BENDING TEICHMÜLLER SPACES AND CHARACTER VARIETIES

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Abstract. We consider the mapping $b_L \colon \mathcal{T} \to \mathcal{X}$ from the Fricke-Teichmüller space $\mathcal{T}$ into the $\mathrm{PSL}_2\mathbb{C}$-character variety $\mathcal{X}$ of the surface, obtained by holonomy representations of bent hyperbolic surfaces along a fixed measured lamination L. We prove that this mapping is an equivariant symplectic real-analytic embedding, and, for almost all measured laminations, proper.

In addition, we show that this "bending map" $b_L \colon \mathcal{T} \to \mathcal{X}$ continuously extends to a mapping from the Thurston boundary of $\mathcal{T}$ to the Morgan-Shalen boundary of $\mathcal{X}$, almost everywhere, as the identity map.

Moreover, we complexify this real-analytic subvariety $\operatorname{Im} b_L$ by symplectically embedding it in the product variety $\mathcal{X} \times \mathcal{X}$ by the diagonal mapping twisted by complex conjugation. We geometrically construct a closed $\mathbb{C}$-symplectic complex-analytic subvariety of $\mathcal{X} \times \mathcal{X}$ containing $\operatorname{Im} b_L$ as a half-dimensional real-analytic subvariety.

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1. Introduction

Thurston discovered the bent hyperbolic surfaces τ on the boundary of the convex core of a (geometrically finite) hyperbolic three-dimensional manifold ([Thu81]). The intrinsic metric of the convex surface is hyperbolic, and the surface is bent along a measured lamination, where the bending angles correspond to the transversal measure of the lamination. Such bent surfaces are particularly useful for capturing the global properties of the hyperbolic manifold.

Lifting the convex surface τ to the universal cover $\mathbb{H}^3$ of the hyperbolic three-manifold, we obtain an equivariant bending $\mathbb{H}^2 \to \mathbb{H}^3$ which preserves the (intrinsic) hyperbolic metric of the surface. Then, this bending map is equivariant via a holonomy representation of a surface group into $\mathrm{PSL}_2\mathbb{C}$. Moreover, if τ is π_1-injective (equivalently, incompressible) in the ambient hyperbolic 3-manifold, then the bending map $\mathbb{H}^2 \to \mathbb{H}^3$ is a proper embedding.

In this paper, we utilize this bending construction in a new generalized manner and construct similar equivariant geometry-preserving mappings, in fact, at the level of associated deformation spaces.

1.1. **Holonomy varieties.**

Let Y be a marked Riemann surface structure on a closed oriented surface S of genus g at least two. Let $\mathrm{QD}(Y)$ denote the space of the holomorphic quadratic differentials on Y, which is a complex vector space of dimension $3g-3$. Then $\mathrm{QD}(Y)$ is identified with the space $\mathcal{P}_Y$ of all $\mathbb{C}\mathrm{P}^1$-structures on Y, and this correspondence yields the *Schwarzian parameterization* of $\mathbb{C}\mathrm{P}^1$-structures (see [Dum09] for example).

Let

$$\mathrm{Hol}\colon \mathcal{P} \to \mathcal{X}$$

be the holonomy map from the deformation space $\mathcal{P}$ of all $\mathbb{C}\mathrm{P}^1$-structures on S to the $\mathrm{PSL}_2\mathbb{C}$-character variety $\mathcal{X}$ of S. Recall that the character variety $\mathcal{X}$ is an affine algebraic variety. Its smooth part has Goldman's complex symplectic structure invariant under the action of the mapping class group; see [Gol84]. Many interesting properties of this mapping, associated with the Schwarzian parametrization, have been discovered, and particularly the following holds.

Theorem 1.1. *The restriction of the holonomy map to* $\mathcal{P}_Y \cong \mathrm{QD}(Y)$ *is a proper Lagrangian complex-analytic embedding into* $\mathcal{X}$.

The injectivity of Theorem 1.1 is due to Poincaré [Poi84]; the properness is due to Kapovich [Kap95] (see [GKM00] for the full proof; see also [Dum17, Tan99]); the Lagrangian property is proven by Kawai [Kaw96]. On the other hand, the entire holonomy map $\mathrm{Hol}\colon \mathcal{P} \to \mathcal{X}$ of $\mathbb{C}\mathrm{P}^1$-structures is neither injective nor proper (see [Hej75]).

By Theorem 1.1, for every marked Riemann surface structure Y, the vector space $\mathrm{QD}(Y) \cong \mathbb{C}^{3g-3}$ is properly embedded onto a half-dimensional smooth subvariety of $\mathcal{X}$. We call this image, associated with the Schwarzian parametrization, the Poincaré holonomy variety of Y. In particular, the holonomy variety of Y contains the Bers slice of Y as a bounded pseudo-convex domain.

The Morgan-Shalen compactification of the character variety $\mathcal{X}$ consisting of certain $\pi_1(S)$-actions of metric trees ([CS83, MS84]). Dumas investigated the asymptotic behavior of the proper mapping $\mathrm{Hol}\,|\mathcal{P}(X)$. Namely, she showed that $\mathrm{Hol}\,|\mathcal{P}(X)$ extends to the ray compactification of the vector space $\mathrm{QD}(X)$ almost everywhere in a natural manner.

Theorem 1.2 (Corollary E in [Dum17])**.** *Let* $q \in \mathrm{QD}(X) \setminus \{0\}$ *be a generic direction. Let* V *be the vertical measured foliation of* q*, and let* $\tilde{V}$ *be the pull-back measured foliation of* V *to the universal cover* $\tilde{X}$*. Then* $\mathrm{Hol}(tq)$ *converges to the* $\pi_1(S)$*-action on the metric tree dual to* $\tilde{V}$ *as* $t \to \infty$.

Moreover, $\mathrm{Hol}\,|\mathcal{P}_X$ *continuously extends the full measure set of the ray-compactification boundary* $\partial\mathrm{QD}(X)$ *to the mapping to the Morgan-Shalen boundary of* $\mathcal{X}$ *in a natural manner.*

1.2. **Real bending varieties.** Recall that $\mathbb{C}\mathrm{P}^1$ is the ideal boundary of the hyperbolic three-space $\mathbb{H}^3$, and the automorphism group $\mathrm{PSL}_2\mathbb{C}$ of $\mathbb{C}\mathrm{P}^1$ is identified with the group of orientation-preserving isometries of $\mathbb{H}^3$. Utilizing this correspondence in a sophisticated manner, Thurston gave another parametrization of $\mathcal{P}$, so that $\mathbb{C}\mathrm{P}^1$-structures correspond to equivariant pleated surfaces in $\mathbb{H}^3$ (§3.1.1). In this paper, we first yield an analogue of Theorem 1.1 by specific slices in the Thurston parametrization of $\mathbb{C}\mathrm{P}^1$-structures.

In fact, Tanigawa [Tan97], Wolf-Scannell [SW02], Dumas-Wolf [DW08] considered the $\mathbb{C}\mathrm{P}^1$-structures with a fixed bending measured lamination and analyzed their conformal structures. In this paper, as in the holonomy variety, we instead consider the holonomy representations of such $\mathbb{C}\mathrm{P}^1$-structures.

For a measured lamination L on a hyperbolic surface τ, we obtain an equivariant pleated surface in $\mathbb{H}^3$ by bending the universal cover of τ, the hyperbolic plane $\mathbb{H}^2$, along the inverse-image $\tilde{L}$ of L in $\mathbb{H}^2$, and the pleated surface $\tilde{\tau} \cong \mathbb{H}^2 \to \mathbb{H}^3$ is equivariant via a representation $\pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$. (See §3.1 for details.) Let $\mathcal{T}$ be the space of marked hyperbolic structures on S, the Fricke-Teichmüller space; then $\mathcal{T}$ is diffeomorphic to $\mathbb{R}^{6g-6}$ as a smooth manifold. The Weil-Petersson form gives a symplectic structure on $\mathcal{T}$, and Goldman extended it to a complex symplectic structure on the smooth part of $\mathcal{X}$ ([Gol84]). For a measured lamination L on S, let $b_L \colon \mathcal{T} \to \mathcal{X}$ be the map taking $\tau \in \mathcal{T}$ to the holonomy representation $\pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ of the pleated surface given by τ and L.

This mapping is closely related to the Thurston parametrization of $\mathcal{P}$ (Theorem 3.1), and the following theorem is an analogue of Theorem 1.1 in the Thurston parametrization.

Theorem A (Theorems 4.1, 15.4, Lemma 3.2)**.** *Let L be an arbitrary measured lamination on S. Then, the bending map $b_L \colon \mathcal{T} \to \mathcal{X}$ is a real-analytic symplectic embedding, and it is equivariant by the subgroup of the mapping class group $\mathcal{G}_L$ of S preserving L.*

Moreover, b_L is proper if and only if L contains no periodic leaves of weight π modulo 2π.

This preservation of the symplectic structure of $\mathcal{T}$ by b_L resembles the preservation of the (intrinsic) hyperbolic metric by the bending map $\mathbb{H}^2 \to \mathbb{H}^3$, and the equivariant property is also analogous. Moreover, by Theorem A, the real bending map b_L is a proper mapping for almost all measured laminations L. In addition, for exceptional laminations, we explicitly characterize the non-properness in the Fenchel-Nielsen coordinates (Theorem 6.1).

Depending on $L \in \mathcal{ML}$, the stabilizer $\mathcal{G}_L$ can be a large subgroup and, on the other hand, it can be the trivial subgroup of the mapping class group MCG (Remark 3.3).

We next consider the asymptotic behavior of $b_L \colon \mathcal{T} \to \mathcal{X}$. Namely, we give an analogue of Theorem 1.2 for the real bending map b_L. Recall that the Thurston boundary of the Teichmüller space is canonically embedded in the Morgan-Shalen boundary (see [Kap01, §11.16]). In this paper, the "boundary map" of b_L is the identity for almost all the points.

Theorem B. *Let $V \in \mathrm{PMF} = \partial_{th}\mathcal{T}$ be a measured foliation such that every singular leaf is a tripod, i.e. a union of three rays with a common endpoint. For every $L \in \mathcal{ML}$ and every sequence $\tau_i \in \mathcal{T}$ converging to V, the bent representation $b_L(\tau_i)$ converges to the $\pi_1(S)$-action on the dual metric tree of $\tilde{V}$ as $i \to \infty$. (Theorem 7.1.)*

Note that a full-measure set of measured foliations satisfies the assumption that every singular leaf is a tripod.

1.3. **Complex bending varieties.** Historically, a real-analytic deformation determined by a measured lamination or a measured foliation (an equivalent object) often has a significant complexification: A Teichmüller geodesic in the Teichmüller space $\mathcal{T}$ is determined by a measured foliation on a Riemann surface, and its complexification is a Teichmüller disk in $\mathcal{T}$ A measured lamination on a hyperbolic surface yields a real-analytic earthquake line in $\mathcal{T}$ ([Thu86, Ker85]), and an earthquake disk is its complexification ([McM98]).

We aim to geometrically complexify the real-analytic embedding $b_L\colon \mathcal{T} \to \mathcal{X}$ in Theorem A, and obtain a complex-analytic mapping from a closed complex-analytic variety. It is plausible that such complexifications of the real bending varieties $\operatorname{Im} b_L$ in a common analytic space will lead us to discover intersecting properties of the original real-analytic varieties.

We first explain the domain of the complexified bending map. Given a representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$, if a holonomy $\rho(\ell) \in \mathrm{PSL}_2\mathbb{C}$ along a loop ℓ is either hyperbolic or elliptic, then one can certainly bend ρ along ℓ as the axis of $\rho(\ell)$ gives the axis of bending deformation. However, it is *not* clear at all if one can define bending if $\rho(\ell)$ is parabolic or the identity.

Therefore, given a weighted multiloop M on S, we introduce an appropriate closed analytic set X_M consisting of certain (double) framed representations, so that the framing determines the bending axes even when the holonomy along some loops of M is trivial (§8). In fact, this modification of $\mathcal{X}$ essentially occurs only in a complex-analytic subvariety of $\mathcal{X}$ disjoint from $\mathcal{T}$: Namely, when specific subvarieties are removed from X_M and $\mathcal{X}$, the map forgetting the framing induces a finite-to-one holomorphic covering map from X_M to $\mathcal{X}$ (see §8.3). In particular, there is a canonical embedding of the Fricke-Teichmüller space $\mathcal{T}$ into X_M as a real-analytic smooth subvariety. In addition, we can pull back the complex symplectic structure on $\mathcal{X}$ to X_M minus a subvariety.

We next explain the target space. Notice that the Fricke-Teichmüller space $\mathcal{T}$ is a component of the real slice of the character variety $\mathcal{X}$. Moreover, the real bending map $b_L\colon \mathcal{T} \to \mathcal{X}$ is in the complex affine variety $\mathcal{X}$ (i.e. its tangent spaces contain no complex lines). Therefore, it is necessary to enlarge the ambient space in order to obtain nontrivial and different complexifications for different bending laminations.

When the $\mathrm{PSL}_2\mathbb{C}$ Lie algebra $\mathfrak{psl}_2\mathbb{C}$ is regarded as a real Lie algebra, its complexification is isomorphic to $\mathfrak{psl}_2\mathbb{C} \oplus (\mathfrak{psl}_2\mathbb{C})^*$, where $*$ denotes the complex conjugate. Thus, for a representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$, we consider the diagonal representation $\pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C}$ twisted by conjugation (of matrix entries), defined by $\gamma \mapsto (\rho(\gamma), \rho(\gamma)^*)$. Then, given a representation framed along loops of M, we can appropriately bend it along the axes determined by their framings, where the bending happens in the space of representations into $\mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C}$. Then we obtain the complex bending map $B_M\colon X_M \to \mathcal{X} \times \mathcal{X}$. (See §9 for details.) Let

$$\Delta^* = \{(\rho_1, \rho_2)\colon \pi_1(S) \to \mathcal{X} \times \mathcal{X} \mid \rho_1 = \rho_2^*\},$$

the anti-holomorphic diagonal in $\mathcal{X} \times \mathcal{X}$. Define $\psi\colon \mathcal{X} \to \Delta^* \subset \mathcal{X} \times \mathcal{X}$ by $\rho \mapsto (\rho, \rho^*)$. Let ω be Goldman's complex symplectic structure on $\mathcal{X}$. Then, the average of pull-back complex symplectic structures $\frac{1}{2}(\mathrm{pr}_1^*\omega + \mathrm{pr}_2^*\omega)$ is a complex symplectic structure on $\mathcal{X} \times \mathcal{X}$, where pr_1 and pr_2 are projections $\mathcal{X} \times \mathcal{X} \to \mathcal{X}$ to the first factor and the second factor, respectively. Then the diagonal embedding $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ preserves the $\mathbb{C}$-symplectic structure.

Each hyperbolic surface $\tau \in \mathcal{T}$ corresponds to a discrete faithful representation $\pi_1(S) \to \mathrm{PSL}_2, \mathbb{R}$ whose image consists of hyperbolic elements except the identity. Choose the orientation of each loop m of M (*oriented multiloop*). Then, the hyperbolic element $\rho(m)$ has a unique repelling fixed point and attracting fixed point on $\mathbb{C}\mathrm{P}^1$, and we have a canonical embedding $\iota_M\colon \mathcal{T} \to X_M$ by adding the information of the fixed points.

Theorem C (Complexified bending maps along multiloops)**.** *Let M be a weighted oriented multiloop on S, i.e. a measured lamination only with periodic leaves. Then $B_M\colon X_M \to \mathcal{X} \times \mathcal{X}$ is a complex-analytic mapping, such that*

1. *the restriction of B_M of $\mathcal{T}$ is a real-analytic embedding into Δ^*;*
2. *$\psi \circ b_M\colon \mathcal{T} \to \mathcal{X} \times \mathcal{X}$ coincides with $B_M \circ \iota_M$ to $\mathcal{T}$ (Figure 1);*
3. *B_M is complex symplectic in the complement of a proper subvariety of X_ℓ;*
4. *B_M is equivariant by the action of the subgroup of the mapping class group preserving M.*

(The complex-analyticity is proven in Theorem 12.1. For (1), see Proposition 13.1. For (2), see Proposition 13.1. For (3), see Theorem 15.5; For (4), see Lemma 9.2.) The removed subvariety in (3) consists of the framed representations such that at least one loop of M has trivial holonomy.

$$\begin{array}{ccc} X_M & \xrightarrow{B_M} & \mathcal{X} \times \mathcal{X} \\ {\scriptstyle \iota_M} \big\uparrow & & {\scriptstyle \psi} \big\uparrow \\ \mathcal{T} & \xrightarrow{\quad b_M \quad} & \mathcal{X} \end{array}$$

FIGURE 1. The commutative diagram describing the complexification B_M of the real-analytic bending map b_M.

Moreover, the properness of Theorem A is also carried over to complexified bending maps for a dense subset of $\mathcal{ML}$.

Theorem D. *If ℓ is a non-separating oriented loop with weight not equal to π modulo 2π, then, the bending map $B_\ell \colon X_\ell \to \mathcal{X} \times \mathcal{X}$ is a proper mapping. (Theorem 14.1.)*

Therefore, under the assumption of Theorem D, the image of B_ℓ is a closed analytic subvariety in $\mathcal{X} \times \mathcal{X}$ (complex bending variety). Thus, via ψ, $\operatorname{Im} b_\ell$ is properly embedded in the real-analytic subvariety of the closed analytic set $\operatorname{Im} B_\ell$, and ψ preserves the $\mathbb{R}$-symplectic structure of $\operatorname{Im} b_\ell$.

On the other hand, the complex bending map B_M is *not* proper or injective in general. However, B_M is injective and proper "almost everywhere": If an analytic subset is removed from the domain X_M and a subvariety is removed from the target $\mathcal{X} \times \mathcal{X}$, then B_M becomes injective and proper (Theorem 10.1, Theorem 11.1).

Next, we consider this complexification of a general bending map b_L. A *quasi-Fuchsian representation* $\pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ is a discrete faithful representation such that the limit set of its image $\operatorname{Im} \rho$ is a Jordan curve in $\mathbb{CP}^1$. The set $\mathcal{QF}$ of quasi-Fuchsian representations is called the *quasi-Fuchsian space*, and its real slice is the Teichmüller space $\mathcal{T}$. It is straightforward to similarly define the complexified bending map B_L on the quasi-Fuchsian space $\mathcal{QF}$ in $\mathcal{X}$, since, for every quasi-Fuchsian representation, every geodesic lamination is realized by the pleating locus of an equivariant pleated surface.

Theorem E. *For every measured lamination L on S, let ℓ_i be a sequence of non-separating weighted loops converging to L as $i \to \infty$. Then, up to a subsequence, the closed $\mathbb{C}$-analytic set $\operatorname{Im} B_{\ell_i}$ converges to a closed $\mathbb{C}$-analytic set in $\mathcal{X} \times \mathcal{X}$ as $i \to \infty$ which is $\mathbb{C}$-symplectic on the smooth part.*

Moreover, the closed $\mathbb{C}$-analytic set $\lim_{i\to\infty} \operatorname{Im} B_{\ell_i}$ contains a unique irreducible connected component $\mathcal{B}_L$ containing $B_L(\mathcal{QF})$, such that $B_L = \psi \circ b_L$ on $\mathcal{T}$. (§16.)

$$\begin{array}{ccc} \mathcal{QF} & \xrightarrow{B_L} & \mathcal{B}_L \subset \mathcal{X} \times \mathcal{X} \\ \uparrow & & \uparrow \psi \\ \mathcal{T} & \xrightarrow{b_L} & \mathcal{X} \end{array}$$

FIGURE 2. The commutative diagram describing the complexification of $\operatorname{Im} b_L$.

1.4. **Outline of the paper.** In §3, we explain some basic notions used in this paper. In particular, we recall that a measured lamination on a hyperbolic surface induces an equivariant locally convex pleated surface $\mathbb{H}^2 \to \mathbb{H}^3$, then we define the real bending map $b_L \colon \mathcal{T} \to \mathcal{X}$ for a measured lamination. In §4, we show the injectivity of the real bending map. In §5, we prove the properness of the real bending map for most of the measured laminations L. On the other hand, in Theorem 6.1, we characterize the non-properness of the bending map. §7.1, we prove Theorem B.

In §8, we introduce the space of representations double-framed along a weighted multiloop M on S (the framed character variety X_M). Then, in §9, we define the complex bending map from the framed character variety $\mathcal{X}_M$ to the product character variety $\mathcal{X} \times \mathcal{X}$. For the definition, a more general type of bending deformation is introduced. In fact, when a representation framed along M is bent along M, accordingly, the hyperbolic space $\mathbb{H}^3$ is equivariantly "bent" inside the $\mathbb{H}^3 \times \mathbb{H}^3$ (§9.4). In §10, we show that the complex bending map is injective almost everywhere. In §11, we show that the complex bending map is proper almost everywhere. In §12, using the "almost-everywhere" injectivity, we prove the analyticity of the complex bending map on the entire domain. In §13, we show that the complex bending map is a complexification of the real bending map. In §14, we prove that the complex bending map is, indeed, genuinely a proper mapping when M is a single non-separating loop of the weight *not* equal to π. In §15, we show that the real bending map is symplectic and the complex bending map is complex symplectic. In §16, we prove Theorem E.

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3. Preliminaries

3.1. **Bending deformation.** ([Thu81], [EM87].) Thurston discovered that the boundary of the convex core of a hyperbolic three-manifold is a hyperbolic surface bent along a measured lamination ([Thu81]). More generally, one can bend a hyperbolic surface along an arbitrary measured lamination and obtain a holonomy representation from the surface fundamental group into $\mathrm{PSL}_2\mathbb{C}$ as follows.

We shall first describe basic bending maps when the bending locus is a single loop. Let τ be a hyperbolic structure on S, and let ℓ be a geodesic loop on τ with weight $w > 0$. The union $\tilde{\ell}$ of all lifts of ℓ to the universal cover $\mathbb{H}^2$ of τ is a set of disjoint geodesics, each with weight w, and it is invariant under the deck transformation. We call the union $\tilde{\ell}$ the total lift of ℓ.

Put the universal cover $\mathbb{H}^2$ in the three-dimensional hyperbolic space $\mathbb{H}^3$ as a totally geodesic hyperbolic plane. By this embedding, the isometric deck transformations of $\mathbb{H}^2$ extend to an isometric action on $\mathbb{H}^3$, and we obtain a representation of $\rho_\tau\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$. Note that, as S is oriented, the orientation of the universal cover $\mathbb{H}^2$ determines a normal direction of the plane. Thus we can bend $\mathbb{H}^2$ along every geodesic α of $\tilde{\ell}$ by angle w so that the normal direction is in the exterior. Thus we obtain a bending map $\beta\colon \mathbb{H}^2 \to \mathbb{H}^3$, which is totally geodesic on every complement of $\mathbb{H}^2 \setminus \tilde{\ell}$. The map β is unique up to an orientation-preserving isometry of $\mathbb{H}^3$. Moreover, β is equivariant by its holonomy representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$. This ρ is called a bending deformation of ρ_τ of L.

If C_1, C_2 are components of $\mathbb{H}^2 \setminus \tilde{\ell}$ such that C_1, C_2 are adjacent along a geodesic α of $\tilde{\ell}$. Let G_1 and G_2 be the subgroups of $\pi_1(S)$ which preserve C_1 and C_2, respectively. If β is normalized so that $\beta_\tau = \beta$ on C_1, then the restriction of β to G_2 is the conjugation of the restriction of ρ_τ to G_2 by the elliptic isometry with the axis α by angle w.

More generally, given an arbitrary measured lamination L on τ, we can take a sequence of weighted loops ℓ_i converging to L as $i \to \infty$. For each i, let $\rho_i\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ be the bending deformation of ρ_τ along ℓ_i. Then ρ_i converges to a representation $\pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ as

$i \to \infty$ if ρ_i are appropriately normalized by $\mathrm{PSL}_2\mathbb{C}$. This limit is the bending deformation of ρ_τ along L, and it is unique up to conjugation by an element of $\mathrm{PSL}_2\mathbb{C}$.

3.1.1. *Equivariant property of the real bending map.* The equivariant property of $b_L\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ in Theorem A can directly be proven from the definition of the bending map. Here, we show this property in a broader context.

A $\mathbb{C}\mathrm{P}^1$-structure on S is a $(\mathbb{C}\mathrm{P}^1, \mathrm{PSL}_2\mathbb{C})$-structure. That is, an atlas of charts mapping open subsets of S into $\mathbb{C}\mathrm{P}^1$ with translation maps in $Aut(\mathbb{C}\mathrm{P}^1) = \mathrm{PSL}_2\mathbb{C}$. (General references about $\mathbb{C}\mathrm{P}^1$-structures are[Dum09, Kap01, Gol22]). Recall that $\mathbb{C}\mathrm{P}^1$ is the ideal boundary of the hyperbolic space $\mathbb{H}^3$, and $\mathrm{PSL}_2\mathbb{C}$ is the group of orientation-preserving isometries of $\mathbb{H}^3$. Using equivariant bending maps described above, Thurston gave a parametrization of the deformation space $\mathcal{P}$ of $\mathbb{C}\mathrm{P}^1$-structures by corresponding them with holonomy-equivariant pleated surfaces in $\mathbb{H}^3$.

Theorem 3.1 (Thurston, [KP94, KT92])**.** *The following canonical identification holds by a (tangential) homeomorphism,*

$$\mathcal{P} = \mathcal{T} \times \mathcal{ML}.$$

Then $b_L(\tau) = \mathrm{Hol}(\tau, L)$ where $(\tau, L) \in \mathcal{T} \times \mathcal{ML}$ denote the $\mathbb{C}\mathrm{P}^1$-structure in Thurston coordinates.

Lemma 3.2. *For $L \in \mathcal{ML}$, let $\mathcal{G}_L$ be the subgroup of* MCG *which preserves L. Then, the real bending map $b_L\colon \mathcal{T} \to \mathcal{X}$ is $\mathcal{G}_L$-equivariant.*

Remark 3.3. *If L is a multiloop, then $\mathcal{G}_L$ contains the subgroup of* MCG *generated by Dehn twists along loops not intersecting L (but including the loops of L). On the other hand, for almost all L in $\mathcal{ML}$, $\mathcal{G}_L$ is the trivial group, since* MCG *is a countable group.*

Proof. The MCG-action on $\mathcal{P}$ is given by marking change and on $\mathcal{X}$ by precomposing induces isomorphisms $\pi_1(S) \to \pi_1(S)$. Then the holonomy map $\mathrm{Hol}\colon \mathcal{P} \to \mathcal{X}$ is MCG-equivariant (see, for example, [Gol06]).

By Thurston's parametrization, For $\tau \in \mathcal{T}$ and $h \in \mathrm{MCG}$, $h(\tau, L) = (\tau, L)$.

$$h \cdot b_L(\tau) = h \cdot \mathrm{Hol}(\tau, L) = \mathrm{Hol}(h, L) = b_L(h\tau).$$

Thus the desired equivariant property holds. □

3.2. **Quasi-geodesics in the hyperbolic space.** We first recall the definition of quasi-isometries. Let $(X, d_X), (Y, d_Y)$ be metric spaces, where d_X, d_Y are the distance functions. Then, for $P > 1, Q > 0$, a mapping $f\colon X \to Y$ is a (P,Q)-*quasi-isometry* if, for all $x_1, x_2 \in X$,

$$P^{-1} d_X(x_1, x_2) - Q < d_Y(f(x_1), f(x_2)) < P\, d_X(x_1, x_2) + Q.$$

In this section, we discuss some conditions for a piecewise geodesic curve in $\mathbb{H}^3$ to be a quasi-geodesic.

3.2.1. *Quasi-geodesics in* $\mathbb{H}^3$. Let c be a bi-infinite piecewise geodesic curve in $\mathbb{H}^3$. Let s_i $(i \in \mathbb{Z})$ be the geodesic segments of c indexed along c, so that s_i and s_{i+1} are adjacent geodesic segments for every $i \in \mathbb{Z}$ and $c = \cup_{i \in \mathbb{Z}} s_i$.

Lemma 3.4. *For every $\epsilon > 0$, there are (large) $R > 0$ and (small) $\delta > 0$, such that if* $\operatorname{length} s_i > R$ *for all $i \in \mathbb{Z}$ and the (interior) angle between arbitrary adjacent geodesic segment s_i, s_{i+1} is at least $\pi - \delta$, then c is a $(1+\epsilon)$-biLipschitz embedding.*

Proof. This lemma follows from [CEG87, I.4.2.10] and [EMM04, Proof of Theorem 4.2]. □

Proposition 3.5. *For all $\epsilon > 0$ and $\epsilon' \in (0, \pi)$, there are $R > 0$ and $Q > 0$, such that if* $\operatorname{length} s_i > R$ *for all $i \in \mathbb{Z}$ and the angle between arbitrary every pair of adjacent geodesic segments is at least $\pi - \epsilon'$, then c is a $(1+\epsilon, Q)$-quasi-isometric embedding.*

Proof. For each $i \in \mathbb{Z}$, let x_i be the common endpoint of s_{i-1} and s_i, so that x_i is a non-smooth point of c. Pick $r > 0$ and we assume that $R > 2r$. Let x_i^- be the point on s_{i-1} such that $d(x_i^-, x_i) = r$. Let x_i^+ be the point on s_i such that $d(x_i, x_i^+) = r$. Then, we replace two geodesic segments $[x_i^-, x_i] \cup [x_i, x_i^+]$ of c with the single geodesic segment $[x_i^-, x_i^+]$; see Figure 3. Let c_r be the piecewise geodesic in $\mathbb{H}^3$ obtained from c by applying this replacement for every $i \in \mathbb{Z}$.

By basic hyperbolic geometry, the following holds.

Lemma 3.6. *For every $\delta > 0$, there is $r_\delta > 0$ satisfying the following: For every $r > r_\delta$ and every $R > 3r$, then the angle at every non-smooth point of c_r is more than $\pi - \delta$.*

Then Lemma 3.4 and Lemma 3.6 imply the proposition. [3.5]

3.3. **Angles between geodesic laminations.** (See [Bab15].)

Let τ be a hyperbolic surface. If two geodesics ℓ_1, ℓ_2 on τ intersect in a point p, then let $\angle_p(\ell_1, \ell_2)$ denote the angle between them which takes a value in $[0, \pi/2]$. More generally, let λ_1 and λ_2 be geodesic laminations on τ. Then the angle $\angle_\tau(\lambda_1, \lambda_2) \in [0, \pi/2]$ be the supremum of $\angle_p(\ell_1, \ell_2)$ over all leaves $\ell_1 \in \lambda_1$ and $\ell_2 \in \lambda_2$ intersecting a point p.

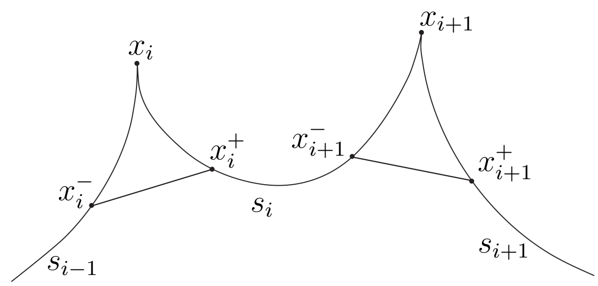

FIGURE 3. A modification of a piecewise geodesic

3.4. **The Morgan-Shalen compactification.** (See [Dum17] [Kap01].) We describe a compactification of the $\mathrm{PSL}(2,\mathbb{C})$-character variety of S.

$$\chi \to \mathbb{R}_{>0}^{\pi_1(S)}/\mathbb{R}_{>0}$$

be the mapping defined by

$$\rho \mapsto (\log|\operatorname{tr}\rho(\gamma)| + 2)_{\gamma\in\pi_1(S)}.$$

Then the image is relatively compact in the infinite-dimensional projective space $\mathbb{R}_{>0}^{\pi_1(S)}/\mathbb{R}_{>0}$, and the compactification in the projective space is called the Morgan-Shalen compactification of χ.

A boundary point p of the Morgan-Shalen compactification corresponds to a minimal small $\pi_1(S)$-action on a metric tree T_p. Namely, if a sequence $\rho_i\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \in \chi$ converges to p, then there is a sequence $r_i > 0$ converging to 0 such that

- there is a sequence of points $x_i \in \mathbb{H}^3$ such that $\pi_1(S)$-action on $\mathbb{H}^3$ with the base point x_i converges to the $\pi_1(S)$-action on T_p when the metric of $\mathbb{H}^3$ is scaled by r_i, and
- the translation lengths of $\rho(\gamma)$ for $\gamma \in \pi_1(S)$ scaled by γ_i converge to the translation lengths on T_p by the $\pi_1(S)$-action.

A Teichmüller space $\mathcal{T}$ can be regarded as the space of marked hyperbolic structures on S. Thus $\mathcal{T}$ can be regarded as a component of the space of discrete faithful representations $\pi_1(S) \to \mathrm{PSL}_2\mathbb{R}$ up to conjugation. The Thurston compactification of the Teichmüller space is the compactification given by the projective length spectra of the translation lengths ([FLP79]). Thus, via Bonahon's interpretation via geodesic currents, the Thurston boundary can be regarded as a part of the Morgan-Shalen boundary, by embedding $\mathcal{T}$ into χ as a component of a real slice. (See [Kap01, §11.16].)

3.5. **Complex analytic geometry.** ([Gol84].) We recall a standard theorem about a complex-analytic set.

Theorem 3.7 (Removable Singularity Theorem; see for example [Tay02], §3.3.2)**.** *Let Y be an analytic set. Let A be a closed subset of Y contained in a proper subvariety of Y. Suppose that $f\colon Y \setminus A \to \mathbb{C}$ is an analytic function which is bounded in a small neighborhood of every point in A. Then f continuously extends to an analytic function on Y.*

3.6. **Goldman's symplectic form.** ([Gol84].) Let $\mathfrak{g}$ be the $\mathrm{PSL}_2\mathbb{C}$-Lie algebra. Then the adjoint representation $\mathrm{Ad}\colon \mathrm{PSL}_2\mathbb{C} \to Aut\mathfrak{g} \subset \mathrm{GL}_3\mathbb{C}$ is induced by the conjugation of $\mathrm{PSL}_2\mathbb{C}$ by $\mathrm{PSL}_2\mathbb{C}$. By $\mathfrak{g}_{\mathrm{Ad}\,\rho}$, we regard $\mathfrak{g}$ as a $\pi_1(S)$-module via the composition of $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$. Then the Zariski tangent space of the representation variety $\mathcal{R}$ at $\rho \in \mathcal{R}$ is Then the vector space of 1-cocycles

$$Z^1(\pi_1(S); \mathfrak{g}_{Ad\rho}) = \{u \in \mathfrak{g}^{\pi_1(S)} \mid u(xy) = u(x) + (\mathrm{Ad}\rho(x))\, u(y)\}.$$

The subspace of 1-coboundaries $B^1(\pi_1(S); \mathfrak{g}_{Ad\rho})$ consists of $u \in \mathfrak{g}^{\pi_1(S)}$, such that there is $u_0 \in \mathfrak{g}$ satisfying $u(x) = u_0 - \mathrm{Ad}(\rho(x))u_0$ for all $x \in \pi_1(S)\}$. Then the Zariski tangent space of $\mathcal{X}$ at ρ is the quotient vector space

$$H^1(\pi_1(S); \mathfrak{g}_{Ad\rho}) = \frac{Z^1(\pi_1(S); \mathfrak{g}_{Ad\rho})}{B^1(\pi_1(S); \mathfrak{g}_{Ad\rho})}.$$

Let $w(\rho)$ denote the bilinear form on the Zariski tangent space obtained by the composition

$$\begin{aligned} H^1(\pi_1(S); \mathfrak{g}_{Ad\rho}) \times H^1(\pi_1(S); \mathfrak{g}_{Ad\rho}) &\overset{\cup}{\to} H^2(\pi_1(S); \mathfrak{g}_{Ad\rho} \otimes \mathfrak{g}_{Ad\rho}) \\ &\overset{\cong}{\to} H^2(\pi_1(S); \mathbb{C}) \cong \mathbb{C}. \end{aligned}$$

Here the first mapping is the cup product, and the second mapping is an isomorphism given by the coefficients pairing by the bilinear form $\mathfrak{B}\colon \mathfrak{g}_{Ad\rho} \otimes \mathfrak{g}_{Ad\rho} \to \mathbb{C}$ given by $(A, B) \to \mathrm{tr}\, AB$. Goldman proved that w is a complex symplectic form on $\mathcal{X}$, i.e. a non-degenerate closed holomorphic $(2, 0)$-form on the character variety $\mathcal{X}$; see [Gol84].

3.7. **Harmonic maps between hyperbolic surfaces.** (See ([Wol91, Min92]; see also [Sak].)

For (marked) Riemann surfaces $X, Y \in \mathcal{T}$, there is a unique harmonic map $h\colon X \to Y$ preserving the marking. Then the Hopf differential of the harmonic map h is a holomorphic quadratic differential q on X. Away from the zeros of q, the differential q gives natural coordinates $w = x + iy$ in $\mathbb{C}$ so that $q = dw$ (see, for example, [FM12]). By these coordinates, the Euclidean structure on $\mathbb{C}$ induces a singular Euclidean metric on X where the zeros of q are the singular points, and the Euclidean structure realizes the conformal structure of X.

The *Beltrami differential* of h is given by

$$\nu_h = \frac{f_{\bar{z}} d\bar{z}}{f_z dz}.$$

Then $|\nu_h(z)| < 1$. Then as Y leaves every compact in $\mathcal{T}$ while X is fixed, $|\nu_h(z)|$ converges to 1. Let

$$G(h) = \log\left(\frac{1}{|\nu(h)|}\right).$$

Let g_Y denote the hyperbolic metric on Y, and let $g = h^*(g_Y)$ be the pull-back metric on X by h. Then g is, in a natural coordinates $x + iy$ given by q,

$$ds^2 = \frac{\cosh G(t) + 1}{2} dx^2 + \frac{\cosh G(t) - 1}{2} dy^2 \tag{1}$$

The L^1-norm $\|q\| = \int_X |\phi_i| dz d\bar{z}$ is the total area of the flat metric. If the r-ball centered at $p \in X$ contains no zeros in the flat metric, then

$$G(h)(p) \leq \sinh^{-1} \frac{\operatorname{Area} X}{2\pi r^2}, \tag{2}$$

where $\operatorname{Area} X$ denotes the hyperbolic area $2\pi(2g - 2)$ of X (Minsky [Min92, Lemma 3.2]).

Therefore, if Y leaves every compact subset in $\mathcal{T}$ while X is fixed, the hyperbolic metric is stretched in the horizontal direction and shrinks in the vertical direction away from the zeros.

More specifically, we let $(Y_i)_{i=1}^{\infty}$ be a sequence in $\mathcal{T}$ converging to a point $[V] \in \mathrm{PML} = \partial\mathcal{T}$ in the Thurston boundary, where PML denotes the space of projective measured foliations on S. Let $h_i \colon X \to Y_i$ be the unique harmonic map, and let q_i be the holomorphic quadratic differential on X given by the Hopf differential of h_i. Let V_i be the vertical measured foliation of q_i, and let H_i be the horizontal measured foliation of $q_i = \phi_i dz^2$. Then its projective class $[V_i]$ converges to $[V]$ as $i \to \infty$ ([Wol91]).

The total Euclidean area $\|q_i\| = \int_X |\phi_i| dz d\bar{z}$ diverges to infinity as $i \to \infty$, and by (1) and (2), h_i stretches X in the horizontal direction H_i so that the Euclidean length and the hyperbolic length are close away from the zeros, and shrinks in the vertical direction V_i more and more.

4. Injectivity of the real bending maps

Let $\mathcal{ML}$ be the space of measured laminations on S. Each pair $(\tau, L) \in \mathcal{T} \times \mathcal{ML}$ induces an equivariant pleated surface $\mathbb{H}^2 \to \mathbb{H}^3$, unique up to $\mathrm{PSL}_2\mathbb{C}$. Let $b\colon \mathcal{T} \times \mathcal{ML} \to \mathcal{X}$ be the holonomy map of the bending maps.

Theorem 4.1. *Fix arbitrary $L \in \mathcal{ML}(S)$. Then the restriction b to $\mathcal{T} \times \{L\}$ is a real-analytic embedding. Moreover, this embedding is proper if and only if L contains no periodic leaf of weight π modulo 2π.*

Let $b_L\colon \mathcal{T} \to \mathcal{X}$ denote the restriction of b to $\mathcal{T} \times \{L\}$. Given a representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$, geodesic lamination λ on S is *realizable* if there is a ρ-equivariant pleated surface $\mathbb{H}^2 \to \mathbb{H}^3$, such that its pleating loci contains λ. Then, for $L \in \mathcal{ML}$, let $N = N_L$ be an open neighborhood of the Fuchsian space $\mathcal{T}$ in the smooth part of $\mathcal{X}$ such that the underlying geodesic lamination $|L|$ is realizable for all $\rho \in \mathcal{X}$. Then, $b_L\colon \mathcal{T} \to \mathcal{X}$ extends to the bending map $\hat{b}_L\colon N_L \to \mathcal{X}$ by bending cocycle ([Bon96]).

Proposition 4.2. *For all $L \in \mathcal{ML}$, $\hat{b}_L\colon N_L \to \mathcal{X}$ is injective.*

Proof. As $|L|$ is realizable on $\operatorname{Im} \hat{b}_L$, we have the unbending map $\hat{b}_{-L}\colon \operatorname{Im} \hat{b}_L \to \mathcal{X}$ by $-L$. Then, clearly, $\hat{b}_{-L} \circ \hat{b}_L$ is the identity map on N_L. Thus $\hat{b}_L$ is injective. □

Proposition 4.3. *The injective map $b_L\colon \mathcal{T} \to \mathcal{X}$ is a real-analytic embedding.*

Proof. (cf. [Ker85].) We regard $\mathcal{T}$ as the Fricke space, i.e. the space of discrete faithful representations into $\mathrm{PSL}(2, \mathbb{R})$ up to conjugation by $\mathrm{PSL}_2\mathbb{R}$. Then, take a small open neighborhood N of $\mathcal{T}$ whose closure is contained in N_L.

If L is a weighted multiloop, the bending map b_L is holomorphic on N as bending transforms the holonomy along a loop by some elliptic elements in a holomorphic manner. In general, pick a sequence of weighted multiloops M_i converging to L as $i \to \infty$. By the injectivity of Proposition 4.2, $\hat{b}_{M_i}\colon N_{M_i} \to N_{M_i}$ is a holomorphic embedding. Then, the holomorphic embedding $\hat{b}_{M_i}|N$ converges uniformly to $b_L|N$ uniformly on compacts as $i \to \infty$. Therefore $\hat{b}_L|N$ is a holomorphic embedding.

Since $\mathcal{T}$ is a real-analytic submanifold of N in $\mathcal{X}$, thus $b_L|\mathcal{T}$ is a real-analytic embedding. □

5. Properness of the bending maps from the Teichmüller spaces

Theorem 5.1. *Let $L \in \mathcal{ML}$. Then, the bending map $b_L\colon \mathcal{T} \to \mathcal{X}$ is proper if and only if L contains no leaves of weight π modulo 2π.*

First, we prove the sufficiency of the condition in Theorem 5.1.

Lemma 5.2. *Fix $L \in \mathcal{ML}$ such that every closed leaf of L contains no leaves of weight π modulo 2π. Let M be the (possibly empty) sublamination of L consisting of the periodic leaves of L. Then, for all $\upsilon, R > 0$, there are finitely many loops $\ell_1, \dots, \ell_n$ on S such that*

- *the lengths of $\ell_1, \dots, \ell_n$ form length parameters of $\mathcal{T}$, and*
- *for each $i = 1, \dots, n$,*
  - *the transversal measure $(L \setminus M)(\ell_i) < \upsilon$, and*
  - *ℓ_i intersects at most one leaf m of M, and their intersection number is at most two.*

Proof. For every $\delta > 0$, there is a pants decomposition $P = P_\delta$ (i.e. a maximal multiloop) on S consisting of

- the loops of M,
- loops which are disjoint from L,
- loops ℓ with $L(\ell) < \delta$ (so that ℓ is a good approximation of a minimal irrational sublamination of L).

By the third condition, if Q is a component of $S \setminus P$, and α is an arc on Q with endpoints on ∂Q, then there is an isotopy of α keeping its endpoints on ∂Q such that $L(\alpha) < 3\delta$. Therefore, if $\delta > 0$ is small enough, for each loop m of P, we can take two loops m_1, m_2 such that

- m_i intersects m at a point or two, and it does not intersect any other loop of P, and
- $(L \setminus M)(m_i) < \upsilon$.

Then we obtain a desired set of loops by adding two such loops for all loops of M. (For length coordinates of $\mathcal{T}$, see [FM12, Theorem 10.7] for example.) □

Proof of the sufficiency of Theorem 5.1. For $\epsilon > 0$, let $\ell_1, \dots, \ell_n$ be the set of loops given by Lemma 5.2. Let τ_i be a sequence in $\mathcal{T}$ which leaves every compact subset. Then, for some $1 \leq k \leq n$, $\text{length}_{\tau_i} \ell_k \to \infty$ as $i \to \infty$ up to a subsequence.

Claim 5.3. *For every $\epsilon > 0$, if $\delta > 0$ is sufficiently small, then*

1. *if $L(\ell_k) < \delta$, then $\beta_i | \tilde{\ell}_k$ is a $(1+\epsilon)$-biLipschitz embedding for sufficiently large i, and*
2. *if ℓ_k intersects a loop m of M, then $\beta_i | \tilde{\ell}_k$ is a $(1+\epsilon, q)$-quasi-isometric embedding for all sufficiently large i, where q only depends on the weight of m.*

Proof. (1) See [Bab10, Lemma 5.3], which was proved based on [CEG87, I.4.2.10].

(2) We straighten ℓ_k and M on $\tau_i \in \mathcal{T}$. From Lemma 5.2, ℓ intersects only one loop m of M, and their intersection number is one or two. We thus assume that $\ell_k \cap m$ consists of two points x_1, x_2 — the proof when the intersection number is one is similar. Then x_1 and x_2 decompose ℓ_k into 2 geodesic segments a_1 and a_2. Since $\mathrm{length}_{\tau_i} \ell_k \to \infty$, the lengths of a_1 and a_2 both goes to ∞ as well. Let $\tilde{\ell}_k$ be the geodesic in $\mathbb{H}^2$ obtained by lifting ℓ_k to the universal cover. Let $\tilde{a}_j$ be a lift of a_j to $\tilde{\ell}_k$, and let $\tilde{x}_j$ and $\tilde{x}_{j+1}$ be its endpoints. For every $\epsilon' > 0$, if $\upsilon > 0$, is sufficiently small, then $\beta_i(\tilde{a}_j)$ is ϵ'-close to the geodesic segment $[\beta_i x_j, \beta_i x_{j+1}]$ connecting its endpoints $\beta_i x_j$ and $\beta_i x_{j+1}$ in the Hausdorff metric. Since every periodic leaf of L has weight not equal to π modulo 2π, there is $\omega > 0$ such that, for every periodic leaf ℓ of L, the distance from the weight of ℓ to the nearest odd multiple of π is at least ω. Therefore, if $\delta > 0$ is sufficiently small, then the angle between $[\beta_i x_j, \beta_i x_{j+1}]$ and $[\beta_i x_{j-1}, \beta_i x_j]$ at x_j is at least $\omega/2$. Let c_i be the piecewise geodesic in $\mathbb{H}^3$ which is a union of the geodesic segments $[\beta_i x_j, \beta_i x_{j+1}]$ over all lifts $\tilde{a}_1, \tilde{a}_2$ of a_1, a_2 to $\tilde{\ell}_k$. Then c_i is ϵ'-Hausdorff close to $\beta_i \tilde{\ell}_k$. Therefore, by Proposition 3.5, we see that c_i is a $(1+\epsilon, q)$-quasi-geodesic. □

By this claim, for large i, the holonomy of $b_L(\tau_i)$ along ℓ_k is hyperbolic, and its translation length diverges to ∞ as $i \to \infty$. Thus $b_L(\tau_i)$ leaves every compact set in $\mathcal{X}$. Thus we have proven the properness. [5.1]

6. Characterization of non-properness

In this section, we explicitly describe how $b_L \colon \mathcal{T} \to \mathcal{X}$ is non-proper when the condition in Theorem 5.1 fails. Let L be a measured lamination on S. Let $m_1, \dots, m_p$ be the periodic leaves of L which have weight π modulo 2π. Then, set $M = m_1 \sqcup \dots \sqcup m_p$. Pick any pants decomposition P of S which contains $m_1, \dots, m_p$. Consider the Fenchel-Nielsen coordinates of $\mathcal{T}$ associated with P. Recall that its length coordinates take values in $\mathbb{R}_{>0}$ and its twist coordinates in $\mathbb{R}$.

Theorem 6.1. *Let τ_i be a sequence in $\mathcal{T}$ which leaves every compact subset. Then $b_L(\tau_i)$ converges in $\mathcal{X}$ if and only if*

- $\mathrm{length}_{\tau_i} m_j \to 0$ *for some $j \in \{1, \dots, p\}$ as $i \to \infty$ (pinched), and*
- *the Fenchel-Nielsen coordinates of τ_i w.r.t. P converge in their parameter spaces as $i \to \infty$, except that the length parameters of the pinched loops go to zero.*

Proof of Theorem 6.1. Let F be a component of $S \setminus M$. Then $b_L(\tau_i)|F$ converges in $\mathcal{X}(F)$ if and only if $\tau_i|F \coloneqq \tau_i|\pi_1(F)$ converges.

Let E and F be adjacent components of $\tilde{S} \setminus \tilde{M}$. Let $\tilde{m}$ be the component of $\tilde{M}$ separating E and F, and let m be the loop of M which lifts to $\tilde{m}$. Let Γ_E and Γ_F be the subgroups of $\pi_1(S)$ preserving E and F, respectively. Then E/Γ_E and F/Γ_F are the components of $S \setminus M$; let $S_E = E/\Gamma_E$ and $S_F = F/\Gamma_F$.

Proposition 6.2. *Let τ_i be a sequence in $\mathcal{T}$, such that the restrictions of τ_i to S_E and to S_F converge in their respective Teichmüller spaces as $i \to \infty$. Pick, for each i, a representative $\xi_i \colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ of $b_L(\tau_i) \in \mathcal{X}$ so that $\xi_i|\Gamma_E$ converges. Then, the restriction $\xi_i|\Gamma_F$ converges if and only if the Fenchel-Nielsen twist parameter along m converges as $i \to \infty$.*

Proof. For each i, let $\beta_i \colon \mathbb{H}^2 \to \mathbb{H}^3$ be the bending map for (τ_i, L) equivariant via ξ_i, so that β_i converges on E. Let M_{τ_i} be the geodesic representative of M on τ_i, and let $\tilde{M}_{\tau_i}$ be the total lift of M_{τ_i} on $\mathbb{H}^2$. Let $\tilde{m}_i$ be the component of $\tilde{M}_{\tau_i}$ corresponding to $\tilde{m}$. Let F_i, E_i be the region on $\tilde{\tau}_i \setminus \tilde{M}_{\tau_i}$ corresponding to F and E, respectively. For each i, pick a geodesic ray r_i in F_i starting from $\tilde{m}_i$ such that r_i is orthogonal to $\tilde{m}_i$ and that r_i does not intersect the total lift $\tilde{L}$ of L.

Let v be the unit tangent vector of r_i at the base point on $\tilde{m}_i$. Since the weight of $\tilde{m}_i$ is π modulo 2π, $d\beta_i(v)$ is tangent to $\beta_i(E_i)$ at a point of $\tilde{m}_i$; see Figure 4, Left. (Suppose, against the hypothesis, that the weight of m_i is not π modulo 2π and $\mathrm{length}_{\tau_i} m_i \to 0$. Let $\alpha_i \in \pi_1(S)$ represent m_i fixing $\tilde{m}_i$. Then $\beta_i(F_i)$ must diverge to the parabolic fixed point of the limit of the hyperbolic element $b_L(\tau_i)\alpha_i$ as $i \to \infty$; therefore $b_L(\tau_i)$ diverges to infinity, which contradicts the other hypothesis.)

First suppose that $\lim_{i\to\infty} \mathrm{length}_{\tau_i} m$ is positive. Then $\xi_i|\Gamma_F$ converges if and only if $\beta_i(r_i)$ converges, which is equivalent to saying the twisting parameter of m converges in $\mathbb{R}$ as $i \to \infty$.

Next suppose that $\lim_{i\to\infty} \mathrm{length}_{\tau_i} m$ is zero. Then the holonomy of m converges to a parabolic element not equal to the identity. Then $\xi_i|\Gamma_F$ converges, if and only if $\beta_i(r_i)$ converges to a geodesic starting from the parabolic fixed point. Since the hyperbolic element $b_L(\tau_i)m$ converges to the parabolic element, and we can take a sequence of the hyperbolic one-parameter subgroups in $\mathrm{PSL}_2\mathbb{C}$ containing $b_L(\tau_i)m$ which converges to the parabolic one-parameter subgroup of $\mathrm{PSL}_2\mathbb{C}$ containing the parabolic element, representing the twisting parameter along m. Therefore, the above convergence is equivalent to saying the twisting parameter of m converges as $i \to \infty$ (Figure 4, Right). $\square$

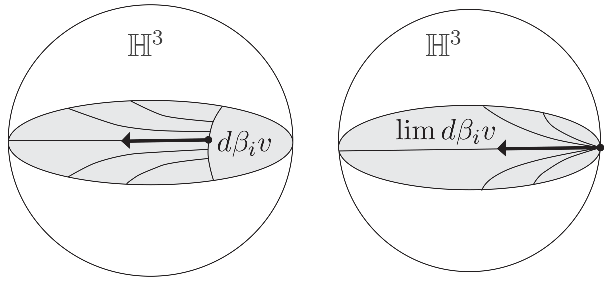

FIGURE 4. The convergence of the twist coordinate under neck-pinching.

The theorem follow from Proposition 6.2 as follows: Suppose that $b_L(\tau_i)$ converges as $i \to \infty$. Then, the hyperbolic structure on every component of $S \setminus M$ must converge. Thus, for each loop m of M, $\mathrm{length}_{\tau_i} m$ limits to a non-negative number. By Proposition 6.2, as $b_L(\tau_i)$ converges, the twist parameters along each loop of M converge. Since τ_i leaves every compact subset, at least one loop of M must be pinched as $i \to \infty$. Hence the two conditions hold.

To prove the other direction, suppose that the lengths of some loops of M limit to zero and all the other Fenchel-Nielsen coordinates with respect to P converge in the parameter space as $i \to \infty$. Let M' be the sub-multiloop of M consisting of the loops whose lengths go to zero. Then, for each component F of $S \setminus M'$, $b_L(\tau_i)|\pi_1(F)$ converges as $i \to \infty$. Therefore, by Proposition 6.2, $b_L(\tau_i)$ converges. This completes the proof. 6.1

7. The boundary map of the real bending map

Theorem 7.1. *Let $[V] \in \mathrm{PML} \cong \partial\mathcal{T}$ be a Thurston boundary point. Suppose that every singular leaf of V is a tripod, i.e. three rays sharing a common endpoint. Let $\tau_i \in \mathcal{T}$ be a sequence of hyperbolic surfaces converging to $[V]$.*

Then, for every measured lamination $L \in \mathcal{ML}$, the representation $b_L(\tau_i) \in \mathcal{X}$ converges to the Morgan-Shalen boundary point corresponding to $[V]$ as $i \to \infty$.

Remark 7.2. *Suppose, in contrast, that the projective measured foliation $[V]$ contains a singular leaf that is not a tripod. Then, the limit tree of τ_i may possibly be "folded" into a smaller tree by a "straight map"; thus a limit of $b_L(\tau_i)$ might not coincide with the Morgan-Shalen boundary point of $[V]$, similarly to the folding phenomenon in* [Dum17]*).*

Proof. Let ℓ be an essential simple closed curve on S. As every singular leaf of V is a tripod, ℓ is not contained in a leaf of V.

Pick a marked Riemann surface $X \in \mathcal{T}$ as a base point of the harmonic parametrization of $\mathcal{T}$ ([Wol91] [Hit87]). Then, for each $i = 1, 2, \dots$, there is a unique harmonic map $h_i \colon X \to \tau_i$, preserving the marking. Let q_i be its Hopf differential on X, which is a holomorphic quadratic differential on X. In this manner, $\mathcal{T}$ is parametrized by the complex vector space of dimension $3g - 3$ consisting of holomorphic quadratic differentials on X.

This harmonic map parametrization is compatible with Thurston's boundary of $\mathcal{T}$. For each $i = 1, 2, \dots$, let E_i be the flat surface corresponding to (X, q_i). As τ_i converges to the boundary point $[V]$ as $i \to \infty$, the unit-area surface $E_i/\sqrt{\operatorname{Area} E_i}$ converges to the flat surface E_∞ as $i \to \infty$ so that E_∞ realizes the conformal structure of X and the projective measured foliation $[V]$ as its vertical foliation.

For each $i = 1, 2, \dots$, let ℓ_i be the geodesic loop on the hyperbolic surface τ_i representing ℓ. Similarly, let m_i be a geodesic representative of ℓ on E_i. We also let m_∞ be a geodesic loop on E_∞ realizing ℓ. By taking appropriate representatives m_i, we may assume that m_i on the normalization $E_i/\sqrt{\operatorname{Area} E_i}$ converges to m_∞ on E_∞. Note that as V has no saddle connections, if i is sufficiently large, m_i is transversal to the vertical foliation V_i of E_i except at the singular points.

We divide the proof into the following two cases.

(1) $|L| = |V|$.
(2) $|L| \neq |V|$.

(Case 1) Suppose that $|L| = |V|$. We show that, if i is sufficiently large, every unit-length segment of ℓ_i on τ_i intersects the geodesic representative of L uniformly small amount w.r.t. the transversal measure of L— this implies that the translation length of ℓ does not change much by bending along L.

Since $|L| = |V|$, let V_L denote the measured foliation on S corresponding to L. For $\epsilon > 0$, pick segments $m_{\infty,1} \dots m_{\infty,n}$ of m_∞, such that

- when each of $m_{\infty,1} \dots m_{\infty,n}$ is divided into three segments of equal length, the middle one-third subsegments $m'_{\infty,1} \dots m'_{\infty,n}$ of $m_{\infty,1}, \dots m_{\infty,n}$ cover the geodesic loop m_∞, and
- $V_L(m_{\infty,j}) < \epsilon/3$ for all $j = 1, \dots, n$ and the endpoints of $m_{\infty,j}$ are not at the singular points of E_∞, where $V_L(m_{\infty,j})$ denotes the measure of $m_{\infty,j}$ given by the transversal measure of V_L.

As $|L| = |V|$, let V_i be the measured foliation supported on the vertical foliation of E_i, such that V_i converges to V_L as $i \to \infty$. By the convergence of m_i to m_∞, we cover m_i by open geodesic segments $m_{i,1} \dots m_{i,n}$ which respectively converge to the cover $m_{\infty,1} \dots m_{\infty,n}$ of m_∞ as $i \to \infty$. In addition, if $i > 0$ is sufficiently large, then $V_L(m_{i,j}) < \frac{2\epsilon}{3}$ for all $j = 1, \dots, n$.

Using the asymptotic behavior of harmonic maps described in the preliminaries (§3.7), the h_i-image of m_i is a quasi-geodesic loop on τ_i homotopic to ℓ_i such that the quasi-isometric distortion is bounded in $i = 1, 2, \dots$. Pick $\alpha \in \pi_1(S)$ representing ℓ. Then, let $\tilde{m}_i$ be the lift of m_i to the universal over $\tilde{E}_i$ invariant by α, and similarly $\tilde{\ell}_i$ be the lift of ℓ_i to the universal cover of τ_i invariant by α. Let $\tilde{h}_i$ be the lift of the harmonic map $h_i \colon X \to \tau_i$ to a $\pi_1(S)$-equivariant mapping between their universal covers. The distance between $\tilde{h}_i(\tilde{m}_i)$ and $\tilde{\ell}_i$ is bounded from above uniformly in i by the convergence of $\tau_i \to [V] \in \partial\mathcal{T}$.

Let Z_i be the set of the singular points of the flat surface E_i. For $r > 0$, let N_i^r be the neighborhood of Z_i on E_i corresponding to the r-neighborhood of Z_i on $E_i/\sqrt{\operatorname{Area} E_i}$. Fix a small $r > 0$. Thus, for every $\epsilon > 0$, if $i > 0$ is sufficiently large, then the h_i-image of $m_i \setminus N_i^r$ are ϵ-biLipschitz embedding w.r.t. the horizontal Euclidean length of E_i. Therefore, this h_i-image is contained in an ϵ-neighborhood of ℓ_i (§3.5). Then, by the uniform quasi-isometric property of $\tilde{h}_i | \tilde{m}_i$, we can cover the geodesic loop ℓ_i by geodesic segments $\ell'_{i,1} \dots \ell'_{i,n}$ corresponding to $m'_{\infty,1} \dots m'_{\infty,n}$ such that $\ell'_{i,j}$ is a geodesic segment on τ_i whose endpoints are ϵ-close to the h_i-image of the endpoints of $m_{i,j}$.

Lemma 7.3. *For every $r > 0$, if $i > 0$ is sufficiently large, then $L_i(\ell'_{i,j}) < \epsilon$ and $\operatorname{length}(\ell'_{i,j}) > r$ for all $j = 1, \dots, n$, where $L_i(\ell'_{i,j})$ denote the measure of ℓ'_{ij} given by the transversal measure of L_i.*

Proof. We first prove $L_i(\ell'_{i,j}) < \epsilon$, the main part of the claim. Pick $\alpha \in \pi_1(S)$ representing the loop ℓ. Then, let $\tilde{m}_\infty$ be a (bi-infinite) lift of m_∞ to the universal cover $\tilde{E}_\infty$ invariant by α. For each $j = 1, \dots, n$, pick a lift $\tilde{m}_{\infty,j}$ of the segment $m_{\infty,j}$ to the universal cover $\tilde{E}_\infty$, such that $\tilde{m}_{\infty,j}$ is contained in $\tilde{m}_\infty$. Let T_∞ be the $\mathbb{R}$-tree obtained by collapsing the vertical leaf of $\tilde{E}_\infty$, and let $\Psi_\infty \colon \tilde{E}_\infty \to T_\infty$ denote the quotient map. Then $\tilde{m}_\infty$ is embedded in T_∞ by Ψ since m_∞ is transversal to V_∞.

For each endpoint p of $\tilde{m}_{\infty,j}$ and each component C of $\tilde{E}_\infty \setminus \tilde{m}_\infty$, pick a rectangle R in C with horizontal and vertical edges such that:

- the interior of R contains no singular point of $\tilde{E}_\infty$;
- p is a vertex of R;
- the opposite vertex z of p is a singular point z of $\tilde{E}$;

- $\Psi_\infty(z)$ is contained in $\tilde{m}_\infty$ and close to p, so that the total measure of the leaves of the vertical foliation passing R is small (Figure 7).

Let $R_{j,1}, R_{j,2}$ denote the rectangles for one endpoints of $\tilde{m}_{\infty,j}$ contained in different components of $\tilde{E}_\infty \setminus \tilde{m}_\infty$, and let $R_{j,3}, R_{j,4}$ denote the rectangles for the other endpoints of $\tilde{m}_{\infty,j}$. Similarly, let z_1, z_2 denote the singular points of $\tilde{E}_\infty$ which are vertices of $R_{j,1}, R_{j,2}$ opposite from the endpoint vertex, and let z_3, z_4 denote the singular points of $\tilde{E}_\infty$ which are vertices of $R_{j,3}, R_{j,4}$ opposite from the other endpoint vertex. We may assume that the projections $\Psi_\infty(z_1), \Psi_\infty(z_2), \Psi_\infty(z_3), \Psi_\infty(z_4)$ lie on $\tilde{m}_\infty$ in this order, if necessary, by exchanging z_1 and z_2, and z_3 and z_4.

For each $k = 1, 2, 3, 4$, pick a small tripod neighborhood γ_k of the singular point z_i in the horizontal leaf containing z_i (Figure 7). As $E_i/\sqrt{\operatorname{Area} E_i}$ converges to E_∞, for sufficiently large i, we pick a tripod neighborhoods $\gamma_{i,j,1}, \gamma_{i,j,2}, \gamma_{i,j,3}, \gamma_{i,j,4}$ of the singular points of $\tilde{E}_i$ such that

- $\gamma_{i,j,1}, \gamma_{i,j,2}, \gamma_{i,j,3}, \gamma_{i,j,4}$ converge to $\gamma_{j,1}, \gamma_{j,2}, \gamma_{j,3}, \gamma_{j,4}$ as $i \to \infty$, respectively.

If i is sufficiently large, by the harmonic map h_i, a small neighborhood of $\gamma_{i,j,k}$ maps to a region close to an ideal triangle $\Delta_{i,j,k}$ in a large compact subset in $\tilde{\tau}_i$ [Min92]. Since the interior of $R_{i,j,k}$ contains no singular point , we may assume that $\tilde{\ell}_i$ is a common edge of $\Delta_{i,j,1}, \Delta_{i,j,2}, \Delta_{i,j,3}, \Delta_{i,j,4}$. Then the endpoints of $\tilde{\ell}_i$ are the ideal vertex of $\Delta_{i,j,k}$. Let v_k be the (other) ideal vertex of $\Delta_{i,j,k}$ which is not an endpoint of $\tilde{\ell}_i$. By reordering, we may additionally assume that $\Delta_{i,j,1}$ and $\Delta_{i,j,4}$ are contained in the same component of $\mathbb{H}^2 \setminus \tilde{\ell}_i$ and $\Delta_{i,j,2}$ and $\Delta_{i,j,3}$ are contained in the other component of $\mathbb{H}^2 \setminus \tilde{\ell}_i$.

Let $\tilde{\ell}_{i,j}$ be the α-invariant lift of $\ell_{i,j}$ in $\tilde{\tau}_i \cong \mathbb{H}^2$. Let L_i denote the geodesic measured lamination on the hyperbolic surface τ_i representing L. For sufficiently large i, if a leaf ℓ of $\tilde{L}$ intersects $\tilde{\ell}'_{i,j}$, then an endpoint of ℓ is between v_1 and v_4 and the other endpoint in v_2 and v_3. Since $V_i(m_{i,j}) < \frac{2\epsilon}{3}$, therefore, $L(\ell'_{i,j}) < \epsilon$.

Since $m_{\infty,j}$ is transversal to the vertical foliation and the harmonic map h_i stretches in the horizontal direction more and more, the length of $\ell'_{i,j}$ diverges to ∞ as $i \to \infty$. □

Let β_i be the $b_L(\tau_i)$-equivariant pleated surface $\mathbb{H}^2 \to \mathbb{H}^3$ obtained by bending, in $\mathbb{H}^3$, the universal cover of τ_i along the inverse-image of L. By Lemma 7.3, for every $\epsilon > 0$, if i is sufficiently large, then the

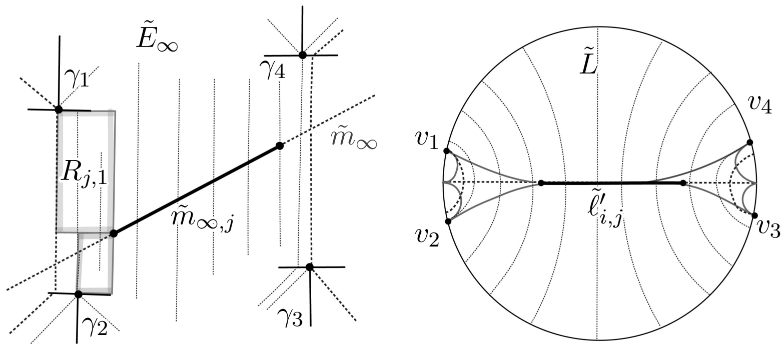

FIGURE 5. The geodesic segment $\tilde{\ell}_{i,j}$ has a small transversal measure.

restriction of $\beta_i \colon \tilde{\tau}_i \to \mathbb{H}^3$ to $\tilde{\ell}$ into $\mathbb{H}^3$ is a $(1-\epsilon, 1+\epsilon)$-biLipschitz embedding ([CEG87, I.4.2.10]). Hence the ratio of the translation length of $b_L(\tau_i)\alpha$ and the τ_i-length of ℓ_i converges to one as $i \to \infty$. Therefore $b_L(\tau_i)$ converges to $[V]$ in the Morgan-Shalen boundary.

(Case 2) Suppose that $|L| \neq |V|$. In this case, we show that every unit segment of the geodesic representative of ℓ on τ_i intersects L uniformly small angle if i is sufficiently large.

Let $\mathcal{L}_\infty$ be the geodesic representative of L on E_∞. Recall that the m_∞ is the geodesic representative of ℓ on the limi flat surface E_∞ of unit area. Consider the $\pi_1(S)$-invariant measured lamination of $\tilde{\tau}_i$ obtained by pulling back L_i on τ_i by the universal covering map. Let $\tilde{L}_i$ be its α-invariant measured lamination on $\tilde{\tau}_i$ consisting of leaves intersecting $\tilde{\ell}_i$.

Proposition 7.4. *For every $\epsilon > 0$, if $i > 0$ is sufficiently large, then $\angle_{\tilde{\tau}_i}(\tilde{\ell}_i, \tilde{L}_i) < \epsilon$.*

(See §3.3 for the definition of the angle $\angle_{\tilde{\tau}_i}(\tilde{\ell}_i, \tilde{L}_i)$.)

Proof. Let μ be a leaf of $\mathcal{L}_\infty$ intersecting $\tilde{m}_\infty$ at a point p_∞.

Pick Euclidean rectangles $R_{\infty,1}, R_{\infty,2}$ in $\tilde{E}_\infty$ such that

- $R_{\infty,1}, R_{\infty,2}$ have horizontal and vertical edges and no singular points in their interiors;
- the interiors of $R_{\infty,1}$ and $R_{\infty,2}$ are contained in different components of $\tilde{E}_\infty \setminus \tilde{m}_\infty$;
- one horizontal edge of $R_{\infty,k} (k = 1, 2)$ is contained in $\tilde{m}_\infty$, and each vertical edge of $R_{\infty,k}$ contains exactly one singular point of $\tilde{E}_\infty$;

- the singular points on the vertical edges of $R_{\infty,k}$ divide the boundary $\partial R_{\infty,k}$ into two piecewise linear curves, and μ passes through $R_{\infty,k}$ and μ intersects each piecewise-linear segment of $\partial R_{\infty,k}$ in a single point (Figure 6).

Since $E_i/\sqrt{\operatorname{Area} E_i}$ converges to E_∞ as $i \to \infty$, for sufficiently large i, we pick Euclidean rectangles $R_{i,1}, R_{i,2}$ in $\tilde{E}_i$ such that $R_{i,j} \to R_i$ as $i \to \infty$. Let $\mathcal{L}_i$ be the geodesic representative of the measured lamination L on the flat surface E_i. By the convergence, the properties of $R_{\infty,1}$ and $R_{\infty,2}$ carry over to $R_{i,1}$ and $R_{i,2}$ for sufficiently large i. Namely, letting μ_i be the leaf of $\tilde{\mathcal{L}}_i$ on $\tilde{E}_i$ corresponding to μ,

- $R_{i,1}, R_{i,2}$ have horizontal and vertical edges and no singular points in their interiors;
- the interiors of $R_{i,1}$ and $R_{i,2}$ are contained in different components of $\tilde{E}_i \setminus \tilde{m}_i$;
- one horizontal edge of $R_{i,k} (k = 1, 2)$ is contained in $\tilde{m}_i$, and each vertical edge of $R_{i,k}$ contains a unique singular point of $\tilde{E}_i$;
- the singular points on the vertical edges of $R_{i,k}$ divide the boundary $\partial R_{i,k}$ into two piecewise geodesic curves, and μ_i passes through $R_{i,k}$ and μ_i intersects each component of $\partial R_{\infty,k}$ minus the singular point in a single point.

Let z_1, z_2 be the singular points of $\partial R_{\infty,1}$ and z_3, z_4 be the singular points of $\partial R_{\infty,2}$. For $k = 1, 2, 3, 4$, let $\gamma_k = \gamma_{\infty,k}$ be a small horizontal tripod neighborhood of z_k (Figure 6). We may assume that the projections of z_1, z_2, z_3, z_4 to $\tilde{m}_i$ along vertical leaves lie on $\tilde{m}_i$ in this order (of indices).

For i large enough, let $z_{i,k}$ be a singular point of the vertical edge of $R_{i,k}$ such that $z_{i,k} \to z_k$ as $i \to \infty$.

Let $\gamma_{i,k}$ be a horizontal tripod neighborhood of $z_{i,k}$ such that $\gamma_{i,k}$ converges to $\gamma_{\infty,k}$ as $i \to \infty$. As in Case One, by the work of Wolf and Minsky ([Wol91, Min92]), if i is sufficiently large, a small neighborhood of $\gamma_{i,k}$ in $\tilde{E}_i/\sqrt{\operatorname{Area} E_i}$ maps to a region in $\tilde{\tau}_i \cong \mathbb{H}^2$ close to an ideal triangle $\Delta_{i,k}$ in a large compact subset. Since $R_{i,k}$ and $R_{\infty,k}$ contain no singular points in their interiors, we may assume that the geodesic $\tilde{\ell}_i$ is a unique common boundary edge of $\Delta_{i,1}, \Delta_{i,2}, \Delta_{i,3}, \Delta_{i,4}$.

Let $v_{i,k}$ be the ideal vertex of $\Delta_{i,k}$ which is not an endpoint of $\tilde{\ell}_i$ for $k = 1, 2, 3, 4$. Then, since the hyperbolic metric stretches in the horizontal direction and shrinks in the vertical direction of E_i (§3.7), the distance between the projections of $v_{i,2}$ and $v_{i,3}$ to $\tilde{\ell}_i$ diverges to infinity.

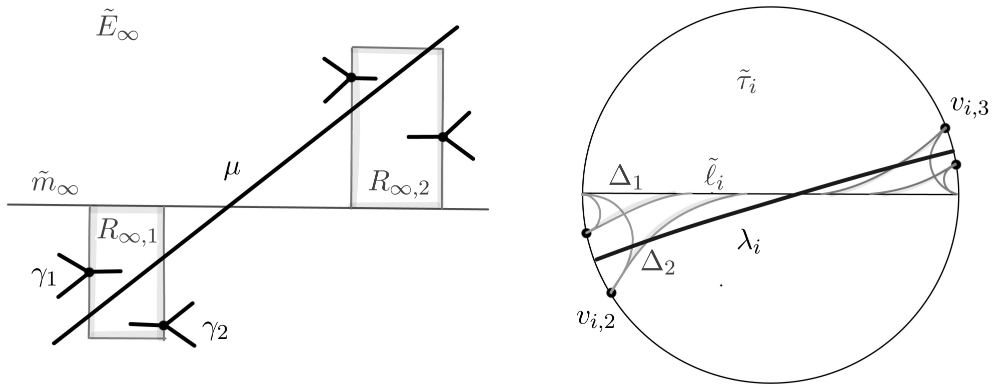

FIGURE 6. $\angle(\mu_i, \tilde{\ell}_i)$ is small for large i.

Let λ_i be the geodesic in $\tilde{\tau}_i$ fellow travels with $h_i(\mu_i)$. The boundary circle $\partial\tilde{\tau}_i \cong \mathbb{S}^1$ with the four points $v_{i,1}, v_{1,2}, v_{i,3}, v_{i,4}$ removed consists of four circular arcs. Then, one endpoint of λ_i is in the circular arc between $v_{i,1}$ and $v_{i,2}$, and the other endpoint is in the circular arc between $v_{i,3}$ and $v_{i,4}$. Since those circular arcs contain the endpoints of $\tilde{\ell}_i$, the divergence of distance above implies $\angle_{\tilde{\tau}_i}(\tilde{\ell}_i, \lambda_i) \to 0$ as $i \to \infty$ (Figure 6).

Suppose that another leaf μ' of $\tilde{\mathcal{L}}_\infty$ is sufficiently close to μ in a large compact subset containing the intersection point p_∞ and the rectangles $R_{\infty,1}$, and $R_{\infty,2}$ in $\tilde{E}_\infty$. Let λ_i' be the leaf of $\tilde{L}_i$ corresponding to μ'. Then, similarly, an endpoint of λ_i' is in the circular arc between $v_{i,1}$ and $v_{i,2}$ and the other endpoint is in the circular arc between $v_{i,3}$ and $v_{i,4}$ for sufficiently larege i. Therefore, by the divergence of the distance between the projections, $\angle_{\tilde{\tau}_i}(\tilde{\ell}_i, \lambda_i') \to 0$ as $i \to \infty$.

Since m_∞ is a closed curve on E_∞, by compactness, we see that $\angle_{\tilde{\tau}_i}(\tilde{\ell}_i, \tilde{L}_i) \to 0$ as $i \to \infty$. □

By Proposition 7.4, $\angle_{\tau_i}(L_i, m_i) \to 0$ as $i \to \infty$. Let $\rho_i = b_L(\tau_i) \colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$. Then, since the angle between L_i and m_i goest to zero, the ratio of $\mathrm{length}_{\tau_i} m_i$ and the translation length of $\rho_i(\alpha)$ converges to one as $i \to \infty$ ([Bab15, Proposition 4.1]). Thus $b_L(\tau_i)$ converge to $[V]$ in the Morgan-Shalen compactification. 7.1

8. Framed character varieties along loops

We have analyzed the real-analytic embedding $b_L\colon \mathcal{T} \to \mathcal{X}$ defined for an arbitrary measured lamination $L \in \mathcal{ML}$. As $\mathcal{T}$ is regarded as the Fricke space, a component of the real slice of the character variety $\mathcal{X}$, one can certainly extend b_L to a holomorphic mapping from a neighborhood of $\mathcal{T}$ in $\mathcal{X}$ into $\mathcal{X}$. However, it does *not* extend to the entire character variety $\mathcal{X}$ for multiple reasons. In particular, for a representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$, if there is no ρ-equivariant pleated surface in $\mathbb{H}^3$ realizing the measured lamination L, then the bending the representation along L does not make sense.

For instance, if a representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ takes a loop ℓ with weight w on S to a parabolic element, then there is no ρ-equivariant pleated surface realizing ℓ. Suppose in addition that the restriction of ρ to each component of $S \setminus \ell$ is non-elementary (which generically holds true). Then, if a sequence of representations ρ_i realizing ℓ converges to ρ, then the bending of ρ_i along ℓ by angle w must diverges in $\mathcal{X}$ as $i \to \infty$.

Therefore, in this section, we modify the character variety $\mathcal{X}$ and obtain a closed complex-analytic set, which will be a domain of the complexified bending map.

For a surface with punctures, Fock and Goncharov introduced a framing of a surface group representation ([FG06]). Their framing assigns a fixed point of peripheral holonomy around each puncture. In fact, this framing was useful for describing the deformation space of $\mathbb{C}\mathrm{P}^1$-structures on a surface with punctures via their framed holonomy representations ([AB20, GM21, Bab25]).

In this paper, we introduce a certain framing along loops which assigns a pair of distinct fixed points of their holonomy. Such framings will be used to determine the axes for bending deformation even when the holonomy along loops is trivial.

8.1. Framing of Representations along a loop.

For simplicity, we first discuss the modification in the case that the bending lamination is a single loop. Let $\mathcal{R}$ be the space of representations $\pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ (without any equivalence relation). Then $\mathcal{R}$ is an affine algebraic variety: Namely, pick a presentation of the fundamental group $\pi_1(S)$, for instance

$$\pi_1(S) = \langle\, a_1, b_1, \ldots, a_g, b_g \mid \Pi_{i=1}^n [a_i, b_i] \,\rangle.$$

Since $\mathrm{PSL}_2(\mathbb{C})$ embeds into $\mathrm{GL}_3(\mathbb{C})$ by the adjoint representation, $\mathrm{PSL}_2(\mathbb{C})$ is a complex affine Lie group sitting in $\mathbb{C}^9$. Then, by the embedding

$\mathcal{R} \to (\mathbb{C}^9)^{2g}$ defined by

$$\rho \mapsto (\rho(a_1), \rho(b_1), \ldots, \rho(a_g), \rho(b_g)) \in (\mathbb{C}^9)^{2g},$$

$\mathcal{R}$ has an affine algebraic structure on cut by the equation corresponding to the relator $\Pi_{i=1}^n [a_i, b_i]$.

Let ℓ be a simple closed curve on S. Let Γ_ℓ be the set of elements in $\pi_1(S)$ whose free homotopy classes are the homotopy class of ℓ on S; clearly, elements in Λ are conjugate to each other by elements in $\pi_1(S)$.

Pick an element $\alpha_\ell \in \Gamma_\ell$. Let $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ be a homomorphism. Suppose that $\rho(\alpha_\ell)$ is *not* a parabolic (but it can be the identity). Then, there is an ordered pair (u, v) of distinct points u, v on $\mathbb{CP}^1$ fixed by α_ℓ pointwise. We can equivariantly extend a pair (u, v) to pairs (u_γ, v_γ) for all representatives $\gamma \in \Gamma_\ell$ so that γ fixes u_γ and v_γ in $\mathbb{CP}^1$. Such an equivariant assignment $(u_\gamma, v_\gamma)_{\gamma \in \Gamma_\ell}$ of ordered fixed points of γ is called a framing of ρ along ℓ. By abuse of notation, we denote this equivariant framing $\{(u_\gamma, v_\gamma)\}_{\gamma \in \Gamma_\ell}$, by (u, v), since it is determined by the initial choice (u, v) for α_ℓ. We call the triple (ρ, u, v) a framed representation. We, later, utilize the equivariant framing to produce the equivariant bending axes (§9.2). Let

$$R_\ell = \left\{ (\rho, u, v) \in \mathcal{R} \times (\mathbb{CP}^1)^2 \,\middle|\, \rho(\alpha_\ell)u = u, \rho(\alpha_\ell)v = v, u \neq v \right\}.$$

Then R_ℓ is a closed analytic subset of $\mathcal{R} \times (\mathbb{CP}^1 \times \mathbb{CP}^1 \setminus D)$, where D is the diagonal $\{(z, z) \mid z \in \mathbb{CP}^1\}$. Note that if $(\rho, u, v) \in R_\ell$, then the $\rho(\alpha_\ell)$ can *not* be a parabolic element, since u, v are distinct fixed points of $\rho(\alpha_\ell)$. On the other hand, $\rho(\alpha_\ell)$ can be either hyperbolic, elliptic, or the identity.

Let $\mathcal{G}_\ell$ be the subgroup of the mapping class group of S which preserves the loop ℓ. Clearly, $\mathcal{G}_\ell$ acts on R_ℓ by marking change.

We now assume that the loop ℓ has a weight in $\mathbb{R}_{>0}$. Suppose, first, that the weight ω of the loop ℓ is not equal to π modulo 2π. Fix any complex number $w \in \mathbb{C}$ with $|w| > 1$. Then, given $(u, v) \in \mathbb{CP}^1 \times \mathbb{CP}^1$ with $u \neq v$, there is a unique hyperbolic element $\gamma_{u,v,w} \in \mathrm{PSL}_2\mathbb{C}$, such that u is the repelling fixed point, v is the attracting fixed point of $\gamma_{u,v,w}$ and that $\gamma_{u,v,w}$ can be conjugated to the hyperbolic element $z \mapsto wz$ by an element of $\mathrm{PSL}_2\mathbb{C}$. Clearly, this mapping $(u, v) \mapsto \gamma_{u,v,w}$ is a biholomorphic mapping onto its image. Then, $(\rho, u, v) \in R_\ell$ biholomorphically corresponds to a unique element $(\rho, \gamma_{u,v,w})$ of $\mathcal{R} \times \mathrm{PSL}_2\mathbb{C}$. Thus $R_\ell \to \mathcal{R} \times \mathrm{PSL}_2\mathbb{C}$ is a biholomorphic map onto its image. Since $\mathrm{PSL}_2\mathbb{C} \cong \mathrm{SO}_3(\mathbb{C}) \subset \mathbb{C}^9$, we see that R_ℓ is biholomorphic to a closed analytic set in a complex vector space of finite dimension. (It is closed, since if $(u, v) \in (\mathbb{CP}^1)^2 \setminus \Delta$ converges to a point in the diagonal Δ, then $\gamma_{u,v,w}$ must leaves every compact subset of $\mathrm{PSL}_2\mathbb{C}$.) Therefore R_ℓ is also a Stein space, as it is a closed analytic subset of a Stein space.

The theory of categorical quotients of Stein manifolds has been developed analogously to GIT-quotients of affine algebraic varieties (see [Sno82]). Let X_ℓ be the categorical quotient (*Stein quotient*) $R_\ell /\!\!/ \mathrm{PSL}_2\mathbb{C}$, which is again Stein. In this quotient, two framed representations (ρ_1, u_1, v_1) and (ρ_2, u_2, v_2) in R_ℓ are identified if and only if every $\mathrm{PSL}_2\mathbb{C}$-invariant analytic function f on R_ℓ takes the same value at (ρ_1, u_1, v_1) and (ρ_2, u_2, v_2); see [Sno82, §3]. We denote, by $[\rho, u, v] \in X_\ell$, the equivalence class of (ρ, u, v).

Next suppose that ℓ has weight π modulo 2π. In this case, the ordering of the framing (u, v) will *not* affect the complexified bending map, and thus we take a slightly stronger quotient. Then, let $\gamma_{u,v}$ be the elliptic element of angle π with the axes connecting u and v. Let $R_\ell/\mathbb{Z}_2$ be the quotient of R_ℓ by the $\mathbb{Z}_2$-action which switches the ordering of the framing, namely, given by $(\rho, u, v) \mapsto (\rho, v, u)$. Consider the map $R_\ell/\mathbb{Z}_2 \to \mathcal{R} \times \mathrm{PSL}_2\mathbb{C}$ defined by $(\rho, u, v) \mapsto (\rho, \gamma_{u,v})$. Thus $R_\ell/\mathbb{Z}_2$ is biholomorphic to a closed analytic set in $\mathcal{R} \times \mathrm{PSL}_2\mathbb{C}$. Similarly, we let X_ℓ be the Stein quotient $(R_\ell/\mathbb{Z}_2) /\!\!/ \mathrm{PSL}_2\mathbb{C}$. The action of $\mathcal{G}_\ell$ on R_ℓ descends to an action on X_ℓ.

8.1.1. *Coordinates for the quotient space of representations framed along a single loop.* We defined the Stein space X_ℓ as a Stein quotient. In this section, we indeed realize X_ℓ as an analytic set in an affine space by identifying it with a subset of a $\mathrm{PSL}_2\mathbb{C}$-character variety $\mathcal{X}(\pi_1(S) * \mathbb{Z})$ of $\pi_1(S) * \mathbb{Z}$. Recall that, for $(\rho, u, v) \in R_\ell$, the element $\gamma_{u,v,w} \in \mathrm{PSL}_2\mathbb{C}$ is a certain hyperbolic element if the weight of the loop ℓ is *not* equal to π modulo 2π and a certain elliptic element of angle π otherwise.

Given $(\rho, u, v) \in R_\ell$, let $\hat\rho = \hat\rho_{u,v,w}$ be the homomorphism from the free product $\pi_1(S) * \mathbb{Z}$ to $\mathrm{PSL}_2\mathbb{C}$, such that every $\gamma \in \pi_1(S)$ maps to $\rho(\gamma)$ and $1 \in \mathbb{Z}$ maps to $\gamma_{u,v,w}$. Then, with respect to the $\mathrm{PSL}_2\mathbb{C}$-action on R_ℓ, we clearly have the following.

Lemma 8.1. (1) *Suppose that the weight of ℓ is not equal to π modulo 2π. Then (ρ_1, u_1, v_1) and (ρ_2, u_2, v_2) are identified by an element of $\mathrm{PSL}_2\mathbb{C}$ if and only if $\hat\rho_1$ and $\hat\rho_2$ are conjugate by $\mathrm{PSL}_2\mathbb{C}$.*

(2) *Suppose that the weight of ℓ is equal to π modulo 2π. Then (ρ_1, u_1, v_1) and (ρ_2, u_2, v_2) are identified by an element of $\mathrm{PSL}_2\mathbb{C} \times \mathbb{Z}_2$ if and only if $\hat\rho_1$ and $\hat\rho_2$ are conjugate by $\mathrm{PSL}_2\mathbb{C}$, where the $\mathbb{Z}_2$-action exchanges the ordering of the framing.*

Let $\hat{\mathcal{R}}$ be the space of representations $\pi_1(S) * \mathbb{Z} \to \mathrm{PSL}_2\mathbb{C}$. Then $\hat{\mathcal{R}}$ is an affine algebraic variety. Suppose that the weight of ℓ is *not* equal to π modulo 2π. We have seen that the mapping $R_\ell \to \mathcal{R} \times \mathrm{PSL}_2\mathbb{C}$ is a biholomorphic map onto its image by the mapping $(\rho, u, v) \mapsto \hat{\rho}$. Let $\hat{\mathcal{R}}_\ell$ be this image. Then $\hat{\mathcal{R}}_\ell$ is the closed analytic subset in $\hat{\mathcal{R}}$ biholomorphic to R_ℓ, and thus in particular it is Stein. Moreover, this biholomorphism $R_\ell \to \hat{\mathcal{R}}_\ell$ is equivariant with respect to the $\mathrm{PSL}_2\mathbb{C}$-action. Thus the Stein space $X_\ell = R_\ell /\!\!/ \mathrm{PSL}_2\mathbb{C}$ is biholomorphic to the subvariety $\hat{\mathcal{R}}_\ell /\!\!/ \mathrm{PSL}_2\mathbb{C}$ of $\mathcal{X}(\pi_1(S) * \mathbb{Z})$.

A similar identification holds in the case when ℓ has weight π modulo 2π. The Stein space $R_\ell/\mathbb{Z}_2$ biholomorphically maps to its image, denoted by $\hat{\mathcal{R}}_\ell$, in $\hat{\mathcal{R}}$ by the mapping $(\rho, u, v) \mapsto \hat{\rho}$. Then $X_\ell = (R_\ell/\mathbb{Z}_2) /\!\!/ \mathrm{PSL}_2\mathbb{C}$ is biholomorphic to the Stein space $\hat{\mathcal{R}}_\ell /\!\!/ \mathrm{PSL}_2\mathbb{C}$.

Let $\gamma \in \pi_1(S) * \mathbb{Z}$. Let $\mathrm{tr}^2(\gamma)$ be the (polynomial) function on $\hat{\mathcal{R}}_\ell$ defined by $(\rho, u, v) \mapsto \mathrm{tr}^2 \rho(\gamma)$. Then $\mathrm{tr}^2(\gamma)$ is a $\mathrm{PSL}_2\mathbb{C}$-equivariant analytic function on $\hat{\mathcal{R}}_\ell$. Then, by [HP04, Corollary 2.3], such trace square functions form coordinates of the Stein quotient $\hat{\mathcal{R}}_\ell /\!\!/ \mathrm{PSL}_2\mathbb{C}$, and they also form coordinates for X_ℓ ($\cong \hat{\mathcal{R}}_\ell /\!\!/ \mathrm{PSL}_2\mathbb{C}$). Therefore we have the following.

Proposition 8.2. *There are finitely many elements $\gamma_1, \gamma_2, \dots \gamma_N$ in $\pi_1(S) * \mathbb{Z}$, such that the analytic mapping $\hat{\mathcal{R}}_\ell \to \mathbb{C}^N$ given by $\mathrm{tr}^2(\gamma_1), \mathrm{tr}^2(\gamma_2)$ $\dots, \mathrm{tr}^2(\gamma_N)$ induces an analytic embedding of X_ℓ into $\mathbb{C}^N$. Thus $\mathrm{tr}^2(\gamma_1)$, $\mathrm{tr}^2(\gamma_2) \dots, \mathrm{tr}^2(\gamma_N)$ form a coordinate ring.*

8.2. **Representations framed along a multi-loop.** In §8.1, we introduced the space of representations $\pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ framed along a single (oriented) loop, constructed a quotient space by the $\mathrm{PSL}_2\mathbb{C}$ action, and realized as an analytic subset of a complex affine space. In this section, we similarly consider the space of representations framed along a weighted multiloop, and then construct its Stein quotient by the action of $\mathrm{PSL}_2\mathbb{C}$.

Let $m_1, \dots m_n$ be non-isotopic essential simple closed curves on S, and let M be their union $m_1 \sqcup m_2 \sqcup \dots \sqcup m_n$. Recall that $\mathcal{R}$ denotes the representation variety $\{\pi_1(S) \to \mathrm{PSL}_2\mathbb{C}\}$. For each $i = 1, \dots, n$, pick a representative $\alpha_i \in \pi_1(S)$ representing m_i. Then, consider the space R_M of tuples $(\rho, (u_i, v_i)_{i=1}^n) \in R \times (\mathbb{C}\mathrm{P}^1)^{2n}$ where

- $\rho \in R$ is a homomorphism $\pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$, and
- $u_i, v_i \in \mathbb{C}\mathrm{P}^1$ are different fixed points of $\rho(\alpha_i)$ for $i = 1, \dots, n$.

As in the case of a single loop, $\rho(\alpha_i)$ are *not* parabolic elements (but can be the identity). Then R_M is a closed analytic subvariety of $R \times$

$(\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1 \setminus \Delta)^n$, where Δ denotes the diagonal as before. Given $(\rho, (u_i, v_i)_{i=1}^n) \in R_M$, we can equivariantly extend (u_i, v_i) to the pairs of fixed points for all representatives of $m_1, \dots, m_n$ in $\pi_1(S)$. We call this extension a framing of ρ along the multiloop M.

8.2.1. *Framed character varieties.* Now we assign a positive number (*weight*) to each loop of M. Let p be the number of components m_i of M, such that the weight of m_i is π modulo 2π. Without loss of generality, we can assume $m_1, \dots, m_n$ are the loops of M with weight π modulo 2π. Then, $\mathbb{Z}_2^p$ acts biholomorphically on R_M by switching the ordering of the fixed points of the framing along $m_1, \dots, m_p$. Note that this $\mathbb{Z}_2^p$-action has no fixed points in R_M.

Fix a complex number $w \in \mathbb{C}$ with $|w| > 1$. As in §8.1.1, let $\gamma_{u_i,v_i,w} \in \mathrm{PSL}_2\mathbb{C}$ be, if the weight of m_i is π modulo 2π, then the elliptic element of angle π whose axis is the geodesic connecting u_i to v_i, and otherwise, the hyperbolic element with the repelling fixed point u_i and the attracting fixed point v_i such that $\gamma_{u_i,v_i,w}$ is conjugate to the dilation $z \mapsto wz$. Then, define the mapping $R_M \to \mathcal{R} \times (\mathrm{PSL}_2\mathbb{C})^m$ by $(\rho, (u_i, v_i)_{i=1}^n) \mapsto (\rho, (\gamma_{u_i,v_i,w})_{i=1}^n)$. This mapping takes $R_M/\mathbb{Z}_2^p$ onto its image $\hat{R}_M$ biholomorphically. Thus $R_M/\mathbb{Z}_2^p$ is a closed analytic set in a finite-dimensional complex vector space. Therefore $R_M/\mathbb{Z}_2^p$ is Stein. The Lie group $\mathrm{PSL}_2\mathbb{C}$ acts analytically on $R_M/\mathbb{Z}_2^p$, by conjugation on ρ. By this action, we obtain its Stein quotient $(R_M/\mathbb{Z}_2^p) /\!\!/ \mathrm{PSL}_2\mathbb{C} =: X_M$. Thus X_M is a Stein space.

The biholomorphic map $R_M/\mathbb{Z}_2^p \to \hat{R}_M$ is equivariant w.r.t. the $\mathrm{PSL}_2\mathbb{C}$-action, X_M is biholomorphic to the corresponding Stein quotient $\hat{R}_M /\!\!/ \mathrm{PSL}_2\mathbb{C}$.

We denote, by $[\rho, (u_i, v_i)]$, the equivalence class of $(\rho, (u_i, v_i)) \in R_M$ in X_M. The subgroup $\mathcal{G}_M$ of MCG acts on R_M, and descends to an action on X_M.

8.2.2. *Coordinates of the quotient space of representations framed along a multiloop.* Let $g_1, g_2, \dots, g_n$ be a standard generating set of the free group $\mathbb{F}^n$ of rank n, so that there are no relators. Every $(\rho, (u_i, v_i)_{i=1}^n) \in R_M$ corresponds to a unique representation $\pi_1(S) * \mathbb{F}^n \to \mathrm{PSL}_2\mathbb{C}$ such that

- $\gamma \in \pi_1(S)$ maps to $\rho(\gamma)$, and
- g_i maps to $\gamma_{u_i,v_i,w}$ for every $i = 1, \dots, n$.

By this correspondence, R_M analytically embed into the space of representations $\pi_1(S) * \mathbb{F}^n \to \mathrm{PSL}_2\mathbb{C}$. As in §8.1.1, by the quotient of the image $\mathcal{R}_M$ by $\mathrm{PSL}_2\mathbb{C}$, [HP04, Corollary 2.3] yields the coordinate ring of $X_M \cong \mathcal{R}_M /\!\!/ \mathrm{PSL}_2\mathbb{C}$.

Proposition 8.3. *There are finitely many elements $\gamma_1, \gamma_2, \ldots, \gamma_N$ of $\pi_1(S) * \mathbb{F}^n$, such that $\mathrm{tr}^2(\gamma_1), \ldots \mathrm{tr}^2(\gamma_N)$ form a coordinate ring of X_M.*

8.3. **The projection from the framed character variety to the character variety.** In this subsection, we explain the relation between the famed character variety X_M and the original character variety χ. Let χ_M^p be the subvariety of χ consisting of representations $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$, such that at least one loop of M corresponds to a parabolic or the identity element of $\mathrm{PSL}_2\mathbb{C}$. Let X_M^p be the subvariety of X_M whose representations are in χ_M^p. Every representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ in $\chi \setminus \chi_M$ has 2^N choices for a framing along M, where are exactly N is the number of components of M. Then the projection map from $\setminus X_M^p$ to $\chi \setminus \chi_M^p$ is a finite holomorphic covering map, and the covering degree is 2^N. Therefore, by the removable singularity theorem (Theorem 3.7), $X_M \to \chi$ is a $\mathbb{C}$-analytic branched covering map.

9. Bending a surface group representation into $\mathrm{PSL}_2\mathbb{C}$ inside the representation space into $\mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C}$

Originally, bending deformation equivariantly bends a totally geodesic $\mathbb{H}^2$ along a measured lamination ([Thu81, EM87]), so that bending is in one direction and the bent $\mathbb{H}^2$ is locally convex. Moreover, one can extend it to an equivariant bending pleated surface along the pleated locus using bending cocycles ([Bon96]). In both cases, bending is done along (bi-infinite) geodesics in $\mathbb{H}^3$ which are embedded in the pleated surfaces.

In this section, we introduce a certain bending deformation of more general equivariant surfaces in $\mathbb{H}^3$. Using such a more general bending, define a complex-analytic bending map $X_M \to \chi \times \chi$ which complexifies the real-analytic bending map $\mathcal{T} \to \chi$.

9.1. **A complexification of the Lie group $\mathrm{PSL}_2\mathbb{C}$ regarded as a real Lie group.** We first recall a complexification of $\mathrm{PSL}_2\mathbb{C}$ when regarded as a real Lie group.

Proposition 9.1 (See Proposition 1.39 in [Zil] for example)**.** *Regard $\mathfrak{psl}_2\mathbb{C}$ as a real Lie algebra. Then the complexification of the Lie algebra $\mathfrak{psl}_2\mathbb{C}$ is isomorphic to $\mathfrak{psl}_2\mathbb{C} \oplus (\mathfrak{psl}_2\mathbb{C})^*$ by the mapping given by $(u, 0) \mapsto (u, Iu)$ and $(0, v) \mapsto (v, -Iv)$, where $(\mathfrak{psl}_2\mathbb{C})^*$ is the complex conjugate of $\mathfrak{psl}_2\mathbb{C}$ and I is the complex multiplication of $\mathfrak{psl}_2\mathbb{C}$.*

As we discussed in §16, we regard $\mathrm{PSL}_2\mathbb{C}$ as a real Lie group, and we complexify $\mathrm{PSL}_2\mathbb{C}$ by

$$\begin{array}{cccc} c\colon \mathrm{PSL}_2\mathbb{C} & \longrightarrow & \mathrm{PSL}_2\mathbb{C}\times\mathrm{PSL}_2\mathbb{C} & \\ \cup & & \cup & , \\ Y & \longmapsto & (Y, Y^*) & \end{array}$$

where Y^* denote the complex conjugate of Y, so that it corresponds to Proposition 9.1. Then c is holomorphic in the first factor and anti-holomorphic in the second factor. Thus c is, in particular, a proper real-analytic embedding of $\mathrm{PSL}(2,\mathbb{C})$ into the complex Lie group $\mathrm{PSL}_2\mathbb{C}\times\mathrm{PSL}_2\mathbb{C}$.

9.2. **Bending framed representations.** We first define a complex bending of representations framed along a single loop. Let ℓ be a loop on S, and we fixed a weight $w > 0$ of ℓ. Fix $\alpha \in \pi_1(S)$ representing ℓ. Let $[(\rho, u, v)] \in X_\ell$, where $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ and u, v are distinct fixed points of $\rho(\alpha)$. Let $(\rho,\rho)\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}\times\mathrm{PSL}_2\mathbb{C}$ denote the diagonal representation given by $\gamma \mapsto (\rho(\gamma), \rho(\gamma))$.

Recall that (u, v) generates a ρ-equivariant framing f along ℓ and Λ_ℓ denotes the subset of $\pi_1(S)$ corresponding to ℓ. That is, for every element $\gamma \in \Lambda_\ell$, an ordered pair $(u_\gamma, v_\gamma) \in \mathbb{CP}^1\times\mathbb{CP}^1$ of different fixed points of $\rho(\gamma)$ is assigned ρ-equivariantly. Consider the oriented geodesic $g_\gamma = (u_\gamma, v_\gamma)$ in $\mathbb{H}^3$ connecting u_γ to v_γ for all $\gamma \in \Lambda_\ell$. Those equivariant geodesics $\{g_\gamma\}$ will be the axes of the bending.

First, we coherently define the direction of the bending so that bending is continuously defined on X_ℓ. Pick any ρ-equivariant (topological) surface $\Sigma\colon \tilde{S} \to \mathbb{H}^3$: For instance, give a triangulation of S, which induces a $\pi_1(S)$-invariant triangulation of $\tilde{S}$; first construct a ρ-equivariant Σ on the one-skeleton of the triangulation on $\tilde{S}$; then equivariantly extend to the interiors of the triangles of $\tilde{S}$. Let $\tilde{\ell}$ be the lift of ℓ to the universal cover $\tilde{S}$ invariant by $\gamma \in \Lambda_\ell$. Then, homotope Σ in $\mathbb{H}^3$ so that Σ takes $\tilde{\ell}$ into the bi-infinite geodesic (u, v) connecting u to v.

We remark that, if $\rho(\alpha)$ is either an elliptic or the identity element, then we can not take Σ to be a pleated surface realizing ℓ. In such a case, the image of $\Sigma(\tilde{\ell})$ is a compact subset of the bi-infinite geodesic (u, v) since Σ is ρ-equivariant.

Then, for every $\theta \in \mathbb{R}$, we can equivariantly bend Σ along the equivariant oriented axes $\{g_\gamma\}$ by angle θ. Then we can accordingly bend the representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ so that the bent surface is equivariant via the bent representation. Here the bending direction is given

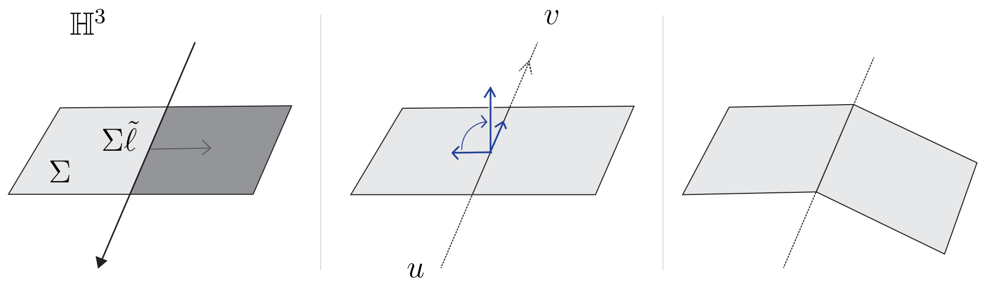

FIGURE 7. Bending direction. Left: The orientation of the loop ℓ determines the rotation side. Middle: The framing of ℓ and the orientation of $\mathbb{H}^3$ determine the direction of the rotation. Right: Bending deformation is determiend by the orientation and the framing of ℓ.

by the orientation of the hyperbolic three-space $\mathbb{H}^3$ and the oriented geodesic (u, v). Namely, those orientations determine the orientation of the plane orthogonal to the geodesic (u, v), and the counter-clockwise direction is the positive bending direction (Figure 7). Thus, if we reverse the order of u and v of the framing (u, v), then the positive bending direction is reversed.

By the orientation of ℓ, we decide which side of $\tilde{\ell}$ is rotated by this bending (Figure 7 Left).

Then, the representation $\pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ obtained by bending ρ by θ is denoted by $b_{\ell,\theta}(\rho, u, v)$. We now define a complex bending map $B_\ell \colon X_\ell \to \mathcal{X} \times \mathcal{X}$ by $B_{\ell,w}(\rho, u, v) = (b_{\ell,w}(\rho, u, v), b_{\ell,-w}(\rho, u, v))$. Note that, in the fast factor and the second factor, the bending ρ is equivariantly done along the same axes and by the same angle, but in the opposite directions (Figure 8).

The bent representation is well-defined up to conjugation by an element of $\mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C}$, and thus $B_\ell(\rho, u, v) \in \mathcal{X} \times \mathcal{X}$ is well-defined. We remark that, if $\rho \colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ is Fuchsian, then the representation of $B_\ell(\rho, u, v)$ in the first factor $\mathcal{X}$ is the complex conjugate of that in the second factor.

For a weighted multiloop M on S, we can similarly define the complex bending map $B_M \colon X_M \to \mathcal{X} \times \mathcal{X}$ as follows. Let $m_1, \ldots, m_n$ be the weighted loops of M. Pick $\gamma_i \in \pi_1(S)$ reprersenting m_i. Let $\tilde{m}_i$ be a γ_i invariant lift of m_i to the universal cover $\tilde{S}$. Let $[\rho, (u_i, v_i)_{i=1}^n] \in X_M$, where (u_i, v_i) be the fixed point of $\rho(\gamma_i)$. Then the oriented geodesic g_i connecting u_i to v_i, equivariantly extends to a system of bending

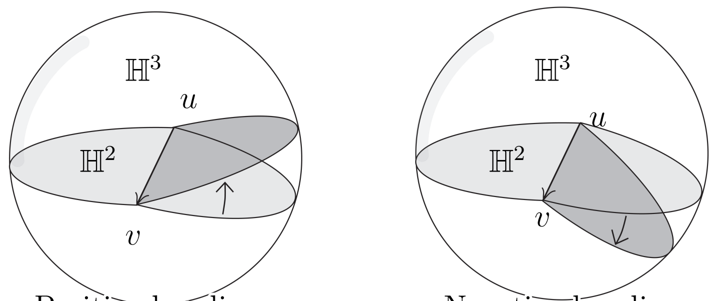

FIGURE 8. Bending in opposite directions in different factors.

axes corresponding to all lifts of m_i to $\tilde{S}$. Find a ρ-equivariant surface $\Sigma\colon \tilde{S} \to \mathbb{H}^3$ such that $\tilde{m}_i$ maps into its corresponding oriented axes g_i.

Let $\theta_1, \ldots, \theta_n$ be real numbers. We can similarly bend the ρ-equivariant surface $\Sigma\colon \tilde{S} \to \mathbb{H}^3$ along the oriented geodesics $g_1, \ldots, g_n$ and their orbit geodesics by angles $\theta_1, \ldots, \theta_n$, respectively, in the positive bending direction (defined by the orientation of $\mathbb{H}^3$ and the orientations of the geodesics). Since we bend Σ in an equivariant manner, the new bent surface $\Sigma^+\colon \tilde{S} \to \mathbb{H}^3$ is also equivariant via a unique representation. We denote the bent representation by

$$b_{(m_i,\theta_i)}(\rho, (u_i, v_i)_{i=1}^n) = b_M^+(\rho, (u_i, v_i)_{i=1}^n)\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}.$$

Similarly, we can bend Σ along the same axes by the same angles but in opposite directions, and we obtain another bent surface $\Sigma^-\colon \tilde{S} \to \mathbb{H}^3$. Then $\Sigma^-$is also equivariant via a unique representation

$$b_M^-(\rho, (u_i, v_i)_{i=1}^n) = b_{(m_i,-w_i)}(\rho, (u_i, v_i)_{i=1}^n)\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}.$$

By combining those two bending of framed representations, we obtain the bending map $B_M\colon X_M \to \chi \times \chi$ by

$$B_M(\rho, (u_i, v_i)_{i=1}^n) = (b_{(m_i,w_i)}(\rho, (u_i, v_i)_{i=1}^n), b_{(m_i,-w_i)_i}(\rho, (u_i, v_i)_{i=1}^n)).$$

Then the mapping $\tilde{S} \to \mathbb{H}^3 \times \mathbb{H}^3$ defined by $x \mapsto (\Sigma^+(x), \Sigma^-(x))$ is equivariant via $B_M(\rho, (u_i, v_i)_{i=1}^n)\colon \pi_1(S) \to \mathrm{PSL}_2 \times \mathrm{PSL}_2\mathbb{C}$.

9.3. **Equivariant property.**

Lemma 9.2. *Let M be a weighted multiloop on S. Let G_M be the subgroup of the mapping class group* $\mathrm{MCG}(S)$*, which preserves M. Then $B_M\colon X_M \to \chi \times \chi$ is G_M-equivariant.*

Proof. Recall that G_M acts on X_M by marking change. Therefore $b_{(m_i,w_i)}\colon X_M \to \chi$ and $b_{(m_i,-w_i)}\colon X_M \to \chi$ are both G_M-equivariant, since the equivariant construction of those mappings respect the G_M-action. Hence B_M is also G_M-equivariant. □

9.4. **Support planes and spaces.** For a marked hyperbolic surface τ homeomorphic to S and a measured lamination L on τ, we have a $\pi_1(S)$-equivariant bending map $\beta_{\tau,L}\colon \mathbb{H}^2 \to \mathbb{H}^3$ which is "locally convex". Letting $\tilde{L}$ be the $\pi_1(S)$-invariant measured lamination on the universal cover $\mathbb{H}^2$ of τ. Then, for each component P of $\mathbb{H}^2 \setminus \tilde{L}$, the mapping $\beta_{\tau,L}$ embeds P into a totally geodesic hyperbolic plane P in $\mathbb{H}^3$. Such a hyperbolic plane is a support plane for $\beta_{L,\tau}$. (See [EM87] for more general support planes.) On the other hand, this equivariant system $\{H_P\}_P$ of totally geodesic hyperbolic planes, indexed by the components, determines the original bending map $b_{\tau,L}\colon \mathbb{H}^2 \to \mathbb{H}^3$.

In §9.2, we bend framed representations $\eta = [\rho, (u_i, v_i)]$ in X_M along a weighted multiloop M, and obtained a representation $\pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C}$. As the symmetric space associated with $\mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C}$ is the product $\mathbb{H}^3 \times \mathbb{H}^3$, we consider a system of supporting hyperbolic three-spaces in the product $\mathbb{H}^3 \times \mathbb{H}^3$ as follows. For every component P of $\tilde{S} \setminus \tilde{M}$, the restriction of Σ^+ to P coincides with the restriction of Σ^- to P composed with an element γ of $\mathrm{PSL}_2\mathbb{C}$. Therefore, the restriction of the surface $(\Sigma^+, \Sigma^-)\colon \tilde{S} \to \mathbb{H}^3 \times \mathbb{H}^3$ to P is contained in a totally geodesic copy H_P of $\mathbb{H}^3$ given by $\{(x, \gamma x) \mid x \in \mathbb{H}^3\} \subset \mathbb{H}^3 \times \mathbb{H}^3$.

Hence, we obtain an equivariant collection of supporting hyperbolic 3-spaces H_P for all components P of $\tilde{S} \setminus \tilde{M}$. We call this collection $\{H_P\}_P$ the (equivariant) bending support system of B_M at η. Let G_P denote the subgroup of $\pi_1(S)$ consisting of the elements preserving the P. Then H_P is preserved by the restriction of the bent representation

$$B_M(\rho, (u_i, v_i)_{i=1}^n)\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C}$$

to the subgroup G_P.

Suppose that P and P' are adjacent components of $\tilde{S} \setminus \tilde{M}$ across a lift $\tilde{m}$ of a loop m of M. Let w be the weight of m. Then, in $\mathbb{H}^3 \times \mathbb{H}^3$, the support spaces H_P and $H_{P'}$ intersect in a geodesic at angle w (complex bending axis), which corresponds to the bending axis in $\mathbb{H}^3$ induced by the framing in the definition of B_M (Figure 9). In particular, if the weight of m is a multiple of π, then $H_P = H_{P'}$. Indeed, for an elliptic element $e \in \mathrm{PSL}_2\mathbb{C}$ with rotation angle π, we have

$$\{(x, x) \in \mathbb{H}^3 \times \mathbb{H}^3 \mid x \in \mathbb{H}^3\} = \{(ex, e^{-1}x) \in \mathbb{H}^3 \times \mathbb{H}^3 \,|\, x \in \mathbb{H}^3)\}.$$

Definition 9.3. *Let $\xi\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C}$ be a representation. A* support system of *ξ with respect to M is an equivariant collection of totally geodesic hyperbolic spaces H_P in $\mathbb{H}^3 \times \mathbb{H}^3$ for all components*

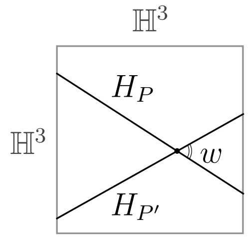

Figure 9. The intersection angle w of totally geodesic copies $H_P, H_{P'}$ of $\mathbb{H}^3$ in $\mathbb{H}^3 \times \mathbb{H}^3$.

P of $\tilde{S} \setminus \tilde{M}$ such that the restriction of ξ to G_P preserves H_P for all components P of $\tilde{S} \setminus \tilde{M}$.

In general, a representation $\pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C}$ may have no support system or many support systems. On the other hand, we will prove that the support system is uniquely determined by $B_M(\rho, (u_i, v_i)_{i=1}^n)$ in most cases; see Lemma 10.2.

10. Complex bending maps are almost injective

In this section, we prove the injectivity of the complex bending map $B_M : X_M \to \chi \times \chi$ when restricted to the complement of certain subvarieties.

Let M be a weighted multiloop on S, and let n be the number of loops of M. Let X_M^p be the subset of X_M consisting of the framed representations $(\rho, (u_i, v_i)_{i=1}^n)$ such that $\operatorname{tr}^2 \rho(m) = 4$ for, at least, one loop m of M, i.e. $\rho(m)$ is either a parabolic element or the identity. As it is an algebraic equation, X_M^p is an analytic subvariety of X_M.

Let X_M^w be the subset consisting of the framed representations (ρ, u, v) such that, for some component F of $S \setminus M$, the restriction of ρ to $\pi_1(F)$ is *weakly reducible*, i.e. the image is, up to a finite index, reducible.

Given a complex Lie subgroup G of $\mathrm{PSL}_2\mathbb{C}$, the set of all representations $\rho : \pi_1(S) \to G$ gives a subvariety of χ. The reducible representations $\pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ form a subvariety of χ. A representation $\rho : \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ is weakly reducible but not reducible, if and only if $\operatorname{Im}\rho$ preserves a pair of points on $\mathbb{C}\mathrm{P}^1$ but it does not fix the pair pointwise. (If $\operatorname{Im}\rho$ preserves a triple of distinct points on $\mathbb{C}\mathrm{P}^1$, then $\operatorname{Im}\rho$ is a finite group.) Thus, the set of weakly reducible representations forms a subvariety of χ.

Thus X_M^w is also an analytic subset of X_M. We prove that the injectivity of the complex bending map holds in the complement of those analytic subsets.

Theorem 10.1. *Let M be a weighted oriented multiloop on S. Then, the complex bending map $B_M\colon X_M \to \mathcal{X} \times \mathcal{X}$ is injective on $X_M \setminus (X_M^p \cup X_M^w)$.*

We first show the uniqueness of the support systems of the complex bending.

Lemma 10.2. *Let $\eta \in X_M \setminus (X_M^w \cup X_M^p)$. Fix a representative $\xi\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C}$ of $B_M(\eta)$. Let P be a component of $\tilde{S} \setminus \tilde{M}$. Then, the support space H_P of ξ is the unique totally geodesic copy of $\mathbb{H}^3$ in $\mathbb{H}^3 \times \mathbb{H}^3$ which contains the bending axes of the boundary components of P.*

Proof. As $\eta \notin X_M^P$, the bending axes of the boundary components of P are uniquely determined by ξ. Let G_P be the subgroup of $\pi_1(S)$ preserving P. Since $\eta \notin X_M^w$, the restriction $\eta|G_P$ is *strongly irreducible* (i.e. not weakly reducible). Therefore one can prove that there is a unique totally geodesic copy of $\mathbb{H}^3$ in $\mathbb{H}^3 \times \mathbb{H}^3$, containing those bending axes, as follows:

Set $B_M(\eta) = (\eta_1, \eta_2)\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C}$, where $\eta_1, \eta_2\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$. Then, since $\eta_i|G_P$ is strongly irreducible, the $\mathrm{PSL}_2\mathbb{C}$-action on $\eta_i|G_P$ by conjugation has the trivial stabilizer for each $i = 1, 2$. Suppose that there are two totally geodesic copies $\{(x, \gamma_1 x) \mid x \in \mathbb{H}^3\}$ and $\{(x, \gamma_2 x) \mid x \in \mathbb{H}^3\}$ of $\mathbb{H}^3$ in $\mathbb{H}^3 \times \mathbb{H}^3$ preserved by $\eta_i(G_P)$, where $\gamma_1, \gamma_2 \in \mathrm{PSL}_2\mathbb{C}$. By the definition of the complexified bending map B_M, we have $\gamma_1 \eta_1 \gamma_1^{-1} = \eta_2$ and $\gamma_2 \eta_1 \gamma_2^{-1} = \eta_2$ when restricted to G_P. Combining those equations, we have $\gamma_2^{-1} \gamma_1 \eta_1 \gamma_1^{-1} \gamma_2 = \eta_1$ on G_P. Hence $\gamma_1 = \gamma_2$, and the two copies of $\mathbb{H}^3$ coincide. □

Lemma 10.2 immediately implies the following.

Corollary 10.3. *Suppose that $\eta_1, \eta_2 \in X_M \setminus (X_M^p \cup X_M^w)$ satisfy $B_M(\eta_1) = B_M(\eta_2) \in \mathcal{X} \times \mathcal{X}$. Let $\xi\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C}$ be a representative of $B_M(\eta_1) = B_M(\eta_2)$. Then, the ξ-equivariant bending support system of B_M at η_1 equivariantly coincides with that at η_2.*

Proof of Theorem 10.1. Suppose that $\eta_1, \eta_2 \in X_M \setminus (X_M^p \cup X_M^w)$ map to the same representation $\pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C}$ in $\mathcal{X} \times \mathcal{X}$ by B_M. Then, let $\xi\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C}$ be a representative of their image.

By Corollary 10.3, the support system of the bending of η_1 equivariantly coincides with that of η_2. Therefore η_1 and η_2 are obtained by unbending ξ exactly in the same manner, and we obtain $\eta_1 = \eta_2$ as follows:

Let $\{H_P\}_P$ denote the support planes of ξ, where P varies over all components P of $\tilde{S} \setminus \tilde{M}$. Recall that, if P and P' are adjacent components of $\tilde{S} \setminus \tilde{M}$ along a lift of a loop m of M, then H_P and H'_P intersect in a geodesic by the angle equal to the weight of m. Take an abstract union $\cup_P H_P$ of the support 3-spaces H_P obtained by gluing adjacent support spaces along the bending geodesic axes. Then we have an ξ-equivariant mapping $\sigma\colon \cup_P H_P \to \mathbb{H}^3 \times \mathbb{H}^3$ by the inclusions $H_P \subset \mathbb{H}^3 \times \mathbb{H}^3$. Note that, letting G_P be the subgroup of $\pi_1(S)$ preserving P in $\tilde{S}$, clearly $\xi(G_P)$ preserves H_P.

Set $\eta_1 = (\rho_1, (u_{1,i}, v_{1,i}))$ and $\eta_2 = (\rho_1, (u_{2,i}, v_{2,i}))$. Then, unbending σ by $-M$, we have an equivariant mapping $\sigma(-M)\colon \cup_P H_P \to \mathbb{H}^3$, and ξ is deformed to an representation of $\pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$. This unbent representation must coincide with ρ_1 and ρ_2 up to conjugation by $\mathrm{PSL}_2\mathbb{C}$, due to the definition of B_M. Moreover, since the endpoints of the bending axes correspond to the framing, we see that $\eta_1 = \eta_2$. 10.1

10.1. **A non-injective example.** We shall see, in an example, the non-injectivity of a complex bending map. Let m be a separating loop on S with some positive weight. Pick a connected subsurface F of S bounded by m. Fix a homomorphism $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ such that $\rho|\pi_1 F$ is the trivial representation. Then, as $\rho(m)$ is in particular the identity, any pair $(u, v) \in \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$ is a framing of ρ along m.

Lemma 10.4. *Fix an arbitrary orientation of m and an arbitrary weight on m. Then $B_m(\rho, (u, v)) = (\rho, \rho) \in \mathcal{X} \times \mathcal{X}$ for all framings (u, v) along m. In particular, B_m is not injective and non-proper.*

Proof. Pick a loop ℓ on S which essentially intersects m exactly in two points (see Figure 10). We can assume, without loss of generality, that the base point of $\pi_1(S)$ is on m. Let γ be an element of $\pi_1(S)$ corresponding to ℓ. Then homotope ℓ so that ℓ is a composition of a loop ℓ_1 on $S \setminus F$ and a loop ℓ_2 on F. Since $\rho|\pi_1(F)$ is trivial, we have $B_m\eta(\gamma_\ell) = B_m\eta(\gamma_{\ell_1})$. We can take a generating set of $\pi_1(S)$ consisting of loops in $S \setminus F$ and loops in F. Therefore $B_m(\rho, (u, v)) = (\rho, \rho)$ in $\mathcal{X} \times \mathcal{X}$.

In particular, as (u, v) may leave every compact in $(\mathbb{C}\mathrm{P}^1)^2$ minus the diagonal, $B_m(\rho, (u, v)) = (\rho, \rho)$ remains true. Therefore B_ℓ is non-proper. □

11. Complex bending maps are almost proper

In this section, we prove the properness of the complex bending map, similarly to the injectivity in §10, in the complement of certain proper

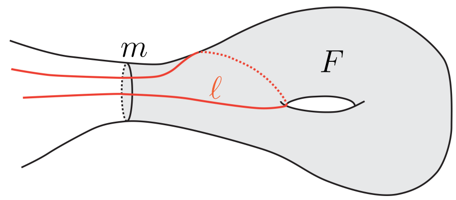

FIGURE 10. A loop ℓ essentialy intersects a bending loop m in two points.

subvarieties. Similarly to X_M^p, we let χ_M^p be the subvariety of the $\mathrm{PSL}_2\mathbb{C}$-character variety χ consisting of representations $\xi\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ such that, for, at least, one loop m of M, its holonomy $\xi(m)$ is parabolic or the identity in $\mathrm{PSL}_2\mathbb{C}$. Similarly to X_M^w, we let χ_M^w be the subvariety of χ such that, consisting of representations $\xi\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ such that, for at least one component F of $S \setminus M$, $\xi|F$ is weakly reducible.

Theorem 11.1. *The restriction of B_M to $X_M \setminus (X_M^p \cup X_M^w)$ is a proper mapping to $(\chi \setminus (\chi_M^p \cup \chi_M^w))^2$.*

Proof. Let $\eta_i \in X_M \setminus (X_M^P \cup X_M^w)$ be a sequence such that $B_M(\eta_i)$ converges to a representation in $(\chi \setminus (\chi_M^p \cap \chi_M^w))^2$ as $i \to \infty$. It suffices to show that η_i also converges in $X_M \setminus (X_M^P \cup X_M^w)$.

Pick a representative $\xi_i\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C}$ of $B_N(\eta_i)$ so that ξ_i converges to $\xi\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C}$, so that its equivalence class ξ is in $(\chi \setminus (\chi_M^p \cap \chi_M^w))^2$. Let $\{H_{i,P}\}$ be the ξ_i-equivariant bending support system of the complex bending of η_i along M, where P varies over all connected components of $\tilde{S} \setminus \tilde{M}$. By the hypothesis, the restriction of ξ to each component of $S \setminus M$ is strongly irreducible. Therefore, by Lemma 10.2, the ξ_i-equivariant support system $\{H_{i,P}\}$ converges to a unique support system $\{H_P\}$ of ξ as $i \to \infty$.

We also show that the bending axes converge.

Claim 11.2. *The ξ_i-equivariant axis system for bending η_i along M in $\mathbb{H}^3 \times \mathbb{H}^3$ converges to a ξ-equivariant axis system as $i \to \infty$.*

Proof. Let m be a loop of M, and let $\tilde{m}$ be a component of $\tilde{M}$ which descends to m. Let $\alpha \in \pi_1(S)$ denote the element preserving $\tilde{m}$ such that the free homotopy class of α is m. Let P, Q denote the adjacent components of $\tilde{S} \setminus \tilde{M}$ separated by $\tilde{m}$. Then $H_{i,P} \cap H_{i,Q}$ is the complex bending axis $g_{i,\tilde{m}}$ for $\tilde{m}$ in $\mathbb{H}^3 \times \mathbb{H}^3$, and also the axis of $\xi_i(\alpha)$. The angle of the intersection of $H_{i,P}$ and $H_{i,Q}$ along the axis is equal to the weight of m. As $\xi_i(m)$ converges to a non-parabolic element $\xi(m)$, the axis $H_{i,P} \cap H_{i,Q}$ converges to the axis of $\xi(\alpha)$ as $i \to \infty$. □

For each $i = 1, 2, \dots,$ let $\{g_{i,\tilde{m}}\}$ denote the ξ_i-equivariant bending axis system in $\mathbb{H}^3 \times \mathbb{H}^3$ of B_M at η_i. Note that η_i is obtained by unbending ξ_i along the axes $g_{i,\tilde{m}}$ by the angles given by the weights M. By the convergence, similarly unbending the limit ξ in $(\mathcal{X} \setminus (\mathcal{X}^p_M \cup \mathcal{X}^w_M))^2$ along the limit bending axis system by the angles given by M, we obtain the limit of η_i as $i \to \infty$. As ξ is in $(\mathcal{X} \setminus (\mathcal{X}^p_M \cup \mathcal{X}^w_M))^2$, thus $\lim_{i\to\infty} \eta_i$ is contained in $X_M \setminus (X^p_M \cup X^w_M)$. 11.1

12. Analyticity of complex bending maps

Theorem 12.1. *For every weighted oriented multiloop M on S, the bending map $B_M \colon X_M \to \mathcal{X} \times \mathcal{X}$ is complex analytic.*

Proof. Recall that X^p_M is the subvariety of the complex-analytic variety X_M consisting of representations such that at least one loop of M is parabolic, and also that X^w_M is the subset of X_M consisting of representations η such that the restriction of η to a component of $S \setminus M$ is weakly reducible. We have shown that the restriction of B_M to $X_M \setminus X^p_M \cup X^w_M$ is injective (Theorem 10.1). We first prove the assertion of Theorem 12.1 for almost everywhere.

Lemma 12.2. *The restriction of B_M to $X_M \setminus (X^p_M \cup X^w_M)$ is complex analytic.*

Proof. Recall that R_M is the space of representations framed along M, and that $R_M /\!\!/ \mathrm{PSL}_2\mathbb{C} = X_M$. Let R^p_M be the subset of R_M consisting of framed representations, such that at least one loop of M is parabolic (or the identity). Let $\eta = (\rho, (u_i, v_i)^n_{i=1})$ be an arbitrary framed representation in $R_M \setminus (R^p_M \cup R^r_M)$, where n is the number of the loops of M. As the closed subvariety $R^p_M \cup R^r_M$ is $\mathrm{PSL}_2\mathbb{C}$-invariant, we can take a $\mathrm{PSL}_2\mathbb{C}$-invariant open neighborhood U of η in $R_M \setminus (R^p_M \cup R^r_M)$. Then, for every framed representation $\zeta \in U$, the stabilizer of ζ in $\mathrm{PSL}_2\mathbb{C}$ is a discrete group, since ζ is *not* in R^r_M, Thus, if we take U appropriately, U is holomorphically a product of $\mathrm{PSL}_2\mathbb{C}$ and an open disk D. Let W be the image of U in X_M. Then, we can biholomorphically identify W in X_M with D in U and take a holomorphic section $s \colon W \to U$.

Pick any component of Q of $\tilde{S} \setminus \tilde{M}$, where $\tilde{M}$ is the inverse image of M in $\tilde{S}$. Let G_Q be the stabilizer of Q in $\pi_1(S)$. By $\mathbb{C}$-bending along M (normalizing so that the restriction to G_Q is unchanged), we obtain a holomorphic mapping $s(W) \to (\mathcal{R} \setminus \mathcal{R}^p_M \cup \mathcal{R}^r_M)^2$ which is a lift of the restriction of B_M to W. Then, for every $\zeta \in s(W)$, its image by this mapping is a pair of strongly irreducible representations in $\mathcal{R}$. Since W is isomorphic to $s(W)$ and the quotient map from $\mathcal{R} \times \mathcal{R}$ to $\mathcal{X} \times \mathcal{X}$

is algebraic, the analyticity of $s(W) \to (\mathcal{R} \setminus \mathcal{R}_M^p \cup \mathcal{R}_M^r)^2$ implies the analyticity of B_M at the equivalent class of η in X_M. $\square$

By Lemma 12.2, $X_M \setminus (X_M^p \cup X_M^w) \to (\mathcal{X} \setminus \mathcal{X}_M^p \cup \mathcal{X}_M^w) \times (\mathcal{X} \setminus \mathcal{X}_M^p \cup \mathcal{X}_M^w)$ is an injective analytic mapping. Since $X_M^p \cup X_M^w$ is an analytic subvariety of X_M, by the removable singularity theorem (Theorem 3.7), the mapping $B_M \colon X_M \to \mathcal{X} \times \mathcal{X}$ is analytic. 12.1

13. The real-bending map sits in the complex-bending map

In this section, we observe that the complex-analytic bending map $B_M \colon X_M \to \mathcal{X} \times \mathcal{X}$ is a natural extension of the real-analytic bending map $b_M \colon \mathcal{T} \to \mathcal{X}$. Recall that Δ^* is the twisted diagonal $\{(\rho, \rho^*) \mid \rho \in \mathcal{X})\}$ and $\psi \colon \mathcal{X} \to \Delta^* \subset \mathcal{X} \times \mathcal{X}$ is the embedding given by $\rho \mapsto (\rho, \rho^*)$.

The forgetful map $X_M \to \mathcal{X}$ restricts to an analytic covering map $X_M \setminus X_M^p \to \mathcal{X} \setminus \mathcal{X}_M^p$ of degree 2^n, where n is the number of loops of M. As the base surface S is oriented, we let $\mathcal{T}$ be the Teichmüller space of S is identified with a unique component of the real slice of $\mathcal{X}$. Since each loop of M is oriented, there is a unique lift of $\mathcal{T}$ to X_M: Namely, given a discrete faithful representation $\rho \colon \pi_1(S) \to \mathrm{PSL}(2, \mathbb{R})$ in $\mathcal{T}$, for each (oriented) loop m of M, assign the reprelling fixed point of $\rho(m) \in \mathrm{PSL}_2\mathbb{C}$ in $\mathbb{C}\mathrm{P}^1$ and the attacting fixed point of $\rho(m)$ in this order as its framing. Let $\iota_M \colon \mathcal{T} \to X_M$ denote this real-analytic embedding.

Proposition 13.1. *Let M be a weighted multiloop on S. Then, the restriction of B_M to $\mathcal{T}$ is a real-analytic embedding into the twisted diagonal Δ^* of $\mathcal{X} \times \mathcal{X}$, such that $B_M \circ \iota_M$ coincides with $\psi \circ b_M \colon \mathcal{T} \to \mathcal{X} \times \mathcal{X}$.*

Proof. Let $b_M^* \colon \mathcal{T} \to \mathcal{X}$ denote the complex conjugate of $b_M \colon \mathcal{T} \to \mathcal{X}$, i.e. the Fuchsian representation $\rho \colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{R}$ maps to the mapping taking $\gamma \in \pi_1(S)$ to $(b_M(\rho)(\gamma))^* \in \mathrm{PSL}_2\mathbb{C}$. When applying the complex bending B_M, a representation into $\mathrm{PSL}_2\mathbb{C}$ is bent in opposite directions in the first and the second factor of $\mathcal{X} \times \mathcal{X}$ (§9.2). Therefore, when applying B_M to a Fuchsian representation, the representation in the second factor is the complex conjugate of the representation in the first factor. Therefore $B_M \circ \iota_M(\rho)$ is $(b_M(\rho), b_M^*(\rho))$ for $\rho \in \mathcal{T}$, as desired. The analyticity of the mapping was already proven in Theorem 12.1. $\square$

14. Properness of the complex bending map along a non-separating loop

Theorem 14.1. *Let ℓ be a non-separating oriented loop on S with weight not equal to π modulo 2π. Then, the complex bending map $B_\ell\colon X_\ell \to \mathcal{X} \times \mathcal{X}$ is proper.*

Corollary 14.2. *The image of B_ℓ is a closed analytic set in $\mathcal{X} \times \mathcal{X}$.*

Remark 14.3. *By Theorem 4.1, the properness of the complex bending map B_M fails if M contains a loop of weight π modulo 2π. Nonetheless, it is still plausible that the image of B_M is a closed analytic subset of $\mathcal{X} \times \mathcal{X}$ for every weighted multiloop M on S as long as the weight of each loop is not equal to π modulo 2π.*

Pick $\theta \in (0, \pi)$. Let

$$E_\theta = \{(\gamma, e) \in \mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C} \mid e \text{ is elliptic of rotation angle } \theta\}.$$

Clearly, for every $(\gamma, e) \in \mathcal{E}_\theta$, $\mathrm{tr}^2\, e$ is a fixed constant in $(0, 4)$ only depending on θ. Thus E_θ is a smooth affine algebraic variety. Then $\mathrm{PSL}_2\mathbb{C}$ acts on $\mathcal{E}_\theta$ by conjugating both parameters γ and e simultaneously. Let $\mathcal{E}_\theta$ be the GIT-quotient $E_\theta /\!\!/ \mathrm{PSL}_2\mathbb{C}$. Then $\mathcal{E}_\theta$ is an affine algebraic variety. Then the following holds.

Lemma 14.4. *The analytic mapping $E_\theta /\!\!/ \mathrm{PSL}_2\mathbb{C} \to \mathbb{C}^2$ defined by $\phi\colon (\gamma, e) \mapsto (\mathrm{tr}^2\,\gamma, \mathrm{tr}^2\,\gamma e)$ is a proper mapping.*

Proof. The map $\mathrm{SL}_2\mathbb{C}\times\mathrm{SL}_2\mathbb{C}/\!\!/\mathrm{SL}_2\mathbb{C} \to \mathbb{C}^2$ given by $(\gamma, e) \mapsto (\mathrm{tr}\,\gamma, \mathrm{tr}\, e, \mathrm{tr}\,\gamma e)$ is a biholomorphic map (see for example, [Gol09]).

Let (α_i, e_i) be a sequence in $\mathcal{E}_\theta \subset \mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C} /\!\!/ \mathrm{PSL}_2\mathbb{C}$ which leaves every compact as $i \to \infty$. Pick any lift $(\tilde\alpha_i, \tilde e_i) \in \mathrm{SL}_2\mathbb{C} \times \mathrm{SL}_2\mathbb{C} /\!\!/ \mathrm{SL}_2\mathbb{C}$ of (α_i, e_i) for each i. Then $(\tilde\alpha_i, \tilde e_i)$ also leaves every compact set as $i \to \infty$.

By a basic trace identity, we have $\mathrm{tr}\,\tilde\alpha_i\tilde e_i + \mathrm{tr}\,\tilde\alpha_i\tilde e_i^{-1} = \mathrm{tr}\,\tilde\alpha_i\,\mathrm{tr}\,\tilde e_i$. Therefore, since $\mathrm{tr}\,\tilde e_i$ is a fixed non-zero constant, up to a subsequence, either $\mathrm{tr}\,\tilde\alpha_i$ or $\mathrm{tr}\,\tilde\alpha_i\tilde e_i$ diverges to ∞ as $i \to \infty$. Thus the image $\phi(\alpha_i, e_i)$ leaves every compact in $\mathbb{C}^2$ as $i \to \infty$. □

Since ℓ is non-separating, we can pick a generating set $\{\gamma_1, \ldots, \gamma_{2g}\}$ of $\pi_1(S)$ such that $\gamma_1, \ldots, \gamma_{2g}$ correspond to loops on S intersecting ℓ exactly once. Let $\eta_i = [\rho_i, (u_i, v_i)] \in X_\ell$ be a sequence which leaves every compact in X_ℓ.

Let $w(\ell)$ denote the weight of ℓ, and let $e_i \in \mathrm{PSL}_2\mathbb{C}$ be the elliptic element by angle $w(\ell)$ along the geodesic from u_i to v_i. Then we can normalize $(\rho_i, (u_i, v_i))$, by an element of $\mathrm{PSL}_2\mathbb{C}$, so that $e_i \in \mathrm{PSL}_2\mathbb{C}$ is independent of i; let e denote this elliptic element in $\mathrm{PSL}_2\mathbb{C}$.

As η_i leaves every compact and $\gamma_1, \ldots, \gamma_n$ form a generating set of $\pi_1(S)$, then there is $k \in \{1, \ldots, n\}$ such that, up to a subsequence, $\rho_i(\gamma_k)$ leaves every compact subset as $i \to \infty$.

Then, since γ_k intersects ℓ exactly at once, by the properness of Lemma 14.4, the image $B_\ell(\eta_i)\gamma_k$ also leaves every compact set as $i \to \infty$. (The hypothesis of the weight of ℓ not being π corresponding to the rotation angle of e not being π in Lemma 14.4.) This immediately implies the properness of B_ℓ, and completes the proof of Theorem 14.1.

15. Symplectic property

In this section, we prove the symplectic property of the bending maps. Complex Fenchel-Nielsen coordinates on the quasi-Fuchsian space are introduced by [Kou94] and [Tan94], and the coordinates holomorphically extend to most part of the character variety $\mathcal{X}$. We explicitly explain the subset of $\mathcal{X}$ where the complex Fenchel-Nielsen coordinates are defined.

Let M be a maximal multiloop on S. Then M contains $3g-3$ loops, where g is the genus of S. Let $\mathcal{X}_M^h$ be the (Euclidean) open full-measure subset of $\mathcal{X}$ consisting of $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ such that

- all loops of M are hyperbolic, and
- for each component P of $S \setminus M$, the restriction of ρ to $\pi_1(P)$ is irreducible.

Pick (real) Fenchel-Nielsen coordinates on the Teichmüller-Fricke space $\mathcal{T}$ with respect to M (see [FM12] for example). Let $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$. For each $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ in $\mathcal{X}_M^h$, let $\ell_i \in \mathbb{C}_+/2\pi I\mathbb{Z}$ be the complex translation length of $\rho(m_i)$: When we $\ell_i = x_i + Iy_i$ in real and imaginary coordinates, $x_i \in \mathbb{R}_{>0}$ is the (real) translation length and the $y_i \in \mathbb{R}$ is the rotation angle of the screw motion of the hyperbolic element $\rho(m_i)$.

Clearly, for real representations $\pi_1(S) \to \mathrm{PSL}_2\mathbb{R}$, their length parameters $\ell_1, \ldots, \ell_{3g-3}$ are all real numbers. Let $\tau_i \in \mathbb{C}/2\pi I\mathbb{Z}$ be the twist coordinate along ℓ_i which complexifies the Fenchel-Nielsen twist coordinate, so that the imaginary direction is the direction of bending deformation (where I denotes the imaginary unit). Similarly, for real representations $\pi_1(S) \to \mathrm{PSL}_2\mathbb{R}$, their twist parameters $\tau_1, \ldots, \tau_{3g-3}$ are all real numbers.

Lemma 15.1. *$\mathcal{X}_M^h$ is a (Zariski) open dense subset of $\mathcal{X}$ and biholomorphic to $(\mathbb{C}_+/2\pi I\mathbb{Z})^{3g-3} \bigoplus (\mathbb{C}/2\pi I\mathbb{Z})^{3g-3}$ by $(\ell_1, \ell_2, \ldots, \ell_{3g-3}, \tau_1, \tau_2, \ldots, \tau_{3g-3})$.*

Proof. The mapping $\mathcal{X}_M^h \to (\mathbb{C}_+/2\pi I\mathbb{Z})^{3g-3} \bigoplus (\mathbb{C}/2\pi I\mathbb{Z})^{3g-3}$ is a holomorphic mapping, as the coordinates are given by traces of loops.

Given a pair of pants P, the irreducible representations $\pi_1(P)$ are algebraically parametrized by the holonomy traces of the three boundary components of P ([Vog89] [Fri96]; see also [Gol09]). Now let P be a component of $S \setminus M$. Then $\rho \in \chi_M^h$, the $\rho|\pi_1(P)$ is parametrized by the complex length coordinates of the boundary components of P.

For a loop m_i of M, let F be the component of $S \setminus (M \setminus \ell)$ which contains M. Then the representation on $\pi_1(F) \to \mathrm{PSL}_2\mathbb{C}$ is determined by the twisting parameter τ_i of m_i and the length parameters ℓ_i of m_i and the boundary loops of F. We see that the mapping is biholomorphic. □

Due to Platis [Pla01] and Goldman [Gol04], the complex Fenchel-Nielsen coordinates yield Darboux coordinates for Goldman's complex symplectic structure.

$$w_G = \Sigma_{i=1}^{3g-3} d\ell_{m_i}^{\mathbb{C}} \wedge dt_{m_i}^{\mathbb{C}}.$$

(see Loustau [Lou15] for details). To be concrete and self-contained, we first explain the Darboux coordinates on χ_M^h.

Lemma 15.2. *Let $M = m_1 \sqcup m_2 \sqcup \cdots \sqcup m_{3g-3}$ be a maximal multiloop on S. Then $w_G = \Sigma_{i=1}^{3g-3} d\ell_{m_i}^{\mathbb{C}} \wedge dt_{m_i}^{\mathbb{C}}$ on χ_M^h.*

Proof. The symplectic structure w_G is a complex symplectic structure, so that the 2-form changes holomorphically in χ. On the Fricke-Teichmüller space space, $w_G|\mathcal{T}$ is given by $\Sigma d\ell_{m_i}^{\mathbb{R}} \wedge dt_{m_i}^{\mathbb{R}}$. Therefore, since the complex Fenchel-Nielsen coordinates are holomorphic coordinates (Lemma 15.1), $w_G = \Sigma d\ell_{m_i}^{\mathbb{C}} \wedge dt_{m_i}^{\mathbb{C}}$ on χ_M^h. □

Then these Darboux coordinates on χ_M^h give the symplectic property of the real bending map.

Proposition 15.3. *If M is a weighted multiloop on S, then $b_M \colon \mathcal{T} \to \chi$ is a symplectic embedding.*

Proof. As M may *not* be maximal, we pick a maximal multiloop M' on S containing M. Set $m_1, m_2, \ldots, m_{3g-3}$ to be the loops of M'. Let $w_1, w_2, \ldots w_{3g-3} \in \mathbb{R}_{\geq 0}$ be the weight of the loops of M' (so that, if ℓ_i is *not* a loop of the original multiloop M, then $w_i = 0$). The Teichmüller-Fricke space $\mathcal{T}$ is a component of the real slice of χ_M^h. In the Darboux coordinates of Lemma 15.2, the real bending map $b_M \colon \mathcal{T} \to \chi$ extends to $\hat{b}_M \colon \chi_M^h \to \chi_M^h$ by the translation

$$(\ell_1, \ldots, \ell_{3g-3}, \tau_1, \ldots, \tau_{3g-3}) \mapsto (\ell_1, \ldots, \ell_{3g-3}, \tau_1 + w_1 I, \ldots, \tau_{3g-3} + w_{3g-3} I).$$

As it is a translation in the Darboux coordinates, $b_M \colon \mathcal{T} \to \chi$ is clearly a symplectic embedding. □

By the limiting argument, all real bending maps are symplectic.

Theorem 15.4. *For every $L \in \mathcal{ML}$, $b_L\colon \mathcal{T} \to \mathcal{X}$ is a symplectic embedding w.r.t. Goldman's symplectic structure.*

Proof. Let ℓ_i be a sequence of weighted loops which converges to L in $\mathcal{ML}$ as $i \to \infty$. (Recall that $b_{\ell_i}\colon \mathcal{T} \to \mathcal{X}$ is a real-analytic embedding.) For each $\tau \in \mathcal{T}$, the tangent space of b_{ℓ_i} at τ converges to the tangent space of b_L at τ. By Proposition 15.3, $b_{\ell_i}\colon \mathcal{T} \to \mathcal{X}$ is a symplectic embedding for each $i = 1, 2, \ldots$. Therefore, by the continuity of the symplectic structure w_G, the limit b_L is also symplectic at τ. □

15.1. Symplectic property for complex bending map.

As $X_M \setminus X_M^p \to \mathcal{X} \setminus \mathcal{X}_M^p$ is an analytic covering map, $X_M \setminus X_M^p$ has a pull-back symplectic structure.

A representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ is reductive, if the Zariski-closure of the image $\mathrm{Im}\,\rho \subset \mathrm{PSL}_2\mathbb{C}$ is reductive. (That is, the maximal normal unipotent subgroup of $\mathrm{Im}\,\rho$ is the trivial group.) Then a representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ is non-reductive, if and only if $\mathrm{Im}\,\rho$ is conjugate to a subgroup consisting of upper triangular matrices which contains at least one (non-identity) parabolic element. Let X_M^r be the set of framed representations $\eta = [\rho, (u_i, v_i)]$ of X_M such that ρ is a reductive representation other than the trivial representation.

Theorem 15.5. *The restriction of B_M to $X_M^r \setminus X_M^p$ is a complex symplectic map.*

Proof. We show that the restriction of $b_M^{\pm}\colon X_M^r \to \mathcal{X}$ is symplectic on $\mathcal{X}_M^h$. For every framed representation in R_M^r, its $\mathrm{PSL}_2\mathbb{C}$-orbit is a closed subset of R_M and biholomorphic to $\mathrm{PSL}_2\mathbb{C}$. Therefore, the reductive part X_M^r is contained in the smooth part of the framed character variety X_M.

Recall that $\mathcal{X}_M^h$ is the subset of character variety $\mathcal{X}$ consisting of $\pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ such that every loop of M maps to a hyperbolic element by ρ and for every component F of $S \setminus M$, the restriction of ρ to the fundamental group of F is irreducible.

Let X_M^h denote the subset of X_M^r consisting of framed representations whose representations are in $\mathcal{X}_M^h$. Then X_M^h is a (Euclidean) open dense full-measure subset of X_M. The complex bending map B_M is symplectic on X_M^h by Lemma 15.2. Therefore, by continuity, B_M is symplectic on $X_M^r \setminus X_M^p$. □

$$\begin{array}{ccc} X_\ell \setminus X^p_\ell & \xrightarrow{b_M} & \chi \setminus \chi^p \subset \chi \\ & \overset{B_M}{\searrow} & \big\downarrow b^-_M \times b^+_M \\ & & \chi \times \chi \end{array}$$

Figure 11. A local commutative diagram describing the complexification of the real bending map.

16. The general complex bending variety

Let L be a non-empty measured lamination on S. Let ℓ_i be a sequence of non-separating weighted oriented loops on S converging to L as $i \to \infty$. By Corollary 14.2, the image of $B_{\ell_i}: X_{\ell_i} \to \chi \times \chi$ is a closed complex-analyticsubset of $\chi \times \chi$.

Theorem 16.1. *The analytic set* $\operatorname{Im} B_{\ell_i}$ *converges, up to a subsequence, to a closed complex-analytic subset of* $\chi \times \chi$ *as* $i \to \infty$.

Proof. By Theorem 15.5, the bending maps $b^{\pm}_{\ell_i}: X_{\ell_i} \to \chi$ is a complex symplectic mapping on $X^r_{\ell_i} \setminus X^p_M \to \chi$.

Claim 16.2. *Let ℓ be an essential simple closed curve with weight w not equal to π modulo 2π. Then $b^{\pm}_\ell: X_\ell \to \chi$ is two-to-one mapping on $X^r_{\ell_i} \setminus X^p_{\ell_i}$.*

Proof. Let $\rho: \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ be a representation in $\chi^r_\ell \setminus \chi^p_\ell$. Let $\alpha \in \pi_1(S)$ be an element representing ℓ. As $\rho(\alpha)$ is *not* a parabolic element or the identity, pick a framing (u, v) of ℓ, where u, v are the fixed points of $\rho(\alpha)$.

Since b^+_ℓ and b^-_ℓ bend each representation in opposite directions, they are inverse to each other, when the framing is kept: Namely b^+_ℓ takes $b^-(\rho, (u, v)) \in \chi$ with the framing (u, v) back to $(\rho, (u, v)) \in X_\ell$. Similarly b^-_ℓ takes $b^+(\rho, (u, v)) \in \chi$ with the framing (u, v) back to $(\rho, (u, v)) \in X_\ell$. Therefore the inverse image $(b^+_\ell)^{-1}(\rho)$ consists of $(b^-(\rho, (u, v)), (u, v))$ and $(b^+(\rho, (u, v)), (v, u))$. Moreover, the above inverse relation of b^+_ℓ and b^-_ℓ implies that there are no other framed representations mapping to ρ by b^+_ℓ. Hence b^+_ℓ is a two-to-one mapping on $X^r_{\ell_i} \setminus X^p_{\ell_i}$.

One can similarly prove that b^-_ℓ is a two-to-one mapping on $X^r_{\ell_i} \setminus X^p_{\ell_i}$. □

As B_{ℓ_i} is complex symplectic almost everywhere, it preserves the complex volume (i.e. Jacobian is one). Therefore, since B_{ℓ_i} is a two-to-one mapping almost everywhere (see §8.3), the volume of the analytic set $\operatorname{Im} B_{\ell_i}$ is locally finite in $\mathcal{X} \times \mathcal{X}$ and uniformly bounded in i. Hence, up to a subsequence, the closed $\mathbb{C}$-analytic set $\operatorname{Im} B_{\ell_i}$ converges to a closed $\mathbb{C}$-analytic set in $\mathcal{X} \times \mathcal{X}$ as $i \to \infty$ by Bishop's theorem [Chi89, Corollary in p205]). 16.1

Remark 16.3. *Since* $\operatorname{Im} B_{\ell_i}$ *is symplectic in the smooth part, the closed* $\mathbb{C}$*-analytic set in the limit is also* $\mathbb{C}$*-symplectic in the smooth part.*

Let $\mathcal{QF}$ be the quasi-Fuchsian space, which contains the Fricke space $\mathcal{T}$. Then, the domain X_{ℓ_i} of B_{ℓ_i} contains $\mathcal{QF}$ for all $i = 1, 2, \ldots$.

Let $\mathcal{QF}_i$ be the open subset of X_{ℓ_i} so that the restriction of $b^+_{\ell_i}$ to the Fuchsian space $\mathcal{T}$ in $\mathcal{QF}_i$ is the real bending map b_{ℓ_i}. Moreover, L is realizable for all quasi-Fuchsian representations, i.e. there is a ρ-equivariant pleated surface $\tilde{S} \to \mathbb{H}^3$ whose pleating lamination contains the geodesic lamination supporting L. Therefore the $\mathbb{R}$-analytic bending map $b_L \colon \mathcal{T} \to \mathcal{X}$ extends to a holomorphic mapping $b_L \colon \mathcal{QF} \to \mathcal{X}$. Similarly to the complex bending map $B_M \colon X_M \to \mathcal{X} \times \mathcal{X}$ for a weighted multiloop, we can define $B_L \colon \mathcal{QF} \to \mathcal{X} \times \mathcal{X}$ by bending $\rho \colon \mathcal{QF} \to \mathcal{X} \times \mathcal{X}$ by L and by $-L$,

$$\rho \mapsto (b_L(\rho), b_{-L}(\rho)).$$

Then B_L complex analytically embeds $\mathcal{QF}$ into $\mathcal{X} \times \mathcal{X}$. Therefore $B_{\ell_i} | \mathcal{QF}_i$ converges to $B_L | \mathcal{QF}$ as $i \to \infty$. By the identity theorem for analytic sets ([FG02, §5.1.1]), the limit of Theorem 16.1 contains the canonical irreducible component $\mathcal{B}_L$ which contains $B_L(\mathcal{QF})$.

Corollary 16.4. *The irreducible component* $\mathcal{B}_L$ *containing the real bending map image* $\psi \circ b_L(\mathcal{T})$ *is independent of the choice of the sequence* ℓ_i *converging to* L *and the subsequence in Theorem 16.1.*

We obtained a unique irreducible closed complex-analytic set $\mathcal{B}_L$ in $\mathcal{X} \times \mathcal{X}$ containing the real-analytic subvariety $\psi \circ b_L(\mathcal{T})$, and it is symplectic on the smooth part. We finished the proof of Theorem E.

For a general measured lamination L, the complex bending variety $\mathcal{B}_L$ is constructed as the limit as above. Since L is realizable for all quasi-Fuchsian representations, $\mathcal{B}_L$ contains quasi-Fuchsian representations bent in opposite directions along L analogues to the complex bending map along a weighted multiloop defined in §9.2. We can more generally hope that generic representations in $\mathcal{B}_L$ have similar properties.

Question 16.5. *Let* L *be a measured lamination on* S *(containing a non-periodic leaf). Let* $\mathcal{B}_L$ *be the complex bending variety of* L *in* $\mathcal{X} \times \mathcal{X}$*. Let* $\eta = (\rho_1, \rho_2)$ *be a generic point in* $\mathcal{B}_L$*. Are there a* ρ_1*-equivariant pleated surface* $\beta_1 \colon \mathbb{H}^2 \to \mathbb{H}^3$ *and a* ρ_2*-equivariant pleated* *surface* $\beta_2\colon \mathbb{H}^2 \to \mathbb{H}^3$ *both realising* $|L|$*, such that* ρ_2 *is obtained by bending* ρ_1 *along the geodesic lamination* $|L|$*?*

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