BENDING TEICHMÜLLER SPACES AND CHARACTER VARIETIES

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ABSTRACT. We consider the mapping $b_L\colon \mathcal{T} \to \chi$ of the Fricke-Teichmüller space $\mathcal{T}$ into the $\mathrm{PSL}_2\mathbb{C}$-character variety χ of the surface, obtained by holonomy representations of bent hyperbolic surfaces along a fixed measured lamination L. We prove that this mapping is an equivariant symplectic real-analytic embedding and, for most of the measured laminations, proper. Therefore b_L is

- a reminiscent of an equivariant pleated surface $\mathbb{H}^2 \to \mathbb{H}^3$, and moreover
- an analogue of a Poincaré holonomy variety ($\mathfrak{sl}_2\mathbb{C}$-oper) in the Thurston parametrization of $\mathbb{C}\mathrm{P}^1$-structures.

In addition, if the measured lamination L is a weighted multiloop, we construct a complexification of b_L, using the product variety $\chi \times \chi$, by a new type of bending deformation, so that this complexification retains similar properties.

CONTENTS

1. Introduction

Thurston discovered the bent hyperbolic surfaces τ on the boundary of the convex core of a (geometrically finite) hyperbolic three-dimensional manifolds ([Thu81]). Indeed, the intrinsic metric of the convex surface is hyperbolic, and the surface is bent along a measured lamination, where the bending angles correspond to the transversal measure of the lamination. Such bent surfaces are particularly useful for capturing the global properties of the hyperbolic manifold.

Lifting the convex surface τ to the universal cover $\mathbb{H}^3$ of the hyperbolic manifold, we obtain an equivariant bending $\mathbb{H}^2 \to \mathbb{H}^3$ which preserves the (intrinsic) hyperbolic metric of the surface. Then, this bending map is equivariant via a holonomy representation of a surface group into $\mathrm{PSL}_2\mathbb{C}$. Moreover, if τ is π_1-injective (equivalently incompressible) in the ambient hyperbolic 3-manifold, then the bending map $\mathbb{H}^2 \to \mathbb{H}^3$ is a proper embedding.

In this paper, utilizing the bending construction, moreover, in a new generalized manner, we construct similar equivariant geometry-preserving mappings, in fact, at the level of associated deformations spaces.

1.1. Holonomy varieties.

Let Y be a marked Riemann surface structure on a closed oriented surface S of genus g at least two. Let $\mathrm{QD}(Y)$ be the space of the holomorphic quadratic differentials on Y, which is a complex vector space of dimension $3g-3$. Then $\mathrm{QD}(Y)$ is identified with the space $\mathcal{P}_Y$ of all $\mathbb{C}\mathrm{P}^1$-structures on Y, and this correspondence yields the *Schwarzian parameterization* of $\mathbb{C}\mathrm{P}^1$-structures (see [Dum09] for example).

Let

$$\mathrm{Hol}\colon \mathcal{P} \to \mathcal{X}$$

be the holonomy map from the deformation space $\mathcal{P}$ of all $\mathbb{C}\mathrm{P}^1$-structures on S to the the $\mathrm{PSL}_2\mathbb{C}$-character varieties $\mathcal{X}$ of S. Recall that the character variety $\mathcal{X}$ is an affine algebraic variety, and it has Goldman's complex symplectic structure invariant under the action of the mapping class group; see [Gol84]. Many interesting properties of this mapping, associated with the Schwarzian parametrization, have been discovered, and particularly the following holds.

Theorem 1.1. *The restriction of the holonomy map to $\mathcal{P}_Y \cong \mathrm{QD}(Y)$ is a proper Lagrangian complex-analytic embedding into $\mathcal{X}$.*

On the other hand, the entire holonomy map $\mathrm{Hol}\colon \mathcal{P} \to \mathcal{X}$ of $\mathbb{C}\mathrm{P}^1$-structures is neither injective nor proper (see [Hej75]).

The injectivity of Theorem 1.1 is due to Poincaré [Poi84]. The properness is due to Kapovich [Kap95] (see [GKM00] for the full proof; see also [Dum17, Tan99]). The Lagrangian property is proven by Kawai [Kaw96].

By Theorem 1.1, for every marked Riemann surface structure Y, the vector space $QD(Y) \cong \mathbb{C}^{3g-3}$ is property embedded onto a half-dimensional smooth subvariety of χ. We call this image, associated with the Schwarzian parametrization, the Poincaré holonomy variety of Y. In particular, the holonomy variety of Y contains the Bers slice of Y as a bounded pseudo-convex domain.

1.2. **Real bending varieties.** Recall that $\mathbb{C}\mathrm{P}^1$ is the ideal boundary of the hyperbolic three-space $\mathbb{H}^3$, and the automorphism group $\mathrm{PSL}_2\mathbb{C}$ of $\mathbb{C}\mathrm{P}^1$ is identified with the group of orientation preserving isometries of $\mathbb{H}^3$. Utilizing this correspondence in a sophisticated manner, Thurston gave another parametrization of $\mathcal{P}$, so that $\mathbb{C}\mathrm{P}^1$-structures correspond to equivariant pleated surfaces in $\mathbb{H}^3$ (§3.1.1). In this paper, we first yield an analogue of Theorem 1.1 by specific slices in the Thurston parametrization of $\mathbb{C}\mathrm{P}^1$-structures.

In fact, Tanigawa [Tan97], Wolf-Scannel [SW02], Dumas-Wolf [DW08] considered the $\mathbb{C}\mathrm{P}^1$-structures with a fixed bending measured lamination and analyzed their conformal structures. In this paper, as in the holonomy variety, we instead consider the holonomy representation of those $\mathbb{C}\mathrm{P}^1$-structures.

For a measured lamination L on a hyperbolic surface τ, we obtain an equivariant pleated surface in $\mathbb{H}^3$ by bending the universal cover of τ, the hyperbolic plane $\mathbb{H}^2$ along the inverse-image $\tilde{L}$ of L in $\mathbb{H}^2$, and the pleated surface $\tilde{\tau} \cong \mathbb{H}^2 \to \mathbb{H}^3$ is equivariant via a representation $\pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$. (See §3.1 for details.) Let $\mathcal{T}$ be the space of marked hyperbolic structures on S, the Fricke-Teichmüller space; then $\mathcal{T}$ is diffeomorphic to $\mathbb{R}^{6g-6}$ as a smooth manifold. The Weil-Peterson form gives a symplectic structure on $\mathcal{T}$, and Goldman extended it to a complex-symplectic structure on χ ([Gol84]). For a measured lamination L on S, let $b_L \colon \mathcal{T} \to \chi$ be the map taking $\tau \in \mathcal{T}$ to the holonomy representation $\pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ of the pleated surface given by τ and L.

This mapping is closely related to the Thurston parametrization of $\mathcal{P}$ (Theorem 3.1), and the following theorem is an analogue of Theorem 1.1 in the Thurston parametrization.

Theorem A (Theorems 4.1, 14.4, Lemma 3.2)**.** *Let L be an arbitrary measured lamination on S. Then, the bending map $b_L\colon \mathcal{T} \to \mathcal{X}$ is a real-analytic symplectic embedding, and it is equivariant by the subgroup of the mapping class group $\mathcal{G}_L$ of S preserving L.*

Moreover, b_L is proper if and only if L contains no periodic leaves of weight π modulo 2π.

On the other hand, the conservation of the symplectic structure of $\mathcal{T}$ by b_L resembles the conservation of the hyperbolic metric by the bending map $\mathbb{H}^2 \to \mathbb{H}^3$, and the equivariant property resembles that of the bending map. Moreover, by Theorem A, the real bending map b_L is a proper mapping for almost all measured laminations L. In addition, for exceptional laminations, we explicitly characterize the non-properness in the Fenchel-Nielsen coordinates (Theorem 6.1).

The stabilizer $\mathcal{G}_L$ can be a large subgroup and, on the other hand, can be the trivial subgroup of the mapping class group MCG depending on $L \in \mathcal{ML}$ (Remark 3.3).

1.3. **Complex bending varieties.** Historically, a real analytic deformation determined by a measure lamination or a measured foliation (an equivalent object) has a significant complexification: A Teichmüller geodesic in the Teichmüller space $\mathcal{T}$ is determined by a measured foliation on a Riemann surface, and its complexification is a Teichmüller disk in $\mathcal{T}$. A measured lamination on a hyperbolic surface yields a real-analytic earthquake line in $\mathcal{T}$ ([Thu86, Ker85]), and an earthquake disk is its complexification ([McM98]).

We aim to geometrically complexify the real-analytic embedding $b_L\colon \mathcal{T} \to \mathcal{X}$ in Theorem A, and obtain a complex-analytic mapping from a closed complex analytic variety. It is plausible that such complexifications of the real bending varieties $\operatorname{Im} b_L$ in a common analytic space will lead us to discover intersecting properties of the original real analytic varieties.

We first explain the domain of the complexified bending map. Given a representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$, if a holonomy $\rho(\ell) \in \mathrm{PSL}_2\mathbb{C}$ along a loop ℓ is either hyperbolic or elliptic, then one can certainly bend ρ along ℓ as the axis of $\rho(\ell)$ gives the axis of bending deformation. However, it is *not* clear if one can bend if $\rho(\ell)$ is parabolic or the identity.

Thus, given a weighted oriented multiloop M on S, we introduce an appropriate closed analytic set X_M consisting of certain (double) framed representations, so that the framing determines the bending axes even when the holonomy along some loops of M is trivial (§7).

In fact, this modification of $\mathcal{X}$ essentially occurs only in a complex-analytic subvariety of $\mathcal{X}$ disjoint from $\mathcal{T}$, so that the map forgetting the framing induces a finite-to-one covering map from X_M to $\mathcal{X}$ when specific subvarieties are removed from X_M and $\mathcal{X}$. (To be more precise, X_M is a finite covering map of a blow-up of the complement of some subvariety of $\mathcal{X}$.) In particular, there is a canonical embedding of the Fricke-Teichmüller space $\mathcal{T}$ into X_M as a real-analytic smooth subvariety. In addition, we can pull back the complex symplectic structure on $\mathcal{X}$ to X_M minus a subvariety.

We next explain the target space. Notice that the Fricke-Teichmüller space $\mathcal{T}$ is a component of the real slice of the character variety $\mathcal{X}$. Moreover, the real bending map $b_L\colon \mathcal{T} \to \mathcal{X}$ is *totally real* in the complex affine variety $\mathcal{X}$ (i.e. its tangent spaces contain no complex lines). Therefore, in order to obtain nontrivial complexifications and also to obtain different complexifications (as images) for different bending laminations, it is necessary to enlarge the ambient space.

When the $\mathrm{PSL}_2\mathbb{C}$-Lie algebra $\mathfrak{psl}_2\mathbb{C}$ is regarded as a real Lie algebra, its complexification is isomorphic to $\mathfrak{psl}_2\mathbb{C} \oplus (\mathfrak{psl}_2\mathbb{C})^*$, where $*$ denotes the complex conjugate. Thus, for a representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$, we consider the diagonal representation $\pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C}$ twisted by conjugation, defined by $\gamma \mapsto (\rho(\gamma), \rho(\gamma)^*)$. Then, given a representation framed along loops of M, we can appropriately bend it along the axes determined by their framings, where the bending happens in the space of representations into $\mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C}$. Then we obtain the complex bending map $B_M\colon X_M \to \mathcal{X} \times \mathcal{X}$. (See §8 for details.) Let

$$\Delta^* = \{(\rho_1, \rho_2)\colon \pi_1(S) \to \mathcal{X} \times \mathcal{X} \mid \rho_1 = \rho_2^*\},$$

the anti-holomorphic diagonal in $\mathcal{X} \times \mathcal{X}$. Define $\psi\colon \mathcal{X} \to \Delta^* \subset \mathcal{X} \times \mathcal{X}$ by $\rho \mapsto (\rho, \rho^*)$.

Theorem B (Complexification)**.** *Let M be an oriented weighted multiloop on S. Then $B_M\colon X_M \to \mathcal{X} \times \mathcal{X}$ is a complex analytic mapping, such that*

1. *the restriction of B_M of $\mathcal{T}$ is a real-analytically embeds into Δ^*;*
2. *$\psi \circ b_M\colon \mathcal{T} \to \mathcal{X} \times \mathcal{X}$ coincides with the restriction of B_M to $\mathcal{T}$ (Figure 1);*
3. *B_M is complex-symplectic in the complement of a subvariety of X_ℓ;*
4. *B_M is equivariant by the action of the subgroup of the mapping class group preserving M.*

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{b_M} & \chi \\ \big\downarrow & & \big\downarrow \psi \\ X_M & \xrightarrow{B_M} & \chi \times \chi \end{array}$$

FIGURE 1. The commutative diagram for the complexification B_M of the real analytic bending map b_M.

(The complex-analyticity is proven in Theorem 11.1. For (1), see Proposition 12.1. For (2), see Proposition 12.1. For (3), see Theorem 14.5; For (4), see Lemma 8.2.) we remark that the removed subvariety in (3) consists of the framed representations such that at least one loop of M has trivial holonomy.

The complex bending map B_M is *not* proper or injective in general. However, B_M is injective and proper "almost everywhere": If an analytic subset is removed from the domain X_M and a subvariety is removed from the target $\chi \times \chi$, then B_M becomes injective and proper (Theorem 9.1, Theorem 10.1). Indeed, in certain cases, the complex bending map is genuinely proper.

Theorem C. *If ℓ is a weighted oriented non-separating loop of weight not equal to π modulo 2π, then, the bending map $B_\ell \colon X_\ell \to \chi \times \chi$ is a proper mapping. (Theorem 13.1.)*

Therefore, under the assumption of Theorem C, the image of B_ℓ is a closed analytic subvariety in $\chi \times \chi$ (complex bending variety). Moreover, it is plausible that such subvarieties are well-defined for all measured laminations on S:

Conjecture 1.2. *For every measured lamination L on S, let ℓ_i be a sequence of non-separating weighted loops converging to L. Then the closed analytic set $\operatorname{Im} B_{\ell_i}$ converges to a closed analytic set in $\chi \times \chi$ as $i \to \infty$.*

1.4. **Outline of the paper.** The preliminary section (§3) explains some basic notions for this paper. In particular, we recall that a measured lamination on a hyperbolic surface induces an equivariant locally convex pleated surface $\mathbb{H}^2 \to \mathbb{H}^3$, then we define the real bending map $b_L \colon \mathcal{T} \to \chi$ for a measured lamination. In §4, we show the injectivity of the real bending map. In §5, we prove the properness of the real bending map for most of the measured laminations L. On the other hand, in §6.1, for particular types of measured laminations, we characterize the non-properness of the bending map.

In §7, we introduce the space of representations double-framed along a weighted oriented multiloop M on S (the framed character variety X_M). Then, in §8, we define the complex bending map from the framed character variety χ_M to the product character variety $\chi \times \chi$. For the definition, a more general type of bending deformation is introduced In fact, when a representation framed along M is bent along M, accordingly, the hyperbolic space $\mathbb{H}^3$ is equivariantly "bent" inside the $\mathbb{H}^3 \times \mathbb{H}^3$ (§8.4). In §9, we show that the complex bending map is injective almost everywhere. In §10, we show the complex bending map is a proper mapping almost everywhere. In §11, using the "almost-everywhere" injectivity, we prove the analyticity of the complex bending map on the entire domain. In §12, we show that the complex bending map is a complexification of the real bending map. In §13, we show that the complex bending map is, indeed, genuinely a proper mapping when M is a single non-separating loop of the weight *not* equal to π.

Lastly, in §14, we prove the real is symplectic and the complex bending map is complex symplectic.

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3. Preliminaries

3.1. **Bending deformation.** ([Thu81], [EM87].) Thurston discovered that the boundary of the convex core of a hyperbolic-three manifold is a hyperbolic surface bent along a measured lamination ([Thu81]). More generally, one can bend a hyperbolic surface along an arbitrarily measured lamination and obtain a holonomy representation from the surface fundamental group into $\mathrm{PSL}_2\mathbb{C}$ as follows.

We shall first describe basic bending maps when the bending locus is a single loop. Let τ be a hyperbolic structure on S, and let ℓ be a geodesic loop on τ with weight $w \geq 0$. The union $\tilde{\ell}$ of all lifts of ℓ to the universal cover $\mathbb{H}^2$ of τ is a set of disjoint geodesics, each with weight w, and it is invariant under the deck transformation. We call the union $\tilde{\ell}$ the total lift of ℓ.

Put the universal cover $\mathbb{H}^2$ in the three-dimensional hyperbolic space $\mathbb{H}^3$ as a totally geodesic hyperbolic plane. By this embedding, the isometric deck transformations of $\mathbb{H}^2$ extend to an isometric action on $\mathbb{H}^3$, and we obtain a representation of $\rho_\tau \colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$. Note that, as S is oriented, the orientation of the universal cover $\mathbb{H}^2$ determines a normal direction of the plane. Then we can bend $\mathbb{H}^2$ along every geodesic α of $\tilde{\ell}$ by angle w so that the normal direction is in the exterior. Thus we obtain a bending map $\beta\colon \mathbb{H}^2 \to \mathbb{H}^3$, which is totally geodesic on every complement of $\mathbb{H}^2 \setminus \tilde{\ell}$. The map β is unique up to an orientation-preserving isometry of $\mathbb{H}^3$. Moreover, β is equivariant by its holonomy representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$. This ρ is called a bending deformation of ρ_τ.

If C_1, C_2 are components of $\mathbb{H}^2 \setminus \tilde{\ell}$ such that C_1, C_2 are adjacent along a geodesic α of $\tilde{\ell}$. Let G_1 and G_2 be the subgroups of $\pi_1(S)$ which preserve C_1 and C_2, respectively. If β is normalized so that $\beta_\tau = \beta$ on C_1, then the restriction of β to G_2 is the conjugation of the restriction of ρ_τ to G_2 by the elliptic isometry with the axis α by angle w.

More generally, given an arbitrary measured lamination L on τ, we can take a sequence of weighted loops ℓ_i converging to L as $i \to \infty$. For each i, let $\rho_i\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ be the bending deformation of ρ_τ along ℓ_i. Then ρ_i converges to a representation $\pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ as $i \to \infty$ if ρ_i are appropriately normalized by $\mathrm{PSL}_2\mathbb{C}$. This limit is the bending deformation of ρ_τ along L, and it is unique up to conjugation by an element of $\mathrm{PSL}_2\mathbb{C}$.

3.1.1. *Equivariant property of the real bending map.* The equivariant property of $b_L\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ in Theorem A can directly be proven from the definition of the bending map. Here we give a proof in a broader context.

A $\mathbb{C}\mathrm{P}^1$-structure on S is a $(\mathbb{C}\mathrm{P}^1, \mathrm{PSL}_2\mathbb{C})$-structure. That is, an atlas of charts mapping open subsets of S into $\mathbb{C}\mathrm{P}^1$ with translation maps in $Aut(\mathbb{C}\mathrm{P}^1) = \mathrm{PSL}_2\mathbb{C}$. (General references about $\mathbb{C}\mathrm{P}^1$-structures are[Dum09, Kap01, Gol22]). Recall that $\mathbb{C}\mathrm{P}^1$ is the ideal boundary of the hyperbolic space $\mathbb{H}^3$, and $\mathrm{PSL}_2\mathbb{C}$ is the group of orientation-preserving isometrics of $\mathbb{H}^3$. Using equivariant bending maps described above, Thurston gave a parametrization of the deformation space $\mathcal{P}$ of $\mathbb{C}\mathrm{P}^1$-structures by corresponding them with holonomy-equivariant pleated surfaces in $\mathbb{H}^3$.

Theorem 3.1 (Thurston, [KP94, KT92])**.**

$$\mathcal{P} = \mathcal{T} \times \mathcal{ML}.$$

Then $b_L(\tau) = \mathrm{Hol}(\tau, L)$ where $(\tau, L) \in \mathcal{T} \times \mathcal{ML}$ denote the $\mathbb{C}\mathrm{P}^1$-structure in Thurston coordinates.

Lemma 3.2. *For $L \in \mathcal{ML}$, let $\mathcal{G}_L$ be the subgroup of* MCG *which preserves L. Then, the real bending map $b_L\colon \mathcal{T} \to \mathcal{X}$ is $\mathcal{G}_L$-equivariant.*

Remark 3.3. *If L is a multiloop, then $\mathcal{G}_L$ contains the subgroup of* MCG *generated by Dehn twists along loops not intersecting L (but including the loops of L). On the other hand, for almost all L in $\mathcal{ML}$, $\mathcal{G}_L$ is the trivial group, since* MCG *is a countable group.*

Proof. The MCG-action on $\mathcal{P}$ is given by marking change and on $\mathcal{X}$ by precomposing induces isomorphisms $\pi_1(S) \to \pi_1(S)$. Then the holonomy map $\mathrm{Hol}\colon \mathcal{P} \to \mathcal{X}$ is MCG-equivariant (see, for example, [Gol06]).

By the Thurson's parametrization, For $\tau \in \mathcal{T}$ and $h \in \mathrm{MCG}$, $h(\tau, L) = (\tau, L)$.

$$h \cdot b_L(\tau) = h \cdot \mathrm{Hol}(\tau, L) = \mathrm{Hol}(h, L) = b_L(h\tau).$$

Thus the desired equivariant property holds. □

3.2. **Quasi-geodesics in the hyperbolic space.** We first recall the definition of quasi-isometries. Let $(X, d_X), (Y, d_Y)$ be metric spaces, where d_X, d_Y are the distance functions. Then, for $P, Q > 0$, a mapping $f\colon X \to Y$ is a (P, Q)-*quasiisometry* if, for all $x_1, x_2 \in X$,

$$P^{-1} d_X(x_1, x_2) - Q < d_Y(f(x_1), f(x_2)) < P\, d_X(x_1, x_2) + Q.$$

In this section, we discuss certain conditions for a piecewise geodesic curve in $\mathbb{H}^3$ to be a quasi-geodesic.

3.2.1. *Quasi-geodesics in $\mathbb{H}^3$.* Let c be a bi-infinite piecewise geodesic curve in $\mathbb{H}^3$. Let $s_i\,(i \in \mathbb{Z})$ be the maximal geodesic segments of c indexed along c, so that s_i and s_{i+1} are adjacent geodesic segments for every $i \in \mathbb{Z}$ and $c = \cup_{i\in\mathbb{Z}} s_i$.

Lemma 3.4. *For every $\epsilon > 0$, there are $R > 0$ and $\delta > 0$, such that if* $\mathrm{length}\, s_i > R$ *for all $i \in \mathbb{Z}$ and the angle between arbitrary adjacent geodesic segment s_i, s_{i+1} is at least $\pi - \delta$, then c is a $(1 + \epsilon)$-bilipschitz embedding.*

Proof. This lemma follows from [CEG87, I.4.2.10]. □

Proposition 3.5. *If every $\epsilon > 0$ and $\epsilon' > 0$, there are $R > 0$ and $Q > 0$, such that if* $\mathrm{length}\, s_i > R$ *for all $i \in \mathbb{Z}$ and the angle between arbitrary every pair of adjacent geodesic segments is at least ϵ', then c is an $(1 + \epsilon, Q)$-quasi-isometric embedding.*

Proof. For each $i \in \mathbb{Z}$, let x_i be the common endpoint of s_{i-1} and s_i, so that x_i is a non-smooth point of c. Let $0 < r < R/2$. Let x_i^- be the point on s_{i-1} such that $d(x_i^-, x_i) = r$. Let x_i^+ be the point on s_i such that $d(x_i, x_i^+) = r$. Then, we replace two geodesic segments $[x_i^-, x_i] \cup [x_i, x_i^+]$ of c with the single geodesic segment $[x_i^-, x_i^+]$. Let c_r be the piecewise geodesic in $\mathbb{H}^3$ obtained from c by applying this replacement for every $i \in \mathbb{Z}$.

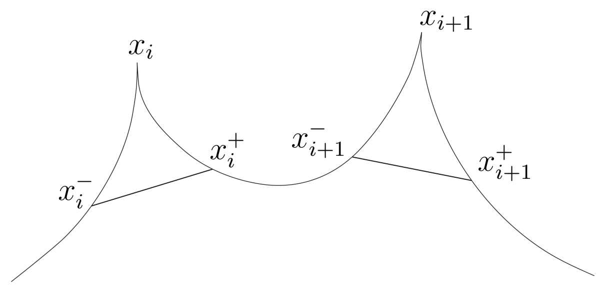

FIGURE 2.

By basic hyperbolic geometry, the following holds.

Lemma 3.6. *For every $\delta > 0$, if $r > 0$ is sufficiently large, then the angle at every non-smooth point of c_r is more than $\pi - \delta$.*

Then Lemma 3.4 and Lemma 3.6 imply the proposition. 3.5

3.3. **Complex analytic geometry.** ([Gol84].) We recall a standard theorem about a complex analytic set.

Theorem 3.7 (Removable Singularity Theorem; see for example [Tay02], §3.3.2)**.** *Let Y be an analytic set. Let A be a closed subset of Y contained in a proper subvariety of Y. Suppose that $f\colon Y \setminus A \to \mathbb{C}$ is an analytic function which is bounded in a small neighborhood of every point in A. Then f continuously extends to an analytic function on Y.*

3.4. **Goldman's symplectic form.** ([Gol84]) Let $\mathfrak{g}$ be the $\mathrm{PSL}_2\mathbb{C}$-lie algebra. Then the adjoint representation $\mathrm{Ad}\colon \mathrm{PSL}_2\mathbb{C} \to Aut\mathfrak{g} \subset \mathrm{GL}_3\mathbb{C}$ is induced by the conjugation of $\mathrm{PSL}_2\mathbb{C}$ by $\mathrm{PSL}_2\mathbb{C}$. By $\mathfrak{g}_{\mathrm{Ad}\,\rho}$, we regard $\mathfrak{g}$ as a $\pi_1(S)$-module via the composition of $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$. Then the Zariski tangent space of the representation variety $\mathcal{R}$ at $\rho \in \mathcal{R}$ is Then the vector space of 1-cocycles

$$Z^1(\pi_1(S); \mathfrak{g}_{Ad\rho}) = \{u \in \mathfrak{g}^{\pi_1(S)} \mid u(xy) = u(x) + (\mathrm{Ad}\rho(x))\, u(y)\}.$$

The subspace of 1-coboundaries $B^1(\pi_1(S); \mathfrak{g}_{Ad\rho})$ consists of $u \in \mathfrak{g}^{\pi_1(S)}$, such that there is $u_0 \in \mathfrak{g}$ satisfying $u(x) = u_0 - \mathrm{Ad}(\rho(x))u_0$ for all $x \in \pi_1(S)\}$. Then the Zariski tangent space of $\mathcal{X}$ at ρ is the quotient vector space

$$H^1(\pi_1(S); \mathfrak{g}_{Ad\rho}) = \frac{Z^1(\pi_1(S); \mathfrak{g}_{Ad\rho})}{B^1(\pi_1(S); \mathfrak{g}_{Ad\rho})}.$$

Let $w(\rho)$ denote the bilinear form on the Zariski tangent space obtained by the composition

$$\begin{aligned} H^1(\pi_1(S); \mathfrak{g}_{Ad\rho}) \times H^1(\pi_1(S); \mathfrak{g}_{Ad\rho}) &\xrightarrow{\cup} H^2(\pi_1(S); \mathfrak{g}_{Ad\rho} \otimes \mathfrak{g}_{Ad\rho}) \\ &\xrightarrow{\cong} H^2(\pi_1(S); \mathbb{C}) \cong \mathbb{C}. \end{aligned}$$

Here the first mapping is the cup product, and the second mapping is an isomorphism given by the coefficients pairing by the bilinear form $\mathfrak{B}\colon \mathfrak{g}_{Ad\rho} \otimes \mathfrak{g}_{Ad\rho} \to \mathbb{C}$ given by $(A, B) \to \operatorname{tr} AB$. Goldman proved that w is a complex-symplectic form on $\mathcal{X}$, i.e. a non-degenerate closed holomorphic $(2, 0)$-form on the character variety $\mathcal{X}$; see [Gol84].

4. Injectivity of the real bending maps

Let $\mathcal{ML}$ be the space of measured laminations on S. Each pair $(\tau, L) \in \mathcal{T} \times \mathcal{ML}$ induces an equivariant pleated surface $\mathbb{H}^2 \to \mathbb{H}^3$, unique up to $\mathrm{PSL}_2\mathbb{C}$. Let $b\colon \mathcal{T} \times \mathcal{ML} \to \mathcal{X}$ be the holonomy map of the bending maps.

Theorem 4.1. *Fix arbitrary $L \in \mathcal{ML}(S)$. Then the restriction b to $\mathcal{T} \times \{L\}$ is a real-analytically embedding. Moreover, this embedding is proper if and only if L contains no periodic leaf of weight π modulo 2π.*

Let $b_L\colon \mathcal{T} \to \mathcal{X}$ denote the restriction of b to $\mathcal{T} \times \{L\}$. Given a representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$, geodesic lamination λ on S is *realizable* if there is a ρ-equivariant pleated surface $\mathbb{H}^2 \to \mathbb{H}^3$, such that its pleating loci contains λ. Then, for $L \in \mathcal{ML}$, let $N = N_L$ be an open neighborhood of the Fuchsian space $\mathcal{T}$ in the smooth part of $\mathcal{X}$ such that the underlying geodesic lamination $|L|$ is realizable for all $\rho \in \mathcal{X}$. Then, $b_L\colon \mathcal{T} \to \mathcal{X}$ extends to the bending map $\hat{b}_L\colon N_L \to \mathcal{X}$ by bending cocycle ([Bon96]).

Proposition 4.2. *For all $L \in \mathcal{ML}$, $\hat{b}_L\colon N_L \to \mathcal{X}$ is injective.*

Proof. As $|L|$ is realizable on $\operatorname{Im} \hat{b}_L$, we have the unbending map $\hat{b}_{-L}\colon \operatorname{Im} \hat{b}_L \to \mathcal{X}$ by $-L$. Then, clearly, $\hat{b}_{-L} \circ \hat{b}_L$ is the identity map on N_L. Thus $\hat{b}_L$ is injective. □

Proposition 4.3. *The injective map $b_L\colon \mathcal{T} \to \mathcal{X}$ is a real-analytic embedding.*

Proof. (cf. [Ker85].) We regard $\mathcal{T}$ as the Fricke space, i.e. the space of discrete faithful representations into $\mathrm{PSL}(2, \mathbb{R})$ up to conjugation by $\mathrm{PSL}_2\mathbb{R}$. Then, take a small open neighborhood N of $\mathcal{T}$ whose closure is contained in N_L.

If L is a weighted multiloop, the bending map b_L is holomorphic on N as bending transforms the holonomy along a loop by some elliptic elements in a holomorphic manner. In general, pick a sequence of weighted multiloops M_i converging to L as $i \to \infty$. By the injectivity of Proposition 4.2, $\hat{b}_{M_i}\colon N_{M_i} \to N_{M_i}$ is a holomorphic embedding. Then, the holomorphic embedding $\hat{b}_{M_i}|N$ converges uniformly to $b_L|N$ uniformly on compacts as $i \to \infty$. Therefore $\hat{b}_L|N$ is a holomorphic embedding.

Since $\mathcal{T}$ is a real-analytic submanifold of N in $\mathcal{X}$, thus $b_L|\mathcal{T}$ is a real-analytic embedding. □

5. Properness of the bending maps from the Teichmüller spaces

Theorem 5.1. *Let $L \in \mathcal{ML}$. Then, the bending map $b_L\colon \mathcal{T} \to \mathcal{X}$ is proper if and only if L contains no leaves of weight π modulo 2π.*

First, we prove the sufficiency of the condition in Theorem 5.1.

Lemma 5.2. *Fix $L \in \mathcal{ML}$ such that every closed leaf of L contains no leaves of weight π modulo 2π. Let M be the (possibly empty) sublamination of L consisting of the periodic leaves of L. Then, for all $\upsilon, R > 0$, there are finitely many loops $\ell_1, \dots, \ell_n$ on S such that*

- *the lengths of $\ell_1, \dots, \ell_n$ form length coordinates of $\mathcal{T}$, and*
- *for each $i = 1, \dots, n$,*
  - *the transversal measure $(L \setminus M)(\ell_i) < \upsilon$, and*
  - *ℓ_i intersects at most one leaf m of M, and the intersection number is at most two.*

Proof. For every $\delta > 0$, there is a pants decomposition $P = P_\delta$ (i.e. a maximal multiloop) on S consisting of

- the loops of M,
- loops which are disjoint from L,
- loops ℓ with $L(\ell) < \delta$ (so that ℓ is a good approximation of a minimal irrational sublamination of L).

By the third condition, if Q is a component of $S \setminus P$, and α is an arc on Q with endpoints on ∂Q, then there is an isotopy of α keeping its endpoints on ∂Q such that $L(\alpha) < 3\delta$. Therefore, if $\delta > 0$ is small enough, for each loop m of P, we can take two loops m_1, m_2 such that

- m_i intersects m at a point or two, and it does not intersect any other loop of P, and
- $(L \setminus M)m_i < \upsilon$.

Then we obtain a desired set of loops by adding such two loops for all loops of M. (For length coordinates of $\mathcal{T}$, see [FM12, Theorem 10.7] for example.) $\square$

Proof of the sufficiently of Theorem 5.1. For $\epsilon > 0$, let $\ell_1, \ldots, \ell_n$ be the set of loops given by Lemma 5.2. Let τ_i be a sequence in $\mathcal{T}$ which leaves every compact. Then, for some $1 \leq k \leq n$, $\operatorname{length}_{\tau_i} \ell_k \to \infty$ as $i \to \infty$ up to a subsequence.

Claim 5.3. *For every $\epsilon > 0$, if $\delta > 0$ is sufficiently small, then*

1. *if $L(\ell_k) < \delta$, then $\beta_i | \tilde{\ell}_k$ is a $(1+\epsilon)$-bilipschitz embedding for sufficiently large i, and*
2. *if ℓ_k intersects a loop m of M, then $\beta_i | \tilde{\ell}_k$ is $(1+\epsilon, q)$-quasi-isometric embedding for all sufficiently large i, where q only depends on the weight of m.*

Proof. (1) See [Bab10, Lemma 5.3], which was proved based on [CEG87, I.4.2.10].

(2) We straighten ℓ_k and M on $\tau_i \in \mathcal{T}$. From Lemma 5.2, ℓ intersects only one loop m of M, and their intersection number is one or two. We thus assume that $\ell_k \cap m$ consists of two points x_1, x_2 — the proof when the intersection number is one is similar. Then x_1 and x_2 decompose ℓ_k into 2 geodesic segments a_1 and a_2. Since $\operatorname{length}_{\tau_i} \ell_k \to \infty$, the lengths of a_1 and a_2 both goes to ∞ as well. Let $\tilde{\ell}_k$ be the geodesic in $\mathbb{H}^2$ obtained by lifting ℓ_k to the universal cover. Let $\tilde{a}_j$ be a lift of a_j to $\tilde{\ell}_k$, and let $\tilde{x}_j$ and $\tilde{x}_{j+1}$ be its endpoints. For every $\epsilon' > 0$, if $\upsilon > 0$, is sufficiently small, then $\beta_i(\tilde{a}_j)$ is ϵ'-close to the geodesic segment $[\beta_i x_j, \beta_i x_{j+1}]$ connecting its endpoints $\beta_i x_j$ and $\beta_i x_{j+1}$ in the Hausdorff metric. Since every periodic leaf of L has weight not equal to π modulo 2π, there is $\omega > 0$ such that, for every periodic leaf ℓ of L, the distance from the weight of ℓ to the nearest odd multiple of π is at least ω. Therefore, if $\delta > 0$ is sufficiently small, then the angle between $[\beta_i x_j, \beta_i x_{j+1}]$ and $[\beta_i x_{j-1}, \beta_i x_j]$ at x_j is at least $\omega/2$. Let c_i be the piecewise geodesic in $\mathbb{H}^3$ which is a union of the geodesic segments $[\beta_i x_j, \beta_i x_{j+1}]$ over all lifts $\tilde{a}_1, \tilde{a}_2$ of a_1, a_2 to $\tilde{\ell}_k$. Then c_i is ϵ'-hausdorff close to $\beta_i \tilde{\ell}_k$. Therefore, by Proposition 3.5, we see that c_i is a $(\epsilon, 1+q)$-quasigeodesic. $\square$

By this claim, for large i, the holonomy of $b_L \tau_i$ along ℓ_k is hyperbolic and its translation length diverges to ∞ as $i \to \infty$. Thus $b_L(\tau_i)$ leaves every compact in $\mathcal{X}$. Thus we have proven the properness. $\boxed{5.1}$

6. Characterization of non-properness

In this section, we explicitly describe how $b_L\colon \mathcal{T} \to \mathcal{X}$ is non-proper when the condition in Theorem 5.1 fails. Let L be a measured lamination on S. Let $m_1, \dots, m_p$ be the periodic leaves of L which have weight π modulo 2π. Then, set $M = m_1 \sqcup \dots \sqcup m_p$. Pick any pants decomposition P of S which contains $m_1, \dots, m_p$. Consider the Fenchel-Nielsen coordinates of $\mathcal{T}$ associated with P. Recall that its length coordinates take values in $\mathbb{R}_{>0}$ and its twist coordinates in $\mathbb{R}$.

Theorem 6.1. *Let τ_i be a sequence $\mathcal{T}$ which leaves every compact. Then $b_L(\tau_i)$ converges in $\mathcal{X}$ if and only if*

- $\operatorname{length}_{\tau_i} m_j \to 0$ *for some $j \in \{1, \dots, p\}$ as $i \to \infty$ (pinched) , and*
- *the Fenchel-Nielsen coordinates of τ_i w.r.t. P converge in their parameter spaces as $i \to \infty$, except that the length parameters of the pinched loops go to zero.*

We immediately have

Corollary 6.2. *The image sequence $b_L(\tau_i)$ is bounded in $\mathcal{X}$ if and only if*

- $\operatorname{length}_{\tau_i} m_j \to 0$ *as $i \to \infty$ for, at least, one $j \in \{1, \dots, p\}$, and*
- *the Fenchel-Nielsen coordinates of τ_i w.r.t. P are bounded in their parameter spaces as $i \to \infty$, except that the length parameters of the pinched loops (but including the twisting parameters of the pinched loops).*

Proof of Theorem 6.1. Let F be a component of $S \setminus M$. Then $b_L(\tau_i)|F$ converges in $\mathcal{X}(F)$ if and only if $\tau_i|F \coloneqq \tau_i|\pi_1(F)$ converges.

Let E and F be adjacent components of $\tilde{S} \setminus \tilde{M}$. Let $\tilde{m}$ be the component of $\tilde{M}$ separating E and F, and let m be the loop of M which lifts to $\tilde{m}$. Let Γ_E and Γ_F be the subgroups of $\pi_1(S)$ preserving E and F, respectively. Then E/Γ_E and F/Γ_F are the components of $S \setminus M$; let $S_E = E/\Gamma_E$ and $S_F = F/\Gamma_F$.

Proposition 6.3. *Let τ_i be a sequence of $\mathcal{T}$, such that the restriction of τ_i to S_E and to S_F converge in their respective Teichmüller spaces as $i \to \infty$. Pick, for each i, a representative $\xi_i\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ of $b_L(\tau_i) \in \mathcal{X}$ so that $\xi_i|\Gamma_E$ converges. Then, the restriction $\xi_i|\Gamma_F$ converges if and only if the Fenchel-Nielsen twisting parameter along m converges as $i \to \infty$.*

Proof. For each i, let $\beta_i\colon \mathbb{H}^2 \to \mathbb{H}^3$ be the bending map for (τ_i, L) equivariant via ξ_i, so that β_i converges on E. Let M_{τ_i} be the geodesic

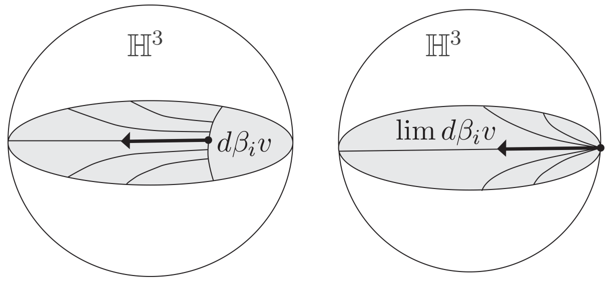

FIGURE 3. The convergence of the twist coordinate under neck-pinching.

representative of M on τ_i, and let $\tilde{M}_{\tau_i}$ be the total lift of M_{τ_i} on $\mathbb{H}^2$. Let $\tilde{m}_i$ be the component of $\tilde{M}_{\tau_i}$ corresponding to $\tilde{m}$. Let F_i, E_i be the region on $\tilde{\tau}_i \setminus \tilde{M}_{\tau_i}$ corresponding to F and E, respectively. For each i, pick a geodesic ray r_i in F_i starting from $\tilde{m}_i$ such that r_i is orthogonal to $\tilde{m}_i$ and that r_i is does not intersect the total lift $\tilde{L}$ of L.

Let v be the unit tangent vector of r_i at the base point on $\tilde{m}_i$. Since the weight of $\tilde{m}$ is π, $d\beta_i(v)$ is tangent to $\beta_i E_i$ at a point of $\tilde{m}$ (Figure 3, Left).

First suppose that $\lim_{i\to\infty} \operatorname{length}_{\tau_i} m$ is positive. Then $\xi_i|\Gamma_F$ converges if and only if $\beta_i(r)$ converges, which is equivalent to saying the twisting parameter of m converges in $\mathbb{R}$ as $i \to \infty$.

Next suppose that $\lim_{i\to\infty} \operatorname{length}_{\tau_i} m$ is zero. Then the holonomy of m converges to a parabolic element not equal to the identity. Then $\xi_i|\Gamma_F$ converges, if and only if $\beta_i(r)$ converges to a geodesic starting from the parabolic fixed point. This is equivalent to saying the twisting parameter of m converges as $i \to \infty$ (Figure 3, Right). □

The theorem follow from Proposition 6.3 as follows: Suppose that $b_L(\tau_i)$ converges as $i \to \infty$. Then, the hyperbolic structure on every component of $S \setminus M$ must converge. Thus, for each loop m of M, $\operatorname{length}_{\tau_i} m$ limits to a non-negative number. By Proposition 6.3, as $b_L(\tau_i)$ converges, the twist parameters along each loop of M converge. Since τ_i leaves every compact, at least one loop of M must be pinched as $i \to \infty$. Hence the two conditions hold.

To prove the other direction, suppose that the lengths of some loops of M limit to zero and all the other Fenchel-Nielsen coordinates with respect to P converge in the parameter space as $i \to \infty$. Let M' be the sub-multiloop of M consisting of the loops whose lengths go to zero. Then, for each component F of $S \setminus M'$, $b_L(\tau_i)|\pi_1(F)$ converges as $i \to \infty$. Therefore, by Proposition 6.3, $b_L(\tau_i)$ converges. This completes the proof. 6.1

7. Framed character varieties along loops

We have analyzed the real analytic embedding $b_L\colon \mathcal{T} \to \mathcal{X}$ defined for an arbitrary measured lamination $L \in \mathcal{ML}$. As $\mathcal{T}$ is regarded as the Fricke space, a component of the real slice of the character variety $\mathcal{X}$, one can certainly extend b_L to a holomorphic mapping from a neighborhood of $\mathcal{T}$ in $\mathcal{X}$ to $\mathcal{X}$. However, it does *not* extend to the entire character variety $\mathcal{X}$ for multiple reasons. Thus, in this section, we modify the character variety $\mathcal{X}$ and obtain a closed complex analytic set, which will be a domain of the complexification.

For a surface with punctures, Fock and Goncharov introduced a framing of a surface group representation ([FG06]). Their framing assigns a fixed point of peripheral holonomy around each puncture. In particular, the framing was useful to describe the deformation space of $\mathbb{C}\mathrm{P}^1$-structures on a surface with punctures by their framed holonomy representations ([AB20, GM21, Bab]).

In this paper, we introduce a certain framing along loops which assigns a pair of distinct fixed points of their holonomy. Such framings will be used to determine the axes for bending deformation even if the holonomy along loops is trivial.

7.1. **Framing of Representations along a loop.**

For simplicity, we first discuss the modification in the case that the bending lamination is a single loop. Let $\mathcal{R}$ be the space of representations $\pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ (without any equivalence relation). Then $\mathcal{R}$ is an affine algebraic variety: Namely, pick a presentation of the fundamental group $\pi_1(S)$, for instance

$$\pi_1(S) = \langle\, a_1, b_1, \dots, a_g, b_g \mid \Pi_{i=1}^n [a_i, b_i]\,\rangle.$$

Since $\mathrm{PSL}_2(\mathbb{C})$ embeds into $\mathrm{GL}_3(\mathbb{C})$ by the adjoint representation, $\mathrm{PSL}_2(\mathbb{C})$ is a complex affine Lie group sitting in $\mathbb{C}^9$. Then, by the embedding $\mathcal{R} \to (\mathbb{C}^9)^{2g}$ defined by

$$\rho \mapsto (\rho(a_1), \rho(b_1), \dots, \rho(a_g), \rho(b_g)) \in (\mathbb{C}^9)^{2g},$$

$\mathcal{R}$ has an affine algebraic structure on cut by the equation corresponding to the relator $\Pi_{i=1}^n [a_i, b_i]$.

Let ℓ be an oriented simple closed curve on S. The orientation will later be used to determine the bending direction in §8.2 in order to define the complexification. Let Λ_ℓ be the set of elements in $\pi_1(S)$ whose free homotopy classes are the homotopy class of ℓ on S; clearly, elements in Λ are conjugate to each other by elements in $\pi_1(S)$.

Pick an element $\alpha \in \Gamma_\ell$. Let $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ be a homomorphism. Suppose that $\rho(\alpha_\ell)$ is *not* a parabolic (but it can be the identity). Then, there is an ordered pair (u, v) of distinct points u, v on $\mathbb{C}\mathrm{P}^1$ which are fixed by α_ℓ pointwise. We can equivariantly extends a pair (u, v) to pairs (u_γ, v_γ) for all representatives $\gamma \in \Lambda_\ell$ so that γ fixes u_γ and v_γ in $\mathbb{C}\mathrm{P}^1$. Such an equivariant assignment $(u_\gamma, v_\gamma)_{\gamma\in\Lambda_\ell}$ of ordered fixed points of γ is called a framing of ρ along ℓ. By abuse of notation, we denote this equivariant framing $\{(u_\gamma, v_\gamma)\}_{\gamma\in\Lambda_\ell}$, by (u, v), since it is determined by the initial choice (u, v) for α_ℓ. We call the triple (ρ, u, v) a framed representation. In order to produce the equivariant bending axes (later), we utilize the equivariant framing. Let

$$R_\ell = \left\{ (\rho, u, v) \in \mathcal{R} \times (\mathbb{C}\mathrm{P}^1)^2 \;\middle|\; \rho(\alpha_\ell)u = u, \rho(\alpha_\ell)v = v, u \neq v \right\}.$$

Then R_ℓ is a closed analytic subset of $\mathcal{R} \times (\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1 \setminus D)$, where D is the diagonal $\{(z, z) \mid z \in \mathbb{C}\mathrm{P}^1\}$. Note that if $(\rho, u, v) \in R_\ell$, then the $\rho(\alpha_\ell)$ can *not* be a parabolic element, since u, v are distinct fixed points of $\rho(\alpha_\ell)$. On the other hand, $\rho(\alpha_\ell)$ can be the identity.

Let $\mathcal{G}_\ell$ be the subgroup of the mapping class group of S which preserves the oriented loop ℓ. Clearly, $\mathcal{G}_\ell$ acts on R_ℓ by marking change.

We now assume that the oriented loop ℓ has a weight w in $\mathbb{R}_{>0}$. Suppose, first, that the weight of the oriented loop ℓ is not equal to π modulo 2π. Fix any complex number $w \in \mathbb{C}$ with $|w| > 1$. Then, given $(u, v) \in \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$ with $u \neq w$, there is a unique hyperbolic element $\gamma_{u,v,w} \in \mathrm{PSL}_2\mathbb{C}$, such that u is the repelling fixed point, v is the attracting fixed point of $\gamma_{u,v,w}$ and that $\gamma_{u,v,w}$ can be conjugated to the hyperbolic element $z \mapsto wz$ by an element of $\mathrm{PSL}_2\mathbb{C}$. Clearly, this mapping $(u, v) \mapsto \gamma_{u,v,w}$ is a biholomorphic mapping onto its image. Then, $(\rho, u, v) \in R_\ell$ biholomorphically corresponds to a unique element $(\rho, \gamma_{u,v,w})$ of $\mathcal{R} \times \mathrm{PSL}_2\mathbb{C}$. Thus $R_\ell \to \mathcal{R} \times \mathrm{PSL}_2\mathbb{C}$ is a biholomorphic map onto its image. Since $\mathrm{PSL}_2\mathbb{C} \cong \mathrm{SO}_3(\mathbb{C}) \subset \mathbb{C}^9$, we see that R_ℓ is biholomorphic to a closed analytic set in a complex vector space of finite dimension. (It is closed, since if $(u, v) \in (\mathbb{C}\mathrm{P}^1)^2 \setminus \Delta$ converges to a point in the diagonal Δ, then $\gamma_{u,v,w}$ must leaves every compact subset of $\mathrm{PSL}_2\mathbb{C}$.) Therefore R_ℓ is also a Stein space, as it is a closed analytic subset of a Stein space.

The theory of categorical quotients of Stein manifolds has been developed analogously to GIT-quotients affine algebraic varieties (see [Sno82]). We let X_ℓ be the categorical quotient (*Stein quotient*) $R_\ell /\!\!/ \mathrm{PSL}_2\mathbb{C}$, which is again Stein. In this quotient, two framed representations (ρ_1, u_1, v_1) and (ρ_2, u_2, v_2) in R_ℓ are identified if and only if every $\mathrm{PSL}_2\mathbb{C}$-invariant analytic function f on R_ℓ takes the same value at

(ρ_1, u_1, v_1) and (ρ_2, u_2, v_2); see [Sno82, §3]. We denote, by $[\rho, u, v]$, the equivalence class of (ρ, u, v) in X_ℓ.

Next suppose that ℓ has weight π modulo 2π. In this case, the ordering of the framing (u, v) will *not* affect the complexified bending map, and thus we take a slightly stronger quotient. Then, let $\gamma_{u,v}$ be the elliptic element of angle π with the axes connecting u and v. Let $R_\ell/\mathbb{Z}_2$ be the quotient of R_ℓ by the $\mathbb{Z}_2$-action which switches the ordering of the framing, namely, given by $(\rho, u, v) \mapsto (\rho, v, u)$. Consider the map $R_\ell/\mathbb{Z}_2 \to \mathcal{R} \times \mathrm{PSL}_2\mathbb{C}$ defined by $(\rho, u, v) \mapsto (\rho, \gamma_{u,v})$. Thus $R_\ell/\mathbb{Z}_2$ is biholomorphic to a closed analytic set in $\mathcal{R} \times \mathrm{PSL}_2\mathbb{C}$. Similarly, we let X_ℓ be the stein quotient $(R_\ell/\mathbb{Z}_2) /\!\!/ \mathrm{PSL}_2\mathbb{C}$. The action of $\mathcal{G}_\ell$ on R_ℓ descends to an action on X_ℓ.

7.1.1. *Coordinates for the quotient space of representations framed along a single loop.* We defined the Stein space X_ℓ as a Stein quotient. In this section, we indeed realize X_ℓ as an analytic set in an affine space by identifying it with a subset of a $\mathrm{PSL}_2\mathbb{C}$-character variety $\mathcal{X}(\pi_1(S) * \mathbb{Z})$ of $\pi_1(S) * \mathbb{Z}$. Recall that, for $(\rho, u, v) \in R_\ell$, the element $\gamma_{u,v,w} \in \mathrm{PSL}_2\mathbb{C}$ is a certain hyperbolic element if the weight of the oriented loop ℓ is *not* equal to π modulo 2π and a certain elliptic element of angle π otherwise.

Given $(\rho, u, v) \in R_\ell$, let $\hat{\rho} = \hat{\rho}_{u,v,w}$ be the homomorphism from the free product $\pi_1(S) * \mathbb{Z}$ to $\mathrm{PSL}_2\mathbb{C}$, such that every $\gamma \in \pi_1(S)$ maps to $\rho(\gamma)$ and $1 \in \mathbb{Z}$ maps to $\gamma_{u,v,w}$. Then, with respect to the $\mathrm{PSL}_2\mathbb{C}$-action on R_ℓ, we clearly have the following.

Lemma 7.1. (1) *Suppose that the weight of ℓ is not equal to π modulo 2π. Then (ρ_1, u_1, v_1) and (ρ_2, u_2, v_2) are identified by an element of* $\mathrm{PSL}_2\mathbb{C}$ *if and only if $\hat{\rho}_1$ and $\hat{\rho}_2$ are conjugate by* $\mathrm{PSL}_2\mathbb{C}$.

(2) *Suppose that the weight of ℓ is equal to π modulo 2π. Then (ρ_1, u_1, v_1) and (ρ_2, u_2, v_2) are identified by an element of* $\mathrm{PSL}_2\mathbb{C} \times \mathbb{Z}_2$ *if and only if $\hat{\rho}_1$ and $\hat{\rho}_2$ are conjugate conjugate by* $\mathrm{PSL}_2\mathbb{C}$, *where the $\mathbb{Z}_2$-action exchanges the ordering of the framing.*

Let $\hat{\mathcal{R}}$ be the space of representations $\pi_1(S) * \mathbb{Z} \to \mathrm{PSL}_2\mathbb{C}$. Then $\hat{\mathcal{R}}$ is an affine algebraic variety. Suppose that the weight of ℓ is *not* equal to π modulo 2π. We have seen that the mapping $R_\ell \to \mathcal{R} \times \mathrm{PSL}_2\mathbb{C}$ is a biholomorphic map onto its image by the mapping $(\rho, u, v) \mapsto \hat{\rho}$. Let $\hat{\mathcal{R}}_\ell$ be this image. Then $\hat{\mathcal{R}}_\ell$ is the closed analytic subset in $\hat{\mathcal{R}}$ biholomorphic to R_ℓ, and thus in particular it is Stein. Moreover, this biholomorphism $R_\ell \to \hat{\mathcal{R}}_\ell$ is equivariant with respect to the $\mathrm{PSL}_2\mathbb{C}$-action. Thus the Stein space $X_\ell = R_\ell /\!\!/ \mathrm{PSL}_2\mathbb{C}$ is biholomorphic to the subvariety $\hat{\mathcal{R}}_\ell /\!\!/ \mathrm{PSL}_2\mathbb{C}$ of $\chi(\pi_1(S) * \mathbb{Z})$.

A similar identification holds in the case when ℓ has weight π modulo 2π. The Stein space $R_\ell/\mathbb{Z}_2$ biholomorphically maps to its image, denoted by $\hat{\mathcal{R}}_\ell$, in $\hat{\mathcal{R}}$ by the mapping $(\rho, u, v) \mapsto \hat{\rho}$. Then $X_\ell = (R_\ell/\mathbb{Z}_2) /\!\!/ \mathrm{PSL}_2\mathbb{C}$ is biholomorphic to the stein space $\hat{\mathcal{R}}_\ell /\!\!/ \mathrm{PSL}_2\mathbb{C}$.

Let $\gamma \in \pi_1(S) * \mathbb{Z}$. Let $\mathrm{tr}^2(\gamma)$ be the (polynomial) function on $\hat{\mathcal{R}}_\ell$ defined by $(\rho, u, v) \mapsto \mathrm{tr}^2 \rho(\gamma)$. Then $\mathrm{tr}^2(\gamma)$ is a $\mathrm{PSL}_2\mathbb{C}$-equivariant analytic function on $\hat{\mathcal{R}}_\ell$. Then, by [HP04, Corollary 2.3], such trace square functions form coordinates of the Stein quotient $\hat{\mathcal{R}}_\ell /\!\!/ \mathrm{PSL}_2\mathbb{C}$, and they also form coordinates for $X_\ell (\cong \hat{\mathcal{R}}_\ell /\!\!/ \mathrm{PSL}_2\mathbb{C})$.

Proposition 7.2. *There are finitely many elements $\gamma_1, \gamma_2, \dots \gamma_N$ in $\pi_1 S * \mathbb{Z}$ such that the analytic mapping $\hat{\mathcal{R}}_\ell \to \mathbb{C}^N$ given by $\mathrm{tr}^2(\gamma_1), \dots, \mathrm{tr}^2(\gamma_N)$ induces an analytic embedding of X_ℓ into $\mathbb{C}^N$. Thus $\mathrm{tr}^2(\gamma_1), \dots, \mathrm{tr}^2(\gamma_N)$ form a coordinate ring.*

7.2. **Representations framed along a multi-loop.** In §7.1, we introduced the space of representations $\pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ framed along a single (oriented) loop, constructed a quotient space by the $\mathrm{PSL}_2\mathbb{C}$ action, and realized as an analytic subset of a complex affine space. In this section, we similarly consider the space of representations framed along a weighted multiloop, and then construct its Stein quotient by the action of $\mathrm{PSL}_2\mathbb{C}$.

Let $m_1, \dots m_n$ be non-isotopic essential simple closed curves on S, and let M be their union $m_1 \sqcup m_2 \sqcup \dots \sqcup m_n$. Recall that $\mathcal{R}$ denotes the representation variety $\{\pi_1(S) \to \mathrm{PSL}_2\mathbb{C}\}$. For each $i = 1, \dots, n$, let Λ_{m_i} denote the set of elements in $\pi_1(S)$ whose free homotopy classes are the homotopy class of m_i. Pick a representative $\alpha_i \in \Lambda_{m_i}$. Then, consider the space R_M of tuples $(\rho, (u_i, v_i)_{i=1}^n) \in R \times (\mathbb{C}\mathrm{P}^1)^{2n}$ where

- $\rho \in R$ is a homomorphism $\pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$, and
- $u_i, v_i \in \mathbb{C}\mathrm{P}^1$ are different fixed points of $\rho(\alpha_i)$ for $i = 1, \dots, n$.

As in the case of a single loop, $\rho(\alpha_i)$ are *not* parabolic elements (but can be the identity). Then R_M is a closed analytic subvariety of $R \times (\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1 \setminus \Delta)^n$, where Δ denotes the diagonal as before. Given $(\rho, (u_i, v_i)_{i=1}^n) \in R_M$, we can equivariantly extend (u_i, v_i) to the pairs of fixed points for all representatives of $m_1, \dots, m_n$ in $\pi_1(S)$. We call this extension a framing of ρ along the multiloop M.

Now we assign a positive number (*weight*) to each loop of M. Let p be the number of components m_i of M, such that the weight of m_i is π modulo 2π. Without loss of generality, we can assume $m_1, \dots, m_n$

are the loops of M with weight π modulo 2π. Then, $\mathbb{Z}_2^p$ acts biholomorphically on R_M by switching the ordering of the fixed points of the framing along $m_1, \dots, m_n$. Note that this $\mathbb{Z}_2^p$-action has no fixed points in R_M.

Fix a complex number $w \in \mathbb{C}$ with $|w| > 1$. As in §7.1.1, let $\gamma_{u_i,v_i,w} \in \mathrm{PSL}_2\mathbb{C}$ be, if the weight of m_i is π modulo 2π, then the elliptic element of angle π whose axis is the geodesic connecting u_i to v_i, and otherwise, the hyperbolic element with the repelling fixed point u_i and the attracting fixed point v_i such that $\gamma_{u_i,v_i,w}$ is conjugate to the dilation $z \mapsto wz$. Then, define the mapping $R_M \to \mathcal{R} \times (\mathrm{PSL}_2\mathbb{C})^m$ by $(\rho, (u_i, v_i)_{i=1}^n) \mapsto (\rho, (\gamma_{u_i,v_i,w})_{i=1}^n)$. This mapping takes $R_M/\mathbb{Z}_2^p$ onto its image $\hat{R}_M$ biholomorphically. Thus $R_M/\mathbb{Z}_2^p$ is a closed analytic set in a finite-dimensional complex vector space. Therefore $R_M/\mathbb{Z}_2^p$ is Stein. The Lie group $\mathrm{PSL}_2\mathbb{C}$ acts analytically on $R_M/\mathbb{Z}_2^p$, by conjugation on ρ. By this action, we obtain its Stein quotient $(R_M/\mathbb{Z}_2^p) /\!/ \mathrm{PSL}_2\mathbb{C} =: X_M$. Thus X_M is a Stein space.

The biholomorphic map $R_M/\mathbb{Z}_2^p \to \hat{R}_M$ is equivariant w.r.t. the $\mathrm{PSL}_2\mathbb{C}$-action, X_M is biholomorphic to the corresponding Stein quotient $\hat{R}_M /\!/ \mathrm{PSL}_2\mathbb{C}$.

We denote, by $[\rho, (u_i, v_i)]$, the equivalence class of $(\rho, (u_i, v_i)) \in R_M$ in X_M. The subgroup $\mathcal{G}_M$ of MCG acts on R_M, and descends to an action on X_M.

7.2.1. *Coordinates of the quotient space of representations framed along a multiloop.* Let $g_1, g_2, \dots, g_n$ be a standard generating set of the free group $\mathbb{F}^n$ of rank n, so that there are no relators. Every $(\rho, (u_i, v_i)_{i=1}^n) \in R_M$ corresponds to a unique representation $\pi_1(S) * \mathbb{F}^n \to \mathrm{PSL}_2\mathbb{C}$ such that

- $\gamma \in \pi_1(S)$ maps to $\rho(\gamma)$, and
- g_i maps to $\gamma_{u_i,v_i,w}$ for every $i = 1, \dots, n$.

By this correspondence, R_M analytically embed into the space of representations $\pi_1(S) \times \mathbb{F}^n \to \mathrm{PSL}_2\mathbb{C}$. As in §7.1.1, by the quotient of the image $\mathcal{R}_M$ by $\mathrm{PSL}_2\mathbb{C}$, [HP04, Corollary 2.3] yields the coordinate ring of $X_M \cong \mathcal{R}_M /\!/ \mathrm{PSL}_2\mathbb{C}$.

Proposition 7.3. *There are finitely many elements $\gamma_1, \gamma_2, \dots, \gamma_N$ of $\pi_1(S)$ corresponding to simple closed curves, such that* $\mathrm{tr}^2(\gamma_1), \dots \mathrm{tr}^2(\gamma_n)$ *form a coordinate ring of X_M.*

8. Bending a surface group representation into $\mathrm{PSL}_2\mathbb{C}$ inside the representation space into $\mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C}$

Originally bending deformation equivariantly bends a totally geodesic $\mathbb{H}^2$ along a measured lamination ([Thu81, EM87]), so that bending is in one-direction and the bent $\mathbb{H}^2$ is locally convex. Moreover, one can extend it to an equivariant bending pleated surface along the pleated locus using bending cocycles ([Bon96]). In both cases, bending is done along (bi-infinite)geodesics in $\mathbb{H}^3$ which are embedded in the pleated surfaces.

In this section, we introduce a new type of bending of more topological equivariant surfaces in $\mathbb{H}^3$. Using such more general bending, define a complex-analytic bending map $X_M \to \mathcal{X} \times \mathcal{X}$ which complexifies the real-analytic bending map $\mathcal{T} \to \mathcal{X}$.

8.1. A complexification of the Lie group $\mathrm{PSL}_2\mathbb{C}$ regarded as a real Lie group.

We first recall a complexification of $\mathrm{PSL}_2\mathbb{C}$ when regarded as a real Lie group.

Proposition 8.1 (See Proposition 1.39 in [Zil] for example)**.** *Regard $\mathfrak{psl}_2\mathbb{C}$ as a real Lie algebra. Then the complexification of the Lie algebra $\mathfrak{psl}_2\mathbb{C}$ is isomorphic to $\mathfrak{psl}_2\mathbb{C} \oplus (\mathfrak{psl}_2\mathbb{C})^*$ by the mapping given by $(u, 0) \mapsto (u, Iu)$ and $(0, v) \mapsto (v, -Iv)$, where $(\mathfrak{psl}_2\mathbb{C})^*$ is the complex conjugate of $\mathfrak{psl}_2\mathbb{C}$ and I is the complex multiplication of $\mathfrak{psl}_2\mathbb{C}$.*

We regard $\mathrm{PSL}_2\mathbb{C}$ as a real Lie group, and we complexify $\mathrm{PSL}_2\mathbb{C}$ by

$$\begin{array}{rcl} c\colon \mathrm{PSL}_2\mathbb{C} & \longrightarrow & \mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C} \\ \Psi & & \Psi \\ Y & \longmapsto & (Y, Y^*) \end{array},$$

so that it corresponds to Proposition 8.1. Then c is holomorphic in the first factor and anti-holomorphic in the second factor. Thus c is, in particular, a proper real-analytic embedding of $\mathrm{PSL}(2, \mathbb{C})$ into the complex Lie group $\mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C}$.

8.2. Bending framed representations.

We first define a complex bending of representations framed along a single loop. Let ℓ be an oriented loop ℓ on S, and we fixed a weight $w > 0$ of ℓ. Fix $\alpha \in \pi_1(S)$ representing of ℓ. Let $[(\rho, u, v)] \in X_\ell$, where $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ and (u, v) is a pair of fixed points of $\rho(\alpha)$. Let $(\rho, \rho)\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C}$ denote the diagonal representation given by $\gamma \mapsto (\rho(\gamma), \rho(\gamma))$.

Recall that (u, v) generates a ρ-equivariant farming f along ℓ and Λ_ℓ denotes the subset of $\pi_1(S)$ corresponding to ℓ. That is, for every

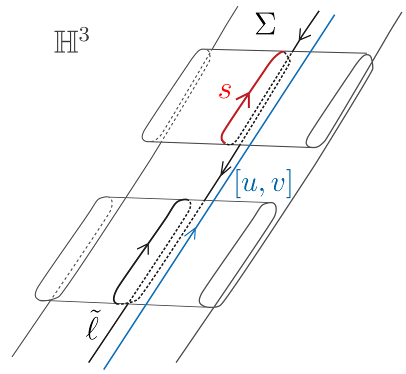

FIGURE 4. A schematic of Σ after the homotopy.

element $\gamma \in \Lambda_\ell$, an ordered pair $(u_\gamma, v_\gamma) \in \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$ of different fixed points of $\rho(\gamma)$ is assigned ρ-equivariantly. Consider the oriented geodesic $g_\gamma = (u_\gamma, v_\gamma)$ in $\mathbb{H}^3$ connecting u_γ to v_γ for all $\gamma \in \Lambda_\ell$. Those equivariant geodesics $\{g_\gamma\}$ will be the axes of the bending.

First, we coherently define the direction of the bending so that bending is continuously defined on X_ℓ. Pick any ρ-equivariant piecewise smooth surface $\Sigma \colon \tilde{S} \to \mathbb{H}^3$. Then, as S is oriented, at every smooth point x of Σ, there are a normal direction of Σ and the hyperbolic plane tangent to Σ at x.

Let $\tilde{\ell}$ be the lift of ℓ to the universal cover $\tilde{S}$ invariant by $\gamma \in \Lambda_\ell$. Then, homotope Σ in $\mathbb{H}^3$ equivariantly and piecewise smoothly so that

(1) Σ takes $\tilde{\ell}$ into the bi-infinite geodesic (u, v), and
(2) $\Sigma \colon \tilde{S} \to \mathbb{H}^3$ is a local embedding at every point $x \in \tilde{\ell} \setminus V$ for some an γ-invariant discrete set V in $\tilde{\ell}$ (See Figure 4).

In particular, each component of $\tilde{\ell} \setminus V$ is embedded in $[u, v]$ by Σ. Moreover, as Σ is ρ-equivariant, there is at least one segment s of $\tilde{\ell} \setminus V$ whose direction matches the oriented geodesic $[u, v]$ by Σ. Then, the orientation of the surface near the segment determines the positive direction of the bending, so that the normal direction is the exterior; see Figure 5. In the special case that ρ is Fuchsian, Σ can be taken to be an equivariant embedding onto a totally geodesic hyperplane if the orientation of ℓ matches with the ordering of the framing.

The choice of the homotopy of Σ will *not* affect the resulting bending map determined by the normal direction. If Σ' is another ρ-equivariant piecewise smooth homotopy to Σ satisfying (1) and (2), then there is a piecewise smooth homotopy between Σ and Σ' preserving the conditions and taking the normal vector on Σ to the normal vector on Σ'.

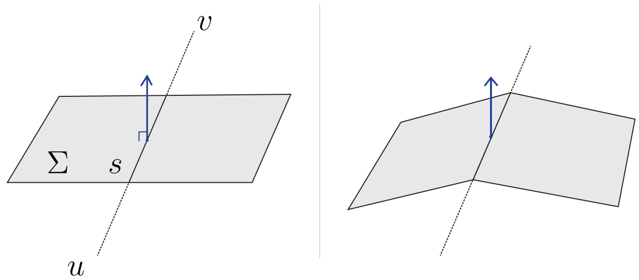

FIGURE 5. Bending by a small angle w.r.t. the normal direction.

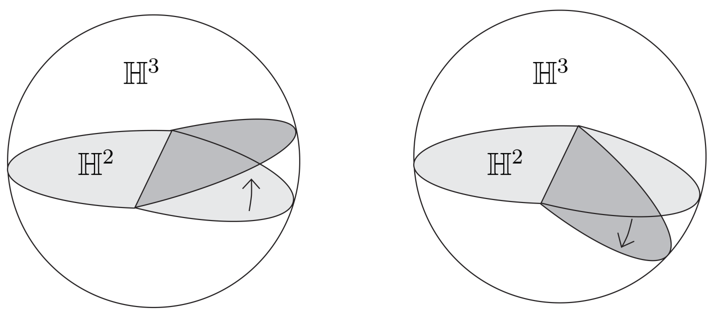

FIGURE 6. Bending in opposite directions in different factors.

Then, for every $\theta \in \mathbb{R}$, we can bend the ρ-equivariant surface $\Sigma\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ along the equivariant axes $\{g_\gamma\}$ by angle θ with respect to the positive bending direction defined above. As we bend Σ in an equivariant manner, the bent surface $\tilde{S} \to \mathbb{H}^3$ remains equivariant via a unique representation; we denote the induced representation by $b_{\ell,\theta}(\rho, u, v)\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$. We now define a complex bending map $B_\ell\colon X_\ell \to \mathcal{X} \times \mathcal{X}$ by $B_{\ell,w}(\rho, u, v) = (b_{\ell,w}(\rho, u, v), b_{\ell,-w}(\rho, u, v))$. Note that, in the fast factor and the second factor, the bending ρ is equivariantly done along the same axes and by the same angle, but in the opposite directions (Figure 6).

The bent representation is well-defined up to conjugation by an element of $\mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C}$, and thus $B_\ell(\rho, u, v) \in \mathcal{X} \times \mathcal{X}$ is well-defined. We remark that, if $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ is Fuchsian, then the representation of $B_\ell(\rho, u, v)$ in the first factor $\mathcal{X}$ is the complex conjugate of that in the second factor.

For an oriented weighted multiloop M on S, we can similarly define the complex bending map $B_M\colon X_M \to \mathcal{X} \times \mathcal{X}$ as follows. Let $m_1, \dots, m_n$ are the weighted loops of M. Pick $\gamma_i \in \Lambda_{m_i}$. Let $\tilde{m}_i$ be a γ_i invariant lift of m_i to the universal cover $\tilde{S}$. Let $[\rho, (u_i, v_i)_{i=1}^n] \in X_M$, where (u_i, v_i) be the fixed point of $\rho(\gamma_i)$. Then the oriented geodesic g_i connecting u_i to v_i, equivariantly extends to a system of bending axes corresponding to all lifts of m_i to $\tilde{S}$. Find a ρ-equivariant piecewise smooth surface $\Sigma\colon \tilde{S} \to \mathbb{H}^3$ such that $\tilde{m}_i$ maps to its corresponding axes g_i, and there is an γ_i invariant discrete subset V_i of $\tilde{m}_i$ such that Σ is a local embedding at every point on $\tilde{m}_i \setminus V_i$. Then, there is a segment s_i of $\tilde{m}_i \setminus V_i$ such that its orientation matches with the orientation of $g_i = (u_i, v_i)$. The normal direction of Σ at a point in s_i determines the positive bending direction of Σ along g_i. Similarly, one can show the positive bending direction does not depend on the choice of Σ by property-preserving homotopy between different choices of Σ.

Let $\theta_1, \ldots, \theta_n$ be real numbers. We can bend the ρ-equivariant surface $\Sigma\colon \tilde{S} \to \mathbb{H}^3$ along the geodesics $g_1, \ldots, g_n$ and their orbit geodesics by angles $\theta_1, \ldots, \theta_n$, respectively, in the positive bending direction defined above. Since we bend Σ in an equivariant manner, the new bent surface $\Sigma^+\colon \tilde{S} \to \mathbb{H}^3$ is also equivariant via a unique representation. We denote the bend representation by

$$b_{(m_i, \theta_i)}(\rho, (u_i, v_i)_{i=1}^n) = b_M^+(\rho, (u_i, v_i)_{i=1}^n)\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}.$$

Similarly, we can bend Σ along the same axes by the same angles but in opposite directions, and we obtain another bent surface $\Sigma^-\colon \tilde{S} \to \mathbb{H}^3$. Then $\Sigma^-$is also equivariant via a unique representation

$$b_M^-(\rho, (u_i, v_i)_{i=1}^n) = b_{(m_i, -w_i)}(\rho, (u_i, v_i)_{i=1}^n)\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}.$$

By combing those two bending of framed representations, we obtain the bending map $B_M\colon X_M \to \chi \times \chi$ by

$$B_M(\rho, (u_i, v_i)_{i=1}^n) = (b_{(m_i, w_i)}(\rho, (u_i, v_i)_{i=1}^n), b_{(m_i, -w_i)_i}(\rho, (u_i, v_i)_{i=1}^n)).$$

Then the mapping $\tilde{S} \to \mathbb{H}^3 \times \mathbb{H}^3$ defined by $x \mapsto (\Sigma^+(x), \Sigma^-(x))$ is equivariant via $B_M(\rho, (u_i, v_i)_{i=1}^n)\colon \pi_1(S) \to \mathrm{PSL}_2 \times \mathrm{PSL}_2\mathbb{C}$.

8.3. **Equivariant property.**

Lemma 8.2. *Let M be a weighted oriented multiloop on S. Let G_M be the subgroup of the mapping class group* $\mathrm{MCG}(S)$*, which preserves M. Then $B_M\colon X_M \to \chi \times \chi$ is G_M-equivariant.*

Proof. Recall that G_M acts on X_M by marking change. Then $b_{(m_i, w_i)}\colon X_M \to \chi$ and $b_{(m_i, -w_i)}\colon X_M \to \chi$ are both G_M-equivariant, since the action is marking chnage. Therefore B_M is also G_M-equivariant. □

8.4. **Support planes and spaces.** For a marked hyperbolic surface τ homeomorphic to S and a measured lamination L on τ, we have a $\pi_1(S)$-equivariant bending map $\beta_{\tau,L}\colon \mathbb{H}^2 \to \mathbb{H}^3$ which is "locally convex". Letting $\tilde{L}$ be the $\pi_1(S)$-invariant measured lamination on the universal cover $\mathbb{H}^2$ of τ. Then, for each component P of $\mathbb{H}^2 \setminus \tilde{L}$, the mapping $\beta_{\tau,L}$ embeds P into a totally geodesic hyperbolic plane P in $\mathbb{H}^3$. Such a hyperbolic plane is a support plane for $\beta_{L,\tau}$. (See [EM87] for more general support planes.) On the other hand, this equivariant system $\{H_P\}_P$ of totally geodesic hyperbolic planes, indexed by the components, determines the original bending map $b_{\tau,L}\colon \mathbb{H}^2 \to \mathbb{H}^3$.

In §8, we bend framed representations $\eta = [\rho, (u_i, v_i)]$ in X_M along a weighted oriented multiloop M defined in , and obtained a representation $\pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C}$. As the symmetric space associated with $\mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C}$ is the product $\mathbb{H}^3 \times \mathbb{H}^3$, we consider a system of supporting hyperbolic three-spaces in the product $\mathbb{H}^3 \times \mathbb{H}^3$as follows. For every component P of $\tilde{S} \setminus \tilde{M}$, the restriction of Σ^+ to P coincides with the restriction of Σ^- to P composed with an element γ of $\mathrm{PSL}_2\mathbb{C}$. Therefore, the restriction of the surface $(\Sigma^+, \Sigma^-)\colon \tilde{S} \to \mathbb{H}^3 \times \mathbb{H}^3$ to P is contained in a totally geodesic copy H_P of $\mathbb{H}^3$ given by $\{(x, \gamma x) \mid x \in \mathbb{H}^3\} \subset \mathbb{H}^3 \times \mathbb{H}^3$.

Hence, we obtain an equivariant collection of supporting hyperbolic 3-spaces H_P for all components P of $\tilde{S} \setminus \tilde{M}$. We call this collection $\{H_P\}_P$ the (equivariant) bending support system of B_M at η. Let G_P denote the subgroup of $\pi_1(S)$ consisting of the elements preserving the P. Then H_P is preserved by the restriction of the bend representation

$$B_M(\rho, (u_i, v_i)_{i=1}^n)\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C}$$

to the subgroup G_P.

Suppose that P and P' are adjacent components of $\tilde{S} \setminus \tilde{M}$ across a lift $\tilde{m}$ of a loop m of M. Let w be the weight of m. Then, in $\mathbb{H}^3 \times \mathbb{H}^3$, the support spaces H_P and $H_{P'}$ intersect in a geodesic at angle w (complex bending axis), which corresponds to the bending axis in $\mathbb{H}^3$ induced by the framing in the definition of B_M (Figure 7). In particular, if the weight of m is a multiple of π, then $H_P = H_{P'}$. Indeed, for an elliptic element $e \in \mathrm{PSL}_2\mathbb{C}$ with rotation angle π, we have

$$\{(x, x) \in \mathbb{H}^3 \times \mathbb{H}^3 \mid x \in \mathbb{H}^3\} = \{(ex, e^{-1}x) \in \mathbb{H}^3 \times \mathbb{H}^3 \,|\, x \in \mathbb{H}^3)\}.$$

Definition 8.3. *Let $\xi\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C}$ be a representation. A* support system of ξ *with respect to M is an equivariant collection of totally geodesic hyperbolic planes H_P for all components P of $\tilde{S} \setminus \tilde{M}$*

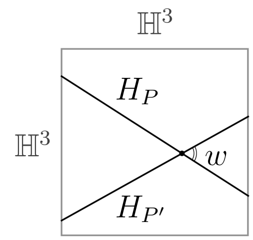

FIGURE 7. The intersection angle w of totally geodesic copies $H_P, H_{P'}$ of $\mathbb{H}^3$ in $\mathbb{H}^3 \times \mathbb{H}^3$.

such that the restriction of ξ *to* G_P *preserves* H_P *for all components* P *of* $\tilde{S} \setminus \tilde{M}$.

In general, a representation $\pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C}$ may have no support system or many support systems. On the other hand, we will prove that the support system is uniquely determined by $B_M(\rho, (u_i, v_i)_{i=1}^n)$ in most cases; see Lemma 9.2.

9. Complex bending maps are almost injective

In this section, we prove the injectivity of the complex bending map $B_M \colon X_M \to \chi \times \chi$ when restricted to the complement of certain subvarieties.

Let M be an oriented weighted multiloop on S, and let n be the number of the loops of M. Let X_M^p be the subset of X_M consisting of the framed representations $(\rho, (u_i, v_i)_{i=1}^n)$ such that $\mathrm{tr}^2 \rho(m) = 4$ for, at least, one loop m of M, i.e. $\rho(m)$ is either a parabolic element or the identity. As it is an algebraic equation, X_M^p is an analytic subvariety of X_M.

Let X_M^r be the subset consisting of the framed representations (ρ, u, v) such that, for some component F of $S \setminus M$, the restriction of ρ to $\pi_1(F)$ is *weakly reducible*, i.e. the image is, up to a finite index, reducible. The set of weakly reducible representations $\pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ is contained in a subvariety of χ. Thus X_M^r is also an analytic subset of X_M. We prove that the injectivity of the complex bending map holds in the complement of those analytic subsets.

Theorem 9.1. *Let* M *be a weighted oriented multiloop on* S. *Then, the complex bending map* $B_M \colon X_M \to \chi \times \chi$ *is injective on* $X_M \setminus (X_M^p \cup X_M^r)$.

We first show a uniqueness of the support systems of the complex bending.

Lemma 9.2. *Let $\eta \in X_M \setminus (X_M^r \cup X_M^p)$. Fix a representative $\xi\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C}$ of $B_M(\eta)$. Let P be a component of $\tilde{S} \setminus \tilde{M}$. Then, the support space H_P of ξ is the unique totally geodesic copy of $\mathbb{H}^3$ in $\mathbb{H}^3 \times \mathbb{H}^3$ which contains the bending axes of the boundary components of P.*

Proof. As $\eta \notin X_M^P$, the bending axes of the boundary components of P are uniquely determined by ξ. Let G_P be the subgroup of $\pi_1(S)$ preserving P. Then, as $\eta | G_P$ is *strongly irreducible* (i.e. not weakly reducible), there is a unique totally geodesic copy of $\mathbb{H}^3$ in $\mathbb{H}^3 \times \mathbb{H}^3$, containing those bending axes. □

Lemma 9.2 immediately implies the following.

Corollary 9.3. *Suppose that $\eta_1, \eta_2 \in X_M \setminus (X_M^p \cup X_M^r)$ satisfy $B_M(\eta_1) = B_M(\eta_2) \in \mathcal{X} \times \mathcal{X}$. Let $\xi\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C}$ be a representative of $B_M(\eta_1) = B_M(\eta_2)$. Then, the ξ-equivariant bending support system of B_M at η_1 equivariantly coincides with that at η_2.*

Proof of Theorem 9.1. Suppose that $\eta_1, \eta_2 \in X_M \setminus (X_M^p \cup X_M^r)$ map to the same representation in $\mathcal{X} \times \mathcal{X}$ by B_M. Then, let $\xi\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C}$ be a representative of their image.

By Corollary 9.3, the support system of the bending of η_1 equivariantly coincides with that of η_2. Therefore η_1 and η_2 are obtained by unbending ξ exactly in the same manner; hence $\eta_1 = \eta_2$. [9.1]

9.1. **A non-injective example.** We shall see, in an example, the non-injectivity of a complex bending map. Let m be a separating loop on S with some positive weight. Pick a connected subsurface F of S bounded by m. Fix a homomorphism $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ such that $\rho | \pi_1 F$ is the trivial representation. Then, as $\rho(m)$ is in particular the identity, any pair $(u, v) \in \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$ is a framing of ρ along m.

Lemma 9.4. *Fix an arbitrary orientation of m and an arbitrary weight on m. Then $B_m(\rho, (u, v)) = (\rho, \rho) \in \mathcal{X} \times \mathcal{X}$ for all framings (u, v) along m. In particular, B_m is not injective.*

Proof. Pick a loop ℓ on S which essentially intersects m exactly in two points (see Figure 8). We can assume, without loss of generality, that the base point of $\pi_1(S)$ is on m. Let γ be an element of $\pi_1(S)$ corresponding to ℓ. Then homotope ℓ so that ℓ is a composition of a loop ℓ_1 on $S \setminus F$ and a loop ℓ_2 on F. Since $\rho | \pi_1(F)$ is trivial, we have $B_m \eta(\gamma_\ell) = B_m \eta(\gamma_{\ell_1})$. We can take a generating set of $\pi_1(S)$ consisting of loops in $S \setminus F$ and loops in F. Therefore $B_m(\rho, (u, v)) = (\rho, \rho)$ in $\mathcal{X} \times \mathcal{X}$. □

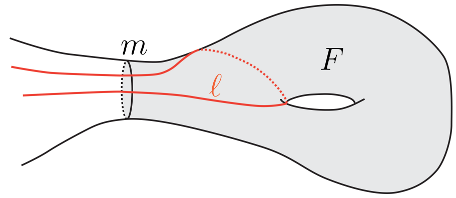

Figure 8.

10. Complex bending maps are almost proper

In this section, we prove the properness of the complex bending map, similarly to the injectivity in §9, in the complement of certain proper subvarieties. Similarly to X_M^p, we let $\mathcal{X}_M^p$ be the subvariety of the $\mathrm{PSL}_2\mathbb{C}$-character variety $\mathcal{X}$ consisting of representations $\xi\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C}$ such that, for, at least, one loop m of M, its holonomy $\xi(m)$ is parabolic (possibly the identity) in each factor of $\mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C}$ (equivalently in at least, one factor). Similarly to X_M^r, we let $\mathcal{X}_M^r$ be the subvariety of $\mathcal{X}$ such that, consisting of representations $\xi\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C}$ such that, for at least one component F of $S \setminus M$, $\xi|F$ is weakly reducible in each factor (equivalently, in one factor).

Theorem 10.1. *The restriction of B_M to $X_M \setminus (X_M^p \cup X_M^r)$ is a proper mapping to $(\mathcal{X} \setminus (\mathcal{X}_M^p \cup \mathcal{X}_M^r))^2$.*

Proof. Let $\eta_i \in X_M \setminus (X_M^P \cup X_M^r)$ be a sequence such that $B_M(\eta_i)$ converges to a representation in $(\mathcal{X} \setminus (\mathcal{X}_M^p \cap \mathcal{X}_M^r))^2$ as $i \to \infty$. It suffices to show that η_i also converges in $X_M \setminus (X_M^P \cup X_M^r)$.

Pick a representative $\xi_i\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C}$ of $B_N(\eta_i)$ so that ξ_i converges to $\xi_\infty\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C}$, so that its equivalence class $[\xi_\infty]$ is in $(\mathcal{X} \setminus (\mathcal{X}_M^p \cap \mathcal{X}_M^r))^2$. Let $\{H_{i,P}\}$ be the ξ_i-equivariant bending support system of the complex bending of η_i along M, where P varies over all connected components of $\tilde{S} \setminus \tilde{M}$. By the hypothesis, the restriction of ξ_∞ to each component of $S \setminus M$ is strongly irreducible. Therefore, by Lemma 9.2, the ξ_i-equivariant support system $\{H_{i,P}\}$ converges to a unique support system $\{H_P\}$ of ξ_∞ as $i \to \infty$.

We also show that the bending axes also converge.

Claim 10.2. *The ξ_i-equivariant axes system for bending η_i along M in $\mathbb{H}^3 \times \mathbb{H}^3$ converges to a ξ-equivariant axis system as $i \to \infty$.*

Proof. Let m be a loop of M, and let $\tilde{m}$ be a component of $\tilde{M}$ which descends to m. Let $\alpha \in \pi_1(S)$ denote the element preserving $\tilde{m}$ such that the free homotopy class α is m. Let P, Q denote the adjacent components of $\tilde{S} \setminus \tilde{M}$ separated by $\tilde{m}$. Then $H_{i,P} \cap H_{i,Q}$ is the complex bending axis $g_{i,\tilde{m}}$ for $\tilde{m}$ in $\mathbb{H}^3 \times \mathbb{H}^3$, and also the axis of $\xi_i(\alpha)$. The angle of the intersection of $H_{i,P}$ and $H_{i,Q}$ along the axis is equal to the weight of m. As $\xi_i(m)$ converges to a non-parabolic element $\xi(m)$, the axis $H_{i,P} \cap H_{i,Q}$ converges to the axis of $\xi(\alpha)$ as $i \to \infty$. □

For each $i = 1, 2, \ldots$, let $\{g_{i,\tilde{m}}\}$ denote the ξ_i-equivariant bending axis system in $\mathbb{H}^3 \times \mathbb{H}^3$ of B_M at η_i. Note that η_i is obtained by unbending ξ_i along the axes $g_{i,\tilde{m}}$ by the angles given by the weights M. By the convergence, similarly unbending the limit ξ in $(\mathcal{X} \setminus (\mathcal{X}_M^p \cup \mathcal{X}_M^r))^2$ along the limit bending axis system by the angles given by M, we obtain the limit of η_i as $i \to \infty$. As ξ is in $(\mathcal{X} \setminus (\mathcal{X}_M^p \cup \mathcal{X}_M^r))^2$, thus $\lim_{i\to\infty} \eta_i$ is contained in $X_M \setminus (X_M^p \cup X_M^r)$. 10.1

11. Analyticity of complex bending maps

Theorem 11.1. *For every weighted oriented multiloop M on S, the bending map $B_M \colon X_M \to \mathcal{X} \times \mathcal{X}$ is complex analytic.*

Proof. Recall that X_M^p is the subvariety of the complex-analytic variety X_M consisting of representations such that at least one loop of M is parabolic, and also that X_M^r is the subset of X_M consisting of representations η such that the restriction of η to a component of $S \setminus M$ is weakly reducible. We have shown that the restriction of B_M to $X_M \setminus X_M^p \cup X_M^r$ is injective (Theorem 9.1). We first prove the assertion of Theorem 11.1 for almost everywhere.

Lemma 11.2. *The restriction of B_M to $X_M \setminus (X_M^p \cup X_M^r)$ is complex analytic.*

Proof. Recall that R_M is the space of representations framed along M, and that $R_M /\!\!/ \mathrm{PSL}_2\mathbb{C} = X_M$. Let R_M^p be the subset of R_M consisting of framed representations, such that, at least, one loop of M is parabolic (or the identity). Let $\eta = (\rho, (u_i, v_i)_{i=1}^n)$ be an arbitrary framed representation in $R_M \setminus (R_M^p \cup R_M^r)$, where n is the number of the loops of M. As the closed subvariety $R_M^p \cup R_M^r$ is $\mathrm{PSL}_2\mathbb{C}$-invariant, we can take a $\mathrm{PSL}_2\mathbb{C}$-invariant open neighborhood U of η in $R_M \setminus (R_M^p \cup R_M^r)$. Then, for every framed representation $\zeta \in U$, the stabilizer of ζ in $\mathrm{PSL}_2\mathbb{C}$ is a discrete group, since ζ is *not* in R_M^r, Thus, if we take U appropriately, U is holomorphically a product of $\mathrm{PSL}_2\mathbb{C}$ and an open disk D. Let W be the image of U in X_M. Then, we can biholomorphically identify W in X_M with D in U and take a holomorphic section $s \colon W \to U$.

Pick any component of Q of $\tilde{S} \setminus \tilde{M}$, where $\tilde{M}$ is the inverse image of M in $\tilde{S}$. Let G_Q be the stabilizer of Q in $\pi_1(S)$. By $\mathbb{C}$-bending along M (normalizing so that the restriction to G_Q unchanged), we obtain a holomorphic mapping $s(W) \to (\mathcal{R} \setminus \mathcal{R}_M^p \cup \mathcal{R}_M^r)^2$ which is a lift of the restriction of B_M to W. Then, for every $\zeta \in s(W)$, its image by this mapping is a pair of strongly irreducible representations in $\mathcal{R}$. Since W is isomorphic to $s(W)$ and the quotient map from $\mathcal{R} \times \mathcal{R}$ to $\mathcal{X} \times \mathcal{X}$ is algebraic, the analyticity of $s(W) \to (\mathcal{R} \setminus \mathcal{R}_M^p \cup \mathcal{R}_M^r)^2$ implies the analycity of B_M at the equivalent class of η in X_M. □

By Lemma 11.2, $X_M \setminus (X_M^p \cup X_M^r) \to (\mathcal{X} \setminus \mathcal{X}_M^p \cup \mathcal{X}_M^r) \times (\mathcal{X} \setminus \mathcal{X}_M^p \cup \mathcal{X}_M^r)$ is an injective analytic mapping. Since $X_M^p \cup X_M^r$ is an analytic subvariety of X_M, by the removable singularity theorem (Theorem 3.7), the mapping $B_M \colon X_M \to \mathcal{X} \times \mathcal{X}$ is analytic. 11.1

12. The real-bending map sits in the complex-bending map

In this section, we observe that the complex-analytic bending map $B_M \colon X_M \to \mathcal{X} \times \mathcal{X}$ is a natural extension of the real-analytic bending map $b_M \colon \mathcal{T} \to \mathcal{X}$. Recall that Δ^* is the twisted diagonal $\{(\rho, \rho^*) \mid \rho \in \mathcal{X})\}$ and $\psi \colon \mathcal{X} \to \Delta^* \subset \mathcal{X} \times \mathcal{X}$ is the embedding given by $\rho \mapsto (\rho, \rho^*)$.

The forgetful map $X_M \to \mathcal{X}$ restricts to an analytic covering map $X_M \setminus X_M^p \to \mathcal{X} \setminus \mathcal{X}_M^p$ of degree 2^n, where n is the number of the loops of M. As the base surface S is oriented, we let $\mathcal{T}$ be the Teichmüller space of S is identified with a unique component of the real slice of $\mathcal{X}$. Then, by the choice of framings, there are 2^n ways to lift the Fricke-Teichmüller space $\mathcal{T}$ to X_M. Given an oriented weighted M on S, there is a unique lift of $\mathcal{T}$ to X_M such that, for each loop m of M, the ordering of the fixed points of the framing along m coincides with the orientation of M. Let $\iota_M \colon \mathcal{T} \to X_M$ be this real-analytic embedding.

Proposition 12.1. *Let M be a weighted oriented multiloop on S. Then, the restriction of B_M to $\mathcal{T}$ is a real-analytic embedding into the twisted diagonal Δ^* of $\mathcal{X} \times \mathcal{X}$, such that $B_M \circ \iota_M$ coincides with $\psi \circ b_M \colon \mathcal{T} \to \mathcal{X} \times \mathcal{X}$.*

Proof. Let $b_M^* \colon \mathcal{T} \to \mathcal{X}$ denote the complex conjugate of $b_M \colon \mathcal{T} \to \mathcal{X}$, i.e. the Fuchsian representation $\rho \colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{R}$ maps to the mapping taking $\gamma \in \pi_1(S)$ to $(b_M(\rho)(\gamma))^* \in \mathrm{PSL}_2\mathbb{C}$. When applying the complex bending B_M, a representation into $\mathrm{PSL}_2\mathbb{C}$ is bent in opposite directions in the first and the second factor of $\mathcal{X} \times \mathcal{X}$ (§8.2). Therefore, when applying B_M to a Fuchsian representation, the representation in the second factor is the complex conjugate of the representation in the first factor. Therefore $B_M \circ \iota_M(\rho)$ is $(b_M(\rho), b_M^*(\rho))$ for $\rho \in \mathcal{T}$, as desired. The analyticity of the was already proven in Theorem 11.1. □

13. Properness of the complex bending map along a non-separating loop

Theorem 13.1. *Let ℓ be an oriented non-separating loop with weight not equal to π modulo 2π. Then, the complex bending map $B_\ell\colon X_\ell \to \mathcal{X} \times \mathcal{X}$ is proper.*

Corollary 13.2. *The image of B_ℓ is a closed analytic set in $\mathcal{X} \times \mathcal{X}$.*

Pick $\theta \in (0, \pi)$. Let

$$E_\theta = \{(\gamma, e) \in \mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C} \mid e \text{ is elliptic of rotation angle } \theta\}.$$

Clearly, for every $(\gamma, e) \in \mathcal{E}_\theta$, $\mathrm{tr}^2\, e$ is a fixed constant in $(0, 4)$ only depending on θ. Thus E_θ is a smooth affine algebraic variety. Then $\mathrm{PSL}_2\mathbb{C}$ acts on $\mathcal{E}_\theta$ by conjugating both parameters γ and e simultaneously. Let $\mathcal{E}_\theta$ be the GIT-quotient $E_\theta /\!\!/ \mathrm{PSL}_2\mathbb{C}$. Then $\mathcal{E}_\theta$ is an affine algebraic variety. Then the following holds.

Lemma 13.3. *The analytic mapping $E_\theta /\!\!/ \mathrm{PSL}_2\mathbb{C} \to \mathbb{C}^2$ defined by $\phi\colon (\gamma, e) \mapsto (\mathrm{tr}^2\, \gamma, \mathrm{tr}^2\, \gamma e)$ is a proper mapping.*

Proof. The map $\mathrm{SL}_2\mathbb{C} \times \mathrm{SL}_2\mathbb{C} /\!\!/ \mathrm{SL}_2\mathbb{C} \to \mathbb{C}^2$ given by $(\gamma, e) \mapsto (\mathrm{tr}\, \gamma, \mathrm{tr}\, e, \mathrm{tr}\, \gamma e)$ is a biholomorphic map (see for example, [Gol09]).

Let (α_i, e_i) be a sequence in $\mathcal{E}_\theta \subset \mathrm{PSL}_2\mathbb{C} \times \mathrm{PSL}_2\mathbb{C} /\!\!/ \mathrm{PSL}_2\mathbb{C}$ which leaves every compact as $i \to \infty$. Pick any lift $(\tilde{\alpha}_i, \tilde{e}_i) \in \mathrm{SL}_2\mathbb{C} \times \mathrm{SL}_2\mathbb{C} /\!\!/ \mathrm{SL}_2\mathbb{C}$ of (α_i, e_i) for each i. Then $(\tilde{\alpha}_i, \tilde{e}_i)$ also leaves every compact as $i \to \infty$.

By a basic trace identity, we have $\mathrm{tr}\, \tilde{\alpha}_i \tilde{e}_i + \mathrm{tr}\, \tilde{\alpha}_i \tilde{e}_i^{-1} = \mathrm{tr}\, \tilde{\alpha}_i\, \mathrm{tr}\, \tilde{e}_i$. Therefore, since $\mathrm{tr}\, \tilde{e}_i$ is a fixed non-zero constant, up to a subsequence, either $\mathrm{tr}\, \tilde{\alpha}_i$ or $\mathrm{tr}\, \tilde{\alpha}_i \tilde{e}_i$ diverges to ∞ as $i \to \infty$. Thus the image $\phi(\alpha_i, e_i)$ leaves every compact in $\mathbb{C}^2$ as $i \to \infty$. □

Pick a generating set $\{\gamma_1, \dots, \gamma_{2g}\}$ of $\pi_1(S)$ such that $\gamma_1, \dots, \gamma_{2g}$ correspond to loops on S intersecting ℓ exactly once. Let $\eta_i = [\rho_i, (u_i, v_i)] \in X_\ell$ be a sequence which leaves every compact in X_ℓ.

Let $w(\ell)$ denote the weight of ℓ, and let $e_i \in \mathrm{PSL}_2\mathbb{C}$ be the elliptic element by angle $w(\ell)$ along the geodesic from u_i to v_i. Then we can normalize $(\rho_i, (u_i, v_i))$, by an element of $\mathrm{PSL}_2\mathbb{C}$, so that $e_i \in \mathrm{PSL}_2\mathbb{C}$ is independent on i. Let e be the independent elliptic element in $\mathrm{PSL}_2\mathbb{C}$.

As η_i leaves every compact and $\gamma_1, \dots, \gamma_n$ form a generating set of $\pi_1(S)$, then there is $k \in \{1, \dots, n\}$ such that, up to a subsequence, $\rho_i(\gamma_k)$ leaves every compact as $i \to \infty$. Then, since γ_k intersect ℓ exactly at once, by the properness of Lemma 13.3, the image $B_\ell(\eta_i)\gamma_k$ also leaves every compact as $i \to \infty$. This immediately implies the properness of B_ℓ.

14. Symplectic property

In this section, we prove the symplectic property of the bending maps. Complex Fenchel-Nielsen coordinates on the quasi-Fuchsian space are introduced by [Kou94] and [Tan94], and the coordinates holomorphically extend to most part of the character variety $\mathcal{X}$. We explicitly explain the subset of $\mathcal{X}$ where the complex Fenchel-Nielsen coordinates are defined.

Let M be a maximal multiloop on S. Then M contains $3g-3$ loops, where g is the genus of S. Let $\mathcal{X}^h_M$ be the (Euclidean) open full-measure subset of $\mathcal{X}$ consisting of $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ such that

- all loops of M are hyperbolic, and
- for each component P of $S \setminus M$, the restriction of ρ to $\pi_1(P)$ is irreducible.

Pick (real) Fenchel-Nielsen coordinates on the Teichmüller-Fricker space $\mathcal{T}$ with respect to M (see [FM12] for example). Let $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \mathrm{Re} z > 0\}$. For each $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ in $\mathcal{X}^h_M$, let $\ell_i \in \mathbb{C}_+/2\pi I\mathbb{Z}$ be the complex translation length of $\rho(m_i)$: When we $\ell_i = x_i + Iy_i$ in real and imaginary coordinates, $x_i \in \mathbb{R}_{\geq 0}$ is the (real) translation length and the $y_i \in \mathbb{R}$ is the rotation angle of the screw motion of the hyperbolic element $\rho(m_i)$.

Clearly, for real representations $\pi_1(S) \to \mathrm{PSL}_2\mathbb{R}$, their length parameters $\ell_1, \dots, \ell_{3g-3}$ are all real numbers. Let $\tau_i \in \mathbb{C}/2\pi I\mathbb{Z}$ be the twist coordinate along ℓ_i which complexifies the Fenchel-Nielsen twist coordinate, so that the imaginary direction is the direction of bending deformation (where I denotes the imaginary unite.). Similarly, for real representations $\pi_1(S) \to \mathrm{PSL}_2\mathbb{R}$, their twist parameters $\tau_1, \dots, \tau_{3g-3}$ are all real numbers.

Lemma 14.1. *Then $\mathcal{X}^h_M$ is a (Zariski) open dense subset of $\mathcal{X}$ and biholomorphic to $(\mathbb{C}_+/2\pi I\mathbb{Z})^{3g-3} \bigoplus (\mathbb{C}/2\pi I\mathbb{Z})^{3g-3}$ by $(\ell_1, \dots, \ell_{3g-3}, \tau_1, \dots, \tau_{3g-3})$.*

Proof. The mapping $\mathcal{X}^h_M \to (\mathbb{C}_+/2\pi I\mathbb{Z})^{3g-3} \bigoplus (\mathbb{C}/2\pi I\mathbb{Z})^{3g-3}$ is a holomorphic mapping, as the coordinates are given by traces of loops.

Given a pair of pants P, the irreducible representations $\pi_1(P)$ are algebraically parametrized by the holonomy traces of the three boundary components of P ([Vog89] [Fri96]; see also [Gol09]). Now let P be a component of $S \setminus M$. Then $\rho \in \mathcal{X}^h_M$, the $\rho|\pi_1(P)$ is parametrized by the complex length coordinates of the boundary components of P.

For a loop m_i of M, let F be the component of $S \setminus (M \setminus \ell)$ which contains M. Then the representation on $\pi_1(F) \to \mathrm{PSL}_2\mathbb{C}$ is determined by the twisting parameter τ_i of m_i and the length parameters

ℓ_i of m_i and the boundary loops of F. We see that the mapping is biholomorphic. □

Due to Platis [Pla01] and Goldman [Gol04], the complex Fenchel-Nielsen coordinates yield Darboux coordinates for Goldman's complex symplectic structure.

$$w_G = \Sigma_{i=1}^{3g-3} d\ell_{m_i}^{\mathbb{C}} \wedge dt_{m_i}^{\mathbb{C}}.$$

(see Loustau [Lou15] for details). To be concrete and self-contained, we first explain the Darboux coordinates on $\mathcal{X}_M^h$.

Lemma 14.2. *Let $M = m_1 \sqcup m_2 \sqcup \cdots \sqcup m_{3g-3}$ be a maximal multiloop on S. Then $w_G = \Sigma_{i=1}^{3g-3} d\ell_{m_i}^{\mathbb{C}} \wedge dt_{m_i}^{\mathbb{C}}$ on $\mathcal{X}_M^h$*

Proof. The symplectic structure w_G is a complex symplectic structure, so that the 2-form changes holomorphically in $\mathcal{X}$. On the Fricke-Teichmüller space space, $w_G|\mathcal{T}$ is given by $\Sigma d\ell_{m_i}^{\mathbb{R}} \wedge dt_{m_i}^{\mathbb{R}}$. Therefore, since the complex Fenchel-Nielsen coordinates are holomorphic coordinates (Lemma 14.1), $w_G = \Sigma d\ell_{m_i}^{\mathbb{C}} \wedge dt_{m_i}^{\mathbb{C}}$ on $\mathcal{X}_M^h$. □

Then this Darboux coordinates on $\mathcal{X}_M^h$ gives the symplectic property of the real bending map.

Proposition 14.3. *If M is a weighted multiloop on S, then $b_M \colon \mathcal{T} \to \mathcal{X}$ is a symplectic embedding.*

Proof. As M may *not* be maximal, we pick a maximal multiloop M' on S containing M. Set $m_1, m_2, \ldots, m_{3g-3}$ to be the loops of M'. Let $w_1, w_2, \ldots w_{3g-3} \in \mathbb{R}_{\geq 0}$ be the weight of the loops of M' (so that, if ℓ_i is *not* a loop of the original multiloop M, then $w_i = 0$). The Teichmüller-Fricke space $\mathcal{T}$ is a component of the real slice of $\mathcal{X}_M^h$. In the Darboux coordinates of Lemma 14.2, the real bending map $b_M \colon \mathcal{T} \to \mathcal{X}$ extends to $\hat{b}_M \colon \mathcal{X}_M^h \to \mathcal{X}_M^h$ by the translation

$$(\ell_1, \ldots, \ell_{3g-3}, \tau_1, \ldots, \tau_{3g-3}) \mapsto (\ell_1, \ldots, \ell_{3g-3}, \tau_1 + w_1 I, \ldots, \tau_{3g-3} + w_{3g-3} I).$$

As it is a translation in the Darboux coordinates, $b_M \colon \mathcal{T} \to \mathcal{X}$ is clearly a symplectic embedding. □

By the limiting argument, all real bending maps are symplectic.

Theorem 14.4. *For every $L \in \mathcal{ML}$, $b_L \colon \mathcal{T} \to \mathcal{X}$ is a symplectic embedding w.r.t. Goldman's symplectic structure.*

Proof. Let ℓ_i be a sequence of weighted loops which converges to L in $\mathcal{ML}$ as $i \to \infty$. (Recall that $b_{\ell_i} \colon \mathcal{T} \to \mathcal{X}$ is a real analytic embedding.) For each $\tau \in \mathcal{T}$, the tangent space of b_{ℓ_i} at τ converges to the tangent space of b_L at τ. By Proposition 14.3, $b_{\ell_i} \colon \mathcal{T} \to \mathcal{X}$ is a symplectic embedding for each $i = 1, 2, \ldots$. Therefore, by the continuity of the symplectic structure w_G, the limit b_L is also symplectic at τ. □

14.1. **Symplectic property for complex bending map.** As $X_M \setminus X_M^p \to \mathcal{X} \setminus \mathcal{X}_M^p$ is an analytic covering map, $X_M \setminus X_M^p$ has a pull-back symplectic structure.

A representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ is reductive, if the Zariski-closure of the image $\operatorname{Im}\rho \subset \mathrm{PSL}_2\mathbb{C}$ is reductive. (That is, the maximal normal unipotent subgroup of $\operatorname{Im}\rho$ is the trivial group.) Then a representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ is non-reductive, if and only if $\operatorname{Im}\rho$ is conjugate to a subgroup consisting of upper triangular matrices which contains at least one (non-identity) parabolic element. Let X_M^r be the set of framed representations $\eta = [\rho, (u_i, v_i)]$ of X_M such that ρ is a reductive representation other than the trivial representation.

Theorem 14.5. *The restriction of B_M to $X_M^r \setminus X_M^p$ is a complex-symplectic map.*

Proof. We show that the restriction of $b_M^{\pm}\colon X_M^r \to \mathcal{X}$ is symplectic on $\mathcal{X}_M^h$.

Let $\eta = [\rho, (u_i, v_i)]$ be a framed representation of $X_M^r \setminus X_M^p$, so that $\rho\colon \pi_1(S) \to \mathrm{PSL}_2\mathbb{C}$ is in the representation variety $\mathcal{R}$. Note that the fixed point set of ρ in $\mathbb{CP}^1$ is either the empty set or two points on $\mathbb{CP}^1$ as $\rho \notin R_M^p$. Thus ρ is reductive. Therefore, the tangent cone and the Zariski tangent space of $\mathcal{R}$ coincide at ρ (see Goldman [Gol85, Theorem 3]). As ρ is reductive, ρ has a closed $\mathrm{PSL}_2\mathbb{C}$-orbit. Therefore, we can apply Luna's slice theorem ([ByBCM02] Theorem 15.5), and the neighborhood of $[\rho]$ is $\mathcal{X}$ is isomorphically embedded in a finite quotient of an algebraic slice through ρ in X_M^p. Thus the Zariski tangent space of $\mathcal{X}$ at the projection $[\rho] \in \mathcal{X}$ also coincides with the tangent cone of $\mathcal{X}$ at $[\rho]$.

Recall that$\mathcal{X}_M^h$ is an open dense subset of $\mathcal{X}$. Therefore, the Zariski tangent space changes continuously in ρ, and the symplectic property of $b_M^{\pm}$ on $\mathcal{X}_M^h$, given by Lemma 14.2, continuously extends to the entire $X_M^r \setminus X_M^p$. Hence, as $\eta \mapsto (b_M^+(\eta), b_M^-(\eta))$, the restriction of B_M to $X_M^p \setminus X_M^r$ is symplectic. (Figure 9.) □

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$$
\begin{array}{ccc}
X_\ell \setminus X_\ell^p & \xrightarrow{b_M} & \chi \setminus \chi^p \subset \chi \\
& \searrow^{B_M} & \downarrow{\scriptstyle b_M^- \times b_M^+} \\
& & \chi \times \chi
\end{array}
$$

FIGURE 9. A local commutative diagram for the complexification of the real bending map.

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