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AN ENERGY METHOD FOR THE EQUATIONS OF MOTION OF COMPRESSIBLE VISCOUS AND HEAT-CONDUCTIVE FLUIDS

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ABSTRACT

A priori estimates for solutions of the quasilinear hyperbolic-parabolic equations governing the initial value problem describing the motion of compressible, viscous and heat-conductive, Newtonian fluids are derived by means of a new energy method. This technique enables us to simplify and unify our previous results on the global existence in time and uniqueness of smooth solutions of these equations for sufficiently smooth and "small" initial data and to obtain their rate of decay.

AMS (MOS) Subject Classifications: 35B40, 35K55, 35M05, 76.35

Key Words: Energy method, Initial value problem, Equation of motion, Compressible, viscous and heat-conductive fluids, Global solutions in time

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SIGNIFICANCE AND EXPLANATION

The motion of compressible, viscous and heat-conductive Newtonian fluids is described by a system of partial differential equations which is of hyperbolic-parabolic type and highly nonlinear. One of the first mathematical problems associated with this system is the initial value problem. We obtain the existence of a unique smooth global solution in time for the initial value problem and also the decay rate of the solution as time tends to infinity. Since the system is quasilinear with respect to the unknowns: density, velocity and temperature, we need to assume that the initial data are close to the constant equilibrium state. The purpose of this paper is to obtain a priori estimates for the solutions of these equations by means of a new energy method. This technique, although still necessarily laborious, enables us to simplify and unify our previous results, described briefly in the abstract and obtained jointly with T. Nishida (see, e.g. MRC TSR #1991).

AN ENERGY METHOD FOR THE EQUATIONS OF MOTION OF COMPRESSIBLE VISCOUS AND HEAT-CONDUCTIVE FLUIDS

Akitaka Matsumura*

§1. Introduction and Main Theorem.

In previous papers [1], [2], we have investigated the global solution in time of the initial value problem for the following equations governing the motion of isotropic Newtonian fluids;

$$
(1.1)\quad \begin{cases} \rho_t + (\rho u^j)_{x_j} = 0 \ , \\ u^i_t + u^j u^i_{x_j} + \frac{1}{\rho} p_{x_i} = \frac{1}{\rho}\left(\mu(u^i_{x_j} + u^j_{x_i}) + \mu' u^k_{x_k}\delta^{ij}\right)_{x_j}, \ i = 1,2,3, \\ \theta_t + u^j \theta_{x_j} + \frac{\theta p_\theta}{\rho c_V} u^j_{x_j} = \frac{1}{\rho c_V}\left((\kappa\theta_{x_j})_{x_j} + \Psi\right) \ , \end{cases}
$$

with the initial data

$$
(1.2)\qquad (\rho,u,\theta)(0,x) = (\rho_0,u_0,\theta_0)(x) \ ,
$$

where $t > 0$, $x = (x_1,x_2,x_3) \in R^3$, ρ is the density, $u = (u^1,u^2,u^3)$ is the velocity, θ is the absolute temperature, $p = p(\rho,\theta)$ is the pressure, $\mu = \mu(\rho,\theta)$ and $\mu'(\rho,\theta)$ are viscous coefficients, $\kappa = \kappa(\rho,\theta)$ is the coefficient of heat conduction, $c_V = c_V(\rho,\theta)$ is the heat capacity at constant volume, and $\Psi = \frac{1}{2}\left(u^j_{x_k} + u^k_{x_j}\right)^2 + \mu'\left(u^j_{x_j}\right)^2$ is dissipation

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function. We shall consider the solutions only in a neighbourhood of any fixed constant state $(\rho, u, \theta) = (\bar{\rho}, 0, \bar{\theta})$ where $\bar{\rho}, \bar{\theta}$ are any positive constants. Moreover, we shall make the following natural assumptions on the hyperbolic-parabolic system (1.1) throughout this paper;

(i) p, c_V, μ, μ' and κ are smooth functions of $O = \{(\rho, u, \theta) : |\rho - \bar{\rho}|, |u|, |\theta - \bar{\theta}| < \gamma_0\}$,

(ii) p_ρ, p_θ, c_V, μ, $\kappa > 0$ and $\mu' + \frac{2}{3}\mu > 0$ in O,

where $\gamma_0 < \min(\bar{\rho}, \bar{\theta})$.

In [1], we succeeded in obtaining a global solution in time of (1.1), (1.2) by using energy methods which were rather technical and complicated under the assumptions that the fluid is an ideal and polytropic gas and that

$$(\rho_0 - \bar{\rho},\ u_0,\ \theta_0 - \bar{\theta}) \text{ is suitably small in } H^3 ,$$

where H^k represents the usual Sobolev's spaces with the norm $\|\cdot\|_k$. We also proved the decay of the solution to the constant state $(\bar{\rho}, 0, \bar{\theta})$, but we were not able to estimate the decay rate of the solution. In [2], we obtained both the global solution in time of the original problem (1.1), (1.2) and its decay rate to the constant state by using a calculation of the decay rate for the linearized equations, together with the energy estimates. Moreover, we had to investigate the precise properties of the spectrum of linearized equations, and assume that

$$(\rho_0 - \bar{\rho},\ u_0,\ \theta_0 - \bar{\theta}) \text{ is suitably small in } H^4 \cap L^1 .$$

The purpose of this paper is to employ a different energy method to handle the nonlinearity. The present approach is simpler and less technical than the energy form used in [1], moreover we do not require all of the precise properties of the linearized equations obtained laboriously in [2]. However, we note that because of the relative roughness of the new method, the coefficients of the various estimates might be more rough than those in [1] and [2]. By making use of this approach, we obtain both the existence of global smooth solution in time as well as its decay rate to the constant state for the general case (1.1), (1.2) under the assumption that

$$(\rho_0 - \bar{\rho},\ u_0,\ \theta_0 - \bar{\theta}) \text{ is suitably small in } H^3 \quad ,$$

(the previous approach required $H^4 \cap L^1$). This method will be applied to an initial boundary value problem for (1.1), (1.2) in a subsequent paper.

The main result is:

Theorem 1.1. Consider the initial value problem (1.1), (1.2) and suppose the initial data $(\rho_0 - \bar{\rho}, u_0, \theta_0 - \bar{\theta}) \in H^3$. Then there exist positive constants ε_0 and C_0 such that if $\|\rho_0 - \bar{\rho}, u_0, \theta_0 - \bar{\theta}\|_3 < \varepsilon_0$, the problem (1.1), (1.2) has the unique global solution in time satisfying

$$\rho - \bar{\rho} \in C^0(0, +\infty; H^3) \cap C^1(0, +\infty; H^2) \quad ,$$

$$(u, \theta - \bar{\theta}) \in C^0(0, +\infty; H^3) \cap C^1(0, +\infty; H^1) \quad ,$$

and

$$(1.3) \qquad \sup_x |(\rho-\bar{\rho}, u, \theta - \bar{\theta})(t)| < C_0(1+t)^{-\frac{3}{4}} \|\rho_0-\bar{\rho}, u_0, \theta_0 - \bar{\theta}\|_3 \quad .$$

§2. Proof of Theorem

First rewrite the problem (1.1), (1.2) by the changes of variables: $\rho \to \bar{\rho} + \rho$, $u \to u$, $\theta \to \bar{\theta} + \theta$, $p(\bar{\rho}+\rho, \bar{\theta}+\theta) \to p(\rho,\theta)$, $\mu(\bar{\rho}+\rho, \bar{\theta}+\theta) \to \mu(\rho,\theta)$ and so on resulting in

$$
(2.1) \quad \begin{cases}
\rho_t + (\bar{\rho} + \rho)u^j_{x_j} + u^j \rho_{x_j} = 0 \ , \\[2mm]
u^i_t + u^j u^i_{x_j} + p_\rho \rho_{x_i} + \dfrac{p_\theta}{\bar{\rho}+\rho}\, \theta_{x_i} = \\[2mm]
\qquad = \dfrac{1}{\bar{\rho}+\rho} \left(\mu(u^i_{x_j} + u^j_{x_i}) + \mu' u^k_{x_k} \delta^{ij}\right)_{x_j} \ , \\[2mm]
\theta_t + u^j \theta_{x_j} + \dfrac{(\bar{\theta}+\theta) p_\theta}{(\bar{\rho}+\rho) c_v} u^j_{x_j} = \dfrac{1}{(\bar{\rho}+\rho) c_v} \left((\kappa \theta_{x_j})_{x_j} + \Psi\right) \ ,
\end{cases}
$$

$$
(2.2) \qquad (\rho, u, \theta)(0,x) = (\rho_0, u_0, \theta_0)(x) \ .
$$

Define a positive constant E_0 by the Sobolev's lemma so that for $\|f\|_2 < E_0$ we have $\sup|f| < C\|f\|_2 < \gamma_0$. Denote $D^k f = \{\partial^k f / \partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}$ for all α, $\alpha_1 + \alpha_2 + \alpha_3 = k\}$ and define $\|\cdot\| = \|\cdot\|_0$. Then the solution of (1.1), (1.2) is sought in the set of functions $X(0, +\infty; E)$ for some $E < E_0$, where for $0 < t_1 < t_2 < +\infty$,

$X(t_1, t_2; E) = \{(\rho,u,\theta) \ ; \ \rho \in C^0(t_1, t_2; H^3) \cap C^1(t_1, t_2; H^2)$,

$D\rho \in L^2(t_1, t_2; H^2)$, $(1+t)^{1/2} D^2 \rho \in L^2(t_1, t_2; L^2)$,

$(1+t) D^3 \rho \in L^2(t_1, t_2; L^2)$,

$(u,\theta) \in C^0(t_1, t_2; H^3) \cap C^1(t_1, t_2; H^1)$,

$D(u,\theta) \in L^2(t_1, t_2; H^3)$, $(1+t)^{1/2} D^2(u,\theta) \in L^2(t_1, t_2; L^2)$,

$(1+t) D^3(u,\theta) \in L^2(t_1, t_2; H^1)$,

and $N(t_1, t_2) < E \quad (E < E_0)\}$.

We define $N(t_1, t_2)$ by

$$N^2(t_1, t_2) = \sup_{t_1<t<t_2} (\|(\rho,u,\theta)(t)\|_3^2 + t\,\|D(\rho,u,\theta)(t)\|^2 + t^2 \cdot$$

$$\cdot \|D^2(\rho,u,\theta)(t)\|_1^2) + \int_{t_1}^{t_2} \|D\rho(\tau)\|_2^2 + \|D(u,\theta)(\tau)\|_3^2 +$$

$$+ \tau\|D^2(\rho,u,\theta)(\tau)\|^2 + \tau^2(\|D^3\rho(\tau)\|^2 + \|D^3(u,\theta)(\tau)\|_1^2)d\tau \quad .$$

We prove Theorem 1.1 by a combination of a local existence result and a priori estimate for the solution in X.

Theorem 2.1. (local existence) Consider the initial value problem (2.1) for $t > t_1$ with the initial data at $t = t_1$ as

$$(2.3) \qquad (\rho,u,\theta)(t_1) \in H^3 \quad .$$

Then there exist positive constants ε_1, $C_1(\varepsilon_1 C_1 < E_0)$ and τ which are independent of t_1 such that if $N(t_1, t_1) < \varepsilon_1$, the problem (2.1)-(2.3) has the unique solution

$$(\rho,u,\theta) \in X(t_1, t_1 + \tau; C_1 N(t_1,t_1)) \quad .$$

The proof of Theorem 2.1 is given in the same way as in [1].

Theorem 2.2. (a priori estimates) Suppose that the initial value problem (2.1), (2.2) has a solution

$$(\rho,u,\theta) \in X(0,T; E)$$

for some $T > 0$ *and some* $E < E_0$. *Then there exist three positive constants* ε_2 *and* ε_3 $(\varepsilon_2, \varepsilon_3 < \varepsilon_1)$ *and* C_2 $(C_2 \varepsilon_3 \sqrt{1+C_1^2} < E_0)$ *which are independent of* T *such that if* $\|\rho_0, u_0, \theta_0\|_3 < \varepsilon_2$ *and* $E < \varepsilon_3$, *then the solution satisfies the a priori estimate*

$$(\rho, u, \theta) \in X(0,T;\ C_2 \|\rho_0, u_0, \theta_0\|_3) \quad .$$

Proof of Theorem 2.2. Take

$$\varepsilon_0 = \min\left(\varepsilon_1, \varepsilon_2, \frac{\varepsilon_3}{C_1}, \frac{\varepsilon_1}{C_2}, \frac{\varepsilon_3}{C_2\sqrt{1+C_1^2}}\right) \quad .$$

We may use the standard continuation arguments of a local solution on $[0, n\tau]$, $n = 1,2,\ldots$ to get the global solution. In fact, by the local existence theorem, the definition of ε_0 and the assumption $\|\rho_0, u_0, \theta_0\|_3 < \varepsilon_0$, we have a local solution

$$(\rho, u, \theta) \in X(0,\tau;\ C_1 \|\rho_0, u_0, \theta_0\|_3) \quad .$$

By $C_1\|\rho_0, u_0, \theta_0\|_3 < C_1\varepsilon_0 < \varepsilon_3$ and $\|\rho_0, u_0, \theta_0\|_3 < \varepsilon_2$, the a priori estimate gives

$$(\rho, u, \theta) \in X(0,\tau;\ C_2 \|\rho_0, u_0, \theta_0\|_3) \quad .$$

Then by $C_2\|\rho_0, u_0, \theta_0\|_3 < C_2\varepsilon_0 < \varepsilon_1$ and the local existence theorem with $t_1 = \tau$, we have again

$$(\rho, u, \theta) \in X(\tau, 2\tau;\ C_1 C_2 \|\rho_0, u_0, \theta_0\|_3) \quad .$$

Noting that

$$N^2(0,2\tau) < N^2(0,\tau) + N^2(\tau,2\tau) \quad ,$$

we also have

$$(\rho,u,\theta) \in X(0,2\tau;\ C_2\sqrt{1+C_1^2}\ \|\rho_0,u_0,\theta_0\|_3) \quad .$$

Now by $C_2\sqrt{1+C_1^2}\ \|\rho_0,u_0,\theta_0\|_3 < C_0\sqrt{1+C_1^2}\ \varepsilon_0 < \varepsilon_3$, the a priori estimate shows that

$$(\rho,u,\theta) \in X(0,2\tau;\ C_2\|\rho_0,u_0,\theta_0\|_3) \quad .$$

Thus we can continue to use the same arguments on $[n\tau,\ (n+1)\tau]$ and $[0,\ (n+1)\tau]$ successively $n = 2,3,\ldots$. Finally the estimate (1.3) follows from Nirenberg's inequality [3]

(2.4) $$\sup_x|\rho,u,\theta| < C\|\rho,u,\theta\|^{\frac{1}{4}}\ \|D^2(\rho,u,\theta)\|^{\frac{3}{4}} \quad .$$

§3. A Priori Estimates

We present here an energy method to obtain a priori estimates for small solutions of equations with dissipatin. First we rewrite the system (2.1) so that all the nonlinear terms appear at the right hand side of equations;

$$
(3.1)\quad \begin{cases} L^0 \equiv \rho_t + \bar{\rho} u^j_{x_j} = f^0 , \\ L^i \equiv u^i_t + p_1 \rho_{x_i} + p_2 \theta_{x_i} - \tilde{\mu} u^i_{x_j x_j} - (\tilde{\mu}+\tilde{\mu}') u^j_{x_i x_j} = f^i , \\ L^4 \equiv \theta_t + p_3 u^j_{x_j} - \tilde{\kappa} \theta_{x_j x_j} = f^4 , \end{cases}
$$

where

$$
p_1 = p_\rho(0,0)/\bar{\rho},\ p_2 = p_\theta(0,0)/\bar{\rho},\ p_3 = \bar{\theta} p_\theta(0,0)/\bar{\rho}\, C_V(0,0) ,
$$

$$
\tilde{\mu} = \mu(0,0)/\bar{\rho},\ \tilde{\mu}' = \mu'(0,0)/\bar{\rho},\ \tilde{\kappa} = \kappa(0,0)/\bar{\rho}\, C_V(0,0) ,
$$

$$
f^0 \equiv -\rho u^j_{x_j} - u^j \rho_{x_j} ,
$$

$$
\begin{aligned} f^i \equiv & - u^j u^i_{x_j} + (p_1 - p_\rho/(\bar{\rho}+\rho)\rho_{x_i} + (p_2 - p_\theta/(\bar{\rho}+\rho))\theta_{x_i} + \\ & + (\mu_\rho \rho_{x_j} + \mu_\theta \theta_{x_j})(u^i_{x_j} + u^j_{x_i})/(\bar{\rho}+\rho) + \\ & + (\mu'_\rho \rho_{x_i} + \mu'_\theta \theta_{x_i}) u^j_{x_j}/(\bar{\rho}+\rho) + (\mu/(\bar{\rho}+\rho) - \tilde{\mu}) u^i_{x_j x_j} + \\ & + ((\mu+\mu')/(\bar{\rho}+\rho) - (\tilde{\mu}+\tilde{\mu}')) u^j_{x_i x_j} , \end{aligned}
$$

and

$$
\begin{aligned} f^4 \equiv & - u^j \theta_{x_j} + (p_3 - (\bar{\theta}+\theta) p_\theta/(\rho+\rho) C_V) u^j_{x_j} + (\kappa_\rho \rho_{x_j} + \kappa_\theta \theta_{x_j}) \theta_{x_j}/(\bar{\rho}+\rho) C_V \\ & + (\kappa/(\bar{\rho}+\rho) C_V - \tilde{\kappa}) \theta_{x_j x_j} + \Psi/\rho C_V . \end{aligned}
$$

Define, for $k, \ell, m = 0,1,2,3,$

$$
A^k(t) \equiv \int \frac{p_1}{\bar{\rho}} D^k f^0 \cdot D^k \rho + D^k f^i \cdot D^k u^i + \frac{p_2}{p_3} D^k f^4 \cdot D^k \theta \, dx ,
$$

$$
B^m(t) \equiv \int D^m f^0_{x_i} \cdot D^m \rho_{x_i} + \frac{\bar{\rho}}{2\tilde{\mu}+\tilde{\mu}'} D^m f^i \cdot D^m \rho_{x_i} \, dx ,
$$

$$
C^\ell(t) \equiv \int \frac{\bar{\rho}}{2\tilde{\mu}+\tilde{\mu}'} D^\ell u^i \cdot D^\ell f^0_{x_i} \, dx .
$$

Then we have the following:

Lemma 3.1. There exist positive constants ν and C which are independent of t such that

$$(i)_k \quad \|D^k(\rho,u,\theta)(t)\|^2 + \nu \int_0^t \|D^{k+1}(u,\theta)(\tau)\|^2\, d\tau \le$$

$$\le C(\|D^k(\rho_0,u_0,\theta_0)\|^2 + \int_0^t |A^k(\tau)| d\tau), \quad 0 \le k \le 3 \quad ,$$

$$(ii)_m \quad \|D^m\rho(t)\|^2 + \nu \int_0^t \|D^m\rho(\tau)\|^2\, d\tau \le$$

$$\le C(\|D^m\rho_0\|^2 + \|D^{m-1}u_0\|^2 + \|D^{m-1}u(t)\|^2 +$$

$$+ \int_0^t \|D^m(u,\theta)(\tau)\|^2 + |B^{m-1}(\tau)| + |C^{m-1}(\tau)| d\tau), \quad 1 \le m \le 3 \quad ,$$

$$(iii)_{\ell,k} \quad t^\ell\|D^k(\rho,u,\theta)(t)\|^2 + \nu \int_0^t \tau^\ell \|D^{k+1}(u,\theta)(\tau)\|^2 d\tau$$

$$\le C(\int_0^t \tau^{\ell-1}\|D^k(\rho,u,\theta)(\tau)\|^2 + |\tau^\ell A^k(\tau)| d\tau), \quad \ell = 1,2, \ k = 1,2,3 \quad ,$$

$$(iv)_{\ell,m} \quad t^\ell\|D^m\rho(t)\|^2 + \nu \int_0^t \tau^\ell\|D^m\rho(\tau)\|^2 d\tau \le$$

$$\le C(t^\ell\|D^{m-1}u(t)\|^2 + \int_0^t \tau^{\ell-1}\|D^m\rho(\tau)\|^2 + \tau^{\ell-1}\|D^{m-1}u(\tau)\|^2 +$$

$$+ \tau^\ell\|D^m(u,\theta)(\tau)\|^2 + |\tau^\ell C^{m-1}(\tau)| + |\tau^\ell B^{m-1}(\tau)| d\tau \quad ,$$

$$\ell = 1,2, \ k = 2,3 \quad .$$

Lemma 3.2. There exist a positive constant C which is independent of t such that

$$(3.2)\quad N^2(0,t) \le C(\|\rho_0,u_0,\theta_0\|_3^2 + \int_0^t \sum_{\ell=0}^{2} (1+\tau^\ell)(|A^\ell(\tau)| + |B^\ell(\tau)| + |C^\ell(\tau)|) + (1+\tau^2)|A^3(\tau)|d\tau) \quad .$$

Proof of Lemma 3.1. For $(i)_k$ and $(iii)_{\ell,k}$, we may estimate the equality

$$\int_0^t \tau^\ell \int \frac{p_1}{\bar\rho} D^k(L^0) \cdot D^k\rho + D^k(L^i) \cdot D^k u^i + \frac{p_2}{p_3} D^k(L^4) \cdot D^k\theta \, dxd\tau$$

$$= \int_0^t \tau^\ell A^k(\tau)d\tau \quad ,$$

which implies after integrating by parts

$$t^\ell \int \frac{p_1}{2\bar\rho} |D^k\rho|^2 + \frac{1}{2} |D^k u|^2 + \frac{p_2}{2p_3} |D^k\theta|^2 dx \Big|_{t=0}^{t=t} +$$

$$+ \int_0^t \tau^\ell \int \tilde\mu |D^{k+1}u|^2 + (\tilde\mu+\tilde\mu')|D^k u^j_{x_j}|^2 + \frac{\tilde\kappa p_2}{p_3} \tau^\ell |D^{k+1}\theta|^2 dxd\tau$$

$$= \int_0^t \ell\tau^{\ell-1}(1-\delta^{\ell,0}) \int \frac{p_1}{2\bar\rho} |D^k\rho|^2 + \frac{1}{2} |D^k u|^2 + \frac{p_2}{2p_3} |D^k\theta|^2 dx +$$

$$+ \tau^\ell A^k(\tau)d\tau \quad ,$$

where $\delta^{i,j}$ represents Kronecker's delta. These inequalities prove $(i)_k$ and $(iii)_{\ell,k}$ easily.

Next for $(ii)_m$ and $(iv)_{\ell,m}$ we may estimate the equality

$$\int_0^t \tau^\ell \int D^{m-1}(L^0_{x_i}) \cdot D^{m-1}\rho_{x_i} + \frac{\bar\rho}{2\tilde\mu+\tilde\mu'} D^{m-1}(L^i) \cdot D^{m-1}\rho_{x_i} \, dxd\tau$$

$$= \int_0^t \tau^\ell B^{m-1}(\tau)d\tau \quad ,$$

which implies after integrating by parts

$$t^\ell \int \frac{|D^m\rho|^2}{2} + \frac{\bar\rho}{2\tilde\mu+\tilde\mu'} D^m u^i \cdot D^m\rho_{x_i} dx\Big|_{t=0}^{t=t} +$$

$$+ \int_0^t \tau^\ell \int \frac{\bar\rho}{2\tilde\mu+\tilde\mu'} |D^m\rho|^2 + \frac{\bar\rho p_2}{2\tilde\mu+\tilde\mu'}, D^{m-1}\theta_{x_i} \cdot D^{m-1}\rho_{x_i} -$$

$$- \frac{\bar\rho}{2\tilde\mu+\tilde\mu'} |D^{m-1}u^i_{x_i}|^2 dxd\tau =$$

$$= \int_0^t \ell\tau^{\ell-1}(1-\delta^{0,\ell}) \int \frac{|D^m\rho|^2}{2} + \frac{\bar\rho}{2\tilde\mu+\tilde\mu'} D^m u^i \cdot D^m\rho_{x_i} \, dx +$$

$$+ C^{m-1}(\tau)\tau^\ell + B^{m-1}(\tau)\tau^\ell \, d\tau \quad .$$

These equalites imply $(ii)_m$ and $(iv)_{\ell,m}$.

<u>Proof of Lemma 3.2</u>. For any positve constant $\varepsilon > 0$, consider the form

$$(3.3) \qquad \sum_{k=0}^{3} (i)_k + \varepsilon \sum_{m=1}^{3} (ii)_m + \varepsilon^2(iii)_{1,1} + \varepsilon^3(iv)_{1,2} +$$

$$+ \varepsilon^4(iii)_{2,2} + \varepsilon^5(iv)_{2,3} + \varepsilon^6(iii)_{2,3} \quad .$$

By taking ε suitably small in (3.3), we can easily prove (3.2) by (3.3).

Next let us estimate the nonlinear terms.

Lemma 3.3. Suppose $(\rho,u,\theta) \in X(0,T;\ E)$ for some $E < E_0$. Then, for any positive constant ε, there exists a positive constant $C(\varepsilon)$ which is independent of t such that

$$\int_0^t \sum_{\ell=0}^{2} (1+\tau^{\ell})(|A^{\ell}(\tau)| + |B^{\ell}(\tau)| + |C^{\ell}(\tau)|) + (1+\tau^2)A^3(\tau)d\tau$$

$$< \varepsilon N^2(0,t) + EC(\varepsilon)N^2(0,t) \quad .$$

Before proving Lemma 3.3, we note that Lemmas 3.2 and 3.3 easily imply the desired a priori estimates. In fact, we may first choose ε so small and next choose E so small that we have

$$N^2(0,t) < C\|\rho_0,u_0,\theta_0\|_3^2 \quad \text{for} \quad E < \varepsilon_2 \quad .$$

Proof of Lemma 3.3. Because there are many terms to estimate, we pick up some examples. The remaining terms will be estimated in the same way. First let us pick up

$$(3.4) \qquad \int_0^t \int f^0\rho \ dxd\tau$$

in $\int_0^t A^0(\tau)d\tau$. By using Nirenberg's inequality (2.4) efficiently we estimate (3.4) as follows;

$$|\int_0^t \int f^0\rho \ dxd\tau| = |\int_0^t \int - \rho^2 u^j_{x_j} - u^j\rho\rho_{x_j} \ dxd\tau|$$

$$= |\int_0^t \int \frac{\rho^2}{2} u^j_{x_j} \ dxd\tau|$$

$$< \frac{1}{2} \int_0^t (\sup_x|\rho|) \|\rho\| \, \|Du\| d\tau$$

$$< C \int_0^t \|\rho\|^{\frac{5}{4}} \|Du\| \, \|D^2\rho\|^{\frac{3}{4}} d\tau$$

$$< CE\left(\sup_\tau \|\rho\|^2\right)^{\frac{1}{8}} \left(\sup_\tau (1+\tau)\|Du\|^2\right)^{\frac{1}{2}} \left(\sup_\tau (1+\tau)^2 \|D^2\rho\|^2\right)^{\frac{3}{8}}$$

$$\times \int_0^t (1+\tau)^{-\frac{5}{4}} d\tau$$

$$< CEN^2(0,t) \quad .$$

Next let us consider the quantity

$$(3.5) \qquad \int_0^t \int \tau^2 D^3(\rho u^j)_{x_j} \cdot D^3\rho \, dxd\tau$$

appearing in $\int_0^t \tau^2 A^3(\tau)d\tau$ or $\int_0^t \tau^2 B^2(\tau)d\tau$. We estimate (3.5) as follows;

$$(3.5) = \int_0^t \int \tau^2 (\rho_{x_j} u^j + \rho u^j_{x_j})_{x_k x_\ell x_m} \cdot \rho_{x_\ell x_k x_m} \, dxd\tau$$

$$= \int_0^t \int \tau^2 u^j D^3 \rho_{x_j} \cdot D^3\rho + 3\tau^2 \rho_{x_j x_k} u^j_{x_\ell x_m} \cdot \rho_{x_k x_\ell x_m} +$$

$$+ 3\tau^2 \rho_{x_j x_k x_\ell} u^j_{x_m} \rho_{x_k x_\ell x_m} + \tau^2 \rho_{x_j} D^3 u^j \cdot D^3\rho +$$

$$+ \tau^2 (D^3\rho \cdot D^3\rho) u^j_{x_j} + 3\tau^2 \rho_{x_k} u^j_{x_j x_\ell x_m} \rho_{x_k x_\ell x_m} +$$

$$+ 3\tau^2 \rho_{x_k x_j} u^\ell_{x_\ell x_m} \rho_{x_k x_j x_m} + \tau^2 \rho \, D^3 u^j_{x_j} \cdot D^3\rho \, dxd\tau$$

$$\equiv I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 \quad ;$$

$$|I_1| = |\int_0^t \int \tau^2 u^j \left(\frac{|D^3\rho|^2}{2}\right)_{x_j} dxd\tau|$$

$$= |\int_0^t \int - \tau^2 u^j_{x_j} \frac{|D^3\rho|^2}{2} dxd\tau|$$

$$< CE \int_0^t \tau^2 \|D^3\rho(\tau)\|^2 d\tau < CEN^2(0,t) \quad ,$$

$$|I_2|,\ |I_7| < \int_0^t \int \varepsilon \tau^2 |D^3\rho|^2 + \frac{9\tau^2}{4\varepsilon} |D^2\rho|^2 \ |D^2u|^2 \ dxd\tau$$

$$< \varepsilon N^2(0,t) + \frac{C}{\varepsilon} \int_0^t \tau^2 \left(\int |D^2\rho|^4 dx\right)^{\frac{1}{2}} \left(\int |D^2u|^4 dx\right)^{\frac{1}{2}} d\tau$$

$$< \varepsilon N^2(0,t) + \frac{C}{\varepsilon} \int_0^t \tau^2 \|D^2\rho\|_1^2 \ \|D^2u\|_1^2 \ d\tau$$

$$< \varepsilon N^2(0,t) + \frac{C}{\varepsilon} \left(\sup_\tau (1+\tau^2)\|D^2\rho\|_1^2\right)\left(\sup_\tau (1+\tau^2)\|D^2u\|_1^2\right) \times$$

$$\times \int_0^t \tau^2 (1+\tau^2)^{-2} d\tau$$

$$< \varepsilon N^2(0,t) + C\varepsilon^{-1} EE_0 N^2(0,t) \quad ,$$

$|I_3|$, $|I_4|$, $|I_5|$ and $|I_6|$ are estimated in the same way as $|I_1|$,

$$|I_8| = |\int_0^t \int \tau^2 \ \rho \ D^3\rho \cdot D^3 u^j_{x_j} \ dxd\tau|$$

$$< \int_0^t \tau^2 \sup_x |\rho| \ \|D^3\rho\| \ \|D^4u\| d\tau$$

$$< \int_0^t \varepsilon \tau^2 \|D^4u\|^2 + C \ \varepsilon^{-1} \tau^2 \|\rho\|^{\frac{1}{2}} \|D^2\rho\|^{\frac{3}{2}} \|D^3\rho\|^2 d\tau$$

$$< \varepsilon N^2(0,t) + CEE_0 \ \varepsilon^{-1} N^2(0,t) \int_0^t \tau^2 (1+\tau)^{-\frac{7}{2}} d\tau$$

$$< \varepsilon N^2(0,t) + C(\varepsilon)EN^2(0,t) \quad .$$

Proceeding in this manner proves Lemma 3.3.

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A priori estimates for solutions of the quasilinear hyperbolic-parabolic equations governing the initial value problem describing the motion of compressible, viscous and heat-conductive, Newtonian fluids are derived by means of a new energy method. This technique enables us to simplify and unify our previous results on the global existence in time and uniqueness of smooth solutions of these equations for sufficiently smooth and "small" initial data and to obtain their rate of decay.

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