that $\psi\colon \mathsf{P} \sqcup \mathsf{P}^* \to \mathsf{T} \sqcup \mathsf{T}^*$ is the uniformization map and $\Psi\colon Q \to (\mathsf{T} \sqcup \mathsf{T}^*)^2$ is defined by

$\Psi(C, D) = (\psi(C), \psi(D))$. Then, by Theorem 10.3, Ψ is an open holomorphic map. In this section, we prove the completeness of Ψ.

Lemma 12.1. *The open map $\Psi\colon Q \to (\mathsf{T} \sqcup \mathsf{T}^*)^2$ has a* local path lifting property*. That is, for every $z \in Q$, there is a neighborhood W of $\Psi(z)$ such that if path α_t, $0 \leq t \leq 1$ in W satisfies $\zeta(z) = \alpha_0$, then there is a lift $\tilde{\alpha}_t$ of α_t to Q with $\tilde{\alpha}_0 = z$.*

Proof. Since Ψ is an open map and $\dim Q = \dim(\mathsf{T} \sqcup \mathsf{T}^*)^2$, Ψ is a locally branched covering map. Then, for every $z \in Q$, there is an open neighborhood V of z in Q and a finite group G_z biholomorphically acting V, such that Ψ is G_z-invariant, and Ψ induces the biholomorphic map $V/G_z \to \Psi(V)$.

For $g \in G_z \setminus \{id\}$, let $F_g \subset W$ be the (pointwise) fixed point set of g. Clearly F_g is a proper analytic subset, and thus $F \coloneqq \cup_{g \in G_z \setminus \{id\}} F_g$ is an analytic subset strictly contained in W.

For every path $\alpha\colon [0,1] \to V$ with $\alpha(0) = \Psi(z)$, we can take a one-parameter family of paths α_t $(t \in [0,1])$ in W with $\alpha_t(0) = \Psi(z)$ such that $\alpha_1 = \alpha$ and, for $t < 1$, α_t is disjoint from $\Psi(F)$ (since $\Psi(F)$ has complex codimension, at least, one.)

Then, for $t < 1$, α_t continuously lifts a path $\tilde{\alpha}_t\colon [0,1] \to Q$, and α_t converges to a desired lift of α_1 as $t \to 1$. ☐

Now we are ready to prove the completeness.

Theorem 12.2. $\Psi\colon Q \to (\mathsf{T} \sqcup \mathsf{T}^*)^2 \setminus \Delta$ *is a complete map, where Δ is the diagonal set.*

The completeness of Theorem 12.2 immediately implies the following:

Corollary 12.3. $\Psi\colon Q \to (\mathsf{T} \sqcup \mathsf{T}^*)^2 \setminus \Delta$ *is surjective onto a connected component of* $(\mathsf{T} \sqcup \mathsf{T}^*)^2 \setminus \Delta$.

Proof of Theorem 12.2. By Lemma 12.1, Ψ has a local path lifting property. Thus, suppose that $(X_t, Y_t)\colon [0,1] \to (\mathsf{T} \sqcup \mathsf{T}^*)^2 \setminus \Delta$ $(0 \leq t \leq 1)$ be a path and there is a (partial) lift $(C_t, D_t)\colon [0,1) \to Q$ of the path (X_t, Y_t). For each $t \in [0,1]$, let $\rho_t \in \mathcal{X}$ denote the common holonomy of C_t and D_t. By the continuity, the orientations of Riemann surfaces in the pairs obviously remain the same along ρ_t for all $t > 0$.

First we, in addition, suppose that there is an increasing sequence $0 \leq t_1 < t_2 < \dots$ converging to 1, such that ρ_{t_i} converges to a representation in $\mathcal{X}$. By Corollary 10.2 or, in the case of opposite orientations, Theorem 11.1, $\mathcal{X}_{X_1} \cap \mathcal{X}_{Y_1}$ is a discrete subset of $\mathcal{X}$. Then, since (X_t, Y_t) converges to a point (X_1, Y_1) in $\mathsf{T} \times \mathsf{T}$, and $\mathcal{X}_{X_t}$ and $\mathcal{X}_{Y_t}$ change continuously in $t \in [0,1]$, every neighborhood $\mathcal{X}_{X_1} \cap \mathcal{X}_{Y_1}$ contains ρ_{t_i} for all sufficiently large i. Thus the sequence ρ_{t_i} converges to a point in $\mathcal{X}_{X_1} \cap \mathcal{X}_{Y_1}$. Since $\mathcal{X}_{X_t} \cap \mathcal{X}_{Y_t}$ is a discrete set in $\mathcal{X}$ which continuously changes in $t \in [0,1]$, we indeed have a genuine convergence.

Lemma 12.4. *ρ_t converges to ρ_1 as $t \to 1$.*

By this lemma, (C_t, D_t) converges in $(\mathsf{P} \sqcup \mathsf{P}^*) \times (\mathsf{P} \sqcup \mathsf{P}^*)$ as $t \to 1$, so that the partial lift (C_t, D_t) extends to $t = 1$.

Thus it is suffices to show the addition assumption always holds:

Proposition 12.5. *There is a compact subset K in $\mathcal{X}$, such that, for every $t > 0$, there is $t' > 0$ such that $\rho_{t'} \in K$.*

Proof. The proof is essentially the same as the proof of Theorem 10.1 or Theorem 11.1, which states that each component of $\chi_X \cap \chi_Y$ is a bounded subset of χ.

For $0 \leq t < 1$, let V_{C_t} and V_{D_t} be the vertical measured foliations of C_t and D_t, respectively. Suppose, to the contrary, that ρ_t leaves every compact subset of χ. As (X_t, Y_t) converges to $(X, Y) \in (\mathsf{T} \sqcup \mathsf{T}^*)^2 \setminus \Delta$ and $\mathrm{Hol}(C_t) = \mathrm{Hol}(D_t)$, similarly to Theorem 9.8 or Theorem 11.3, for t close to 1, we can construct a $\mathbb{Z}$-weighed train track Γ_t on S, such that

- the intersection function $[\Gamma_t] \colon \mathsf{C} \to \mathbb{Z}$ is continuous in t, and
- for every closed curve α on S, there is a constant $q_\alpha > 0$ and a function $\epsilon_t > 0$ converging to 0, such that, for all sufficiently large $t > 0$, $[\Gamma_t](\alpha)$ is $(1 + \epsilon_t, q_\alpha)$-quasi-isometric to $V_{C_t}(\alpha) - V_{D_t}(\alpha)$ if the orientation of X and Y are the same and to $V_{C_t}(\alpha) + V_{D_t}(\alpha)$ if the orientation of X and Y are different.

The first condition implies that the intersection number is constant in t, whereas the second condition implies that the intersection number with some closed curve α diverges to infinity as $t \to 1$. This is a contradiction. □

12.2

Last we remark the behavior of Hol near the diagonal Δ.

Proposition 12.6. *Let $(C_t, D_t), 0 \leq t < 1$ be a path in B, such that $\Psi(C_t, D_t)$ converges to a diagonal point (X, X) of $(\mathsf{T} \sqcup \mathsf{T}^*)^2$. Then $\mathrm{Hol}\, C_t = \mathrm{Hol}\, D_t$ leaves every compact in χ as $t \to 1$.*

Proof. Recall that $\mathrm{Hol} \colon \overline{\mathsf{P}} \to \chi$ is a locally biholomorphic map. Therefore, for every compact subset K in χ, if a neighborhood U of X in $\overline{\mathsf{T}}$ is sufficiently small, for every $Y, Z \in U$, $\chi_Y \cap \chi_Z$ is disjoint from K. Then the assertion is immediate. □

12.1. **Cardinalities of the intersections.** By the surjectivey of Corollary 12.3 and the existence of non-quasi-Fuchsian components of B in $\mathsf{P} \sqcup \mathsf{P}^*$ in Lemma 2.5, we immediately have the following:

Corollary 12.7. *Let X, Y be distinct marked Riemann surface structures on S with any orientations. If the orientations of X and Y are opposite, the intersection $\chi_X \cap \chi_Y$ contains, at least, two points, if the orientations of X and Y are opposite, $\chi_X \cap \chi_Y$ contains, at least, one point.*

13. A proof of the simultaneous uniformization theorem

In this section, using Theorem A, we give a new proof simultaneous uniformization theorem without using the measurable Riemann mapping theorem. Recall QF is the quasi-Fuchsian space, and it is embedded in $\mathsf{B}/\mathbb{Z}_2$.

Given (C, D) in QF, the universal covers $\tilde{C}, \tilde{D}$ are the connected components of $\mathbb{C}\mathrm{P}^1$ minus its equivariant Jordan curve equivariant via $\mathrm{Hol}\, C = \mathrm{Hol}\, D$.

Lemma 13.1. QF *is a union of connected components of* $\mathsf{B}/\mathbb{Z}_2$.

Proof. As being a quasi-isometric embedding is an open condition, QF is an open subset of B. Thus, it suffices to show that QF is closed.

Let $(C_i, D_i) \in \mathsf{P} \times \mathsf{P}^*$ be a sequence in QF which converges to $(C, D) \in \mathsf{P} \times \mathsf{P}^*$. Let $\rho_i \colon \pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$ be the quasi-Fuchsian representation of C_i and D_i. We show that the holonomy $\rho \colon \pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$ of the limits C and D is also quasi-Fuchsian. Let $\tilde{C}_i$ and $\tilde{D}_i$

be the universal covers of C_i and D_i, respectively. Then $\tilde{C}_i$ and $\tilde{D}_i$ are the components of $\mathbb{C}\mathrm{P}^1$ minus the ρ_i-equivariant Jourdan curve. Let $f_i\colon \tilde{C}_i \cup \mathbb{S}^1 \to \mathbb{C}\mathrm{P}^1$ and $g_i\colon \tilde{D}_i \cup \mathbb{S}^1 \to \mathbb{C}\mathrm{P}^1$ be the extensions of the embeddings to their boundary circles by a theorem of Carathéodory. Let h_i be the homeomorphism $\tilde{C}_i \cup \mathbb{S}^1 \cup \tilde{D}_i \to \mathbb{C}\mathrm{P}^1$.

Since embeddings $\operatorname{dev} C_i$ converge to $\operatorname{dev} C$ uniformly on compact as $i \to \infty$, the limit $\operatorname{dev} C$ is also an embedding. Then, by the convergence of corresponding convex pleated surfaces in $\mathbb{H}^3$, the equivariant property implies that $\operatorname{dev} C$ extends to the boundary circle continuously and equivariantly. Similarly, since the embedding $\operatorname{dev} D_i$ converging to $\operatorname{dev} D$, then $\operatorname{dev} D$ is also an embedding, and $\operatorname{dev} D$ extends to the boundary circle continuously and equivariantly. Therefore h_i converges to a continuous map

$$h\colon \mathbb{S}^2 \cong \tilde{C} \cup \mathbb{S}^1 \cup \tilde{D} \to \mathbb{C}\mathrm{P}^1$$

such that the restriction of h to $\tilde{C} \sqcup \tilde{D}$ is an embedding.

The domain and the target of h are both homeomorphic to $\mathbb{S}^2$. Therefore, if $h|\mathbb{S}^1$ is not a Jordan curve on $\mathbb{C}\mathrm{P}^1$, then there is a point $z \in \mathbb{C}\mathrm{P}^1$, such that $h^{-1}(z)$ is a single segment of $\mathbb{S}^1$. By the equivariant property, $h^{-1}(z) = \mathbb{S}^1$, and $\operatorname{Im} h$ is a wedge of two copies of $\mathbb{S}^2$, which is a contradiction. □

The following asserts that the diagonal of $\mathsf{T} \times \mathsf{T}^*$ corresponds to the Fuchsian representations.

Lemma 13.2. *Let $X \in \mathsf{T}$. Let $\eta\colon \pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$ be a quasi-Fuchsian representation, such that the ideal boundary of $\mathbb{H}^3/\mathrm{Im}\rho$ realizes the marked Riemann surface X and its complex conjugate $X^* \in \mathsf{T}^*$. Then η is the Fuchsian representation $\pi_1(S) \to \mathrm{PSL}(2, \mathbb{R})$ such that $X = \mathbb{H}^2/\mathrm{Im}\eta$.*

Proof. By the Riemann uniformization theorem, the universal covers $\tilde{X}$ and $\tilde{X}^*$ are the upper and the lower half planes. Then, by identifying their ideal boundaries equivariantly, we obtain $\mathbb{C}\mathrm{P}^1$ so that the universal covers $\tilde{X}$ and $\tilde{X}^*$ are round open disks. Let $(C, D) \in \mathsf{P} \times \mathsf{P}^*$ be the pair corresponding to η, such that the complex structure of C is X and the complex structure of D is on X^*.

On the other hand, the universal covers $\tilde{C}$ and $\tilde{D}$ are connected components of $\mathbb{C}\mathrm{P}^1 \setminus \Lambda(\eta)$, where $\Lambda(\eta)$ is the η-equivariant Jordan curve in $\mathbb{C}\mathrm{P}^1$. Thus, there is a η-equivariant homeomorphism $\phi\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$, such that ϕ restricts to a biholomorphism from $\mathbb{C}\mathrm{P}^1 \setminus \mathbb{R} \cup \{\infty\} \to \mathbb{C}\mathrm{P}^1 \setminus \Lambda(\eta)$.

Then, by Morera's theorem for the line integral along triangles (see [SS03] for example), ϕ is a genuine biholomorphic map $\mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$. Therefore ϕ is a Möbius transformation, and therefore η is conformally conjugate to the Fuchsian representation uniformizing X. □

Proposition 13.3. *QF is a single connected component of $\mathsf{B}/\mathbb{Z}_2$.*

Proof. By Lemma 13.1, QF is the union of some connected components of B. By Theorem A, for every component Q of QF, the image $\Psi(Q)$ contains the diagonal $\{(X, X^*)\}$ of $\mathsf{T} \times \mathsf{T}^*$. Then, by Lemma 13.2, every diagonal pair $(X, X^*) \in \mathsf{T} \times \mathsf{T}^*$ corresponds to a unique point in QF. Therefore QF is connected. □

Last we reprove the simultaneous uniformization theorem.

Theorem 13.4. *QF is biholomorphic to $\mathsf{T} \times \mathsf{T}^*$ by Ψ.*

Proof. By Theorem A, Ψ is a complete local branched covering map. Since Ψ is surjective, by Lemma 13.2, Ψ is a degree-one over the diagonal $\{(X, \overline{X}) \mid X \in \mathsf{T}\}$, and the diagonal corresponds to the Fuchsian space.

The set of ramification points of Ψ is an analytic set. The Fuchsian space is a totally real subspace of dimension $6g-6$. Therefore, if the ramification locus contains the Fuchsian space, then the locus must have the complex dimension $6g-6$, the full dimension. This is a contradiction as Ψ is a locally branched covering map. Therefore, $\mathsf{QF} \to \mathsf{T} \times \mathsf{T}^*$ has a degree one, and thus it is biholomorphic. □

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