Some Results by Energy Methods on Large-Time Behavior of Viscous Gas

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"Nonlinear Evolution Equations and Related Topics"
Kyoto Universiy, March 9-11, 2015

Introduction

A model system of viscous gas

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ u_t + u \cdot \nabla u + \frac{1}{\rho} \nabla p = \frac{\mu}{\rho} \Delta u + \frac{\mu + \lambda}{\rho} \nabla (\nabla \cdot u), \\ p = p(\rho) = a \rho^\gamma, \end{cases} \tag{1}$$

where $t \geq 0,\ x \in R^n (n = 1, 2, 3), \mu > 0, \mu + \lambda > 0, a > 0, \gamma \geq 1$.

Consider the Cauchy problem for (1) with the initial data

$$(\rho, u)(0) = (\rho_0, u_0). \tag{2}$$

Two papers :

- **Nishida-M (1980) , J. Math. Kyoto Univ.,**
  asymptotic stability of constant states in $\boldsymbol{R}^3, \boldsymbol{R}^2$

- **Nishihara-M (1985) , Japan J. Appl. Math.,**
  asymptotic stability of traveling waves in $\boldsymbol{R}^1$

Topic 1. (joint work with T. Maeda)

- **Nishida-M (1980),** $\boldsymbol{x} \in \boldsymbol{R}^2$ **or** $\boldsymbol{R}^3$, $\bar{\rho} > 0$

$$(\rho_0 - \bar{\rho}, u_0) \in H^3, \text{ small} \Longrightarrow (\bar{\rho}, 0) \text{ is asymptotically stable.}$$

Since then, Hoff ($\bar{\rho} > 0$), Feireisl, Lions ($\bar{\rho} = 0$), . . . , Z.Xin-J.Li-X.Huang ($\bar{\rho} \geq 0$), . . .

Our present aim

Consider the asymptotic stability of an unbounded state

$$\rho = P = \frac{\bar{\rho}}{1+t}, \quad u = U = (\frac{x_1}{1+t}, 0).$$

Theorem 1. *Suppose* $x \in R^2$, $p = a\rho$, *and* $a - \dfrac{(2\mu+\lambda)}{\bar{\rho}} > 0$.

Then, there exists a $\varepsilon_0 > 0$ *such that if* $\|\rho_0 - \bar{\rho}, u_0 - (x_1, 0)\|_{H^2} \leq \varepsilon_0$, *the Cauchy problem* (1),(2) *has a unique global solution in time* (ρ, u), *satisfying* $(\rho - P, u - U) \in C([0, +\infty); H^2)$ *and*

$$\sup_{x \in R^2} | \rho(t,x) - \frac{\bar{\rho}}{1+t} | \leq C(1+t)^{-\frac{3}{2}},$$

$$\sup_{x \in R^2} | u(t,x) - (\frac{x_1}{1+t}, 0) | \leq C(1+t)^{-\frac{1}{2}}.$$

Remarks

- The proof is given by a combination of changing the variable x_1 along a characteristic curve and using a time-weighted energy method .
- For R^3, the similar results hold in H^3 with the asymptotics

$$\sup_{x\in R^3} \mid \rho - P \mid \leq C(1+t)^{-2}, \quad \sup_{x\in R^3} \mid u - U \mid \leq C(1+t)^{-1}.$$

- Open problems
  - Isentropic case : $p = a\rho^\gamma,\ (\gamma > 1)$
  - Full system case : $p = R\rho\theta,\ e = \frac{R}{\gamma-1}\theta$

$$\rho = P = \frac{\bar{\rho}}{1+t}, \quad u = U = (\frac{x_1}{1+t}, 0), \quad \theta = \frac{2\mu+\lambda}{R\bar{\rho}}.$$

Topic 2. (joint work with Yang Wang)

Asymptotic stability of traveling wave solutions in R

System in Lagrange coordinates :

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = (\mu \dfrac{u_x}{v})_x, \\ p = p(v) = av^{-\gamma}. \end{cases}$$

Traveling wave solution (viscous shock wave):

$$(v, u) = (V, U)(x - st), \quad (V, U)(\pm\infty) = (v_\pm, u_\pm)$$

It exists under the Rankine-Hugoniot and entropy conditions.

Known results (μ : a positive constant)

- **Nishihara-M (1985)**

  $\exists C(v_-, \gamma) > 0$ with $C \to \infty$ as $\gamma \to 1$ such that if

$$|v_+ - v_-| \leq C(v_-, \gamma),$$

then $(V, U)(x - st)$ is asymptotically stable for small initial perturbations with integral zero, that is,

$$\int (v_0 - V)(x)\,dx = \int (u_0 - U)(x)\,dx = 0.$$

- For $\gamma = 1$, any large viscous shock wave is OK!.
- For $\gamma > 1$, a restriction on the amplitude is imposed.

- **Mascia-Zumbrun(2004), Liu-Zeng(2009)**

$$|v_+ - v_-| : \text{suitably small} \Longrightarrow \text{asymptotic stability}$$

for general initial perturbations whose integrals are not necessarily zero.

- **Barker-Humpherys-Laffite-Rudd-Zumbrun (2008), Humpherys-Laffite-Zumbrun (2010)**

$$|v_+ - v_-| : \text{suitably large} \Longrightarrow \text{asymptotic stability}$$

They also carried out numerical studies which indicate the asymptotic stability for intermediate amplitude as well.

Our present aim Consider the case $\mu = \mu(v) > 0$.

In the Chapman-Enskog expansion theory in rarefied gas dynamics (cf. Chapman-Cowling (1970)), the viscosity coefficient is given by a function of the absolute temperature θ.

Typical two examples :

$$\begin{cases} \mu = \bar{\mu}\,\theta^{\frac{1}{2}}, & \text{Hard sphere Model}, \\ \mu = \bar{\mu}\,\theta^{\frac{1}{2}+\frac{2}{(s-1)}}, & \text{Cut-off inverse power force Model}, \end{cases}$$

where $s\,(\geq 5)$ and $\bar{\mu}\,(> 0)$ are some constants.

The above two models are unified as

$$\mu = \bar{\mu}\,\theta^{\beta} \qquad (\beta \geq \frac{1}{2}).$$

Since our model is isentropic,

$$\boldsymbol{p = R\frac{\theta}{v} = a\, v^{-\gamma},} \quad (\boldsymbol{R} : \textit{gas constant})$$

which implies

$$\boldsymbol{\theta = \frac{a}{R} v^{-(\gamma-1)}.}$$

Hence,

$$\boldsymbol{\mu = \mu_0\, v^{-(\gamma-1)\beta}} \qquad (\boldsymbol{\beta \geq \frac{1}{2},\ \mu_0 = \bar{\mu}(\frac{a}{R})^{\beta}}).$$

Thus, we assume

$$(\boldsymbol{A}) \qquad \mu = \mu(v) = \mu_0\, v^{-\alpha} \qquad (\alpha \geq \frac{1}{2}(\gamma - 1),\ \mu_0 > 0).$$

Cauchy problem :

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = \mu_0\left(\frac{u_x}{v^{\alpha+1}}\right)_x, \quad \left(\alpha \geq \frac{1}{2}(\gamma-1)\right) \\ p = p(v) = av^{-\gamma} \end{cases} \tag{3}$$

with the initial and far field conditions

$$\begin{cases} (v,u)(0,x) = (v_0,u_0)(x), \\ \lim_{x\to\pm\infty}(v,u)(t,x) = (v_\pm,u_\pm). \end{cases} \tag{4}$$

Assumptions on initial data

$$(v_0 - V, u_0 - U) \in H^1 \cap L^1,$$

$$\inf_{x \in R} v_0(x) > 0,$$

$$\int (v_0 - V)(x)\,dx = \int (u_0 - U)(x)\,dx = 0.$$

Setting

$$\phi_0(x) = \int_{-\infty}^{x} (v_0 - V)(y)\,dy, \quad \psi_0(x) = \int_{-\infty}^{x} (u_0 - U)(y)\,dy,$$

we further assume

$$(\phi_0,\ \psi_0) \in L^2. \quad (\Rightarrow (\phi_0,\ \psi_0) \in H^2)$$

Theorem 2. *Suppose* $\alpha \geq \frac{1}{2}(\gamma - 1)$. *Then, there exists a* $\varepsilon_0 > 0$ *such that if* $\|\phi_0, \psi_0\|_2 \leq \varepsilon_0$, *the Cauchy problem* (3),(4) *has a unique global solution in time* (v, u), *satisfying* $(v - V, u - U) \in C([0, \infty); H^1)$ *and*

$$\sup_{x \in R} |(v, u)(x, t) - (V, U)(x - st)| \to 0 \quad (t \to \infty).$$

Remarks

- **In the proof, the essetial *a priori* estimate is given by a technical weighted energy method, "double step weighted energy method", developed by Mei-M (1997) and Hashimoto-M (2007).**

- **Open problem**

  Full system case :

$$\mu = \bar{\mu}\,\theta^{\beta},\ \lambda = \bar{\lambda}\,\theta^{\beta},\ \kappa = \bar{\kappa}\,\theta^{\beta} \qquad (\beta \geq \frac{1}{2}).$$

Sketch of the proof of Theorem 1

Write $x = (x, y) \in R^2$ and assume $\bar{\rho} = 1$, $\mu + \lambda = 0$ for simplicity.

Cauchy problem :

$$\begin{cases} \rho_t + (\rho u_1)_x + (\rho u_2)_y = 0, \\ u_{1t} + (u_1 u_{1x} + u_2 u_{1y}) + \frac{1}{\rho} p_x - \frac{\mu}{\rho}(u_{1xx} + u_{1yy}) = 0, \\ u_{2t} + (u_1 u_{2x} + u_2 u_{2y}) + \frac{1}{\rho} p_y - \frac{\mu}{\rho}(u_{2xx} + u_{2yy}) = 0, \\ p = a\rho \end{cases}$$

with the initial data

$$(\rho, u_1, u_2)(0, x, y) = (\rho_0, u_{1.0}, u_{2.0})(x, y), \quad (x, y) \in \mathrm{R}^2.$$

Change the unknown variables : $(\rho, u_1, u_2) \to (\eta, \phi, \psi)$

$$\rho = \frac{(1+\eta)}{1+t}, \quad u_1 = \frac{x}{1+t} + \phi, \quad u_2 = \psi.$$

System for (η, ϕ, ψ) :

$$\begin{cases} \eta_t + \frac{x}{1+t}\eta_x + ((1+\eta)\phi)_x + ((1+\eta)\psi)_y = 0, \\ \phi_t + \frac{x}{1+t}\phi_x + \frac{1}{1+t}\phi + \phi\phi_x + \psi\phi_y + \frac{a}{1+\eta}\eta_x - \frac{\mu(1+t)}{1+\eta}(\phi_{xx} + \phi_{yy}) = 0, \\ \psi_t + \frac{x}{1+t}\psi_x + \phi\psi_x + \psi\psi_y + \frac{a}{1+\eta}\eta_y - \frac{\mu(1+t)}{1+\eta}(\psi_{xx} + \psi_{yy}) = 0, \\ (\eta, \phi, \psi)(0, x, y) = (\eta_0, \phi_0, \psi_0)(x, y). \end{cases}$$

(15)

Characteristic curve w.r.t. x

$$\begin{cases} \dfrac{dx(t)}{dt} = \dfrac{x(t)}{1+t}, \\ x(0) = \tilde{x} \end{cases} \implies \quad x = x(t) = (1+t)\tilde{x}.$$

Change of variable x : $\quad x = (1+t)\tilde{x}$

$$\frac{\partial}{\partial x} \implies \frac{1}{1+t}\frac{\partial}{\partial \tilde{x}}, \qquad \frac{\partial}{\partial t} + \frac{x}{1+t}\frac{\partial}{\partial x} \implies \frac{\partial}{\partial t}$$

$$\begin{cases} \tilde{\eta}_t + \dfrac{((1+\tilde{\eta})\tilde{\phi})_{\tilde{x}}}{1+t} + ((1+\tilde{\eta})\tilde{\psi})_y = 0, \\ \tilde{\phi}_t + \dfrac{\tilde{\phi}}{1+t} + \dfrac{\tilde{\phi}\tilde{\phi}_{\tilde{x}}}{1+t} + \tilde{\psi}\tilde{\phi}_y + \dfrac{a}{1+t}\dfrac{\tilde{\eta}_{\tilde{x}}}{(1+\tilde{\eta})} - \dfrac{\mu}{1+\tilde{\eta}}\left(\dfrac{\tilde{\phi}_{\tilde{x}\tilde{x}}}{1+t} + (1+t)\tilde{\phi}_{yy}\right) = 0, \\ \tilde{\psi}_t + \dfrac{\tilde{\phi}\tilde{\psi}_{\tilde{x}}}{1+t} + \tilde{\psi}\tilde{\psi}_y + \dfrac{a}{1+\tilde{\eta}}\tilde{\eta}_y - \dfrac{\mu}{1+\tilde{\eta}}\left(\dfrac{\tilde{\psi}_{\tilde{x}\tilde{x}}}{1+t} + (1+t)\tilde{\psi}_{yy}\right) = 0. \end{cases}$$

Reformulated problem :

$$\begin{cases} \eta_t + \dfrac{1}{1+t}\phi_x + \psi_y = N_0, \\ \phi_t + \dfrac{1}{1+t}\phi + \dfrac{a}{1+t}\eta_x - \mu\left(\dfrac{1}{1+t}\phi_{xx} + (1+t)\phi_{yy}\right) = N_1, \\ \psi_t + a\eta_y - \mu\left(\dfrac{1}{1+t}\psi_{xx} + (1+t)\psi_{yy}\right) = N_2, \\ (\eta,\phi,\psi)(0) = (\eta_0,\phi_0,\psi_0) \in H^2. \end{cases} \quad (5)$$

We look for the global solution in time of (5) such that

$$(\eta,\phi,\psi) \in C([0,\infty); H^2), \qquad \sup_{(x,y)\in R^2} |(\eta,\phi,\psi)(t,x,y)| \le C(1+t)^{-\frac{1}{2}}.$$

(17)

$$\boldsymbol{E}(\boldsymbol{t}) := \|(\boldsymbol{\eta},\boldsymbol{\phi},\boldsymbol{\psi})(\boldsymbol{t})\|_2^2 + (1+\boldsymbol{t})^2\|(\boldsymbol{\eta},\boldsymbol{\phi},\boldsymbol{\psi})_y(\boldsymbol{t})\|_1^2 + (1+\boldsymbol{t})^4\|(\boldsymbol{\eta},\boldsymbol{\phi},\boldsymbol{\psi})_{yy}(\boldsymbol{t})\|^2.$$

Proposition 3 (*a priori* **estimate**). *Suppose* $a - \mu > 0$. *Then there exist positive constants* ε_0 *and* C_0 *such that if* $(\eta,\phi,\psi) \in C([0,T];H^2)$ *is the solution of the Cauchy problem* (5) *for some* $T > 0$ *and* $\sup_{t\in[0,T]} E(t) \le \varepsilon_0$ *it holds that*

$$\begin{aligned}
&\|(\eta,\phi,\psi)(t)\|_2^2 + (1+t)^2\|(\eta,\phi,\psi)_y(t)\|_1^2 + (1+t)^4\|(\eta,\phi,\psi)_{yy}(t)\|^2 \\
&\quad + \int_0^t \left(\frac{1}{1+\tau}\|\phi(\tau)\|^2 + \frac{1}{1+\tau}\|(\phi,\psi)_x(\tau)\|_2^2 + (1+\tau)\|(\phi,\psi)_y(\tau)\|_2^2\right) d\tau \\
&\quad + \int_0^t \left((1+\tau)^3\|(\phi,\psi)_{yy}(\tau)\|_1^2 + (1+\tau)^5\|(\phi,\psi)_{yyy}(\tau)\|^2)\right) d\tau \\
&\quad + \int_0^t \left(\frac{1}{1+\tau}\|\eta_x(\tau)\|_1^2 + (1+\tau)\|\eta_y(t)\|_1^2 + (1+\tau)^3\|\eta_{yy}(\tau)\|^2\right) d\tau \\
&\quad \le C_0\|(\eta_0,\phi_0,\psi_0)\|_2^2, \quad t \in [0,T].
\end{aligned}$$

Decay estimates

$$\sup_{(x,y)\in R^2} |\eta(t,x,y)|^2 \leq \iint |(2\eta\eta_x)_y|\, dxdy$$
$$\leq C(\|\eta_x\|\|\eta_y\| + \|\eta\|\|\eta_{xy}\|)$$
$$\leq C(1+t)^{-1}$$

which implies

$$\sup_{(x,y)\in R^2} \mid \rho(t,x,y) - \frac{1}{1+t} \mid \leq C(1+t)^{-\frac{3}{2}},$$

and similarly

$$\sup_{(x,y)\in R^2} \mid u(t,x,y) - (\frac{x}{1+t}, 0) \mid \leq C(1+t)^{-\frac{1}{2}}.$$

Basic energy estimate

Estimate (5) $\cdot$ $(a\eta,\ \phi,\ \psi)$:

$$\frac{1}{2}\frac{d}{dt}\iint(a\eta^2+\phi^2+\psi^2)dxdy$$

$$+\iint\left(\frac{1}{1+t}\phi^2+\frac{\mu}{1+t}(\phi_x^2+\psi_x^2)+\mu(1+t)(\phi_y^2+\psi_y^2)\right)dxdy$$

$$=\iint(a\eta N_0+\phi N_1+\psi N_2)\,dxdy. \tag{6}$$

Estimate $(5)_2 \cdot \eta_x + (5)_3 \cdot (1+t)\eta_y$:

$$\begin{aligned}
&\frac{d}{dt}\iint \left(\phi\eta_x + (1+t)\psi\eta_y + \frac{1}{2a}\psi^2 + \frac{\mu}{2}\left(\eta_x^2 + (1+t)^2\eta_y^2\right)\right) dxdy \\
&\quad + \iint \left(\frac{a}{1+t}\eta_x^2 + (a-\mu)(1+t)\eta_y^2\right) dxdy \\
&\quad - \iint \left(\frac{1}{1+t}\phi_x^2 + 2\phi_x\psi_y + (1+t)\psi_y^2\right) dxdy \\
&\quad + \iint \left(\frac{\mu}{a(1+t)}\psi_x^2 + \frac{\mu}{a}(1+t)\psi_y^2 + \frac{1}{1+t}\phi\eta_x\right) dxdy \qquad (7) \\
&\quad = \iint \Big(\eta_x N_1 + (1+t)\eta_y N_2 - \phi_x N_0 - (1+t)\psi_y N_0 \\
&\qquad\qquad + \frac{1}{a}\psi N_2 + \mu\left(\eta_x N_{0x} + (1+t)^2\eta_y N_{0y}\right)\Big) dxdy.
\end{aligned}$$

By (5), we can estimate

$$\|(\eta,\phi,\psi)(t)\|^2$$

$$+\int_0^t \left(\frac{1}{1+\tau}\|\phi(\tau)\|^2+\frac{1}{1+\tau}\|(\phi,\psi)_x(\tau)\|^2+(1+\tau)\|(\phi,\psi)_y(\tau)\|^2\right)\,d\tau.$$

By combining (5) and (6), we can estimate

$$\int_0^t \left(\frac{1}{1+\tau}\|\eta_x(\tau)\|^2+(1+\tau)\|\eta_y(t)\|^2\right)\,d\tau.$$

Proceed the time-weighted estimates by

$$\partial_y(5)\cdot(1+t)^2(a\eta,\ \phi,\ \psi)_y,\quad \partial_y^2(5)\cdot(1+t)^4(a\eta,\ \phi,\ \psi)_{yy},$$

$$\partial_y(5)_2\cdot(1+t)^2\eta_{xy}+\partial_y(5)_3\cdot(1+t)^3\eta_{yy}.$$

Sketch of the proof of Theorem 2

Existence of viscous shock wave

Setting $\xi = x - st$, and $(v,u) = (V,U)(\xi)$, we have the ODE system

$$\begin{cases} -sV_\xi - U_\xi = 0, \\ -sU_\xi + p(V)_\xi = \mu_0 (\frac{U_\xi}{V^{\alpha+1}})_\xi, \end{cases}$$

with $(V,U)(\pm\infty) = (v_\pm, u_\pm)$. We only treat the case $s > 0$.

Integrating the system, we have

$$\begin{cases} sV + U = sv_\pm + u_\pm, \\ -sU + p(V) - \mu_0 \frac{U_\xi}{V^{\alpha+1}} = -su_\pm + p(v_\pm), \end{cases}$$

and the "Rankine-Hugoniot condition"

$$\begin{cases} -s(v_+ - v_-) - (u_+ - u_-) = 0, \\ -s(u_+ - u_-) + p(v_+) - p(v_-) = 0. \end{cases}$$

Equation of V

$$\begin{cases} V_\xi = \dfrac{V^{\alpha+1}}{\mu_0 s} h(V), \quad \xi \in R, \\ V(\pm\infty) = v_\pm \end{cases}$$

where

$$h(V) := s^2(v_- - V) + p(v_-) - p(V) > 0, \quad V \in (v_\mp, v_\pm),$$

and $h(v_\pm) = 0$. Under the "entropy condition"

$$v_- < v_+, \qquad (i.e.\ u_- > u_+),$$

the solution V uniquely exists up to the shift of ξ, satisfying

$$V_\xi(\xi) > 0, \quad v_- < V(\xi) < v_+, \quad \xi \in R,$$

and U is given by $U = u_\pm + s(v_\pm - V)$.

Change of the variables : $(x,t) \to (\xi = x - st, t)$

$$\begin{cases} v_t - sv_\xi - u_\xi = 0, \\ u_t - su_\xi + p(v)_\xi = \mu_0 (\dfrac{u_\xi}{v^{\alpha+1}})_\xi. \end{cases}$$

Change of the unknown variables : Define $(\phi, \psi)(t, \xi)$ by

$$\phi(t,\xi) = \int_{-\infty}^{\xi} (v - V)(t,\eta)\, d\eta, \quad \psi(t,\xi) = \int_{-\infty}^{\xi} (u - U)(t,\eta)\, d\eta,$$

that is,

$$v = V + \phi_\xi, \quad u = U + \psi_\xi.$$

Cauchy problem for (ϕ, ψ) :

$$\begin{cases} \phi_t - s\phi_\xi - \psi_\xi = 0, \\ \psi_t - s\psi_\xi + p(V + \phi_\xi) - p(V) = \mu_0 \left(\dfrac{(U + \psi_\xi)_\xi}{(V + \phi_\xi)^{\alpha+1}} - \dfrac{U_\xi}{V^{\alpha+1}} \right), \end{cases}$$

with the initial data

$$(\phi, \psi)(0) = (\phi_0, \psi_0) \in H^2.$$

We look for the global solution in time such that

$$(\phi, \psi) \in C([0, \infty); H^2), \quad \sup_{x \in R} |(\phi, \psi)(t, x)| \to 0 \quad (t \to \infty).$$

Proposition 4 (*a priori* estimate).

Suppose $\alpha \geq \frac{1}{2}(\gamma - 1)$. *Then there exist positive constants* ε_0 *and* C_0 *such that if* $(\phi, \psi) \in C([0, T]; H^2)$ *is the solution of the Cauchy problem for some* $T > 0$ *and*

$$\sup_{t \in [0,T]} \|(\phi, \psi)(t)\|_2 \leq \varepsilon_0,$$

it holds that for $t \in [0, T]$

$$\|(\phi, \psi)(t)\|_2^2 + \int_0^t (\|\phi_\xi(\tau)\|_1^2 + \|\psi_\xi(\tau)\|_2^2)\, d\tau \leq C_0 \|\phi_0, \psi_0\|_2^2.$$

A priori estimate

Rewrite the system as

$$\begin{cases} \phi_t - s\phi_\xi - \psi_\xi = 0, \\ \psi_t - s\psi_\xi - K(V)\phi_\xi - \dfrac{\mu_0}{V^{\alpha+1}}\psi_{\xi\xi} = G, \end{cases} \tag{5}$$

where

$$K(V) = -p'(V) + (\alpha+1)\frac{h(V)}{V},$$

and

$$G = -\{p(V+\phi_\xi) - p(V) - p'(V)\phi_\xi\} + \mu_0\{\frac{1}{(V+\phi_\xi)^{\alpha+1}} - \frac{1}{V^{\alpha+1}}\}\psi_{\xi\xi}$$
$$+ V^{\alpha+1}h(V)\{-\frac{1}{(V+\phi_\xi)^{\alpha+1}} + \frac{1}{V^{\alpha+1}} - \frac{\alpha+1}{V^{\alpha+2}}\phi_\xi\}.$$

Recall

$$s > 0, \quad v_- < V < v_+, \quad V_\xi = \frac{V^{\alpha+1}}{\mu_0 s} h(V) > 0, \quad p'(V) = -\gamma \frac{p(V)}{V},$$

which implies

$$K(V) \geq \gamma \frac{p(v_+)}{v_+}.$$

Basic energy estimate

There exists a positive constant C *such that it holds that for* $t \in [0, T]$

$$\| (\phi, \psi)(t) \|^2 + \int_0^t (\| \psi_\xi(\tau) \|^2 + \| \sqrt{V_\xi} \psi(\tau) \|^2) \, d\tau$$

$$\leq C\{\| \phi_0, \psi_0 \|^2 + \int_0^t \int |\psi||G| \, d\xi d\tau\}.$$

Proof

We use a technical weighted energy method, "double step weighted energy method", developed by Mei-M (1997) and Hashimoto-M (2007).

First-Step : Introduce the functions (transform functions)

$$\chi_1 = \chi_1(V) > 0, \quad \chi_2 = \chi_2(V) > 0,$$

and transform the unknown variables

$$\phi = \chi_1(V)\tilde{\phi}, \quad \psi = \chi_2(V)\tilde{\psi}.$$

Then we have

$$\begin{cases} (\chi_1\tilde{\phi})_t - s(\chi_1\tilde{\phi})_\xi - (\chi_2\tilde{\psi})_\xi = 0, \\ (\chi_2\tilde{\psi})_t - s(\chi_2\tilde{\psi})_\xi - K(\chi_1\tilde{\phi})_\xi - \dfrac{\mu_0}{V^{\alpha+1}}(\chi_2\tilde{\psi})_{\xi\xi} = G. \end{cases}$$

Second-Step :

Introduce the another set of functions (weight functions)

$$W_1 = W_1(V) > 0, \quad W_2 = W_2(V) > 0.$$

Multiply the first equation by $W_1\tilde{\phi}$, the second equation by $W_2\tilde{\psi}$ and sum them up. Then we have

$$\begin{aligned}
&\frac{d}{dt}\int \frac{1}{2}(W_1\chi_1\tilde{\phi}^2 + W_2\chi_2\tilde{\psi}^2)\,d\xi \\
&+ \int \frac{s}{2}\{(W_1'\chi_1 - W_1\chi_1')\tilde{\phi}^2 + (W_2'\chi_2 - W_2\chi_2')\tilde{\psi}^2\}V_\xi\,d\xi \\
&+ \int \{(W_1'\chi_2 - KW_2\chi_1')V_\xi\tilde{\phi}\tilde{\psi} + (W_1\chi_2 - KW_2\chi_1)\tilde{\phi}_\xi\tilde{\psi}\}\,d\xi \\
&+ \int \mu_0(\frac{W_2}{V^{\alpha+1}}\tilde{\psi})_\xi(\chi_2\tilde{\psi})_\xi\,d\xi = \int W_2\tilde{\psi}G\,d\xi.
\end{aligned}$$

(31)

In order for the coefficients of the cross terms $\tilde{\phi}\tilde{\psi}$ and $\tilde{\phi}_\xi\tilde{\psi}$ to banish, we impose that

$$W_1'\chi_2 - KW_2\chi_1' = 0, \quad W_1\chi_2 - KW_2\chi_1 = 0.$$

Choose χ_1, χ_2, W_1 and W_2 as

$$\chi_1(V) = W_1(V) = 1, \quad \chi_2(V) = K(V)W(V), \quad W_2(V) = W(V).$$

Then we have

$$\frac{d}{dt}\int \frac{1}{2}(\tilde{\phi}^2 + KW^2\tilde{\psi}^2)\,d\xi + \int (-\frac{s}{2}K'W^2V_\xi\tilde{\psi}^2 + \mu_0\{(\frac{W}{V^{\alpha+1}})'(KW)'V_\xi^2\tilde{\psi}^2$$
$$+ (\frac{KW^2}{V^{\alpha+1}})'V_\xi\tilde{\psi}\tilde{\psi}_\xi + \frac{KW^2}{V^{\alpha+1}}(\tilde{\psi}_\xi)^2\})\,d\xi = \int W\tilde{\psi}G\,d\xi.$$

(32)

It further deduces

$$\frac{d}{dt}\int \frac{1}{2}(\tilde{\phi}^2 + KW^2\tilde{\psi}^2)\, d\xi + \int (A(V)V_\xi\tilde{\psi}^2 + \mu_0\frac{KW^2}{V^{\alpha+1}}(\tilde{\psi}_\xi)^2)\, d\xi$$
$$= \int W\tilde{\psi}G\, d\xi,$$

where

$$A(V) = -\frac{s}{2}K'W^2 + \frac{1}{s}(\frac{W}{V^{\alpha+1}})'(KW)'V^{\alpha+1}h - \frac{1}{2s}((\frac{KW^2}{V^{\alpha+1}})'V^{\alpha+1}h)'.$$

We show the uniform positiveness of $A(V)$ for $V \in [v_-, v_+]$.

Recall

$$p'(V) = -\gamma \frac{p(V)}{V}, \quad h'(V) = -s^2 + \gamma \frac{p(V)}{V}.$$

Representations of K, K' and K'' in terms of h and p :

$$K(V) = \gamma \frac{p(V)}{V} + (\alpha + 1)\frac{h(V)}{V},$$

$$K'(V) = -\gamma(\gamma - \alpha)\frac{p(V)}{V^2} - (\alpha + 1)\frac{s^2}{V} - (\alpha + 1)\frac{h(V)}{V^2},$$

$$K''(V) = \gamma((\gamma - \alpha)(\gamma + 2) - (\alpha + 1))\frac{p(V)}{V^3} + 2(\alpha + 1)\frac{s^2}{V^2} + 2(\alpha + 1)\frac{h(V)}{V^3}.$$

Reformulate $A(V)$ as a quadratic form of h :

$$A(V) = \frac{W}{2s}(A_0(V) + A_1(V)h(V) + A_2(V)h^2(V)),$$

where

$$\begin{aligned} A_0(V) =& \frac{\gamma^2(\gamma+1)p^2}{V^3}W - 2(\frac{\gamma^2 p^2}{V^2} - \gamma s^2 \frac{p}{V})W', \\ A_1(V) =& (\gamma(\gamma+2)(2\alpha+1-\gamma)\frac{p}{V^3} - 2s^2\frac{(\alpha+1)}{V^2})W \\ & - 2(\gamma(2\alpha+1-\gamma)\frac{p}{V^2} - 2s^2\frac{(\alpha+1)}{V})W' - 2\gamma\frac{p}{V}W'', \\ A_2(V) =& 2(\alpha+1)(-\frac{W}{V^3} + \frac{W'}{V^2} - \frac{W''}{V}). \end{aligned}$$

Choice of $W(V)$:

$$W(V) = V. \quad (\implies A_2(V) = 0)$$

Final form of $A(V)$:

$$A(V) = \frac{1}{2s}\{2s^2\gamma p(V) + \gamma^2(\gamma - 1)\frac{p^2(V)}{V}$$
$$+2\gamma^2(\alpha - \frac{1}{2}(\gamma - 1))\frac{p(V)}{V}h(V) + 2(\alpha + 1)s^2 h(V)\}.$$

By the physical assumption $\alpha \geq \frac{1}{2}(\gamma - 1)$,

$$A(V) \geq s\gamma p(v_+), \quad V \in [v_-, v_+].$$

Thus, integrating the inequality, we have

$$\|(\tilde{\phi},\tilde{\psi})(t)\|^2+\int_0^t(\|\tilde{\psi}_\xi(\tau)\|^2+\|\sqrt{V_\xi}\tilde{\psi}(\tau)\|^2)\,d\tau \le C\{\|\tilde{\phi}_0,\tilde{\psi}_0\|^2+\int_0^t\int|\tilde{\psi}||G|\,d\xi d\tau\}.$$

Recalling the transformations $\phi=\tilde{\phi}$, $\psi=VK(V)\tilde{\psi}$, the proof is thus completed.

Thank You !

A remark

We give a remark that our arguments in the previous sections can be extended to the models with more general viscosity coefficient. In the book of Chapman-Cowling, the viscosity coefficient is given in more general form as

$$\mu = \bar{\mu}\,\theta^{\frac{1}{2}} F(\theta).$$

It is noted that $F(\theta) = 1$ for the Hard sphere Model, and $F(\theta) = \theta^{\frac{2}{(s-1)}}$ for the Cut-off inverse power force Model. As a typical model whose F is not a power function of θ, the Sutherland's model is well known where

$$F(\theta) = \frac{1}{1 + \frac{s_0}{\theta}}, \quad (s_0 > 0 : \textit{Sutherland's constant}).$$

Even for the above general case, if we also take

$$\chi_1(V) = W_1(V) = 1, \quad \chi_2(V) = K(V)W(V), \quad W_2(V) = W(V) = V$$

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as in the proof of the Lemma 3, then after long calculations we can have the following :

$$\begin{aligned}
A(V) = \frac{1}{2s}\{ & 2s^2\gamma p(V) + \gamma^2(\gamma-1)\frac{p^2(V)}{V} \\
& + 2s^2 h(V)\left(\frac{\gamma+1}{2} + (\gamma-1)\theta\frac{F'}{F} - (\gamma-1)^2\theta(\theta\frac{F'}{F})'\right)\Big|_{\theta=(a/R)V^{1-\gamma}} \\
& + 2\gamma(\gamma-1)h(V)\frac{p(V)}{V}\left(\gamma\theta\frac{F'}{F} + (\gamma-1)\theta(\theta\frac{F'}{F})'\right)\Big|_{\theta=(a/R)V^{1-\gamma}} \\
& - (\gamma-1)^2\frac{h^2(V)}{V}\left(\gamma\theta(\theta\frac{F'}{F})' + (\gamma-1)\theta^2(\theta\frac{F'}{F})''\right)\Big|_{\theta=(a/R)V^{1-\gamma}} \}.
\end{aligned}$$

Therefore, if it holds

$$\begin{cases}
\frac{\gamma+1}{2} + (\gamma-1)\theta\frac{F'}{F} - (\gamma-1)^2\theta(\theta\frac{F'}{F})' \geq 0, \\
\gamma\theta\frac{F'}{F} + (\gamma-1)\theta(\theta\frac{F'}{F})' \geq 0, \\
\gamma\theta(\theta\frac{F'}{F})' + (\gamma-1)\theta^2(\theta\frac{F'}{F})'' \leq 0,
\end{cases}$$

then the desired *a priori* estimate holds, and eventually the asymptotic stability of the viscous shock follows for small initial perturbations with integral zero. Finally, we note that as for the Sutherland's model the direct computation shows the above condition holds for $\gamma \in [1, 2]$.