Character sheaves

Tanmay Deshpande

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The category $\mathcal{D}_G(G)$

Twists

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Character sheaves on solvable algebraic groups

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Let $\mathbb{F}_q$ be a finite field of characteristic $p$ and let $k = \overline{\mathbb{F}}_q$. We work with algebraic groups $G$ over $k$ equipped with a $q$-Frobenius map $F : G \to G$. Then for each positive integer $m$, we have the finite group $G(\mathbb{F}_q^m) = G^{F^m}$. 

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Our goal is to study the irreducible characters of the finite group $G(\mathbb{F}_q) = G^F$ in terms of certain geometric objects which we will call “character sheaves”.

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Our goal is to study the irreducible characters of the finite group $G(\mathbb{F}_q) = G^F$ in terms of certain geometric objects which we will call “character sheaves”.

Character sheaves on $G$ are supposed to be certain special objects in $\mathcal{D}_G(G)$, the triangulated category of conjugation equivariant $\overline{\mathbb{Q}}_\ell$-complexes on $G$. They are supposed to be a geometric analogue of the notion of irreducible characters.
Remark on disconnected groups

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However, it is important to note that in the disconnected case we should consider not only the original Frobenius $F$, but also all its pure inner forms which are parameterized by the finite set $H^1(F, G)$.

If $g \in G$, then the corresponding inner form of the Frobenius is defined by $gF := \text{ad}(g) \circ F : G \to G$. The finite group $G^{gF}$ is said to be an inner form of the group $G^F$. 
Character sheaves are supposed to lie in the $\mathbb{Q}_\ell$-linear triangulated braided monoidal category $\mathcal{D}_G(G)$ of conjugation equivariant $\mathbb{Q}_\ell$-complexes on $G$. (We fix some prime number $\ell \neq p$.)
The triangulated braided category $\mathcal{D}_G(G)$

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- Each object $C \in \mathcal{D}_G(G)$ has an equivariance structure which defines isomorphisms

$$\phi_C(g, x) : C_x \xrightarrow{\simeq} C_{gxg^{-1}}.$$
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$$\phi_C(g, x) : C_x \xrightarrow{\cong} C_{gxg^{-1}}.$$ 

- Each object $C \in \mathcal{D}_G(G)$ has its associated twist $\theta_C : C \to C$ defined on stalks by

$$\theta_C(x) = \phi_C(x, x) : C_x \to C_x.$$
For $C_1, C_2 \in \mathcal{D}_G(G)$, we have their convolution with compact support

$$C_1 \ast C_2 = \mu!(C_1 oxtimes C_2) = \mu!(p_1^*C_1 \otimes p_2^*C_2).$$

![Diagram](image)
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\[ G \times G \xrightarrow{\mu} G \]
\[ \begin{array}{c}
G \\
\downarrow p_1 \\
\end{array} \quad \begin{array}{c}
G \\
\downarrow p_2 \\
\end{array} \]

- We have braid isomorphisms

\[ \beta_{c_1,c_2} : C_1 \ast C_2 \xrightarrow{\sim} C_2 \ast C_1 \text{ which satisfy} \]

\[ \theta_{c_1 \ast c_2} = \beta_{c_2,c_1} \circ \beta_{c_1,c_2} \circ (\theta_{c_1 \ast c_2}). \]
The sheaf-function correspondence

- If \( C \in \mathcal{D}_G(G) \) is such that we have \( \psi : F^* C \rightarrow C \), then we can define its associated trace of Frobenius function on each pure inner form and obtain a function \( Tr_{C,\psi} \in \text{Fun}([G], F) \).
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- We will say that an object $C \in \mathcal{D}_G(G)$ is simple if $\text{End}(C) = \overline{\mathbb{Q}}_\ell$. If a simple $C$ is $F$-stable, then the function $\text{Tr}_{C,\psi}$ is determined by $C$ up to scaling.
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- **Question:** Given any irreducible character $\chi$ of $G^F$, does there exist a simple $F$-stable $C \in \mathcal{D}_G(G)$ such that $\chi = Tr_{C,\psi}$?
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- **Question:** Given any irreducible character \( \chi \) of \( G^F \), does there exist a simple \( F \)-stable \( C \in \mathcal{D}_G(G) \) such that \( \chi = Tr_{C,\psi} \)?

- **Ans:** Not always. We will see some examples soon.
Geometric and rational conjugacy classes

- We say that $g, h \in G^F$ are geometrically conjugate if $g = xhx^{-1}$ for some $x \in G$, not necessarily in $G^F$. 
Geometric and rational conjugacy classes

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Let $G^F/ \sim$ denote the set of conjugacy classes in the finite group $G^F$. These are called the rational conjugacy classes.

If $g \in G^F$ and $C_G(g)$ is connected then the geometric conjugacy class of $g$ coincides with its rational conjugacy class.
Let $G$ be connected. We have a twist $(G^F/\sim) \xrightarrow{\Theta} (G^F/\sim)$ which permutes the rational conjugacy classes within each geometric conjugacy class.
The twisting map

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$$\Theta : \langle g \rangle \mapsto \langle F(x)^{-1}x \rangle.$$
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  \[ \Theta^* : \text{Fun}(G^F/\sim) \xrightarrow{\cong} \text{Fun}(G^F/\sim). \]
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- In general the twist of an irreducible character can be complicated.
The trace of Frobenius functions and twists

If $C \in \mathcal{D}_G(G)$ is simple, then the twist $\theta_C : C \rightarrow C$ is a scalar.

**Lemma**

Let $C$ be an $F$-stable simple object in $\mathcal{D}_G(G)$ and let $\psi : F^* C \xrightarrow{\sim} C$. Then we have

$$\Theta^*(Tr_C,\psi) = \theta_C \cdot Tr_C,\psi$$

i.e. the associated trace of Frobenius function is an eigenvector for the twisting operator with eigenvalue $\theta_C$. 
Example: Borel subgroup of $SL_2(\mathbb{F}_q)$

Let $B := \left\{ \begin{pmatrix} t & a \\ 0 & t^{-1} \end{pmatrix} \right\}$ be the Borel subgroup of $SL_2$. $B(\mathbb{F}_q)$ has $q + 3$ conjugacy classes:
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$$l \quad -l \quad \begin{pmatrix} u \\ 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} u' \\ 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} -u \\ 0 \\ -1 \end{pmatrix} \quad \begin{pmatrix} -u' \\ 0 \\ -1 \end{pmatrix} \quad \begin{pmatrix} t \\ 0 \\ t^{-1} \end{pmatrix} \quad t \in \mathbb{F}_q^\times \setminus \{\pm 1\}$$
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\[
I \quad -I \quad \begin{pmatrix} u & \alpha \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} u' & -\alpha \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} -u & -\alpha \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} -u' & \alpha \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \quad t \in \mathbb{F}_q^\times \setminus \{\pm 1\}
\]

Its character table is given by: (Here $\varepsilon = \left( \frac{-1}{q} \right)$)

<table>
<thead>
<tr>
<th>Irrep</th>
<th>$I$</th>
<th>$-I$</th>
<th>$u$</th>
<th>$u'$</th>
<th>$-u$</th>
<th>$-u'$</th>
<th>$t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi$</td>
<td>1</td>
<td>$\chi(-1)$</td>
<td>1</td>
<td>1</td>
<td>$\chi(-1)$</td>
<td>$\chi(-1)$</td>
<td>$\chi(t)$</td>
</tr>
<tr>
<td>$\chi_1$</td>
<td>$\frac{q-1}{2}$</td>
<td>$\frac{q-1}{2}$</td>
<td>$\frac{\sqrt{\varepsilon q-1}}{2}$</td>
<td>$-\frac{\sqrt{\varepsilon q-1}}{2}$</td>
<td>$\frac{\sqrt{\varepsilon q-1}}{2}$</td>
<td>$-\frac{\sqrt{\varepsilon q-1}}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>$\frac{q-1}{2}$</td>
<td>$\frac{q-1}{2}$</td>
<td>$-\frac{\sqrt{\varepsilon q-1}}{2}$</td>
<td>$\frac{\sqrt{\varepsilon q-1}}{2}$</td>
<td>$-\frac{\sqrt{\varepsilon q-1}}{2}$</td>
<td>$\frac{\sqrt{\varepsilon q-1}}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>$\frac{q-1}{2}$</td>
<td>$-\frac{q-1}{2}$</td>
<td>$\frac{\sqrt{\varepsilon q-1}}{2}$</td>
<td>$-\frac{\sqrt{\varepsilon q-1}}{2}$</td>
<td>$-\frac{\sqrt{\varepsilon q+1}}{2}$</td>
<td>$\frac{\sqrt{\varepsilon q+1}}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>$\frac{q-1}{2}$</td>
<td>$-\frac{q-1}{2}$</td>
<td>$-\frac{\sqrt{\varepsilon q-1}}{2}$</td>
<td>$\frac{\sqrt{\varepsilon q-1}}{2}$</td>
<td>$\frac{\sqrt{\varepsilon q+1}}{2}$</td>
<td>$-\frac{\sqrt{\varepsilon q+1}}{2}$</td>
<td>0</td>
</tr>
</tbody>
</table>

We see that the last 4 irreducible characters that are not preserved by the twist $\Theta$. 


Example: Almost characters for $B(\mathbb{F}_q)$

However, let us take certain linear combinations of these 4 characters:

\[
\begin{align*}
\chi_1 + \chi_2 + \chi_3 + \chi_4 &= 2q^{-1} - 1 - 1 - 0 \\
\chi_1 - \chi_2 - \chi_3 + \chi_4 &= 20 q^{-1} - 1 - 1 - 1 \\
\chi_1 - \chi_2 - \chi_3 + \chi_4 &= 20 q^{-1} - 1 - 1 - 1 \\
\chi_1 - \chi_2 + \chi_3 - \chi_4 &= 20 q^{-1} - 1 - 1 - 1
\end{align*}
\]

We see that the first 3 “almost characters” above are fixed by $\Theta$, whereas the last one is an eigenvector with eigenvalue -1.

The unitary matrix relating the 4 special characters and almost characters is the $S$-matrix of a certain modular category, namely the Drinfeld double of $\mathbb{Z}/2\mathbb{Z}$. 
Example: Almost characters for $B(\overline{F}_q)$

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```
<table>
<thead>
<tr>
<th>“Almost char”</th>
<th>I</th>
<th>−I</th>
<th>u</th>
<th>u'</th>
<th>−u</th>
<th>Θ</th>
<th>−u'</th>
<th>t</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{x_1+x_2+x_3+x_4}{2}$</td>
<td>$q-1$</td>
<td>0</td>
<td>$-1$</td>
<td>$-1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>$-1$</td>
<td>$-1$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$\frac{x_1-x_2+x_3-x_4}{2}$</td>
<td>0</td>
<td>0</td>
<td>$\sqrt{\varepsilon}q$</td>
<td>$-\sqrt{\varepsilon}q$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
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<td>$\frac{x_1-x_2-x_3+x_4}{2}$</td>
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We see that the first 3 “almost characters” above are fixed by $\Theta$, whereas the last one is an eigenvector with eigenvalue $-1$. 

The unitary matrix

\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 \\
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\end{pmatrix}
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relating the 4 special characters and almost characters is the $S$-matrix of a certain modular category, namely the Drinfeld double of $\mathbb{Z}/2\mathbb{Z}$. 
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<td>$q - 1$</td>
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$$\frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

relating the 4 special characters and almost characters is the $S$-matrix of a certain modular category, namely the Drinfeld double of $\mathbb{Z}/2\mathbb{Z}$. 
Main conjecture

Our main goal is to define a set of some special simple objects in $\mathcal{D}_G(G)$ which we will call character sheaves.

**Conjecture 1**

There exists a (possibly infinite) set $CS(G)$ of isomorphism classes of some special simple objects of $\mathcal{D}_G(G)$ such that
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**Conjecture 1**

There exists a (possibly infinite) set $CS(G)$ of isomorphism classes of some special simple objects of $\mathcal{D}_G(G)$ such that

- The set $CS(G)$ can be partitioned into finite families called $\mathbb{L}$-packets, i.e.

$$CS(G) = \bigsqcup_{\mathbb{L}(G)} L,$$

where each $L$ is a finite set of character sheaves. Associated with each $\mathbb{L}$-packet $L$, there is a modular category $C_L$ whose simple objects are the character sheaves in the $\mathbb{L}$-packet $L$. 
Main conjecture

Conjecture 1 (contd.)

Let $\varphi : \mathcal{D}_G(G) \rightarrow \mathcal{D}_G(G)$ be any braided triangulated autoequivalence. Then $\varphi$ preserves the set $\mathcal{C}S(G)$ and if $L$ is an $\mathcal{L}$-packet of character sheaves, then $\varphi(L)$ is also an $\mathcal{L}$-packet.
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- For each $C \in CS(G)^F$, we fix an isomorphism $\psi_C : F^* C \to C$ such that $|Tr_{C,\psi_C}| = 1$. Then the set $\{Tr_{C,\psi_C}\}_{C \in CS(G)^F} \subset Fun([G], F)$ is an orthonormal basis of $\Theta^*$-eigenvectors.
Main conjecture

Conjecture 1 (contd.)

We have $\mathcal{CS}(G)^F = \bigsqcup_{L \in \mathbb{L}(G)^F} L^F$. The unitary matrix relating the two bases $\{\text{Tr}_C, \psi_C\}_{C \in \mathcal{CS}(G)^F}$ and $\text{Irrep}(G, F)$ of $\text{Fun}([G], F)$ is block diagonal where the blocks correspond to the $F$-stable $\mathbb{L}$-packets.
Main conjecture

Conjecture 1 (contd.)

- We have \( CS(G)^F = \bigsqcup_{L \in \mathbb{L}(G)^F} L^F \). The unitary matrix relating the two bases \( \{ Tr_C, \psi_C \}_{C \in CS(G)^F} \) and Irrep\((G, F)\) of Fun([\(G\], F) is block diagonal where the blocks correspond to the \(F\)-stable \(\mathbb{L}\)-packets.

- If \( L \in \mathbb{L}(G)^F \) then we have the modular autoequivalence \( F_L^* : \mathcal{C}_L \to \mathcal{C}_L \). Then the corresponding block in the change of basis matrix is equal to the crossed \(S\)-matrix associated with the pair \((\mathcal{C}_L, F_L^*)\).
Our approach towards a theory of character sheaves

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- For a minimal idempotent $f$, we look at the full subcategory $f\mathcal{D}_G(G) \subset \mathcal{D}_G(G)$ and we aim to first define the set $CS_f(G)$ of character sheaves and its $\mathbb{L}$-packet decomposition in this full subcategory.
Our approach towards a theory of character sheaves

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- There are some special idempotents $e$ in $\mathcal{D}_G(G)$ which can be described very explicitly. These are known as Heisenberg idempotents.
An auxiliary conjecture

Conjecture 2

(i) A Heisenberg idempotent $e \in \mathcal{D}_G(G)$ is always a minimal idempotent.
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(ii) Let \( f \in \mathcal{D}_G(G) \) be any minimal idempotent. Then there exists a Heisenberg idempotent \( e \in \mathcal{D}_{G'}(G') \) for some \( G' \subset G \) such that \( f \cong \text{ind}_{G'}^G e \) and such that we have a braided triangulated equivalence

\[
\text{ind}_{G'}^G : e \mathcal{D}_{G'}(G') \xrightarrow{\cong} f \mathcal{D}_G(G).
\]
An auxiliary conjecture

Conjecture 2

(i) A Heisenberg idempotent $e \in \mathcal{D}_G(G)$ is always a minimal idempotent.

(ii) Let $f \in \mathcal{D}_G(G)$ be any minimal idempotent. Then there exists a Heisenberg idempotent $e \in \mathcal{D}_{G'}(G')$ for some $G' \subset G$ such that $f \simeq \text{ind}_G^{G'} e$ and such that we have a braided triangulated equivalence

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\text{ind}_G^{G'} : e \mathcal{D}_{G'}(G') \xrightarrow{\simeq} f \mathcal{D}_G(G).
\]

Theorem

(a) Conjecture 2(i) implies 2(ii).
An auxiliary conjecture

**Conjecture 2**

1. A Heisenberg idempotent $e \in \mathcal{D}_G(G)$ is always a minimal idempotent.

2. Let $f \in \mathcal{D}_G(G)$ be any minimal idempotent. Then there exists a Heisenberg idempotent $e \in \mathcal{D}_{G'}(G')$ for some $G' \subset G$ such that $f \cong \text{ind}^G_{G'} e$ and such that we have a braided triangulated equivalence

$$\text{ind}^G_{G'} : e\mathcal{D}_{G'}(G') \cong f\mathcal{D}_G(G).$$

**Theorem**

(a) Conjecture 2(i) implies 2(ii).

(b) Conjecture 2 is true if $G^\circ$ is solvable.