

CHAPTER IV

EQUATIONS OF COMPRESSIBLE FLUIDS

In this chapter, the initial value problem to the equations of motion for compressible, heat-conductive, isotropic Newtonian fluids is investigated. In § 4.2, the existence theorem for a unique local solution in time is established. And the a priori energy estimates are improved in § 4.3. In § 4.4, using the energy estimates alone, we obtain a global solution in time of the equations of polytropic ideal gases (special cases of isotropic Newtonian fluids) for suitably small initial data. Furthermore in § 4.5, using a combination of the decay rate estimates for the solutions of the linearized equations and the energy estimates, we obtain a global solution in time of the original equations of isotropic Newtonian fluids for suitably small initial data.

§ 4.1 Equations and Historical Remarks

The motion of the general isotropic Newtonian fluids is described by the five conservation laws (cf [28] [29]);

$$(4.1) \quad \left\{ \begin{array}{l} \rho_t + (\rho u^j)_{x_j} = 0, \\ u_t^i + u^j u_{x_j}^i + \frac{1}{\rho} p_{x_i} = \frac{1}{\rho} \{ (\mu (u_{x_j}^i + u_{x_i}^j))_{x_j} + (\mu' u_{x_j}^j)_{x_i} \} + f^i, \quad 1 \leq i \leq 3, \\ \theta_t + u^j \theta_{x_j} + \frac{\theta p_{\theta}}{\rho c_V} u_{x_j}^j = \frac{1}{\rho c_V} \{ (\kappa \theta_{x_j})_{x_j} + \Psi \}, \end{array} \right.$$

where $t \geq 0$, $x = (x_1, x_2, x_3) \in R^3$,

ρ : density, $u = (u^1, u^2, u^3)$: velocity, θ : absolute temperature,

$p = p(\rho, \theta)$: pressure, $f = (f^1, f^2, f^3)$: outer force, $\mu = \mu(\rho, \theta)$: viscosity coefficient, $\mu' = \mu'(\rho, \theta)$: second viscosity coefficient, $\kappa = \kappa(\rho, \theta)$: coefficient of heat conduction, $c_V = c_V(\rho, \theta)$: heat capacity at constant volume and $\Psi = \frac{\mu}{2} (u_{x_j}^i + u_{x_i}^j)^2 + \mu' (u_{x_j}^j)^2$: dissipation function. Here and in what follows, we use the summation convention when we are not confused.

The existence theorems of unique local solution in time of (4.1) are obtained by Nash [53], Itaya [15][16] for the initial value problem, and by Tani [70][71][72] for the first initial boundary value problem and the free boundary problem. On the other hand, the existence theorem of global solution in time is not known in general. Recently some one-dimensional model equations are investigated on the global existence in time by Kanel' [21], Itaya [17][18], Tani [69], Kazhikhov and Shelukhin [25] [26]. Precisely speaking, Kanel' obtained the global solution in time for the model equation

$$\begin{cases} v_t = u_x, \\ u_t = -(p(v))_x + \mu(u_x/v)_x, \quad t \geq 0, \quad x \in R^1, \end{cases}$$

with the initial data

$$(v - \bar{v}, u)(x, 0) = (v_0(x) - \bar{v}, u_0(x)) \in H^1(R^1),$$

for some positive constant \bar{v} . It contains the barotropic model in the Lagrangian coordinate : $p = a^2/v^\gamma$, $\gamma = \text{constant} \geq 1$ and $a, \mu = \text{constants} > 0$. Itaya obtained the global solution in time for the isothermal gas model equation

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ u_t + uu_x + \frac{a^2}{\rho} \rho_x = \frac{\mu}{\rho} u_{xx}, \quad t \geq 0, \quad x \in R^1, \quad a, \mu : \text{constants} > 0, \end{cases}$$

with the initial data

$$\begin{cases} \rho(x,0) = \rho_0(x) \in \mathcal{B}^{1+\sigma}, \\ u(x,0) = u_0(x) \in \mathcal{B}^{2+\sigma}, \end{cases}$$

where $0 < \rho_1 \leq \rho_0(x) \leq \rho_2 < \infty$, ρ_1, ρ_2 are some constants, $\sigma \in (0,1)$ and $u_{0,x} \in L^1(\mathbb{R}^1)$. As to the initial-boundary value problem, Tani obtained the global solution in time of the generalized Burger's equation

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ u_t + uu_x = \frac{\mu}{\rho} u_{xx}, \quad t \geq 0, x \in [0,1], \mu : \text{constant} > 0, \end{cases}$$

with the initial and boundary data

$$\begin{cases} u(x,0) = u_0(x) \in \mathcal{B}^{2+\sigma} & (\sigma \in (0,1)), \\ \rho(x,0) = \rho_0(x) \in \mathcal{B}^{1+\sigma} & (0 < \rho_1 \leq \rho_0(x) \leq \rho_2 < \infty), \\ u(0,t) = u(1,t) = 0. \end{cases}$$

Moreover Kazhikhov and Shelukhin obtained the global solution in time of the one-dimensional equation of (4.1)

$$\begin{cases} \rho_t + \rho^2 u_x = 0, \\ u_t + R(\rho\theta)_x = \mu(\rho u_x)_x, \\ \theta_t + \frac{R\rho\theta}{c_V} u_x = \frac{1}{c_V} (\kappa(\rho\theta)_x + \mu\rho u_x^2), \\ t \geq 0, x \in [0,1], \mu, \kappa, c_V \text{ and } R : \text{constants} > 0, \end{cases}$$

with the initial and boundary data

$$(u(x,0), \theta(x,0)) = (u_0(x), \theta_0(x)) \in \mathcal{B}^{2+\sigma} \quad (\sigma \in (0,1)),$$

$$\rho(x,0) = \rho_0(x) \in \mathcal{B}^{1+\sigma} \quad (0 < \rho_1 \leq \rho_0(x) \leq \rho_2 < \infty),$$

$$u(0,t) = u(1,t) = \theta_x(0,t) = \theta_x(1,t) = 0,$$

where the system is described in the Lagrangian coordinates.

For the initial value problem of (4.1), the global solutions in time have not been known even in the one space-dimension. Just recently, Matsumura and Nishida [39][40] investigated the global existence for (4.1) in the three space-dimension. First in [39], they assume the following conditions on (4.1) ;

- i) the fluid is ideal gas : $p = R\rho\theta$, (R : gas constant > 0),
- ii) the gas is polytropic : $e = c_V\theta$, where e represents the internal energy and c_V is constant,
- iii) μ, μ' and κ are positive constants and $f = 0$.

Then we have

$$c_V = \frac{R}{\gamma-1} \quad (\gamma : \text{ratio of specific heats}),$$

by the thermodynamical relations, and consequently the equation (4.1) is written in the form

$$(4.2) \quad \begin{aligned} \rho_t + (\rho u^j)_{x_j} &= 0, \\ u_t^i + u^j u_{x_j}^i + \frac{R\theta}{\rho} \rho_{x_i} + R\theta_{x_i} &= \frac{1}{\rho} \{ (\mu(u_{x_j}^i + u_{x_i}^j)_{x_j} + (\mu' u_{x_j}^j)_{x_i} \}, \\ \theta_t + u^j \theta_{x_j} + (\gamma-1)\theta u_{x_j}^j &= \frac{(\gamma-1)}{R\rho} ((\kappa \theta_{x_j})_{x_j} + \Psi). \end{aligned}$$

Using the energy estimates alone (cf [21][37]), they obtained a global solution in time for the initial value problem (4.2) with the initial data

$$(4.3) \quad (\rho(x, 0), u(x, 0), \theta(x, 0)) = (\rho_0(x), u_0(x), \theta_0(x))$$

satisfying that $(\rho_0 - \bar{\rho}, u_0, \theta_0 - \bar{\theta})$ is suitably small in $H^3(R^3)$ where $(\bar{\rho}, \bar{\theta})$ are some positive constants. Their energy method can also be applied to the following barotropic model equation for ideal isentropic fluids and equation of magnetohydrodynamics :

Barotropic Model

$$\begin{aligned} \rho_t + (\rho u^j)_{x_j} &= 0, \\ u_t^i + u^j u_{x_j}^i + \frac{1}{\rho} p_{x_i} &= \frac{\mu}{\rho} u_{x_j x_j}^i + \frac{\mu + \mu'}{\rho} u_{x_i x_j}^j, \end{aligned}$$

where $p = p(\rho), p'(\rho) > 0$ for $\rho > 0$ and $\mu, \mu' = \text{constants} > 0$:

Equations of Magnetohydrodynamics

$$\begin{aligned} \rho_t + (\rho u^j)_{x_j} &= 0, \\ u_t^i + u^j u_{x_j}^i + \frac{1}{\rho} p_{x_i} &= \frac{\mu}{\rho} u_{x_j x_j}^i + \frac{\mu + \mu'}{\rho} u_{x_i x_j}^j + (J \times B)^i, \\ B_t^i + (\nabla \times E)^i &= 0, \\ J^i &= \sigma_0 (E + u \times B)^i = (\nabla \times (\mu_0^{-1} B))^i, \\ B_{x_j}^j &= 0, \quad p = p(\rho), \end{aligned}$$

where B : magnetic field, E : electric field, J : density of electric current, σ_0 : electric conductivity, μ_0 : magnetic permeability, $\mu, \mu', \mu_0, \sigma_0$: positive constants and $p'(\rho) > 0$ for $\rho > 0$.

Furthermore in [40], using a combination of the decay rate estimates for the solutions of the linearized equations and the energy estimates, they succeeded to obtain a global solution in time of the original

equation (4.1) with the initial data (4.3) under the following assumptions ;

i) $\frac{\partial p}{\partial \rho}, \frac{\partial p}{\partial \theta}, c_V, \mu, \kappa > 0$ and $\mu' + \frac{2}{3} \mu \geq 0$ for $(\rho, \theta) > 0$,

ii) $f = 0$,

iii) $(\rho_0 - \bar{\rho}, u_0, \theta_0 - \bar{\theta})$ is suitably small in $H^4 \cap L^1$.

They also showed the asymptotic behavior of the solution ;

$$\| (\rho - \bar{\rho}, u, \theta - \bar{\theta})(t) \|_2 \leq C(1+t)^{-3/4}.$$

We summarize all these results in this chapter. Refer also to Kawashima-Matsumura-Nishida [24], for the interesting problems on the asymptotic behavior of the solutions of the equations of compressible fluid, especially on its relations to that of the solutions of the equations of incompressible fluid and Boltzmann equations.

4.2 Local Existence

Let us construct a unique local solution in time for the initial value problem (4.1) (4.3) in a neighbourhood of any constant state $(\rho, u, \theta) = (\bar{\rho}, 0, \bar{\theta})$ where $(\bar{\rho}, \bar{\theta})$ are any positive constants. First, rewrite the system (4.1) (4.3) by the change of the unknown and known variables as follows ; $\rho \rightarrow \bar{\rho} + \rho, u \rightarrow u, \theta \rightarrow \bar{\theta} + \theta, p(\bar{\rho} + \rho, \bar{\theta} + \theta) \rightarrow p(\rho, \theta), \mu(\bar{\rho} + \rho, \bar{\theta} + \theta) \rightarrow \mu(\rho, \theta)$ and so on ;

$$(4.1)' \left\{ \begin{array}{l} \rho_t + u^j \rho_{x_j} = G^0, \\ u_t^i - \tilde{\mu} u^i_{x_j x_j} - (\tilde{\mu} + \mu') u^j_{x_i x_j} = G^i, \\ \theta_t - \tilde{\kappa} \theta_{x_j x_j} = G^4, \\ (\rho, u, \theta)(0) = (\rho_0, u_0, \theta_0), x \in R^3, 0 \leq t \leq T, \end{array} \right.$$

where

$$\tilde{\mu} = \tilde{\mu}(\rho, \theta) \equiv \frac{\mu(\rho, \theta)}{\bar{\rho} + \rho}, \quad \tilde{\mu}' = \tilde{\mu}'(\rho, \theta) \equiv \frac{\mu'(\rho, \theta)}{\bar{\rho} + \rho},$$

$$\tilde{\kappa} = \tilde{\kappa}(\rho, \theta) \equiv \frac{\kappa(\rho, \theta)}{(\bar{\rho} + \rho)c_V(\rho, \theta)},$$

$$G^0 = G^0(\rho, u) \equiv -(\bar{\rho} + \rho)u_{x_j}^j,$$

$$G^i = G^i(\rho, u, \theta) \equiv -\frac{p_\rho(\rho, \theta)}{\bar{\rho} + \rho} \rho_{x_i} - \frac{p_\theta(\rho, \theta)}{\bar{\rho} + \rho} \theta_{x_i} - u_{x_j}^j u_{x_j}^i + \\ + \frac{1}{\bar{\rho} + \rho} \{ (\mu(\rho, \theta))_{x_j} (u_{x_j}^i + u_{x_i}^j) + (\mu'(\rho, \theta))_{x_i} (u_{x_j}^j) \},$$

$$G^A = G^A(\rho, u, \theta) \equiv -\frac{(\bar{\theta} + \theta)p_\theta(\rho, \theta)}{(\bar{\rho} + \rho)c_V(\rho, \theta)} u_{x_j}^j - u_{x_j}^j \theta_{x_j} + \\ + \frac{1}{(\bar{\rho} + \rho)c_V(\rho, \theta)} \{ (\kappa(\rho, \theta))_{x_j} \theta_{x_j} + \Psi \}.$$

Then the solutions are sought in a neighbourhood of $(0, 0, 0)$.

Define \mathcal{U} by

$$\mathcal{U} = \{ (\rho, u, \theta) \mid |\rho|, |u|, |\theta| \leq \gamma_1 \},$$

where γ_1 is some positive constant such that $\gamma_1 < \min(\bar{\rho}, \bar{\theta})$. We suppose

Assumption 4.1

i) p, c_V, μ, μ' and κ are smooth in \mathcal{U} .

ii) $\frac{\partial p}{\partial \rho}, \frac{\partial p}{\partial \theta}, c_V, \mu, \kappa \geq v_0$ and $\mu' + \frac{2}{3}\mu \geq 0$ in \mathcal{U}

for some positive constant v_0 .

Let l be a positive integer ≥ 3 . Let us define the set X^l of the solutions. By Lemma 2.1, we can choose a positive constant E_0 such that, if $f \in H^2$ satisfies $\|f\|_2 \leq E_0$, then $\|f\|_{\mathcal{B}^0} \leq \gamma_1$. Then, for $E \leq E_0$ and $0 \leq t_1, t_2 \leq T$, the set $X^l(t_1, t_2; E)$ is defined by

$$\begin{aligned} X^l(t_1, t_2; E) = \{ (\rho, u, \theta) \mid & \rho \in C^0(t_1, t_2; H^l) \cap C^1(t_1, t_2; H^{l-1}), \\ & (u, \theta) \in C^0(t_1, t_2; H^l) \cap C^1(t_1, t_2; H^{l-2}) \cap L_2(t_1, t_2; H^{l+1}) \\ & \text{and } \sup_{t_1 \leq t \leq t_2} \|(\rho, u, \theta)(t)\|_l^2 + \int_{t_1}^{t_2} \|(u, \theta)(t)\|_{l+1}^2 dt \leq E^2 \}. \end{aligned}$$

Now, for $(\eta, v, \zeta) \in X^l(0, \tau; E)$ ($\tau \leq T$), consider the linear problem

$$(4.4) \quad \left\{ \begin{aligned} L_v^0(\rho) &= G^0(\eta, v), \\ L_{\eta, \zeta}^i(u) &= G^i(\eta, v, \zeta), \\ L_{\eta, \zeta}^A(\theta) &= G^A(\eta, v, \zeta), \\ (\rho, u, \theta)(0) &= (\rho_0, u_0, \theta_0), \end{aligned} \right.$$

where

$$\begin{aligned} L_v^0(\rho) &\equiv \rho_t + v^j \rho_{x_j}, \\ L_{\eta, \zeta}^i(u) &\equiv u_t^i - \tilde{\mu}(\eta, \zeta) u_{x_j x_j}^i - (\tilde{\mu} + \tilde{\mu}')(\eta, \zeta) u_{x_i x_j}^j, \\ L_{\eta, \zeta}^A(\theta) &\equiv \theta_t - \tilde{\kappa}(\eta, \zeta) \theta_{x_j x_j}. \end{aligned}$$

In order to apply Propositions 2.5 and 2.11 to (4.4), we prepare the following lemma that is proved in the same way as Lemma 3.1.

Lemma 4.1 Suppose Assumption 4.1 and $(\eta, v, \zeta), (\eta', v', \zeta') \in X^l(0, \tau; E)$. Then we have

$$i) G^0(\eta, v) \in L_2(0, \tau; H^{\underline{L}}) \cap C^0(0, \tau; H^{\underline{L}-1}),$$

$$\int_0^\tau \|G^0(\eta, v)(s)\|_{\underline{L}}^2 ds \leq C(E_0)E^2,$$

$$ii) \tilde{\mu}(\eta, \zeta), \tilde{\mu}'(\eta, \zeta), \tilde{\kappa}(\eta, \zeta) \in C^0(0, \tau; \mathcal{B}^1),$$

$$\inf_{\substack{x \in R^3 \\ 0 \leq t \leq \tau}} (\tilde{\mu}(\eta, \zeta)|\xi|^2|V|^2 + (\tilde{\mu}(\eta, \zeta) + \tilde{\mu}'(\eta, \zeta))|\xi_i V^i|^2) \geq \nu_1 |\xi|^2 |V|^2,$$

$$\inf_{\substack{x \in R^3 \\ 0 \leq t \leq \tau}} \tilde{\kappa}(\eta, \zeta) \geq \nu_1, \text{ for some positive constant } \nu_1 \text{ and all } \xi, V \in R^3,$$

$$iii) D_x \tilde{\mu}, D_x \tilde{\mu}', D_x \tilde{\kappa} \in C^0(0, \tau; H^{\underline{L}-1}),$$

$$\|D_x \tilde{\mu}\|_{\underline{L}-1}, \|D_x \tilde{\mu}'\|_{\underline{L}-1}, \|D_x \tilde{\kappa}\|_{\underline{L}-1} \leq C(E_0)E,$$

$$iv) G^i(\eta, v, \zeta), G^A(\eta, v, \zeta) \in C^0(0, \tau; H^{\underline{L}-1}),$$

$$\|G^i\|_{\underline{L}-1}, \|G^A\|_{\underline{L}-1} \leq C(E_0)E,$$

$$v) \|(v^j - v'^j)\eta_{x_j}\|_{\underline{L}-1} \leq C(E_0)\|v - v'\|_{\underline{L}-1},$$

$$vi) \int_0^\tau \|G^0(\eta, v)(s) - G^0(\eta', v')(s)\|_{\underline{L}-1}^2 ds$$

$$\leq C(E_0) \left(\sup_{0 \leq t \leq \tau} \|\eta(t) - \eta'(t)\|_{\underline{L}-1}^2 + \int_0^\tau \|v - v'(s)\|_{\underline{L}-1}^2 ds \right),$$

$$vii) \|(\tilde{\mu}(\eta, \zeta) - \tilde{\mu}(\eta', \zeta'))D_x^2 v\|_{\underline{L}-2}, \|(\tilde{\mu}'(\eta, \zeta) - \tilde{\mu}'(\eta', \zeta'))D_x^2 v\|_{\underline{L}-2}$$

$$\leq C(E_0)\|(\eta - \eta', \zeta - \zeta')\|_{\underline{L}-1},$$

$$viii) \|G^i(\eta, v, \zeta) - G^i(\eta', v', \zeta')\|_{\underline{L}-2} \quad (1 \leq i \leq 4)$$

$$\leq C(E_0)\|(\eta - \eta', v - v', \zeta - \zeta')\|_{\underline{L}-1}.$$

By virtue of Lemma 4.1, we have

Proposition 4.2 Suppose Assumption 4.1 and $(\rho_0, u_0, \theta_0) \in H^{\underline{l}}$ ($\underline{l} \geq 3$). Then there exist positive constants τ and δ ($\delta < 1$) such that, if $(\eta, \nu, \zeta) \in X^{\underline{l}}(0, \tau; E)$ and $\| \rho_0, u_0, \theta_0 \|_{\underline{l}} \leq \delta E$ for some $E \leq E_0$, then the linear problem (4.4) has a unique solution

$$(\rho, u, \theta) \in X^{\underline{l}}(0, \tau; E) .$$

Proof of Proposition 4.2 By Propositions 2.5 and 2.11, we have a unique solution of (4.4) such that

$$\begin{aligned} \rho &\in C^0(0, \tau; H^{\underline{l}}) \cap C^1(0, \tau; H^{\underline{l}-1}) , \\ (u, \theta) &\in C^0(0, \tau; H^{\underline{l}}) \cap C^1(0, \tau; H^{\underline{l}-2}) \cap L_2(0, \tau; H^{\underline{l}+1}) , \end{aligned}$$

and the energy inequalities

$$\begin{aligned} (4.5) \quad \| \rho(t) \|_{\underline{l}}^2 &\leq 2e^{C(E_0)t} (\| \rho_0 \|_{\underline{l}}^2 + t \int_0^t \| G^0(s) \|_{\underline{l}}^2 ds) \\ &\leq 2e^{C(E_0)\tau} (\| \rho_0 \|_{\underline{l}}^2 + C(E_0)\tau E^2) , \end{aligned}$$

$$\begin{aligned} (4.6) \quad \| (u, \theta)(t) \|_{\underline{l}}^2 + \nu \int_0^t \| (u, \theta)(s) \|_{\underline{l}+1}^2 ds \\ \leq e^{C(E_0)t} (\| u_0, \theta_0 \|_{\underline{l}}^2 + \int_0^t \sum_{i=1}^4 \| G^i(s) \|_{\underline{l}-1}^2 ds) \\ \leq e^{C(E_0)\tau} (\| u_0, \theta_0 \|_{\underline{l}}^2 + \tau C(E_0)E^2) . \end{aligned}$$

It follows from (4.5) and (4.6) that

$$\begin{aligned}
& \sup_{0 < t \leq \tau} \| (\rho, u, \theta)(t) \|_{\underline{L}}^2 + \int_0^\tau \| (u, \theta)(s) \|_{\underline{L}+1}^2 ds \\
& \leq 2(\nu')^{-1} e^{C\tau} \| \rho_0, u_0, \theta_0 \|_{\underline{L}}^2 + \tau C E^2 \\
& \leq (2(\nu')^{-1} e^{C\tau} \delta^2 + \tau C) E^2.
\end{aligned}$$

Therefore, by taking δ and τ so small that

$$2(\nu')^{-1} e^{C\tau} \delta^2 + \tau C \leq 1,$$

we have

$$(\rho, u, \theta) \in X^{\underline{L}}(0, \tau; E).$$

This completes the proof of Proposition 4.2.

Let us construct the approximate sequence $\{(\rho^{(m)}, u^{(m)}, \theta^{(m)})\}_{m=0}^\infty$ for the quasilinear problem (4.1)' as follows ;

$$\begin{cases}
(\rho^{(0)}, u^{(0)}, \theta^{(0)}) = 0 & m = 0 ; \\
\begin{cases}
L_{u^{(m-1)}}^0(\rho^{(m)}) = G^0(\rho^{(m-1)}, u^{(m-1)}), \\
L_{\rho^{(m-1)}, \theta^{(m-1)}}^i(u^{(m)}) = G^i(\rho^{(m-1)}, u^{(m-1)}, \theta^{(m-1)}) \quad (1 \leq i \leq 4), \\
(\rho^{(m)}, u^{(m)}, \theta^{(m)})(0) = (\rho_0, u_0, \theta_0), \quad m \geq 1.
\end{cases}
\end{cases}$$

In the same way as in § 3.2, we have

Theorem 4.3 (Local Existence Theorem) Suppose Assumption 4.1
and $(\rho_0, u_0, \theta_0) \in H^{\underline{L}}$ ($\underline{L} \geq 3$). Then there exist positive constants τ and δ ($\delta < 1$) such that, if $\| \rho_0, u_0, \theta_0 \|_{\underline{L}} \leq \delta E$ for some $E \leq E_0$, then the initial

value problem (4.1)' has a unique solution

$$(\rho, u, \theta) \in X^l(0, \tau; E) .$$

Remark For $l \geq 4$, it follows from Lemma 2.1 that

$$\rho \in C^0(0, \tau; \mathcal{B}^2) \cap C^1(0, \tau; \mathcal{B}^1) ,$$

$$(u, \theta) \in C^0(0, \tau; \mathcal{B}^2) \cap C^1(0, \tau; \mathcal{B}^0) ,$$

i.e., (ρ, u, θ) is an also classical solution. For $l=3$, Lemma 2.1 also implies

$$\rho \in C^0(0, \tau; \mathcal{B}^{1+\sigma}) \cap C^1(0, \tau; \mathcal{B}^\sigma) ,$$

$$(u, \theta) \in C^0(0, \tau; \mathcal{B}^\sigma) , \quad 0 < \sigma < 1/2 ,$$

from which ρ is classical. As to (u, θ) , since the system

$$\begin{cases} u_t^i - \tilde{\mu} u_{x_j x_j}^i - (\tilde{\mu} + \tilde{\mu}') u_{x_i x_j}^j = G^i, \\ \theta_t - \tilde{\kappa} \theta_{x_j x_j} = G^A, \end{cases}$$

is uniformly parabolic in the sence of Petrowski and since

$$\tilde{\mu}, \tilde{\mu}' \in C^0(0, \tau; \mathcal{B}^{1+\sigma}) ,$$

$$G^i, G^A \in C^0(0, \tau; \mathcal{B}^\sigma) ,$$

$$(u_0, \theta_0) \in \mathcal{B}^\sigma ,$$

it follows from the arguments at Chapter 9 in [8] that (u, θ) is classical for $t > 0$. Thus (ρ, u, θ) is also the classical solution of (4.1)' for $t > 0$.

4.3 A Priori Energy Estimates

Consider the initial value problem (4.1)' again ;

$$(4.1)' \left\{ \begin{array}{l} P^0 \equiv \rho_t + u^j \rho_{x_j} + (\bar{\rho} + \rho) u^j_{x_j} = 0 , \\ P^i \equiv u^i_t - \tilde{\mu} u^i_{x_j x_j} - (\tilde{\mu} + \mu') u^j_{x_i x_j} + \tilde{p}_\rho \rho_{x_i} + \tilde{p}_\theta \theta_{x_i} = g^i , \\ P^4 \equiv \theta_t - \tilde{\kappa} \theta_{x_j x_j} + \tilde{p}_3 u^j_{x_j} = g^4 , \\ (\rho, u, \theta)(0) = (\rho_0, u_0, \theta_0) , \end{array} \right.$$

where

$$\tilde{p}_\rho \equiv \frac{p_\rho}{\bar{\rho} + \rho} , \quad \tilde{p}_\theta \equiv \frac{p_\theta}{\bar{\rho} + \rho} , \quad \tilde{p}_3 \equiv \frac{(\bar{\theta} + \theta) p_\theta}{(\bar{\rho} + \rho) c_V} ,$$

$$g^i \equiv -u^j u^i_{x_j} + \frac{1}{\bar{\rho} + \rho} (\mu_{x_j} (u^i_{x_j} + u^j_{x_i}) + \mu'_{x_i} u^j_{x_j}) ,$$

$$g^4 \equiv -u^j \theta_{x_j} + \frac{1}{(\bar{\rho} + \rho) c_V} (\kappa_{x_j} \theta_{x_j} + \Psi) .$$

Of course, we suppose Assumption 4.1. Let E_0 be defined as before. Then

in this section we define $X^l(t_1, t_2; E)$ ($l \geq 3$) by

$$X^l(t_1, t_2; E) = \{ (\rho, u, \theta) \mid \rho \in C^0(t_1, t_2; H^l) \cap C^1(t_1, t_2; H^{l-1}),$$

$$D_x \rho \in L_2(t_1, t_2; H^{l-1}), (u, \theta) \in C^0(t_1, t_2; H^l) \cap C^1(t_1, t_2; H^{l-2}),$$

$$D_x(u, \theta) \in L_2(t_1, t_2; H^l) \text{ and}$$

$$\sup_{t_1 \leq t \leq t_2} \| (\rho, u, \theta)(t) \|_l^2 + \int_{t_1}^{t_2} \| D_x \rho(s) \|_{l-1}^2 + \| D_x(u, \theta)(s) \|_l^2 ds$$

$$\leq E^2 \quad (E \leq E_0) \} .$$

Then we note that, although the above definition of X^l is slightly different from that of § 4.2, the entirely same local existence theorem (Theo-

rem 4.3) holds even for the above X^l . Let us study more detail a-priori energy estimates for (4.1)' than that in the previous sections. First note that, if $(\rho, u, \theta) \in X^l(0, h; E_0)$, we have

$$(4.7) \quad \tilde{\mu}, (\tilde{\mu} + \tilde{\mu}'), \tilde{\kappa}, \tilde{p}_\rho, \tilde{p}_\theta, \tilde{p}_z \geq \nu_3,$$

for some positive constant ν_3 not depending on h . Now suppose $(\rho, u, \theta) \in X^l(0, h; E)$ is a solution of (4.1)' for some $E \leq E_0$. Then we have the following Lemmas :

Lemma 4.4 There exists a constant $C(E_0)$ not depending on h such that for $1 \leq m \leq l$,

$$(4.8) \quad \sum_{i=1}^4 \| D_x^{m-1} g^i(\rho, u, \theta) \| \leq C(E_0) E \| D_x(\rho, u, \theta) \|_{m-1}.$$

Lemma 4.5 There exist positive constants ν_4 and $C(E_0)$ not depending on h such that for $1 \leq m \leq l$,

$$(4.9)_m \quad \| D_x^m(u, \theta)(t) \|^2 + \nu_4 \int_0^t \| D_x^{m+1}(u, \theta)(s) \|^2 ds \\ \leq \| D_x^m(u_0, \theta_0) \|^2 + C(E_0) \left(\int_0^t \| D_x^m(\rho, u, \theta)(s) \|^2 + E \| D_x(\rho, u, \theta)(s) \|_{m-1}^2 ds \right).$$

Lemma 4.6 There exist positive constants ν_5 and $C(E_0)$ not depending on h such that for $1 \leq m \leq l$,

$$(4.10)_m \quad \| D_x^m \rho(t) \|^2 + \nu_5 \int_0^t \| D_x^m \rho(s) \|^2 ds \leq C(E_0) \{ \| \rho_0, u_0 \|_m^2 + \| D_x^{m-1} u(t) \|^2 \\ + \int_0^t (\| D_x^m(u, \theta) \|^2 + E \| D_x(\rho, u, \theta) \|_{m-1}^2 + \| D_x^2 \theta \|_1 \| D_x^m \rho \| \| D_x^{m-1} u \|_1)(s) ds \}.$$

As Lemma 4.4 is shown in the same way as Lemma 3.2, we show Lemmas 4.5 and 4.6.

Proof of Lemma 4.5 We show the statements only for $(\rho, u, \theta) \in X^1(0, h; E) \cap C^1(0, h; H^\infty)$. Using Lemmas 2.1, 2.9 and 3.5, we estimate the equality

$$(4.11) \quad \int - D_x^{m-1}(P^i) \cdot D_x^{m-1}(\Delta u^i) - D_x^{m-1}(P^4) \cdot D_x^{m-1}(\Delta \theta) \, dx \\ = \int - D_x^{m-1} g^i \cdot D_x^{m-1}(\Delta u^i) - D_x^{m-1} g^4 \cdot D_x^{m-1}(\Delta \theta) \, dx$$

as follows ;

$$(4.12) \text{ the right hand side of (4.11)} \leq \sum_i \| D_x^{m-1} g^i \| \| D_x^{m+1} u \| + \| D_x^{m-1} g^4 \| \| D_x^{m+1} \theta \| \\ \leq \varepsilon (\| D_x^{m+1} u \|^2 + \| D_x^{m+1} \theta \|^2) + \varepsilon^{-1} C(E_0) E^2 \| D_x(\rho, u, \theta) \|_{m-1}^2$$

for any positive number ε ;

$$(4.13) \quad \int - D_x^{m-1}(P^i) \cdot D_x^{m-1}(\Delta u^i) \, dx \\ = \int - D_x^{m-1} u_t^i \cdot D_x^{m-1}(\Delta u^i) + \tilde{\mu} |D_x^{m-1} u^i|^2 + (\tilde{\mu} + \tilde{\mu}') D_x^{m-1} u_{x_i x_j}^j \cdot D_x^{m-1} u_{x_k x_k}^i - \\ - \tilde{p}_\rho D_x^{m-1} \rho_{x_i} \cdot D_x^{m-1}(\Delta u^i) - \tilde{p}_\theta D_x^{m-1} \theta_{x_i} \cdot D_x^{m-1}(\Delta u^i) + \\ + (D_x^{m-1}(\tilde{\mu} \Delta u^i) - \tilde{\mu} D_x^{m-1}(\Delta u^i)) \cdot D_x^{m-1}(\Delta u^i) + \\ + (D_x^{m-1}((\tilde{\mu} + \tilde{\mu}') u_{x_i x_j}^j) - (\tilde{\mu} + \tilde{\mu}') D_x^{m-1} u_{x_i x_j}^j) \cdot D_x^{m-1}(\Delta u^i) - \\ - (D_x^{m-1}(\tilde{p}_\rho \rho_{x_i}) - \tilde{p}_\rho D_x^{m-1} \rho_{x_i}) \cdot D_x^{m-1}(\Delta u^i) -$$

$$\begin{aligned}
& - (D_x^{m-1}(\tilde{p}_\theta \theta_{x_i}) - \tilde{p}_\theta D_x^{m-1} \theta_{x_i}) \cdot D_x^{m-1}(\Delta u^i) \, dx \\
& \geq \frac{1}{2} \frac{d}{dt} \int |D_x^m u|^2 \, dx + \nu_3 \int |D_x^{m+1} u|^2 + (\tilde{\mu} + \tilde{\mu}') |D_x^{m-1} u_{x_j x_k}^j|^2 - \\
& \quad - (\tilde{\mu} + \tilde{\mu}')_{x_i} D_x^{m-1} u_{x_j}^j \cdot D_x^{m-1} u_{x_k x_k}^i + (\tilde{\mu} + \tilde{\mu}')_{x_k} D_x^{m-1} u_{x_j}^j \cdot D_x^{m-1} u_{x_i x_k}^i \, dx - \\
& \quad - 2\epsilon \|D_x^{m+1} u\|^2 - \epsilon^{-1} C (\|D_x^m \rho\|^2 + \|D_x^m \theta\|^2) - \epsilon^{-1} CE \|D_x(\rho, u, \theta)\|_{m-1}^2 \\
& \geq \frac{1}{2} \frac{d}{dt} \|D_x^m u\|^2 + \nu_3 \|D_x^{m+1} u\|^2 - 3\epsilon \|D_x^{m+1} u\|^2 - \\
& \quad - C\epsilon^{-1} \|D_x^m(\rho, \theta)\|^2 - C\epsilon^{-1} E \|D_x(\rho, u, \theta)\|_{m-1}^2
\end{aligned}$$

for any positive number ϵ ;

$$\begin{aligned}
(4.14) \quad & \int - D_x^{m-1}(P^4) \cdot D_x^{m-1}(\Delta \theta) \, dx \\
& = \int - D_x^{m-1} \theta_t \cdot D_x^{m-1} \theta_{x_k x_k} + \tilde{\kappa} |D_x^{m-1} \theta_{x_k x_k}|^2 - \tilde{p}_3 D_x^{m-1} u_{x_j}^j \cdot D_x^{m-1}(\Delta \theta) - \\
& \quad - (D_x^{m-1}(\tilde{p}_3 u_{x_j}^j) - \tilde{p}_3 D_x^{m-1} u_{x_j}^j) \cdot D_x^{m-1}(\Delta \theta) \, dx \\
& \geq \frac{1}{2} \frac{d}{dt} \|D_x^m \theta\|^2 + \nu_3 \|D_x^{m+1} \theta\|^2 - 2\epsilon \|D_x^{m+1} \theta\|^2 - \\
& \quad - C\epsilon^{-1} \|D_x^m u\|^2 - C\epsilon^{-1} E \|D_x(\rho, u, \theta)\|_{m-1}^2
\end{aligned}$$

for any positive number ϵ .

Hence, due to (4.12)-(4.14) ,

$$(4.15) \quad \frac{d}{dt} \| D_x^m(u, \theta) \|^2 + 2(\nu_3 - 3\varepsilon) \| D_x^{m+1}u \|^2 + 2(\nu_3 - 2\varepsilon) \| D_x^{m+1}\theta \|^2 \\ - C(E_0)\varepsilon^{-1} \| D_x^m(\rho, u, \theta) \|^2 - C(E_0)\varepsilon^{-1}E \| D_x(\rho, u, \theta) \|_{m-1}^2 \leq 0$$

which implies (4.9)_m by taking $\varepsilon = \nu_3/4$ and integrating (4.15) with respect to t . This completes the proof of Lemma 4.5.

Proof of Lemma 4.6 In this case, we estimate the equality

$$(4.16) \quad \int D_x^{m-1}(P^0)_{x_i} \cdot D_x^{m-1}\rho_{x_i} + \frac{(\bar{\rho}+\rho)^2}{2\mu+\mu'} D_x^{m-1}(P^i) \cdot D_x^{m-1}\rho_{x_i} dx \\ = \int \frac{(\bar{\rho}+\rho)^2}{2\mu+\mu'} D_x^{m-1}g^i \cdot D_x^{m-1}\rho_{x_i} dx$$

as follows ;

$$(4.17) \quad \int \frac{(\bar{\rho}+\rho)^2}{2\mu+\mu'} D_x^{m-1}g^i \cdot D_x^{m-1}\rho_{x_i} dx \\ \leq \varepsilon \| D_x^m\rho \|^2 + CE\varepsilon^{-1} \| D_x(\rho, u, \theta) \|_{m-1}^2$$

for any positive number ε ;

$$(4.18) \quad \int D_x^{m-1}(P^0)_{x_i} \cdot D_x^{m-1}\rho_{x_i} dx \\ = \int D_x^{m-1}\rho_{x_i} \cdot D_x^{m-1}\rho_{x_i} + u^j D_x^{m-1}\rho_{x_j x_i} \cdot D_x^{m-1}\rho_{x_i} + (\bar{\rho}+\rho) D_x^{m-1}u_{x_i x_j}^j \cdot D_x^{m-1}\rho_{x_i} + \\ + D_x^{m-1}(u_{x_i}^j \rho_{x_j}) \cdot D_x^{m-1}\rho_{x_i} + D_x^{m-1}(\rho_{x_i} u_{x_j}^j) \cdot D_x^{m-1}\rho_{x_i} + \\ + (D_x^{m-1}(u_{x_i}^j \rho_{x_i x_j}) - u^j D_x^{m-1}\rho_{x_i x_j}) \cdot D_x^{m-1}\rho_{x_i} dx$$

$$\begin{aligned}
&\geq \frac{1}{2} \frac{d}{dt} \int |D_x^{m-1} \rho_{x_i}|^2 dx + \int u^j (|D_x^{m-1} \rho_{x_i}|^2)_{x_j} + \\
&\quad + (\bar{\rho} + \rho) D_x^{m-1} u_{x_i x_j}^j \cdot D_x^{m-1} \rho_{x_i} dx - CE \| D_x(\rho, u, \theta) \|_{m-1}^2 \\
&\geq \frac{1}{2} \frac{d}{dt} \| D_x^m \rho \|^2 + \int (\bar{\rho} + \rho) D_x^{m-1} u_{x_i x_j}^j \cdot D_x^{m-1} \rho_{x_i} dx - \\
&\quad - CE \| D_x(\rho, u, \theta) \|_{m-1}^2 ;
\end{aligned}$$

$$\begin{aligned}
(4.19) \quad &\int \frac{(\bar{\rho} + \rho)^2}{2\mu + \mu'} D_x^{m-1} (P^i) \cdot D_x^{m-1} \rho_{x_i} dx \\
&= \int \frac{(\bar{\rho} + \rho)^2}{2\mu + \mu'} D_x^{m-1} u_t^i \cdot D_x^{m-1} \rho_{x_i} - \frac{\mu(\bar{\rho} + \rho)}{2\mu + \mu'} D_x^{m-1} u_{x_j x_j}^i \cdot D_x^{m-1} \rho_{x_i} - \\
&\quad - \frac{(\mu + \mu')(\bar{\rho} + \rho)}{2\mu + \mu'} D_x^{m-1} u_{x_i x_j}^j \cdot D_x^{m-1} \rho_{x_i} + \frac{(\bar{\rho} + \rho) p_\rho}{2\mu + \mu'} |D_x^{m-1} \rho_{x_i}|^2 + \\
&\quad + \frac{(\bar{\rho} + \rho) p_\theta}{2\mu + \mu'} D_x^{m-1} \theta_{x_i} \cdot D_x^{m-1} \rho_{x_i} + \frac{(\bar{\rho} + \rho)^2}{2\mu + \mu'} \{ - (D_x^{m-1} (\overset{\sim}{\mu} u_{x_j x_j}^i) - \overset{\sim}{\mu} D_x^{m-1} u_{x_j x_j}^i) - \\
&\quad - (D_x^{m-1} (\overset{\sim}{\mu} + \mu') u_{x_i x_j}^j) - (\overset{\sim}{\mu} + \mu') D_x^{m-1} u_{x_i x_j}^j \} + (D_x^{m-1} (\tilde{p}_\rho \rho_{x_i}) - \tilde{p}_\rho D_x^{m-1} \rho_{x_i}) + \\
&\quad + (D_x^{m-1} (\tilde{p}_\theta \theta_{x_i}) - \tilde{p}_\theta D_x^{m-1} \theta_{x_i}) \} \cdot D_x^{m-1} \rho_{x_i} dx \\
&\geq \frac{d}{dt} \int \frac{(\bar{\rho} + \rho)^2}{2\mu + \mu'} D_x^{m-1} u^i \cdot D_x^{m-1} \rho_{x_i} dx - \int \frac{(\bar{\rho} + \rho)^2}{2\mu + \mu'} {}_t D_x^{m-1} u^i \cdot D_x^{m-1} \rho_{x_i} + \\
&\quad + \frac{(\bar{\rho} + \rho)^2}{2\mu + \mu'} D_x^{m-1} u^i \cdot D_x^{m-1} \rho_{x_i} {}_t dx - \int (\bar{\rho} + \rho) D_x^{m-1} u_{x_i x_j}^j \cdot D_x^{m-1} \rho_{x_i} dx + \\
&\quad + v_6 \| D_x^m \rho \|^2 - C \| D_x^m \theta \|^2 - CE \| D_x(\rho, u, \theta) \|_{m-1}^2
\end{aligned}$$

for some positive constant v_6 ;

$$\begin{aligned}
(4.20) \quad & - \int \left(\frac{(\bar{\rho}+\rho)^2}{2\mu+\mu'} \right)_t D_x^{m-1} u^i \cdot D_x^{m-1} \rho_{x_i} dx \\
& = - \int (D_x^{m-1} u^i \cdot D_x^{m-1} \rho_{x_i}) \left\{ \frac{2(\bar{\rho}+\rho)}{2\mu+\mu'} \rho_t - \left(\frac{\bar{\rho}+\rho}{2\mu+\mu'} \right)^2 (2\mu+\mu') \rho_t + \right. \\
& \quad \left. + (2\mu+\mu') \theta_t \right\} dx \\
& = \int (D_x^{m-1} u^i \cdot D_x^{m-1} \rho_{x_i}) \left\{ \left(\frac{2(\bar{\rho}+\rho)}{2\mu+\mu'} - \left(\frac{\bar{\rho}+\rho}{2\mu+\mu'} \right)^2 (2\mu+\mu') \right) (u^j \rho_{x_j} + (\bar{\rho}+\rho) u_{x_j}^j) + \right. \\
& \quad \left. + \left(\frac{\bar{\rho}+\rho}{2\mu+\mu'} \right)^2 (2\mu+\mu') \theta \left(\tilde{\kappa}_{x_j x_j}^{\theta} - \tilde{p}_3^j u_{x_j}^j + g^4(\rho, u, \theta) \right) \right\} dx \\
& \geq - CE \| D_x(\rho, u, \theta) \|_{m-1}^2 - C \int |\Delta\theta| |D_x^{m-1} u| |D_x^m \rho| dx \\
& \geq CE \| D_x(\rho, u) \|_{m-1}^2 - C \| D_x^2 \theta \|_1 \| D_x^{m-1} u \|_1 \| D_x^m \rho \|_1 ;
\end{aligned}$$

$$\begin{aligned}
(4.21) \quad & - \int \frac{(\bar{\rho}+\rho)^2}{2\mu+\mu'} D_x^{m-1} u^i \cdot D_x^{m-1} \rho_{x_i} dx \\
& = \int \frac{(\bar{\rho}+\rho)^2}{2\mu+\mu'} D_x^{m-1} u^i \cdot D_x^{m-1} (u_{x_i}^j \rho_{x_j} + u^j \rho_{x_j x_i} + \rho_{x_i} u_{x_j}^j + (\bar{\rho}+\rho) u_{x_i x_j}^j) dx \\
& \geq \int \frac{(\bar{\rho}+\rho)^2}{2\mu+\mu'} u^j D_x^{m-1} u^i \cdot D_x^{m-1} \rho_{x_i x_j} + \frac{(\bar{\rho}+\rho)^3}{2\mu+\mu'} D_x^{m-1} u^i \cdot D_x^{m-1} u_{x_i x_j}^j dx \\
& \quad - CE \| D_x(\rho, u) \|_{m-1}^2 \\
& = - \int \left(\frac{(\bar{\rho}+\rho)^2}{2\mu+\mu'} \right)_{x_j} u^j D_x^{m-1} u^i \cdot D_x^{m-1} \rho_{x_i} + \frac{(\bar{\rho}+\rho)^2}{2\mu+\mu'} u_{x_j}^j D_x^{m-1} u^i \cdot D_x^{m-1} \rho_{x_i} + \\
& \quad + \frac{(\bar{\rho}+\rho)^2}{2\mu+\mu'} u^j D_x^{m-1} u_{x_j}^i \cdot D_x^{m-1} \rho_{x_i} dx - \int \left(\frac{(\bar{\rho}+\rho)^3}{2\mu+\mu'} \right)_{x_j} D_x^{m-1} u^i \cdot D_x^{m-1} u_{x_i}^j + \\
& \quad + \frac{(\bar{\rho}+\rho)^3}{2\mu+\mu'} D_x^{m-1} u_{x_j}^i \cdot D_x^{m-1} u_{x_i}^j dx - CE \| D_x(\rho, u) \|_{m-1}^2
\end{aligned}$$

$$\geq -C \|D_x^m u\|^2 - CE \|D_x(\rho, u)\|_{m-1}^2 .$$

Hence, due to (4.17)-(4.21), we have

$$(4.22) \quad \frac{d}{dt} \int \frac{1}{2} |D_x^m \rho|^2 + \frac{(\bar{\rho} + \rho)^2}{2\mu + \mu'} D_x^{m-1} u^i \cdot D_x^{m-1} \rho_{x_i} dx + \nu_6 \|D_x^m \rho\|^2 -$$

$$- C(E_0) \|D_x^m(u, \theta)\|^2 - C(E_0)E \|D_x(\rho, u, \theta)\|_{m-1}^2 -$$

$$- C(E_0) \|D_x^2 \theta\|_1 \|D_x^{m-1} u\|_1 \|D_x^m \rho\| \leq 0 .$$

Noting that

$$\int \frac{1}{2} |D_x^m \rho|^2 + \frac{(\bar{\rho} + \rho)^2}{2\mu + \mu'} D_x^{m-1} u^i \cdot D_x^{m-1} \rho_{x_i} dx$$

$$\geq \frac{1}{4} \|D_x^m \rho\|^2 - C(E_0) \|D_x^{m-1} u\|^2 ,$$

we obtain (4.10)_m by integrating (4.22) with respect to t .

This completes the proof of Lemma 4.6.

Now let α be some positive number. Making

$$\sum_{m=1}^l \alpha^{2m} (4.9)_m + \alpha^{2m-1} (4.10)_m ,$$

we easily have

$$(4.23) \quad \sum_{m=1}^l \{ \alpha^{2m} (1 - \alpha C(E_0)) \|D_x^m(u, \theta)\|^2 + \alpha^{2m-1} \|D_x^m \rho\|^2 +$$

$$+ \int_0^t \alpha^{2m} (\nu_4 - \alpha C(E_0) - \alpha^2 C(E_0)) \|D_x^{m+1}(u, \theta)(s)\|^2 +$$

$$+ \alpha^{2m-1} (\nu_5 - \alpha C(E_0)) \|D_x^m \rho(s)\|^2 ds \} \leq$$

$$\leq C(E_0, \alpha) \{ \| \rho_0, u_0, \theta_0 \|_{\mathcal{L}}^2 + \| u(t) \|^2 + \int_0^t \| D_x(u, \theta)(s) \|^2 ds + \\ + E \int_0^t \| D_x(\rho, u, \theta)(s) \|_{\mathcal{L}-1}^2 ds \} .$$

Therefore, by taking α so small as

$$1 - \alpha C, \nu_4 - \alpha C, -\alpha^2 C, \nu_5 - \alpha C > 0,$$

we have

$$(4.24) \quad \| (\rho, u, \theta)(t) \|_{\mathcal{L}}^2 + \int_0^t (\nu_7 - C''E) (\| D_x \rho \|_{\mathcal{L}-1}^2 + \| D_x(u, \theta) \|_{\mathcal{L}}^2) ds \\ \leq C' (\| \rho_0, u_0, \theta_0 \|_{\mathcal{L}}^2 + \| (\rho, u, \theta)(t) \|^2 + \int_0^t \| D_x(u, \theta) \|^2 ds) .$$

for some positive constants ν_7 and C'' . Now suppose the following

A Priori Estimate 4.1 There exist positive constants $\varepsilon'_1, \varepsilon'_2$ and C' such that, if $E \leq \varepsilon'_1$ and $\| \rho_0, u_0, \theta_0 \| \leq \varepsilon'_2$, it follows

$$\| (\rho, u, \theta)(t) \|^2 + \int_0^t \| D_x(u, \theta)(s) \|^2 ds \\ \leq C' (\| \rho_0, u_0, \theta_0 \| \|^2 + E \int_0^t \| D_x(\rho, u, \theta) \|_{\mathcal{L}-1}^2 ds),$$

where $\| \cdot \|$ denotes some seminorm.

Then it follows from (4.24) that for $E \leq \min(\varepsilon'_1, \frac{\nu_7}{2(C''+C')})$ and $\| \rho_0, u_0, \theta_0 \| \leq \varepsilon'_2$,

$$\begin{aligned} & \| (\rho, u, \theta)(t) \|_{\mathcal{L}}^2 + \int_0^t \| D_x \rho(s) \|_{\mathcal{L}^{-1}}^2 + \| D_x(u, \theta)(s) \|_{\mathcal{L}}^2 ds \\ & \leq C (\| \rho_0, u_0, \theta_0 \|_{\mathcal{L}}^2 + \| \rho_0, u_0, \theta_0 \|_{\mathcal{L}}^2) \end{aligned}$$

which indeed is corresponding to A Priori Estimate in Chapter I.

Thus, if we can show A Priori Estimate 4.1, we can get a global solution in time of the initial value problem (4.1)' for suitably small initial data.

4.4 Global Existence I, Polytropic Ideal Gas

In the previous section, it is showed that a global solution in time of (4.1)' can be obtained if A Priori Estimate 4.1 is shown. In this section, we study an example in which A Priori Estimate 4.1 is derived only by the energy estimates.

On (4.1)', we suppose the following in addition to Assumption 4.1.

Assumption 4.2

i) The fluid is an ideal gas, i.e.,

$$(4.25) \quad p = R(\bar{\rho} + \rho)(\bar{\theta} + \theta)$$

where R represents gas constant > 0 .

ii) The gas is polytropic, i.e.,

$$(4.26) \quad e = c_V(\bar{\theta} + \theta)$$

where c_V is positive constant and e represents the internal energy per unit mass.

Then the thermodynamical relations imply

$$(4.27) \quad c_V = \frac{R}{\gamma - 1},$$

where γ is the ratio of specific heats ≥ 1 . Substituting (4.25)-(4.27) to (4.1)', consider

$$(4.2)' \quad \left\{ \begin{array}{l} \rho_t + ((\bar{\rho} + \rho)u^j)_{x_j} = 0, \\ u_t^i + u^j u_{x_j}^i + \frac{R(\bar{\theta} + \theta)}{\bar{\rho} + \rho} \rho_{x_i} + R\theta_{x_i} = \frac{1}{\bar{\rho} + \rho} \{ (\mu(u_{x_j}^i + u_{x_i}^j))_{x_j} + (\mu' u_{x_j}^j)_{x_i} \}, \\ \theta_t + u^j \theta_{x_j} + (\gamma - 1)(\bar{\theta} + \theta)u_{x_j}^j = \frac{(\gamma - 1)}{R(\bar{\rho} + \rho)} ((\kappa \theta)_{x_j})_{x_j} + \Psi, \\ (\rho, u, \theta)(0) = (\rho_0, u_0, \theta_0). \end{array} \right.$$

Then we have

Theorem 4.9 (Global Existence Theorem) Suppose Assumptions 4.1 and 4.2. Moreover suppose $(\rho_0, u_0, \theta_0) \in H^{\underline{l}}$ ($\underline{l} \geq 3$). Then there exist positive constants ε_0 and C_0 such that, if $\|(\rho_0, u_0, \theta_0)\|_{\underline{l}} \leq \varepsilon_0$, then the initial value problem (4.2)' has a unique solution

$$(\rho, u, \theta) \in X^{\underline{l}}(0, +\infty; C_0 \|(\rho_0, u_0, \theta_0)\|_{\underline{l}})$$

satisfying

$$(4.28) \quad \|(\rho, u, \theta)(t)\|_{\mathcal{B}_0} + \|D_x(\rho, u, \theta)(t)\|_{\underline{l}-2} \rightarrow 0 \text{ as } t \rightarrow +\infty.$$

Proof of Theorem 4.9 First, it is easily checked that we may take $(\bar{\rho}, \bar{\theta}) = (1, 0, 1)$ without loss of generality by regarding $\mu(\bar{\rho})^{-1}, \mu'(\bar{\rho})^{-1}, R\bar{\theta}$ and $\kappa\bar{\theta}(\bar{\rho})^{-1}$ as μ, μ', R and κ . Then the system (4.2)' is written in the form ;

$$(4.29) \left\{ \begin{array}{l} \rho_t + u^j \rho_{x_j} + (1+\rho)u_{x_j}^j = 0, \\ u_t^i + u^j u_{x_j}^i + \frac{R(1+\theta)}{1+\rho} \rho_{x_i} + R\theta_{x_i} = \frac{1}{1+\rho} \{ (\mu(u_{x_j}^i + u_{x_i}^j))_{x_j} + (\mu' u_{x_j}^j)_{x_i} \}, \\ \theta_t + u^j \theta_{x_j} + (\gamma-1)(1+\theta)u_{x_j}^j = \frac{(\gamma-1)}{R(1+\rho)} \{ (\kappa\theta_{x_i})_{x_i} + \Psi \}, \\ (\rho, u, \theta)(0) = (\rho_0, u_0, \theta_0). \end{array} \right.$$

By the previous arguments, it suffices to show the estimates for the solutions $(\rho, u, \theta) \in X^1(0, h; E)$ of (4.29) which imply A Priori Estimate 4.1.

Set

$$s = (1+\theta)/(1+\rho)^{\gamma-1} - 1,$$

and define a $E^0(\rho, u, s)$ by

$$(4.30) \quad E^0(\rho, u, s) = \frac{R}{\gamma-1} ((1+\rho)^\gamma - 1 - \gamma\rho)(1+s) + \\ + \frac{1}{2} (1+\rho)u^i u^i + R\rho + \frac{R}{2(\gamma-1)} (1+\rho)s^2.$$

Then we note

Lemma 4.10 There exist positive constants ρ_2 ($\rho_2 \leq 1/2$) and such that E^0 is positive definite, i.e.,

$$(4.31) \quad v(\rho^2 + |u|^2 + \theta^2) \leq E^0(\rho, u, s) \leq v^{-1}(\rho^2 + |u|^2 + \theta^2) \quad \text{for } |\rho| \leq \rho_2$$

where v and ρ_2 depend only on γ and E_0 .

Proof of Lemma 4.10 First we note that for $|\rho| \leq 1/2$

$$(4.32) \quad \theta^2 \leq 8 \left(\frac{3}{2}\right)^{2(\gamma-1)} (s^2 + (\gamma-1)^2 \rho^2).$$

Next by mean value theorem we have

$$\frac{R}{\gamma-1} ((1+\rho)^\gamma - 1 - \gamma\rho) = \frac{\gamma R}{2} \rho^2 + \frac{\gamma R(\gamma-2)}{3!} \rho^3 (1+\xi\rho)^{\gamma-3},$$

for some $\xi \in (0,1)$. Therefore when $|\rho| \leq 1/2$, (4.30) has the estimate

$$(4.33) \quad \begin{aligned} E^0(\rho, u, s) &\geq R \left\{ \frac{\gamma}{2} \rho^2 + \rho s + \frac{s^2}{2(\gamma-1)} - \frac{\gamma|\gamma-2|}{6} 2^{|\gamma-3|} |\rho|^3 - \right. \\ &\quad \left. - \left(\frac{\gamma}{2} \rho^2 + \frac{\gamma|\gamma-2|}{6} 2^{|\gamma-3|} |\rho|^3 \right) |s| - \frac{1}{2(\gamma-1)} |\rho| s^2 \right\} + \frac{1}{4} u^i u^i \\ &\geq \frac{R}{2} \left\{ \frac{\rho^2}{2} \left(1 - \frac{2\gamma|\gamma-2|}{3} 2^{|\gamma-2|} |\rho| - (\gamma-1)(\gamma-1/2) \left(\gamma + \frac{\gamma|\gamma-2|}{3} 2^{|\gamma-3|} |\rho| \right) \rho^2 \right) + \right. \\ &\quad \left. + \frac{s^2}{2(\gamma-1)(\gamma-1/2)} \left(1 - \frac{2|\rho|}{(\gamma-1/2)} \right) \right\} + \frac{1}{4} u^i u^i \\ &\geq \frac{R}{8} \left(\rho^2 + \frac{s^2}{(\gamma-1)(\gamma-1/2)} \right) + \frac{1}{4} u^i u^i \end{aligned}$$

provided $|\rho| \leq \rho_2 = \rho_2(\gamma)$.

Inequalities (4.32)-(4.33) give the first one in (4.31). The second inequality is trivial.

This completes the proof of Lemma 4.10.

Let us estimate $E^0(\rho, u, s)$. From (4.29), (ρ, u, s) satisfy

$$(4.34) \left\{ \begin{aligned}
& \rho_t + ((1+\rho)u^j)_{x_j} = 0, \\
& u_t^i + u^j u_{x_j}^i + \frac{R}{1+\rho} ((1+\rho)^\gamma (1+s))_{x_i} = \\
& \quad = \frac{1}{1+\rho} \{ (\mu(u_{x_j}^i + u_{x_i}^j))_{x_j} + (\mu' u_{x_j}^j)_{x_i} \}, \\
& s_t + u^j s_{x_j} - \frac{(\gamma-1)}{R} \left(\frac{\kappa}{1+\rho} s_{x_j} + \frac{\kappa(\gamma-1)(1+s)}{(1+\rho)^2} \rho_{x_j} \right)_{x_j} - \\
& \quad - \frac{\kappa\gamma(\gamma-1)}{R} \left(\frac{1}{(1+\rho)^2} s_{x_j} + \frac{(\gamma-1)(1+s)}{(1+\rho)^3} \rho_{x_j} \right) \rho_{x_j} - \\
& \quad - \frac{\gamma-1}{R(1+\rho)^\gamma} \Psi = 0.
\end{aligned} \right.$$

We compute

$$\begin{aligned}
\frac{\partial}{\partial t} E^0(\rho, u, s) &= (1+\rho)u^i u_t^i + \left\{ \frac{1}{2} u^i u^i + \frac{\gamma R}{\gamma-1} (1+s) ((1+\rho)^{\gamma-1} - 1) + \right. \\
& \quad \left. + R s + \frac{R}{2(\gamma-1)} s^2 \right\} \rho_t + \left\{ \frac{R}{\gamma-1} ((1+\rho)^\gamma - (1+\rho)) + \frac{R}{\gamma-1} (1+\rho) s \right\} s_t \\
&= \dots \text{ by use of (4.34)} \dots \\
&\leq \sum_j \left(\right)_{x_j} - \mu u_{x_j}^i u_{x_j}^i - (\mu + \mu') (u_{x_j}^j)^2 - \frac{\kappa(\gamma(1+\rho)^{\gamma-1} - 1)}{1+\rho} s_{x_j} \rho_{x_j} - \\
& \quad - \frac{\kappa(\gamma-1)(1+s)(\gamma(1+\rho)^{\gamma-1} - 1)}{(1+\rho)^2} \rho_{x_j} \rho_{x_j} - \kappa s_{x_j} s_{x_j} - \\
& \quad - \frac{\kappa(\gamma-1)(1+s)}{(1+\rho)} \rho_{x_j} s_{x_j} + CE |D_x(\rho, u, s)|^2
\end{aligned}$$

$$(4.35) \quad \leq \sum_j ()_{x_j} - \bar{\mu} u_{x_j}^i u_{x_j}^i - (\bar{\mu} + \bar{\mu}') (u_{x_j}^j)^2 - \\ - 2\bar{\kappa}(\gamma-1) s_{x_j} \rho_{x_j} - \bar{\kappa} s_{x_j} s_{x_j} + CE |D_x(\rho, u, s)|^2$$

where $\sum_j ()_{x_j}$ means the terms in divergent form of functions of ρ, u, s and their derivatives which will disappear after integration in x , and where $\bar{\mu}, \bar{\mu}'$ and $\bar{\kappa}$ denote $\bar{\mu} = \mu(0,0), \bar{\mu}' = \mu'(0,0)$ and $\bar{\kappa} = \kappa(0,0)$.

In addition to (4.35), we calculate (cf. Proof of Lemma 4.6)

$$\left(\frac{1}{2} \rho_{x_i} \rho_{x_i} + \frac{(1+\rho)^2}{2\mu+\mu'} \rho_{x_i} u^i \right)_t \\ = \left(\rho_{x_i} + \frac{(1+\rho)^2}{2\mu+\mu'} u^i \right) \rho_{x_i t} + \frac{(1+\rho)^2}{2\mu+\mu'} \rho_{x_i} u^i_t + \\ + \left(\frac{(1+\rho)^2}{2\mu+\mu'} \right) \rho_{x_i} u^i \rho_t + \left(\frac{(1+\rho)^2}{2\mu+\mu'} \right) \theta_{x_i} u^i \theta_t \\ = \dots \text{ by use of (4.34) } \dots \\ \leq \sum_j ()_{x_j} - \frac{R(1+\rho)^{\gamma+1}}{2\mu+\mu'} \rho_{x_j} s_{x_j} - \frac{R\gamma(1+\rho)^\gamma(1+s)}{2\mu+\mu'} \rho_{x_j} \rho_{x_j} + \\ + \frac{(1+\rho)^3}{2\mu+\mu'} (u_{x_j}^j)^2 + CE |D_x(\rho, u, s)|^2 + CE |D_x^2 \theta| |D_x \rho| \\ (4.36) \quad \leq \sum_j ()_{x_j} - \frac{R}{2\bar{\mu} + \bar{\mu}'} \rho_{x_j} s_{x_j} - \frac{R\gamma}{2\bar{\mu} + \bar{\mu}'} \rho_{x_j} \rho_{x_j} + \\ + \frac{1}{2\bar{\mu} + \bar{\mu}'} (u_{x_j}^j)^2 + CE |D_x(\rho, u, s)|^2 + CE |D_x^2 \theta| |D_x \rho| .$$

Making

$$\int (4.35) + \beta(4.36) dx \quad (0 < \beta < 1),$$

we obtain the following energy inequality

$$(4.37) \quad \frac{\partial}{\partial t} \int E^0(\rho, u, s) + \left(\frac{1}{2} \rho_{x_i} \rho_{x_i} + \frac{(1+\rho)^2}{2\bar{\mu}+\bar{\mu}'} \rho_{x_i} u^i \right) dx +$$

$$+ \int \bar{\mu} u_{x_j}^i u_{x_j}^i + (\bar{\mu}+\bar{\mu}') (u_{x_j}^j)^2 + 2\bar{\kappa}(\gamma-1) s_{x_j} \rho_{x_j} +$$

$$+ \beta \left(\frac{R}{2\bar{\mu}+\bar{\mu}'} \rho_{x_i} s_{x_i} + \frac{R\gamma}{2\bar{\mu}+\bar{\mu}'} \rho_{x_j} \rho_{x_j} - \frac{1}{2\bar{\mu}+\bar{\mu}'} (u_{x_j}^j)^2 \right) dx$$

$$\leq CE \| D_x(\rho, u, \theta) \|_1^2.$$

Therefore if we take β so small that

$$\beta < \min \left(\frac{(2\bar{\mu}+\bar{\mu}')^2}{8(1+\rho_2)^4}, (\bar{\mu}+\bar{\mu}') (2\bar{\mu}+\bar{\mu}'), \frac{4\bar{\kappa}(2\bar{\mu}+\bar{\mu}')}{R} \right),$$

then

$$\frac{R}{8} \left(\rho^2 + \frac{1}{(\gamma-1)(\gamma-1/2)} s^2 \right) + \frac{1}{8} u^i u^i + \frac{\beta}{4} \rho_{x_i} \rho_{x_i}$$

$$\leq E^0(\rho, u, s) + \beta \left(\frac{1}{2} \rho_{x_i} \rho_{x_i} + \frac{(1+\rho)^2}{2\bar{\mu}+\bar{\mu}'} \rho_{x_i} u^i \right),$$

$$\bar{\mu} u_{x_j}^i u_{x_j}^i + \bar{\kappa} \gamma \left(\frac{1}{(\gamma+1)^2 + \gamma} s_{x_j} s_{x_j} + \rho_{x_j} \rho_{x_j} \right)$$

$$\leq \bar{\mu} u_{x_j}^i u_{x_j}^i + (\bar{\mu}+\bar{\mu}' - \frac{\beta}{2\bar{\mu}+\bar{\mu}'}) (u_{x_j}^j)^2 + \bar{\kappa} s_{x_j} s_{x_j} +$$

$$+ \left(2\bar{\kappa}(\gamma-1) + \frac{\beta R}{2\bar{\mu}+\bar{\mu}'} \right) s_{x_j} \rho_{x_j} + \left(\bar{\kappa}(\gamma-1)^2 + \frac{\beta R \gamma}{2\bar{\mu}+\bar{\mu}'} \right) \rho_{x_j} \rho_{x_j} .$$

Thus after integration in t we obtain

$$\begin{aligned} & \int \frac{R}{8} \left(\rho^2 + \frac{1}{(\gamma-1)(\gamma-1/2)} s^2 \right) + \frac{1}{8} u^i u^i + \frac{\beta}{4} \rho_{x_j} \rho_{x_j} \, dx \\ & + \int_0^t \int \left(\bar{\mu} u_{x_j}^i u_{x_j}^i + \bar{\kappa} \gamma \left(\rho_{x_i} \rho_{x_i} + \frac{1}{(\gamma+1)^2 + \gamma} s_{x_i} s_{x_i} \right) \right) dx d\tau \\ & \leq \int E^0(\rho, u, s) + \beta \left(\frac{1}{2} \rho_{x_i} \rho_{x_i} + \frac{(1+\rho)^2}{2\bar{\mu}+\bar{\mu}'} \rho_{x_i} u^i \right) dx \Big|_{t=0} + \\ & + CE \int_0^t \| D_x(\rho, u, \theta) \|_1^2 \, d\tau \\ & \leq C \| \rho_0, u_0, \theta_0 \|_1^2 + CE \int_0^t \| D_x(\rho, u, \theta) \|_1^2 \, d\tau \end{aligned}$$

which implies

$$\begin{aligned} & \| (\rho, u, s)(t) \|^2 + \| D_x \rho(t) \|^2 + \int_0^t \| D_x(\rho, u, s)(\tau) \|^2 \, d\tau \\ & \leq C \| \rho_0, u_0, \theta_0 \|_1^2 + CE \int_0^t \| D_x(\rho, u, \theta)(\tau) \|_1^2 \, d\tau \end{aligned}$$

that is,

$$\begin{aligned} & \| (\rho, u, \theta)(t) \|^2 + \| D_x \rho(t) \|^2 + \int_0^t \| D_x(\rho, u, \theta)(\tau) \|^2 \, d\tau \\ & \leq C \| \rho_0, u_0, \theta_0 \|_1^2 + CE \int_0^t \| D_x(\rho, u, \theta)(\tau) \|_1^2 \, d\tau . \end{aligned}$$

This is the desired estimate which implies A Priori Estimate 4.1.

Finally the decay estimate (4.28) easily follows from

$$\sup_{t>0} \|(\rho, u, \theta)(t)\|_L^2 + \int_0^\infty \|D_x \rho(\tau)\|_{L^{-1}}^2 + \|D_x(u, \theta)(\tau)\|_L^2 d\tau < +\infty.$$

4.5 Global Existence II, Isotropic Newtonian Fluids

In this section, returning to the original problem

$$(4.1)' \left\{ \begin{array}{l} \rho_t + ((\bar{\rho} + \rho)u^j)_{x_j} = 0, \\ u_t^i + u^j u_{x_j}^i + \frac{1}{\bar{\rho} + \rho} p_{x_i} = \frac{1}{\bar{\rho} + \rho} \{(\mu(u_{x_j}^i + u_{x_i}^j))_{x_j} + (\mu' u_{x_j}^j)_{x_i}\}, \\ \theta_t + u^j \theta_{x_j} + \frac{(\bar{\theta} + \theta)p_{\theta}}{(\bar{\rho} + \rho)c_V} u_{x_j}^j = \frac{1}{(\bar{\rho} + \rho)c_V} ((\kappa \theta_{x_j})_{x_j} + \Psi), \\ (\rho, u, \theta)(0) = (\rho_0, u_0, \theta_0), \end{array} \right.$$

we seek a global solution in time in the set $X^L(0, +\infty; E)$ where for $E \leq E_0$

$$X^L(t_1, t_2; E) \equiv \{(\rho, u, \theta) \mid \rho \in C^0(t_1, t_2; H^L) \cap C^1(t_1, t_2; H^{L-1}), \\ (u, \theta) \in C^0(t_1, t_2; H^L) \cap C^1(t_1, t_2; H^{L-2}) \cap L_2(t_1, t_2; H^{L+1}) \text{ and}$$

$$\sup_{t_1 \leq t \leq t_2} \|(\rho, u, \theta)(t)\|_L^2 + \int_{t_2}^{t_1} \|\rho(s)\|_L^2 + \|(u, \theta)(s)\|_{L+1}^2 ds \leq E^2 \}.$$

Here we note that, although this definition of X^L is slightly different from that in the previous sections, all results hold even for this X^L .

Then we have the following main theorem in this chapter.

Theorem 4.11 (Global Existence Theorem)

Suppose Assumption 4.1

and the initial data

$$(\rho_0, u_0, \theta_0) \in H^l \cap L^1 \text{ for } l \geq 4,$$

and set

$$\Phi_l = \| \rho_0, u_0, \theta_0 \|_l + \| \rho_0, u_0, \theta_0 \|_{L^1}.$$

Then there exist positive constants ε_0 and C_0 such that, if $\Phi_l \leq \varepsilon_0$, then the initial value problem (4.1)' has a unique solution

$$(\rho, u, \theta) \in X^l(0, +\infty; C_0 \Phi_l),$$

and it has the decay rate

$$(4.38) \quad \| (\rho, u, \theta)(t) \|_2 \leq C_0 \Phi_l (1+t)^{-3/4}.$$

In particular, if

$$(4.39) \quad \mu, \mu' \text{ and } \kappa \text{ do not depend on } \rho,$$

then the above assertion holds for $l \geq 3$ also.

By the arguments in the previous sections, it suffices to show the following a priori estimate which implies A Priori Estimate 4.1 and the decay rate estimate (4.38).

Proposition 4.12 Under the same assumptions in Theorem 4.11, there exist positive constants $\varepsilon'_1, \varepsilon'_2$ and C' such that, if $\Phi_l \leq \varepsilon'_1$ and $E \leq \varepsilon'_2$, then the solution $(\rho, u, \theta) \in X^l(0, h; E)$ has the decay rate estimate

$$(4.40) \quad \| (\rho, u, \theta)(t) \|_2 \leq C' \Phi_l (1+t)^{-3/4},$$

for $l = 4$ in general and $l = 3$ in the case of (4.39) where $\varepsilon_1', \varepsilon_2'$ and C' do not depend on h .

Proof of Proposition 4.12 We can show Proposition 4.12 by investigating decay rate for the solutions of the linearized equations. Rewrite the system (4.1)' so that all the nonlinear terms appear at the right hand side of equations ;

$$(4.41) \quad \left\{ \begin{array}{l} \rho_t + \bar{\rho} u_{x_i}^i = f^0, \\ u_t^i + \bar{p}_1 \rho_{x_i} + \bar{p}_2 \theta_{x_i} - \bar{\mu} u_{x_j x_j}^i - (\bar{\mu} + \bar{\mu}') u_{x_i x_j}^j = f^i, \\ \theta_t + \bar{p}_3 u_{x_j}^j - \bar{\kappa} \theta_{x_j x_j} = f^A, \\ (\rho, u, \theta)(0) = (\rho_0, u_0, \theta_0), \end{array} \right.$$

where

$$\begin{aligned} \bar{p}_1 &= \tilde{p}_\rho(0,0) = \frac{p_\rho(0,0)}{\bar{\rho}}, \quad \bar{p}_2 = \tilde{p}_\theta(0,0) = \frac{p_\theta(0,0)}{\bar{\rho}}, \\ \bar{p}_3 &= \tilde{p}_3(0,0) = \frac{\bar{\theta} p_\theta(0,0)}{c_V(0,0) \bar{\rho}}, \quad \bar{\mu} = \tilde{\mu}(0,0) = \frac{\mu(0,0)}{\bar{\rho}}, \\ \bar{\mu}' &= \tilde{\mu}'(0,0) = \frac{\mu'(0,0)}{\bar{\rho}}, \quad \bar{\kappa} = \tilde{\kappa}(0,0) = \frac{\kappa(0,0)}{c_V(0,0) \bar{\rho}}, \end{aligned}$$

and

$$(4.42) \quad \begin{aligned} f^0 &= -\rho u_{x_j}^j - u^j \rho_{x_j}, \\ f^i &= (\tilde{\mu} - \bar{\mu}) u_{x_j x_j}^i + (\tilde{\mu} - \bar{\mu} + \tilde{\mu}' - \bar{\mu}') u_{x_i x_j}^j - (\tilde{p}_\rho - \bar{p}_1) \rho_{x_i} - (\tilde{p}_\theta - \bar{p}_2) \theta_{x_i} - \end{aligned}$$

$$- u^j u_{x_j}^i + (\mu_{x_j} (u_{x_j}^i + u_{x_i}^j) + \mu_{x_i} u_{x_j}^j) / (\bar{\rho} + \rho) ,$$

$$f^A = (\hat{\kappa} - \bar{\kappa}) \theta_{x_j x_j} - (\hat{p}_3 - \bar{p}_3) u_{x_j}^j - u^j \theta_{x_j} + (\kappa_{x_j} \theta_{x_j} + \Psi) / (\bar{\rho} + \rho) c_V .$$

Set

$$\alpha = \sqrt{p_\rho(0,0)} , \quad \beta = \frac{p_\theta(0,0)}{\bar{\rho}} \sqrt{\frac{\bar{\theta}}{c_V(0,0)}} ,$$

$$(4.43) \quad U = \begin{pmatrix} \frac{\sqrt{p_\rho(0,0)}}{\bar{\rho}} & \rho \\ u^i \\ \sqrt{\frac{c_V(0,0)}{\bar{\theta}}} & \theta \end{pmatrix} , \quad F(U) = \begin{pmatrix} \frac{\sqrt{p_\rho(0,0)}}{\bar{\rho}} f^0 \\ f^i \\ \sqrt{\frac{c_V(0,0)}{\bar{\theta}}} f^A \end{pmatrix} ,$$

and

$$(4.44) \quad A = \begin{pmatrix} 0 & -\alpha \partial_{x_j} & 0 \\ -\alpha \partial_{x_i} & \bar{\mu} \delta^{ij} \Delta + (\bar{\mu} + \bar{\mu}') \partial_{x_i x_j}^2 & -\beta \partial_{x_i} \\ 0 & -\beta \partial_{x_j} & \bar{\kappa} \Delta \end{pmatrix}$$

Then we can write (4.41) in the form

$$(4.45) \quad U_t = AU + F(U) ,$$

or in the Fourier transform of (4.45)

$$(4.46) \quad \hat{U}_t = A(\xi) \hat{U} + \hat{F}(U) ,$$

where

$$(4.47) \quad A(\xi) = - \begin{pmatrix} 0 & i\alpha\xi_j & 0 \\ i\alpha\xi_k & \bar{\mu}\delta^{ij}|\xi|^2 + (\bar{\mu}+\bar{\mu}')\xi_k\xi_j & i\beta\xi_k \\ 0 & i\beta\xi_j & \bar{\kappa}|\xi|^2 \end{pmatrix}$$

Now let us analyze the spectrum of $A(\xi)$. The characteristic equation of $A(\xi)$ is given by

$$(4.48) \quad \det | \lambda I - A(\xi) | \\ = (\lambda + \bar{\mu}|\xi|^2)^2 \{ \lambda^3 + (\bar{\kappa} + 2\bar{\mu} + \bar{\mu}')|\xi|^2 \lambda^2 + \\ + (\bar{\kappa}(2\bar{\mu} + \bar{\mu}')|\xi|^4 + (\alpha^2 + \beta^2)|\xi|^2)\lambda + \alpha^2 \bar{\kappa}|\xi|^4 \} \\ \equiv (\lambda + \bar{\mu}|\xi|^2)^2 f(\lambda) .$$

Set $\lambda_3(\xi) = -\bar{\mu}|\xi|^2$ and denote by $\lambda_i(\xi)$ ($0 \leq i \leq 2$) the roots of the equation $f(\lambda) = 0$. Then we have

Lemma 4.13

- i) $\lambda_j(\xi)$ depends on only $|\xi|$ and $\lambda_j(0) = 0$ ($0 \leq j \leq 3$).
- ii) $\operatorname{Re} \lambda_j(\xi) < 0$ for all $|\xi| > 0$ ($0 \leq j \leq 3$).
- iii) $\operatorname{rank} (\lambda_3(\xi)I - A(\xi)) = 3$ for all $|\xi|$ except at most one point of $|\xi| > 0$.
- iv) There exist positive constants $r_1 < r_2$ such that $\lambda_j(\xi)$ ($0 \leq j \leq 3$) has the Taylor (resp. Laurent) series expansion

$$(4.49) \quad \lambda_j(\xi) = \sum_{n=1}^{\infty} |\xi|^n \lambda_j^{(n)} \\ (\text{resp. } \lambda_j(\xi) = \sum_{n=-2}^{\infty} |\xi|^{-n} \lambda_j^{(n)}))$$

for $|\xi| < r_1$ (resp. $|\xi| > r_2$).

The Taylor series expansion has the form

$$(4.50) \left\{ \begin{array}{l} \lambda_0(\xi) = -\frac{\alpha \bar{\kappa}}{\alpha^2 + \beta^2} |\xi|^2 + o(|\xi|^3), \\ \lambda_1(\xi) = i\sqrt{\alpha^2 + \beta^2} |\xi| - \left(\frac{2\bar{\mu} + \bar{\mu}'}{2} + \frac{\bar{\kappa}\beta^2}{2(\alpha^2 + \beta^2)} \right) |\xi|^2 + o(|\xi|^3), \\ \lambda_2(\xi) = \lambda_1^*(\xi) \quad (\lambda^* \text{ represents complex conjugate.}), \\ \lambda_3(\xi) = -\bar{\mu} |\xi|^2, \end{array} \right.$$

and the Laurent series expansion has the form, if $\bar{\kappa} \neq 2\bar{\mu} + \bar{\mu}'$,

$$(4.51) \left\{ \begin{array}{l} \lambda_k(\xi) = -\frac{\alpha^2}{2\bar{\mu} + \bar{\mu}'} + o(|\xi|^{-1}), \\ \lambda_l(\xi) = -\bar{\kappa} |\xi|^2 - \frac{\beta^2}{2\bar{\mu} + \bar{\mu}' - \bar{\kappa}} + o(|\xi|^{-1}), \quad (0 \leq k, l, m \leq 2) \\ \lambda_m(\xi) = -(2\bar{\mu} + \bar{\mu}') |\xi|^2 + \frac{(2\bar{\mu} + \bar{\mu}' - \bar{\kappa})\alpha^2 + (2\bar{\mu} + \bar{\mu}')\beta^2}{(2\bar{\mu} + \bar{\mu}') (2\bar{\mu} + \bar{\mu}' - \bar{\kappa})} + o(|\xi|^{-1}), \\ \lambda_3(\xi) = -\bar{\mu} |\xi|^2, \end{array} \right.$$

if $\bar{\kappa} = 2\bar{\mu} + \bar{\mu}'$,

$$(4.52) \left\{ \begin{array}{l} \lambda_k(\xi) = -\frac{\alpha^2}{\bar{\kappa}} + o(|\xi|^{-1}), \\ \lambda_l(\xi) = -\bar{\kappa} |\xi|^2 + i\beta |\xi| - \frac{\alpha^2}{4\bar{\kappa}} + o(|\xi|^{-1}), \\ \lambda_m(\xi) = \lambda_l^*(\xi), \\ \lambda_3(\xi) = -\bar{\mu} |\xi|^2. \end{array} \right.$$

v) $\lambda_j \neq \lambda_k$, $j \neq k$ for all $|\xi|$ except at most four points of $|\xi| > 0$.

vi) There exist positive constants β_0 and β_1 such that for all $|\xi| \leq r_1$,

$$-\beta_0 |\xi|^2 \leq \operatorname{Re} \lambda_j(\xi) \leq -\beta_1 |\xi|^2 \quad (0 \leq j \leq 3).$$

vii) There exists a positive constant β_2 such that for all $|\xi| \geq r_1$

$$\operatorname{Re} \lambda_j(\xi) < -\beta_2 \quad (0 \leq j \leq 3).$$

viii) The matrix exponential $e^{tA(\xi)}$ has the spectral resolution

$$(4.53) \quad e^{tA(\xi)} = \sum_{j=0}^3 e^{t\lambda_j(\xi)} P_j(\xi)$$

for all $|\xi|$ except at most four points of $|\xi| > 0$.

ix) $P_j(\xi)$ ($0 \leq j \leq 3$) has the estimate

$$\|P_j(\xi)\| \leq C \quad \text{for } |\xi| \leq r_1 \text{ and } |\xi| \geq r_2,$$

where $\|\cdot\|$ denotes the matrix norm.

x) By modification of the right hand side of (4.53), (4.53) makes sense even near the points of a multiple eigenvalue, and $e^{tA(\xi)}$ has the estimate

$$\|e^{tA(\xi)}\| \leq C(1+t)^3 e^{-\beta_2 t} \quad \text{for } |\xi| \geq r_1.$$

xi) For $|\xi| \leq r_1$, $P_j(\xi)$ ($0 \leq j \leq 3$) has the following expansion corresponding to (4.49);

$$P_j(\xi) = \sum_{n=0}^{\infty} |\xi|^{-n} P_j^{(n)}(\omega),$$

where $\omega = \xi |\xi|^{-1}$ and $\{P_j^{(0)}(\omega)\}$ are orthogonal projections and are given by

$$P_0^{(0)}(\omega) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta^{ij} - \omega_i \omega_j & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$P_1^{(0)}(\omega) = \begin{pmatrix} \frac{\beta^2}{\alpha^2 + \beta^2} & 0 & \frac{-\alpha\beta}{\alpha^2 + \beta^2} \\ 0 & 0 & 0 \\ \frac{-\alpha\beta}{\alpha^2 + \beta^2} & 0 & \frac{\alpha^2}{\alpha^2 + \beta^2} \end{pmatrix}$$

$$P_2^{(0)}(\omega) = \begin{pmatrix} \frac{\alpha^2}{2(\alpha^2 + \beta^2)} & -\frac{\alpha\omega_j}{2\sqrt{\alpha^2 + \beta^2}} & \frac{\alpha\beta}{2(\alpha^2 + \beta^2)} \\ -\frac{\alpha\omega_i}{2\sqrt{\alpha^2 + \beta^2}} & \frac{\omega_i \omega_j}{2} & -\frac{\beta\omega_i}{2\sqrt{\alpha^2 + \beta^2}} \\ \frac{\alpha\beta}{2(\alpha^2 + \beta^2)} & -\frac{\beta\omega_j}{2\sqrt{\alpha^2 + \beta^2}} & \frac{\beta^2}{2(\alpha^2 + \beta^2)} \end{pmatrix}$$

$$P_3^{(0)}(\omega) = \begin{pmatrix} \frac{\alpha^2}{2(\alpha^2 + \beta^2)} & \frac{\alpha\omega_j}{2\sqrt{\alpha^2 + \beta^2}} & \frac{\alpha\beta}{2(\alpha^2 + \beta^2)} \\ \frac{\alpha\omega_i}{2\sqrt{\alpha^2 + \beta^2}} & \frac{\omega_i \omega_j}{2} & \frac{\beta\omega_i}{2\sqrt{\alpha^2 + \beta^2}} \\ \frac{\alpha\beta}{2(\alpha^2 + \beta^2)} & \frac{\beta\omega_j}{2\sqrt{\alpha^2 + \beta^2}} & \frac{\beta^2}{2(\alpha^2 + \beta^2)} \end{pmatrix}$$

Proof of Lemma 4.13

i) It is clear.

ii) If we suppose iv) for a moment, it suffices to show that $f(ik) \neq 0$

(k : real). Suppose

$$f(ik) = -ik^3 - (\bar{\kappa} + 2\bar{\mu} + \bar{\mu}')|\xi|^2 k^2 + i(\bar{\kappa}(2\bar{\mu} + \bar{\mu}')|\xi|^4 + (\alpha^2 + \beta^2)|\xi|^2)k + \alpha^2 \bar{\kappa} |\xi|^4 = 0.$$

Then we have

$$\bar{\kappa}(2\bar{\mu} + \bar{\mu}')|\xi|^4 + (\alpha^2 + \beta^2)|\xi|^2 = k^2,$$

and

$$\alpha^2 \bar{\kappa} |\xi|^4 = (\bar{\kappa} + 2\bar{\mu} + \bar{\mu}')|\xi|^2 k^2,$$

which imply

$$\bar{\kappa}(2\bar{\mu} + \bar{\mu}')|\xi|^2 + \alpha^2 + \beta^2 = \frac{\alpha^2 \bar{\kappa}}{\bar{\kappa} + 2\bar{\mu} + \bar{\mu}'} < \alpha^2$$

which is a contradiction. Thus we have ii).

iii)

$$\lambda_3(\xi)I - A(\xi) = \begin{pmatrix} -\bar{\mu}|\xi|^2 & i\alpha\xi_j & 0 \\ i\alpha\xi_k & (\bar{\mu} + \bar{\mu}')\xi_k\xi_j & i\beta\xi_k \\ 0 & i\beta\xi_j & (\bar{\kappa} - \bar{\mu})|\xi|^2 \end{pmatrix}.$$

Considering the minor $\{(\bar{\mu} + \bar{\mu}')\xi_k\xi_j\}$, we can easily see

$$\text{rank}(\lambda_3(\xi)I - A(\xi)) \leq 3.$$

Since $|\xi| > 0$, we may suppose $\xi_1 \neq 0$ without loss of generality. Then

consider the minor

$$M = \begin{pmatrix} -\bar{\mu}|\xi|^2 & i\alpha\xi_1 & 0 \\ i\alpha\xi_1 & (\bar{\mu}+\bar{\mu}')\xi_1^2 & i\beta\xi_1 \\ 0 & i\beta\xi_1 & (\bar{\kappa}-\bar{\mu})|\xi|^2 \end{pmatrix}.$$

Since

$$\det M = |\xi|^2 \xi_1^2 \{ \bar{\kappa}\alpha^2 - (\alpha^2 + \beta^2)\bar{\mu} - \bar{\mu}(\bar{\mu} + \bar{\mu}')(\bar{\kappa} - \bar{\mu}) |\xi|^2 \},$$

$$\text{if } \frac{\alpha^2 \bar{\kappa}}{\alpha^2 + \beta^2} \leq \bar{\mu} \leq \bar{\kappa},$$

$$\det M \neq 0 \quad \text{for all } |\xi| > 0,$$

$$\text{and if } \bar{\mu} > \bar{\kappa} \text{ or } \bar{\mu} < \frac{\alpha^2}{\alpha^2 + \beta^2},$$

$$\det M \neq 0 \quad \text{for } |\xi|^2 \neq \frac{\bar{\kappa}\alpha^2 - \bar{\mu}(\alpha^2 + \beta^2)}{\bar{\mu}(\bar{\mu} + \bar{\mu}')(\bar{\kappa} - \bar{\mu})}$$

which imply iii).

iv) We apply the following lemma (Appendix in [13]) to $f(\lambda) = 0$.

Lemma 4.14 Let $p(\tau, \eta)$ be a polynomial in two variables τ and η which has the form

$$p(\tau, \eta) = c_m(\eta)\tau^m + c_{m-1}(\eta)\tau^{m-1} + \dots + c_0(\eta),$$

where $m \geq 1$ and $c_m(\tau) = 0$. We can then write

$$p(\tau, \eta) = c_m(\eta) \prod_1^m (\tau - \tau_j(\eta)),$$

where each τ_j for some positive integer p is an analytic function of $\eta^{\frac{1}{p}}$