On Elliptic Artin Groups

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§ Introduction

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Properties

• One of generalization of finite or affine root systems.
• The structure of them is described by the elliptic Dynkin diagrams.
• In his original motivation, vertices in an elliptic Dynkin diagram correspond to vanishing cycles and edges describe intersection numbers of them.
• Elliptic root systems appear as “root systems” of toroidal / double loop Lie algebras.
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  - In his original motivation, vertices in an elliptic Dynkin diagram correspond to vanishing cycles and edges describe intersection numbers of them.
  - Elliptic root systems appear as “root systems” of toroidal / double loop Lie algebras.

- K. Saito and Takebayashi (1990):
  They studied the corresponding Weyl groups (the elliptic Weyl groups).
    - Presentations by generators and relations attached to the elliptic Dynkin diagrams are given.
Yamada (2000):
He studied a \( q \)-analogue of the elliptic Weyl groups (the elliptic Hecke algebras) for several types of marked elliptic root systems.

Shiota-S (2011):
They studied elliptic Hecke algebras for (almost) all marked elliptic root systems, and made a complete dictionary between them and Cherednik-Macdonald's double affine Hecke algebras.
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**Remark.** He also studied the corresponding Artin groups (*the elliptic Artin groups*) for some special types of marked elliptic root systems, and relationship between these groups and the fundamental groups of the compliments of the discriminants in the semi-universal deformations of simple elliptic singularities studied by Looijenga, Givental and van der Lek.
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• Shiota-S (2011):
They studied elliptic Hecke algebras for (almost) all marked elliptic root systems, and made a complete dictionary between them and Cherednik-Macdonald’s double affine Hecke algebras.
What are (marked) elliptic root systems?

- **finite case** = a root system lives in vector spaces with a positive definite bilinear form $I$

- **affine case** = a root system lives in vector spaces with a bilinear form $I$ of $\dim(\text{rad}(I)) = 1$

- In an elliptic root system, there are two primitive imaginary roots $\mathbf{1}$ and $\mathbf{2}$.

- An elliptic root system automatically has a $\text{SL}_2(\mathbb{Z})$-symmetry:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $(a, b, c, d) \in \text{SL}_2(\mathbb{Z})$.

- **Aim**: Study this symmetry on the corresponding Hecke algebras and Artin groups.
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**i.e.** In an elliptic root system, there are two primitive imaginary roots (null roots) \( \delta_1 \) and \( \delta_2 \).
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\begin{align*}
\delta_1 & \mapsto a\delta_1 + c\delta_2 \\
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Plan

§1 Review on elliptic root systems

§2 Definition of elliptic Artin groups

§3 $SL_2(\mathbb{Z})$-action

§4 Hidden braid group symmetries on elliptic Artin groups
§1 Review on elliptic root systems

Let $F$ be an $(n+2)$ dim'l real vector space, and $I:F \times F \to \mathbb{R}$ a positive semi-definite symmetric bilinear form with 2-dim'l radical $\text{rad}(I)$.

For a non-isotropic vector $\alpha \in F$ ($\iff I(\alpha, \alpha) \neq 0$), set

$$s_\alpha(u) := u - I(u, \alpha^\vee)\alpha \quad (u \in F'),$$

where $\alpha^\vee := 2\alpha/I(\alpha, \alpha)$.
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where \( \alpha^\vee := 2\alpha/I(\alpha, \alpha) \).

**Definition.** (1) A set of non-isotropic vectors \( R \) is an elliptic root system if
(i) \( Q(R) \otimes \mathbb{R} \cong F \) where \( Q(R) \) is the additive subgroup of \( F \) gen. by \( R \),
(ii) \( s_\alpha(R) = R \) for every \( \alpha \in R \),
(iii) \( I(\alpha, \beta^\vee) \in \mathbb{Z} \) for every \( \alpha, \beta \in R \),
(iv) \( R \) is irreducible.
(2) A subspace \( G \) of \( \text{rad}(I) \) of rank 1 defined over \( \mathbb{Q} \) is called a marking, and a pair \((R, G)\) is called a marked elliptic root system.
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(i) $Q(R) \otimes \mathbb{R} \cong F$ where $Q(R)$ is the additive subgroup of $F$ gen. by $R$,
(ii) $s_\alpha(R) = R$ for every $\alpha \in R$,
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(iv) $R$ is irreducible.

(2) A subspace $G$ of $\text{rad}(I)$ of rank 1 defined over $\mathbb{Q}$ is called a **marking**, and a pair $(R, G)$ is called a **marked elliptic root system**.

Let $\pi_a : R \to R/G$ (resp. $\pi_f : R \to R/\text{rad}(I)$) be a natural projection.

$\Rightarrow R_a := \pi_a(R)$ (resp. $R_f := \pi_f(R)$) is an affine (resp. finite) root system.
Example.

$F_0$: an $n$-dim’l real vector space
$I_0: F_0 \times F_0 \to \mathbb{R}$ a positive definite bilinear form
$R_0$: a finite root system of type $X_n$ ($X = A, B, C, D, E, F, G$)
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$R_0$: a finite root system of type $X_n$ ($X = A, B, C, D, E, F, G$)

Set $F := F_0 \oplus \mathbb{R}\delta_1 \oplus \mathbb{R}\delta_2$, and define $I: F \times F \rightarrow \mathbb{R}$ by

$$I|_{F_0} = I_0 \quad \text{and} \quad \text{rad}(I) = \mathbb{R}\delta_1 \oplus \mathbb{R}\delta_2.$$ 

Then, $R := \{\alpha + n_1\delta_1 + n_2\delta_2 | \alpha \in R_0, n_1, n_2 \in \mathbb{Z}\}$ is an elliptic root system.

Set $G := \mathbb{R}\delta_1$.

⇒ The pair $(R, G)$ is a marked elliptic root system (of type $X_n^{(1,1)}$).
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$\Rightarrow$ The pair $(R, G)$ is a marked elliptic root system (of type $X_n^{(1,1)}$).

- $R_a \cong \{\alpha + n_2\delta_2 \mid \alpha \in R_0, n_2 \in \mathbb{Z}\}$ : the affine root system of type $X_n^{(1)}$.
- $R_f \cong \{\alpha \mid \alpha \in R_0\}$ : the finite root system of type $X_n$. 
Example.

\( F_0 \): an \( n \)-dim’l real vector space
\( I_0 : F_0 \times F_0 \to \mathbb{R} \) a positive definite bilinear form
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\( \Rightarrow \) The pair \((R, G)\) is a marked elliptic root system (of type \( X_n^{(1,1)} \)).

- \( R_a \cong \{\alpha + n_2\delta_2 \mid \alpha \in R_0, n_2 \in \mathbb{Z}\} \) : the affine root system of type \( X_n^{(1)} \).
- \( R_f \cong \{\alpha \mid \alpha \in R_0\} \) : the finite root system of type \( X_n \).

- In this talk, we mainly consider marked elliptic root systems of type \( X_n^{(1,1)} \).
Elliptic Dynkin diagrams

Let \((R, G)\) be a marked elliptic root system.

Take a generator \(\delta_1\) of \(G \cap Q(R)\) (i.e. \(G \cap Q(R) = \mathbb{Z}\delta_1\)).

For \(\alpha \in R\), set \(k_\alpha := \inf\{k \in \mathbb{Z}_{>0} \mid \alpha + k\delta_1 \in R\}\), and \(\alpha^* := \alpha + k_\alpha\delta_1\).
Elliptic Dynkin diagrams

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For $\alpha \in R$, set $k_\alpha := \inf\{k \in \mathbb{Z}_0^+ | \alpha + k\delta_1 \in R\}$, and $\alpha^* := \alpha + k_\alpha\delta_1$.

Fix a subset $\Gamma_a = \{\alpha_0, \cdots, \alpha_n\}$ of $R$ which has the following properties:

- $\pi_a(\Gamma_a) = \{\pi_a(\alpha_0), \cdots, \pi_a(\alpha_n)\}$ form a basis of $R_a$,
- $\delta_a = \sum_{i=0}^n n_i \pi_a(\alpha_i)$ with $n_i \in \mathbb{Z}_0^+$ and $n_0 = 1$.
  ($\delta_a$ is a primitive imaginary root of $R_a$.)

Set $\delta_2 := \sum_{i=0}^n n_i \alpha_i$. 
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  \((\delta_a\) is a primitive imaginary root of \(R_a\).)
Set \(\delta_2 := \sum_{i=0}^{n} n_i\alpha_i\).

For \(0 \leq i \leq n\), set \(m_i := I_R(\alpha_i, \alpha_i)n_i/2k_\alpha_i\).
Here \(I_R\) is a normalization of \(I\) s.t. \(\inf\{I_R(\alpha, \alpha) | \alpha \in R\} = 2\).
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Let \(m_{max} := \max\{m_i \mid 0 \leq i \leq n\}\), and set

\[
\Gamma_{max} := \{\alpha_i \in \Gamma_a \mid m_i = m_{max}\} \quad \text{and} \quad \Gamma_{max}^* := \{\alpha_i^* \mid \alpha_i \in \Gamma_{max}\}.
\]
Define a finite graph \( \Gamma(R, G) \) (the elliptic Dynkin diagram of \( (R, G) \)) by the following way:

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(1) The set of vertices is $\Gamma := \Gamma_a \cup \Gamma^*_{\text{max}}$.

(2) Draw bonds and arrows among vertices according to the same rules for usual root system.

- $\alpha \quad \circ \beta$ if $I(\alpha, \beta) = I(\beta, \alpha) = 0$,
- $\alpha \circ \dashv \beta$ if $I(\alpha, \beta^\vee) = I(\beta, \alpha^\vee) = -1$,
- $\alpha \mu \dashv \beta$ if $I(\alpha, \beta^\vee) = -\mu$ and $I(\beta, \alpha^\vee) = -1$ for $\mu = 2, 3$,
- $\alpha \infty \dashv \beta$ if $I(\alpha, \beta^\vee) = I(\beta, \alpha^\vee) = -2$. 

**Theorem (K.Saito).** The isomorphism classes of marked elliptic root systems are completely classified by their elliptic Dynkin diagrams.
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**Theorem (K. Saito).** The isomorphism classes of marked elliptic root systems are completely classified by their elliptic Dynkin diagrams.
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**Example.** Let us consider the marked elliptic root system of type $B_n^{(1,1)}$.

(i) Let $R_0$ be the finite root system of type $B_n$. By the definition, we have

$$R = \{\alpha_f + n_1\delta_1 + n_2\delta_2 \mid \alpha_f \in R_0, n_1, n_2 \in \mathbb{Z}\} \quad \text{and} \quad G = \mathbb{Z}\delta_1.$$
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(ii) Recall $\pi_a : F \to F/G$.

$$R_a = \pi_a(R) \cong \{ \alpha_f + n_2 \delta_2 \mid \alpha_f \in R_0, n_2 \in \mathbb{Z} \} : \text{the affine root system of type } B_n^{(1)}.$$
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(iii) $\Gamma_a = \{ \alpha_0, \cdots, \alpha_n \} : \text{an usual basis of } R(B_n^{(1)}).$
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\[ B_{n}^{(1)} \]
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$$\delta_2 = 1\alpha_0 + 1\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + 1\alpha_n.$$ 

$$\Rightarrow \quad m_0 = 1, m_1 = 1, m_2 = 2, \ldots, m_{n-1} = 2, m_n = 1.$$ 

$$\Rightarrow \quad \Gamma_{\text{max}} = \{ \alpha_2, \ldots, \alpha_{n-1} \} \quad \text{and} \quad \Gamma_{\text{max}}^* = \{ \alpha_2^*, \ldots, \alpha_{n-1}^* \}.$$

![Diagram of $B_n^{(1)}$ root system]
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(ii) Recall $\pi_a : F \to F/G$.

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(iii) $\Gamma_a = \{\alpha_0, \cdots, \alpha_n\} : \text{an usual basis of } R(B_n^{(1)})$.

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$$\Rightarrow \quad \Gamma_{max} = \{\alpha_2, \cdots, \alpha_{n-1}\} \quad \text{and} \quad \Gamma_{max}^* = \{\alpha_2^*, \cdots, \alpha_{n-1}^*\}.$$ 

\[ \begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{elliptic_root_system.png}
\caption{Elliptic root system of type $B_n^{(1,1)}$.}
\end{figure} \]
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$$R = \{ \alpha_f + n_1 \delta_1 + n_2 \delta_2 \mid \alpha_f \in R_0, n_1, n_2 \in \mathbb{Z} \} \quad \text{and} \quad G = \mathbb{Z} \delta_1.$$

(ii) Recall $\pi_a : F \to F/G$.

$$R_a = \pi_a(R) \cong \{ \alpha_f + n_2 \delta_2 \mid \alpha_f \in R_0, n_2 \in \mathbb{Z} \} : \quad \text{the affine root system of type } B_n^{(1)}.$$

(iii) $\Gamma_a = \{ \alpha_0, \cdots, \alpha_n \} : \text{an usual basis of } R(B_n^{(1)})$.

$$\delta_2 = 1\alpha_0 + 1\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + 1\alpha_n.$$

$$\Rightarrow \quad m_0 = 1, m_1 = 1, m_2 = 2, \cdots, m_{n-1} = 2, m_n = 1.$$

$$\Rightarrow \quad \Gamma_{max} = \{ \alpha_2, \cdots, \alpha_{n-1} \} \quad \text{and} \quad \Gamma_{max}^* = \{ \alpha_2^*, \cdots, \alpha_{n-1}^* \}.$$

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Add bonds and arrows according to the usual rules, and new bonds and arrows.
Elliptic Weyl groups

\[ W(R, G) := \langle s_\alpha \mid \alpha \in R \rangle \subset O(F, I) \] (Elliptic Weyl group).
Elliptic Weyl groups

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Set $$\Lambda := F \oplus \mathbb{R} \gamma_1 \oplus \mathbb{R} \gamma_2,$$ and define an extended symmetric bilinear form $$I_\Lambda : \Lambda \times \Lambda \to \mathbb{R}$$ by

$$I_\Lambda(\alpha_f, \gamma_j) = 0, \quad I_\Lambda(\delta_i, \gamma_j) = \delta_{i,j} \quad (\alpha_f \in R_0, i, j = 1, 2).$$ 

$$\Rightarrow I_\Lambda$$ is a nondegenerate symmetric bilinear form of signature $$(n + 2, 2).$$
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\( \bullet \) \( R \) is considered as a non-isotropic subset of \( \Lambda \).

\( \Rightarrow \) For each \( \alpha \in R \), one can consider the corresponding reflection:

\[
s_{\Lambda, \alpha} : u \mapsto u - I_\Lambda(u, \alpha^\vee)u \quad (u \in \Lambda).
\]

Set

\[
W_\Lambda(R, G) := \langle s_{\Lambda, \alpha} \mid \alpha \in R \rangle \subset O(\Lambda, I_\Lambda).
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Remark.
- \( W(R, G) \) and \( W_\Lambda(R, G) \) are not Coxeter groups.
- Especially, word problems have not solved.

\( \Rightarrow \) Technology in Coxeter groups can not be applied.
Theorem (K. Saito-Takebayashi, Yamada, Shiota-S). $W_{\Lambda}(R, G)$ has a presentation with generators $s_{\Lambda, \alpha}$ ($\alpha \in \Gamma(R, G)$) subject to $(s_{\Lambda, \alpha})^2 = 1$, usual Coxeter relations, and "elliptic Coxeter relations".

relations between two generators.

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![Diagram](image)

Relations between 3 generators

Relations between 4 generators
Example. Recall the elliptic Dynkin diagram of type $B_n^{(1,1)}$. 

There are many subdiagrams isomorphic to $e$. 

Set $r_i := s_i$; $r_i := s_i$; $x_i := r_i r_i r_i$. 

For every subdiagram of the form $f_i f_j f_i$, $r_j (r_i r_i r_i) r_j (r_i r_i r_i) = (r_i r_i r_i) r_j (r_i r_i r_i)$.
Example. Recall the elliptic Dynkin diagram of type $B_n^{(1,1)}$.

There are many subdiagrams isomorphic to $\alpha_i$. 

\[
\begin{array}{cccc}
\alpha_0 & \alpha_2^* & \alpha_3^* & \alpha_{n-1}^* \\
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_{n-1}
\end{array}
\]
Example. Recall the elliptic Dynkin diagram of type $B_n^{(1,1)}$.

![Elliptic Dynkin Diagram](https://via.placeholder.com/150)

There are many subdiagrams isomorphic to ![Isomorphic Subdiagram](https://via.placeholder.com/150).
**Example.** Recall the elliptic Dynkin diagram of type $B_n^{(1,1)}$.

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\alpha_{n-1}
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There are many subdiagrams isomorphic to $\Delta$

Set

$$r_i := s_{\Lambda, \alpha_i}, \quad r_i^* := s_{\Lambda, \alpha_i^*}, \quad \text{and} \quad x_i := r_ir_i^*.$$  

For every subdiagram of the form $\Delta$ we assume

$$r_jx_ir_jx_i = x_ir_jx_ir_j \quad \Leftrightarrow \quad r_j(r_ir_i^*)r_j(r_ir_i^*) = (r_ir_i^*)r_j(r_ir_i^*)r_j.$$
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The attached relation is

$$r_0 x_2 r_0 x_2 = x_2 r_0 x_2 r_0.$$ 

There are many subdiagrams isomorphic to

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$$r_j x_i r_j x_i = x_i r_j x_i r_j \quad \Leftrightarrow \quad r_j (r_i r_i^*) r_j (r_i r_i^*) = (r_i r_i^*) r_j (r_i r_i^*) r_j.$$
o An abelian subgroup $N$ of $W_{\Lambda}(R,G)$

In general, there exists a vertex $\alpha_j \in \Gamma_a \setminus \Gamma_{\text{max}}$. For such a vertex, we define $x_j$ by the following way.
An abelian subgroup $N$ of $W_\Lambda(R, G)$

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**Lemma.** The component $\Gamma(R, G) \setminus (\Gamma_{\max} \cup \Gamma_{\max}^*) = \Gamma_a \setminus \Gamma_{\max}$ is a disjoint union of $A$-type diagrams, say $\Gamma(A_{l_1}), \cdots, \Gamma(A_{l_r})$. 
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Assume $\alpha_j \in \Gamma_a \setminus \Gamma_{max}$, then $\alpha_j$ belongs to a component $\Gamma(A_{l_k}) = \{\alpha_{i_1}, \cdots, \alpha_{i_{l_k}}\}$ for some $k$. Let us consider of the following diagram:

\[
\begin{array}{c}
\alpha_{i_0}^* \\
\downarrow \\
\alpha_{i_0} \\
\end{array} \quad \mu \quad \begin{array}{c}
\alpha_{i_1} \\
\alpha_{i_2} \\
\vdots \\
\alpha_{i_k} \\
\end{array}
\]

where $\alpha_{i_0}$ is a vertex in $\Gamma_{max}$ which is connected to $\Gamma(A_{l_k})$. 

\[\mu = 1, 2^\pm, 3^\pm,\]
An abelian subgroup \( N \) of \( W_{\Lambda}(R, G) \)

In general, there exists a vertex \( \alpha_j \in \Gamma_a \setminus \Gamma_{max} \). For such a vertex, we define \( x_j \) by the following way.

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where \( \alpha_{i_0} \) is a vertex in \( \Gamma_{max} \) which is connected to \( \Gamma(A_{l_k}) \).

By using this diagram we define

\[
x_j := r_j x_{j-1} r_j x_{j-1}^{-1} \quad (1 \leq j \leq l_k)
\]

inductively.
Example. Type $B_{n}^{(1,1)}$. 

$\begin{align*}
\alpha_0 & \quad \alpha_2^* \quad \alpha_3^* \\
\alpha_1 & \quad \alpha_2 & \quad \alpha_3 & \quad \alpha_{n-1} \\
\alpha_1 & \quad \alpha_2 & \quad \alpha_3 & \quad \alpha_{n-1} \\
\end{align*}$
Example. Type $B_{n}^{(1,1)}$: $\Gamma_{\text{max}} = \{\alpha_{2}, \cdots, \alpha_{n-1}\}$.
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- For $2 \leq i \leq n - 1$ ($\iff \alpha_i \in \Gamma_{\text{max}}$: The cyan vertices),

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- For $2 \leq i \leq n-1$ ($\Leftrightarrow \alpha_i \in \Gamma_{\text{max}}$: The cyan vertices),
  $$x_i := r_i r_i^*.$$

- For $i = 0, 1, n$ ($\Leftrightarrow \alpha_i \in \Gamma_a \setminus \Gamma_{\text{max}}$: The red vertices),
  $$x_0 := r_0 x_2 r_0 x_2^{-1} = r_0 (r_2 r_2^*) r_0 (r_2 r_2^*)^{-1},$$
  $$x_1 := r_1 x_2 r_1 x_2^{-1} = r_1 (r_2 r_2^*) r_1 (r_2 r_2^*)^{-1},$$
  $$x_n := r_n x_{n-1} r_n x_{n-1}^{-1} = r_n (r_{n-1} r_{n-1}^*) r_n (r_{n-1} r_{n-1}^*)^{-1}.$$
For $\alpha \in \Gamma_a$, set
\[
\alpha^\dagger := k_\alpha \alpha^\vee.
\]
Hence, $Q((R, G)_a) := \bigoplus_{i=0}^n \mathbb{Z}\alpha_i^\dagger$ forms a root lattice of an affine root system $(R, G)_a$ with a basis $\{\alpha_i^\dagger\}_{i=0}^n$. 
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**Proposition (K.Saito-Takebayashi).** (1) The subgroup $N$ of $W_\Lambda(R, G)$ generated by $x_i$ $(i = 0, \ldots, n)$ is an abelian group.
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**Proposition (K.Saito-Takebayashi).** (1) The subgroup $N$ of $W_\Lambda(R, G)$ generated by $x_i$ ($i = 0, \ldots, n$) is an abelian group.

(2) Let $\delta'_a$ be a primitive imaginary root of the affine root system $(R, G)_a$. Write $\delta'_a = \sum_{i=0}^{n} n'_i \alpha_i^\dagger$ with $n'_{\alpha_0^\dagger} = 1$. Then, the element $x_0^{n'_0} \cdots x_n^{n'_n}$ belongs to the center of $W_\Lambda(R, G)$.
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Hence, $Q((R, G)_a) := \bigoplus_{i=0}^{n} \mathbb{Z} \alpha_i^\dagger$ forms a root lattice of an affine root system $(R, G)_a$ with a basis $\{\alpha_i^\dagger\}_{i=0}^{n}$.

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(3) The elliptic Weyl group $W(R, G)$ is isomorphic to $W_\Lambda(R, G)/\langle x_0^{n'_0} \cdots x_n^{n'_n} \rangle$. Namely, $W(R, G)$ has a presentation with generators $r_\alpha = s_\Lambda, \alpha$ ($\alpha \in \Gamma(R, G)$), and $r_\alpha^2 = 1$, the usual and elliptic Coxeter relations, and $x_0^{n'_0} \cdots x_n^{n'_n} = 1$. 

§2 Elliptic Artin groups

Definition. For a marked elliptic root system \((R,G)\), let \(A(R,G)\) a group with generators \(g_\alpha \) (\(\alpha \in R\)) and the relations obtained form the Coxeter and the elliptic Coxeter relations of \(W_\Lambda(R,G)\) replacing \(r_\alpha\) with \(g_\alpha\). It is celled the elliptic Artin group of type \((R,G)\).
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Proposition(Yamada,Shiota-S). For \(i = 0, \cdots, n\), define elements \(x_i \in A(R, G)\) in the similar way as \(A(R, G)\), replacing \(r_\alpha\) with \(g_\alpha\).

1. The subgroup \(N\) of \(W_\Lambda(R, G)\) generated by all \(x_i\)'s is an abelian group.

2. Recall \(\delta'_a = \sum_{i=0}^{n} n'_i \alpha_i^\dagger\) with \(n'_0 = 1\). Then, the element \(x_0^{n'_0} \cdots x_n^{n'_n}\) belongs to the center of \(A(R, G)\).
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- These groups were essentially appeared in our previous study on “elliptic Hecke algebras” (Shiota-S).
Elliptic Hecke algebras

\(A[A(R, G)]:\) The group ring of \(A(R, G)\) over \(A := \mathbb{Z}[t_\alpha]_{\alpha \in \Gamma}/\mathcal{I}.
\)

\(\mathcal{I}\) is a certain ideal of \(\mathbb{Z}[t_\alpha]_{\alpha \in \Gamma}.
\)

\(\Rightarrow A\) is the Laurent polynomial ring of unequal parameters.
Elliptic Hecke algebras

\( \mathbb{A}[A(R, G)] \): The group ring of \( A(R, G) \) over \( \mathbb{A} := \mathbb{Z}[t_{\alpha}]_{\alpha \in \Gamma}/\mathcal{I} \).

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\( \Rightarrow \mathbb{A} \) is the Laurent polynomial ring of unequal parameters.

Set

\[ \mathbb{H}(R, G) := \mathbb{A}[W_{\wedge}(R, G)]/J, \]

where \( J \) is an ideal generated by

\[ (g_{\alpha} - t_{\alpha})(g_{\alpha} + t_{\alpha}^{-1}) \quad (\alpha \in \Gamma) \]

(Hecke relations).

The elliptic Hecke algebra
Elliptic Hecke algebras

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the elliptic Hecke algebra (Hecke relations).

Theorem (Shiota-S). The elliptic Hecke algebra \( \mathbb{H}(R, G) \) is isomorphic to a subalgebra of a Cherednik-Macdonald’s double affine Hecke algebra.

Remark. DAHA/ell.Hecke \( \cong \mathbb{A}[\text{a certain finite group}] \).
§3 $SL_2(\mathbb{Z})$-action

Since an elliptic root system $R$ has two primitive null roots $\delta_1$ and $\delta_2$, there is a natural action of $SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$. 

Remark. For other types, $\text{Im } \phi$ is isomorphic to the congruence subgroup $\Gamma_0(k)$ or $\Gamma_0(3)$. Here $\Gamma_0(k) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1, \quad c \equiv 0 \pmod{k} \right\}$. 

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§3 $SL_2(\mathbb{Z})$-action

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Recall $\Lambda = F \oplus \mathbb{R}\gamma_1 \oplus \mathbb{R}\gamma_2$, and $I_\Lambda : \Lambda \times \Lambda \to \mathbb{R}$ (non-deg. symm. bilinear form).

$$\Theta(R, G) := \{ \phi \in O(\Lambda, I_\Lambda) \mid \phi(R) = R \}.$$  

$\Rightarrow$ Every $\phi \in \Theta(R, G)$ preserves $Q_0 := \mathbb{Z}\delta_1 \oplus \mathbb{Z}\delta_2$. 

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\[ \Theta_0(R, G) := \{ \varphi \in \Theta(R, G) \mid \det(\varphi|_{Q_0}) = 1 \} . \]

Let $\pi : \Theta_0(R, G) \to \text{Aut}(Q_0)$ be a map defined by $\varphi \mapsto \varphi|_{Q_0}$.
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**Lemma.** If $(R, G)$ is a marked elliptic root system of type $X_n^{(1,1)}$, the image of $\pi$ coincides with $SL(Q_0) \cong SL_2(\mathbb{Z})$. 
§3 $SL_2(\mathbb{Z})$-action

- Since an elliptic root system $R$ has two primitive null roots $\delta_1$ and $\delta_2$, there is a natural action of $SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$.

Recall $\Lambda = F \oplus \mathbb{R}\gamma_1 \oplus \mathbb{R}\gamma_2$, and $I_\Lambda : \Lambda \times \Lambda \to \mathbb{R}$ (non-deg. symm. bilinear form).

$$\Theta(R, G) := \{ \varphi \in O(\Lambda, I_\Lambda) \mid \varphi(R) = R \}.$$ 

$\Rightarrow$ Every $\varphi \in \Theta(R, G)$ preserves $Q_0 := \mathbb{Z}\delta_1 \oplus \mathbb{Z}\delta_2$.

$$\Theta_0(R, G) := \{ \varphi \in \Theta(R, G) \mid \det(\varphi|_{Q_0}) = 1 \}.$$ 

Let $\pi : \Theta_0(R, G) \to Aut(Q_0)$ be a map defined by $\varphi \mapsto \varphi|_{Q_0}$.

**Lemma.** If $(R, G)$ is a marked elliptic root system of type $X_n^{(1,1)}$, the image of $\pi$ coincides with $SL(Q_0) \cong SL_2(\mathbb{Z})$.

**Remark.** For another type, $\text{Im} \pi$ is isomorphic to the congruence subgroup $\Gamma_0(2)$ or $\Gamma_0(3)$. Here $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| c \equiv 0 \text{ mod } N \right\}$. 
• From now on, we assume \( (R, G) \) is of type \( X_n^{(1,1)} \).

\[ \Rightarrow \text{There is an exact sequence} \]

\[ 1 \rightarrow \text{Ker } \pi \rightarrow \Theta_0(R, G) \xrightarrow{\pi} \text{SL}(Q_0) \rightarrow 1. \]

\[ \varphi \mapsto \varphi|_{Q_0} \]
From now on, we assume $(R, G)$ is of type $X_n^{(1,1)}$.

There is an exact sequence

$$1 \rightarrow \text{Ker } \pi \rightarrow \Theta_0(R, G) \xrightarrow{\pi} SL(Q_0) \rightarrow 1.$$ 

Moreover, we have

- The above exact sequence is split. i.e. $\Theta_0(R, G) \cong \text{Ker } \pi \times SL(Q_0)$. Especially, $SL(Q_0) \cong SL_2(\mathbb{Z})$ acts on Ker $\pi$.
- $W_\Lambda(R, G)$ is a normal subgroup of Ker $\pi$ which is invariant under the $SL_2(\mathbb{Z})$-action. i.e.
- Ker $\pi/W_\Lambda(R, G)$ is a finite group (generated by some diagram automorphisms).
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• \(W_{\Lambda}(R, G)\) is a normal subgroup of \(\text{Ker } \pi\) which is invariant under the \(SL_{2}(\mathbb{Z})\)-action. i.e.

• \(\text{Ker } \pi/W_{\Lambda}(R, G)\) is a finite group (generated by some diagram automorphisms).

**Remark.** For other types, similar results hold by replacing \(SL_{2}(\mathbb{Z})\) with \(\Gamma_{0}(2)\) or \(\Gamma_{0}(3)\).
• From now on, we assume \((R, G)\) is of type \(X_n^{(1,1)}\).

⇒ There is an exact sequence

\[
1 \rightarrow \text{Ker} \, \pi \rightarrow \Theta_0(R, G) \xrightarrow{\pi} \text{SL}(Q_0) \rightarrow 1.
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**Remark.** For other types, similar results hold by replacing \(\text{SL}_2(\mathbb{Z})\) with \(\Gamma_0(2)\) or \(\Gamma_0(3)\).

**Q.** These stories work for elliptic Artin groups?
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There is an exact sequence

\[
1 \to \ker \pi \to \Theta_0(R, G) \xrightarrow{\pi} SL(Q_0) \to 1.
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Moreover, we have

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- The above exact sequence is split. \textit{i.e.} \(\Theta_0(R, G) \cong \ker \pi \times SL(Q_0)\).
  Especially, \(SL(Q_0) \cong SL_2(\mathbb{Z})\) acts on \(\ker \pi\).

- \(W_\Lambda(R, G)\) is a normal subgroup of \(\ker \pi\) which is invariant under the \(SL_2(\mathbb{Z})\)-action. \textit{i.e.}

- \(\ker \pi/W_\Lambda(R, G)\) is a finite group (generated by some diagram automorphisms).

\textbf{Remark.} For other types, similar results hold by replacing \(SL_2(\mathbb{Z})\) with \(\Gamma_0(2)\) or \(\Gamma_0(3)\).

\textbf{Q.} These stories work for elliptic Artin groups?

\textbf{A.} YES (at least for of type \(X_n^{(1,1)}\)).
§4 Hidden braid group symmetry

o Preliminaries

(1) Set

\[ S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \]

Then, \( SL_2(\mathbb{Z}) = \langle S, T \mid S^2 = (ST)^3 = (TS)^3, \; S^4 = 1 \rangle. \)
§4 Hidden braid group symmetry

○ Preliminaries

(1) Set

\[ S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \]

Then, \( SL_2(\mathbb{Z}) = \langle S, T \mid S^2 = (ST)^3 = (TS)^3, \ S^4 = 1 \rangle. \)

(2) Let \( U := TST = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \) Then \( T \) and \( U^{-1} \) generate \( SL_2(\mathbb{Z}) \) and they satisfy the braid relation of type \( A_2: \ TU^{-1}T = U^{-1}TU^{-1}. \) Moreover, there exists an exact sequence:

\[
1 \to \langle c \rangle \cong \mathbb{Z} \to A(A_2) \to SL_2(\mathbb{Z}) \to 1 \text{ (central extension).}
\]

\[
a \mapsto T, \quad b \mapsto U^{-1}
\]

Here \( A(A_2) = \langle a, b \mid aba = bab \rangle \) is the Artin group of type \( A_2, \) and \( c := (ab)^6 = (ba)^6. \)
(3) Let \((R, G)\) be a marked elliptic root system of type \(X_{n}^{(1,1)}\). Recall that the modular group \(SL_2(\mathbb{Z})\) acts on \(W_\Lambda(R, G)\). i.e.

\[
\iota : \quad SL_2(\mathbb{Z}) \quad \leftrightarrow \quad Aut(W_\Lambda(R, G))
\]

\[
X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \mapsto \quad X_W.
\]

Note that there exists a natural projection:

\[
\varpi : A(R, G) \quad \rightarrow \quad W_\Lambda(R, G)
\]

where \(i_1 i_2 (R, G)\):

\[
g_{i_1} g_{i_2} \quad \mapsto \quad s_i g_{i_2} s_i g_{i_1}
\]

**Definition.** A group automorphism \(\varphi \in Aut(A(R, G))\) is called a lift of \(\varphi \in Aut(W_\Lambda(R, G))\) if \(\varpi(\varphi(g)) = \varpi(g)\) for every \(g \in A(R, G)\).
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(4) Note that there exists a natural projection:

\[
\varpi : A(R, G) \mapsto W_\Lambda(R, G)
\]

where \(\alpha_{i_1} \cdots \alpha_{i_l} \in \Gamma(R, G)\).

\[
g\alpha_{i_1} \cdots g\alpha_{i_l} \mapsto s\alpha_{i_1} \cdots s\alpha_{i_l}
\]
Let \((R, G)\) be a marked elliptic root system of type \(X_n^{(1,1)}\). Recall that the modular group \(SL_2(\mathbb{Z})\) acts on \(W_\wedge(R, G)\). i.e.
\[
\iota : \quad SL_2(\mathbb{Z}) \quad \hookrightarrow \quad Aut\left(W_\wedge(R, G)\right)
\]

\[
X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \mapsto \quad X_W.
\]

Note that there exists a natural projection:
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\varpi : \quad A(R, G) \quad \rightarrow \quad W_\wedge(R, G)
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where \(\alpha_{i_1} \cdots \alpha_{i_l} \in \Gamma(R, G)\).

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g\alpha_{i_1} \cdots g\alpha_{i_l} \quad \mapsto \quad s\alpha_{i_1} \cdots s\alpha_{i_l}
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**Definition.** A group automorphism \(\varphi \in Aut\left(A(R, G)\right)\) is called a lift of \(\psi \in Aut\left(W_\wedge(R, G)\right)\) if
\[
\varpi(\varphi(g)) = \psi(\varpi(g)) \quad \text{for every } g \in A(R, G).
\]
Theorem. For \((R, G)\) of type \(X_n^{(1,1)}\), there exist group automorphisms \(S_A\) and \(T_A\) of \(A(R, G)\) which have the following properties.

1. They are lifts of \(S_W\) and \(T_W \in \text{Aut}(W_\Lambda(R, G))\), respectively.

2. The following commutative relations are satisfied:
   
   \[(S_A T_A)^3 = (T_A S_A)^3 = S_A^2.\]

3. \(S_A^4 \Delta \text{Inn}(A(R, G))\). More precisely,
   
   \[S_A^4(g) = (g w_0)^2 g (g w_0)^2\]
   for every \(g \in A(R, G)\).

4. Let \(G\) be a subgroup of \(\text{Aut}(A(R, G))\) generated by \(S_A\) and \(T_A\). Then there exists an exact sequence.

\[
1 \rightarrow \mathbb{Z} = \langle S_A \rangle \rightarrow G \rightarrow \text{SL}_2(\mathbb{Z}) \rightarrow 1
\]

Namely, \(G\) is isomorphic to the braid group of type \(A_2\).
Theorem. For \((R, G)\) of type \(X_n^{(1,1)}\), there exist group automorphisms \(S_A\) and \(T_A\) of \(A(R, G)\) which have the following properties.

1. They are lifts of \(S_W\) and \(T_W \in \text{Aut}(W_A(R, G))\), respectively.
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(S_AT_A)^3 = (T_AS_A)^3 = S_A^2.
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1. They are lifts of \(S_W\) and \(T_W \in \text{Aut}(W_\Lambda(R, G))\), respectively.
2. The following commutative relations are satisfied:
   \[(S_A T_A)^3 = (T_A S_A)^3 = S_A^2.\]
3. \(S_A^4 \in \text{Inn}(A(R, G))\). More precisely,
   \[S_A^4(g) = (g w_0)^{-2} g (g w_0)^2 \quad \text{for every } g \in A(R, G).\]

Here \(w_0\) is the longest elem. of \(W(R_0)\), \(g w_0\) is the corresp. elem. in \(A(R, G)\).
Theorem. For \((R, G)\) of type \(X_n^{(1,1)}\), there exist group automorphisms \(S_A\) and \(T_A\) of \(A(R, G)\) which have the following properties.

1. They are lifts of \(S_W\) and \(T_W \in \text{Aut}(W_\wedge(R, G))\), respectively.
2. The following commutative relations are satisfied:
   \[(S_AT_A)^3 = (T_AS_A)^3 = S_A^2.\]
3. \(S_A^4 \in \text{Inn}(A(R, G))\). More precisely,
   \[S_A^4(g) = (g_{w_0})^{-2}g(g_{w_0})^2 \quad \text{for every } g \in A(R, G).\]
Here \(w_0\) is the longest elem. of \(W(R_0)\), \(g_{w_0}\) is the corresp. elem. in \(A(R, G)\).
4. Let \(G\) be a subgroup of \(\text{Aut}(A(R, G))\) generated by \(S_A\) and \(T_A\). Then there exists an exact sequence.

\[
1 \rightarrow \mathbb{Z} \cong \langle S_A^4 \rangle \rightarrow G \rightarrow SL_2(\mathbb{Z}) \rightarrow 1 \quad \text{(central extension)}
\]

Namely, \(G\) is isomorphic to the braid group of type \(A_2\).
(hidden braid group symmetry of \(A(R, G)\)).
Remark. (1) The explicit forms of $S_A$ and $T_A$ are obtained. But, since they deeply depend on “personality” of each $(R, G)$, we omit to give them.
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(2) Our $S_A$ and $T_A$ induce automorphisms of the corresponding elliptic (≡ double affine) Hecke algebra. Namely, the elliptic Hecke algebra of type $X_n^{(1,1)}$ also has the hidden braid group symmetry.
Remark. (1) The explicit forms of $S_A$ and $T_A$ are obtained. But, since they deeply depend on “personality” of each $(R, G)$, we omit to give them.

(2) Our $S_A$ and $T_A$ induce automorphisms of the corresponding elliptic (⇐ double affine) Hecke algebra. Namely, the elliptic Hecke algebra of type $X_n^{(1,1)}$ also has the hidden braid group symmetry.

(3) As well-known, there is an anti-automorphism $\omega$ of a double affine Hecke algebra which is called the duality anti-automorphism or Fourier transformation due to Cherednik. (It plays an important role in his proof of Macdonald’s inner product conjecture.) In our setting, it is a “lift” of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in GL_2(\mathbb{Z}) \setminus SL_2(\mathbb{Z})$, and does not coincide one of $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. 
(4) What is happened in marked elliptic root systems of other types?

→ Work in progress:
  
  We have done only for some low rank cases.
Thank you!