CHAPTER III

QUASILINEAR WAVE EQUATIONS

In this chapter, the initial value problems to the quasilinear wave equations are investigated. The existence theorem for a unique local solution in time is established in § 3.2. In § 3.3, manipulating the energy estimates alone, we obtain a unique global solution in time of the quasilinear dissipative wave equation with suitably small initial data. Furthermore in § 3.4, using a combination of the decay rate estimates for the solutions of the linearized equations and the energy estimates, we obtain a unique global solution in time of the quasilinear wave equation with suitably high nonlinearity and small initial data.

§ 3.1 Equations and Historical Remarks

Let us consider the initial value problem to the quasilinear wave equations

$$(3.1)\qquad u_{tt} - \sum_{i,j=1}^{n} a_{ij}(u,D_{x,t}u)u_{x_ix_j} + b(u,D_{x,t}u) = 0 \qquad x \in R^n, 0\le t\le T,$$

with the initial data

$$(3.2)\qquad u(x,0) = u_0(x)\ ,\ u_t(x,0) = u_1(x)\ ,$$

where we suppose the following :

Assumption 3.1

i) $a_{ij}(y) \in C^\infty(R^{n+2}) \qquad 1\le i,j\le n\ ,$

ii) $a_{ij}(y) = a_{ji}(y)$ $\qquad 1\le i,j\le n.$

iii) There exist positive constants γ_0 and ν (≤ 1) such that

$$\nu|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(y)\xi_i\xi_j \le \nu^{-1}|\xi|^2$$

for all $\xi \in R^n$ and $y \in R^{n+2}$ satisfying $|y| \le \gamma_0$.

iv) $b(y) \in C^\infty(R^{n+2})$.

v) $b(0) = 0$.

In particular,we say "dissipative" in this chapter when b has the form

$$(3.3) \qquad b(u,D_{x,t}u) = \alpha u_t + \tilde{b}(u,D_{x,t}u)$$

where α is some positive constant and $\tilde{b}(y)$ satisfies $\tilde{b}(0) = 0$ and $(D_y\tilde{b})(0) = 0$.

The equations (3.1) describe a mathematical model of vibrations of nonlinear string,film and,generaly speaking,model of wave propergation in a medium with nonlinear structure (cf. [1] [65]). They also have a relation to the equations for heat flow with memory (cf. [4] [34] [35]). In fact,the simplest model for heat flow with memory is represented by

$$(3.4) \qquad \begin{cases} \theta_t = -\displaystyle\int_0^t a(t-s)q_x(s)\,ds \,, \\ q = -\sigma(\theta_x) \qquad\qquad (\ \sigma'(0) > 0\ )\,, \end{cases}$$

where θ and q represent absolute temperature and heat flow respectively. When we take $a(t) = \delta(t)$,the equation (3.4) is reduced to the usual quasilinear heat equation

$$\theta_t - (\sigma(\theta_x))_x = 0 ,$$

and when we take $a(t) = e^{-\alpha t}$,the equation (3.4) is indeed reduced to the quasilinear dissipative wave equation

$$\theta_{tt} - (\sigma(\theta_x))_x + \alpha\theta_t = 0.$$

The local solution in time of the initial value problem (3.1)-(3.2) is investigated in Dionne[5]. So we are interested in a global solution in time in what follows. First note that the quasilinear problem(3.1)-(3.2) dose not generally have smooth global solutions,no matter how smooth the initial data are.(cf [19][20][31]). In this thesis we treat only the smooth global solutions for suitably "small" initial data.
For the semilinear wave equations

$$u_{tt} - \sum_{i,j} a_{ij}(x,t)u_{x_i x_j} + b(u,D_{x,t}u) = 0,$$

there have been many results,for example,Ebihara[6][7],Glassey[10][11], Heinz and Wahl[12],Lions and Strauss[33],Matsumura[36],Morawetz and Strauss[44],Nakao[45]-[52],Rabinowitz[57],Rauch[59],Sattinger[62],Segal [63][64],Strauss[67][68] and Wahl[74]-[76].
On the other hand,for the quasilinear and nonlinear wave equations,there have been a few papers,for example,Rabinowitz[58],Nishida[55],Matsumura [37][38] and Klainerman[27]. To be precise,Rabinowitz obtained the time periodic solution of the nonlinear dissipative wave equation

$$u_{tt} - u_{xx} + \alpha u_t = \varepsilon f(x,t,D_{x,t}u,D^2_{x,t}u) ,$$

$$x \in [0,1],\ t \in R^1,\ \alpha > 0,$$

for suitably small ε. By the arguments of Riemann invariants, Nishida obtained a global solution in time of the dissipative wave equation

$$u_{tt} - (\sigma(u_x))_x + \alpha u_t = 0,\ x \in R^1,\ t \geq 0,\ \alpha > 0,$$

for suitably small initial data. But his argument is not applicable to the multi-space dimensional cases. For the multi-space dimensional cases, Matsumura obtained a global solution in time of the dissipative wave equations (3.1)-(3.3) for suitably small initial data. The proof is based exclusively on the energy estimates alone. These results are summarized in § 3.3. Furthermore in § 3.4, it is proved that, if the quasi-linear wave equation (3.1) has suitably high nonlinearity, a global solution in time of (3.1)-(3.2) is obtained for suitably small initial data. In this case, the proof is based on a combination of the estimates of decay rate for the solutions of the linearized equation and the energy estimates. Just recently, Klainermann obtained a global solution in time of the quasilinear wave equation

$$u_{tt} - \sum_{i,j} a_{ij}(D_{x,t}u)u_{x_i x_j} = 0,\ x \in R^n,\ t \geq 0,$$

for the space dimension $n \geq 6$ and suitably small initial data by using a combination of the decay rate estimates, the energy estimates and the Hörmander's implicit function theorem.

§ 3.2 Local Existence

In this section, let us construct a unique local solution in time of the initial value problem (3.1)-(3.2). Of course, we suppose Assumption 3.1 here. First, by Lemma 2.1, we can choose a positive constant E_0 such that, if $f = \{f^i\}_{i=1}^{n+2} \in H^{[n/2]+1}$ satisfies $\| f \|_{[n/2]+1} \leq E_0$, then we have $\| f \|_{\mathcal{B}^0} \leq \gamma_0$. Let s be a positive integer not less than $[n/2] + 2$. Then the solution is sought in the space $X^s(0,\tau;E)$ for some $E \leq E_0$ and $\tau \leq T$ where, for $0 \leq t_1 \leq t_2 \leq T$, X^s is defined by

$$X^s(t_1,t_2;E) = \{ u \mid u \in C^i(t_1,t_2;H^{s+1-i}) \quad (0 \leq i \leq 2) \quad \text{and}$$

$$\sup_{t_1 \leq t \leq t_2} (\| u(t) \|_{s+1}^2 + \| u_t(t) \|_s^2) \leq E^2 \quad (E \leq E_0) \}.$$

Next, for $v \in X^s(0,\tau;E)$, consider the linear problem

$$(3.5) \quad \begin{cases} L_v(u) \equiv u_{tt} - \sum_{i,j=1}^{n} a_{ij}(v,D_{x,t}v)u_{x_i x_j} = - b(v,D_{x,t}v) , \\ u(x,0) = u_0(x) , \ u_t(x,0) = u_1(x) , \ x \in R^n , \ 0 \leq t \leq \tau. \end{cases}$$

In order to apply Proposition 2.8 to (3.5), we prepair

Lemma 3.1 Suppose Assumption 3.1 and $v,w \in X^s(0,\tau;E)$. Then we have the following :

i) $a_{ij}(v,D_{x,t}v) \in C^0(0,\tau;\mathcal{B}^1) \cap C^1(0,\tau;\mathcal{B}^0)$.

ii) $\nu|\xi|^2 \leq \sum_{i,j} a_{ij}(v,D_{x,t}v)\xi_i\xi_j \leq \nu^{-1}|\xi|^2$,

for all $\xi, x \in R^n$ and $0 \leq t \leq \tau$.

iii) $D_{x,t}a_{ij}(v,D_{x,t}v) \in C^0(0,\tau;H^{s-1})$.

iv) $\| D_{x,t}a_{ij}(v,D_{x,t}v) \|_{s-1} \leq Ch_1(\gamma_0)(1+\| D_{x,t}v \|_s)^{s-1}(\| v_{tt} \|_{s-1} +$

$+ \| D_{x,t}v \|_s) \leq C(E_0)(\| v_{tt} \|_{s-1} + \| D_{x,t}v \|_s)$,

where $h_1(\gamma_0) = \sup_{|y|\leq\gamma_0} \sum_{|\alpha|\leq s} |(\frac{\partial}{\partial y})^\alpha a_{ij}(y)|$.

v) $\| \{a_{ij}(v,D_{x,t}v) - a_{ij}(w,D_{x,t}w)\}v_{x_i x_j} \|_{s-1}$

$\leq Ch_1(\gamma_0)(\| v-w \|_s + \| v_t-w_t \|_{s-1})(1 + \| v,w \|_{s+1} + \| v_t,w_t \|_s)^{s-2}\cdot$

$\cdot\| v \|_{s+1} \leq C(E_0)(\| v-w \|_s + \| v_t-w_t \|_{s-1})$.

vi) $b(v,D_{x,t}v) \in C^0(0,\tau;H^s)$.

vii) $\| b(v,D_{x,t}v) \|_s \leq Ch_2(\gamma_0)(1+\| v \|_{s+1}+\| v_t \|_s)^{s-1}(\| v \|_{s+1} +$

$+ \| v_t \|_s) \leq C(E_0)(\| v \|_{s+1} + \| v_t \|_s)$,

where $h_2(\gamma_0) = \sup_{|y|\leq\gamma_0} \sum_{|\alpha|\leq s} |(\frac{\partial}{\partial y})^\alpha b(y)|$.

viii) $\| b(v,D_{x,t}v) - b(w,D_{x,t}w) \|_{s-1} \leq Ch_2(\gamma_0)(1+\| v,w \|_s+\| v_t,w_t \|_{s-1})^{s-2}\cdot$

$\cdot(\| v-w \|_s + \| v_t-w_t \|_{s-1}) \leq C(E_0)(\| v-w \|_s + \| v_t-w_t \|_{s-1})$.

ix) $\| \sum_{i,j=1}^{n} a_{ij}(v,D_{x,t}v)w_{x_i x_j} - b(v,D_{x,t}v) \|_{s-1}$

$\leq C(E_0)(\| w \|_{s+1} + \| v \|_s + \| v_t \|_{s-1})$

$\leq C_6(E_0)E$.

Proof of Lemma 3.1

i) Since $v \in X^s(0,\tau;E)$ and $s + 1 \geq [n/2] + 3$, Lemma 2.1 asserts

$$v \in C^0(0,\tau;\mathcal{B}^2) \cap C^1(0,\tau;\mathcal{B}^1)$$

which proves i).

ii) It is clear by the definition of E_0 and iii) of Assumption 3.1.

iii)-ix) We can give a proof by using Lemma 2.1. We show only vii) because the others are proved in the same way.

Now

$$\| b(v,D_{x,t}v) \|_s$$

$$\leq \| b(v,D_{x,t}v) \| + \sum_{1\leq|\alpha|\leq s} \| (\frac{\partial}{\partial x})^\alpha b(v,D_{x,t}v) \|$$

$$\leq \| \int_0^1 (y\cdot D_y b)(\tau y)\, d\tau \Big|_{y=(v,D_{x,t}v)} \| +$$

$$+ \sum_{1\leq|\alpha|\leq s} \| \sum_{\substack{1\leq\Sigma|\rho_{\kappa_m}|\leq|\alpha| \\ 1\leq\Sigma l_m\leq|\alpha|}} C_\rho g_\rho(v,D_{x,t}v) \prod_{\kappa_1=1}^{l_1} (\frac{\partial}{\partial x})^{\rho_{\kappa_1}} v \prod_{\kappa_2=1}^{l_2} (\frac{\partial}{\partial x})^{\rho_{\kappa_2}} v_t \cdots$$

$$\cdots \prod_{\kappa_{n+2}=1}^{l_{n+2}} (\frac{\partial}{\partial x})^{\rho_{\kappa_{n+2}}} v_{x_n} \|$$

$$\equiv I_1 + I_2 \quad ,$$

where $g_\rho(y)$ represents one of $\{ (\frac{\partial}{\partial y})^\beta b(y) \; ; \; |\beta| \leq |\alpha| \}$ and $\rho = \rho_{\kappa_1}, \rho_{\kappa_2}, \ldots, \rho_{\kappa_{n+2}}$, $1\leq\kappa_m\leq l_m$ and $1\leq m\leq n+2$.

For I_1, we have easily

$$I_1 \leq Ch_2(\gamma_0)(\| v \|_1 + \| v_t \|) .$$

Let us estimate I_2. Since we can easily have the desired estimates for the terms with $\Sigma l_m = 1$ in I_2, we may suppose $\Sigma l_m \geq 2$. Furthermore , if $|\rho_{\kappa_m}| \leq s - [n/2] - 1$, we can estimate the factor $(\frac{\partial}{\partial x})^{\rho_{\kappa_m}} f$ by Lemma2.1 as

$$\| (\frac{\partial}{\partial x})^{\rho_{\kappa_m}} f \|_{\mathcal{B}^0} \leq C\| f \|_s .$$

Therefore, it suffices to consider only the factors with $|\rho_{\kappa_m}| \geq s - [n/2]$. Now suppose n = odd. Then Lemma 2.1 implies

$$(3.6) \qquad (\frac{\partial}{\partial x})^{\rho_{\kappa_1}} v \in L^{p_{\rho_{\kappa_1}}} , \ (\frac{\partial}{\partial x})^{\rho_{\kappa_2}} v_t \in L^{p_{\rho_{\kappa_2}}} , \ldots\ldots$$

$$(\frac{\partial}{\partial x})^{\rho_{\kappa_{n+2}}} v_{x_n} \in L^{p_{\rho_{\kappa_{n+2}}}} ,$$

where

$$(3.7) \qquad \frac{1}{p_{\rho_{\kappa_m}}} \in [\frac{|\rho_{\kappa_m}|}{n} - \frac{2s-n}{2n} , \frac{1}{2}] \qquad (1 \leq m \leq n+2).$$

Denote the least number of $1/p_\rho$ satisfying (3.7) by $1/P_\rho$. Then it follows from $\Sigma|\rho_{\kappa_m}| \leq |\alpha| \leq s$ and $2 \leq \Sigma l_m$ that

$$\sum_\rho \frac{1}{P_\rho} = \Sigma (\frac{|\rho_{\kappa_m}|}{n} - \frac{2s-n}{2n})$$

$$\leq \frac{|\alpha|}{n} - \frac{2s-n}{2n} \Sigma l_m$$

$$\leq \frac{s}{n} - \frac{2s-n}{n} = \frac{1}{2} - \frac{1}{n}(s - \frac{n}{2}) < \frac{1}{2} .$$

Therefore , we can take positive constants p_ρ as

$$\sum_\rho \frac{1}{p_\rho} = \frac{1}{2}.$$

Thus we can estimate I_2 as

$$I_2 \le Ch_2(\gamma_0)\Sigma \| \prod_{\kappa_1=1}^{l_1} (\frac{\partial}{\partial x})^{\rho_{\kappa_1}} v \cdots\cdots \prod_{\kappa_{n+2}=1}^{l_{n+2}} (\frac{\partial}{\partial x})^{\rho_{\kappa_{n+2}}} v_{x_n} \|$$

$$\le Ch_2(\gamma_0)\Sigma \prod_{\kappa_1=1}^{l_1} (\int |(\frac{\partial}{\partial x})^{\rho_{\kappa_1}} v|^{p_{\rho_{\kappa_1}}} dx)^{1/p_{\rho_{\kappa_1}}} \cdots\cdots$$

$$\cdots\cdots \prod_{\kappa_{n+2}=1}^{l_{n+2}} (\int |(\frac{\partial}{\partial x})^{\rho_{\kappa_{n+2}}} v_{x_n}|^{p_{\rho_{\kappa_{n+2}}}} dx)^{1/p_{\rho_{\kappa_{n+2}}}}$$

$$\le Ch_2(\gamma_0) \sum_{1\le \Sigma l_m \le s} \prod_{\kappa_1=1}^{l_1} \| v \|_{s+1} \prod_{\kappa_2=1}^{l_2} \| v_t \|_s \cdots\cdots \prod_{\kappa_{n+2}=1}^{l_{n+2}} \| v \|_{s+1}$$

$$\le Ch_2(\gamma_0)(1+\| v \|_{s+1}+\| v_t \|_s)^{s-1}(\| v \|_{s+1}+\| v_t \|_s) ,$$

which proves vii). This completes the proof of Lemma 3.1.

By virtue of Lemma 3.1,we have

Proposition 3.2 Suppose Assumption 3.1,$u_0(x) \in H^{s+1}$ and $u_1(x) \in H^s$. Then there exists a positive constant τ such that,if for some $E \le E_0$

$$\begin{cases} v \in X^s(0,\tau;E) , \\ \sup_{0\le t\le\tau} \| v_{tt}(t) \|_{s-1} \le C_6 E , \\ \| u_0 \|_{s+1}^2 + \| u_1 \|_s^2 \le (\frac{\nu}{2} E)^2 , \end{cases}$$

then the initial value problem (3.5) has a unique solution u satisfying

$$\begin{cases} u \in X^s(0,\tau;E) , \\ \sup_{0\le t\le\tau} \| u_{tt}(t) \|_{s-1} \le C_6 E , \end{cases}$$

where $C_6 = C_6(E_0)$ is as in Lemma 3.1.

Proof of Proposition 3.2 By Proposition 2.8 and Lemma 3.1, we have a unique solution of (3.5) satisfying

$$u \in C^i(0,\tau;H^{s+1-i}) \quad (0\le i\le 2) ,$$

and the energy inequality

$$\| u(t) \|_{s+1}^2 + \| u_t(t) \|_s^2$$

$$\le \nu^{-2} e^{C(E_0)t} (\| u_0 \|_{s+1}^2 + \| u_1 \|_s^2 + \int_0^t C(E_0)E^2 \, ds)$$

which implies

$$\begin{aligned} (3.8) \qquad & \sup_{0\le t\le\tau} (\| u(t) \|_{s+1}^2 + \| u_t(t) \|_s^2) \\ & \le \nu^{-2} e^{C(E_0)\tau} (\| u_0 \|_{s+1}^2 + \| u_1 \|_s^2) + C(E_0)E^2\tau \\ & \le (\frac{1}{4} e^{C(E_0)\tau} + C(E_0)\tau)E^2 . \end{aligned}$$

Therefore, taking τ so small in (3.8) that

$$\frac{1}{4} e^{C(E_0)\tau} + C(E_0)\tau \le 1 ,$$

we have

$$(3.9) \qquad u \in X^s(0,\tau;E).$$

Finally, since $u,v \in X^s(0,\tau;E)$, ix) of Lemma 3.1 gives

$$\sup_{0\le t\le\tau} \| u_{tt}(t) \|_{s-1} \le C_6(E_0)E .$$

This completes the proof of Proposition 3.2.

Let us construct the local solution of (3.1)-(3.2). We construct the approximate sequence $\{u^{(m)}\}_{m=0}^{\infty}$ as follows ;

$$u^{(0)} \equiv 0 \qquad (m=0),$$

$$(3.10)\quad \begin{cases} L_{u^{(m-1)}}(u^{(m)}) = - b(u^{(m-1)}, D_{x,t}u^{(m-1)}) & (m\ge 1), \\ u^{(m)} = u_0 \ , \ u_t^{(m-1)} = u_1 \ . \end{cases}$$

Let τ be as in Proposition 3.2. Then, since it evidently holds

$$\begin{cases} u^{(0)} \in X^s(0,\tau;E), \\ \sup\limits_{0\le t\le\tau} \| u_{tt}^{(0)}(t) \|_{s-1} \le C_6 E , \end{cases}$$

Proposition 3.2 asserts for all $m \ge 1$,

$$(3.11)\quad \begin{cases} u^{(m)} \in X^s(0,\tau;E) , \\ \sup\limits_{0\le t\le\tau} \| u_{tt}^{(m)}(t) \|_{s-1} \le C_6 E. \end{cases}$$

Next, it follows from (3.6) that for $m \ge 2$

$$(3.12)\quad \begin{cases} L_{u^{(m-1)}}(u^{(m)} - u^{(m-1)}) = \sum\limits_{i,j} (a_{ij}(u^{(m-1)}, D_{x,t}u^{(m-1)}) - \\ \quad - a_{ij}(u^{(m-2)}, D_{x,t}u^{(m-2)}))u^{(m-1)}_{x_i x_j} - (b(u^{(m-1)}, D_{x,t}u^{(m-1)}) - \\ \quad - b(u^{(m-2)}, D_{x,t}u^{(m-2)})) , \\ (u^{(m)} - u^{(m-1)})(0) = 0 \ , \ (u^{(m)} - u^{(m-1)})_t(0) = 0 \ . \end{cases}$$

By applying Proposition 2.7 with $l = s$ to (3.12) and using Lemma 3.1 and (3.11), we obtain

$$\text{(3.13)} \qquad \|(u^{(m)} - u^{(m-1)})(t)\|_s^2 + \|(u^{(m)} - u^{(m-1)})_t(t)\|_{s-1}^2$$

$$\leq C\int_0^t \|(u^{(m-1)} - u^{(m-2)})(\theta)\|_s^2 + \|(u^{(m-1)} - u^{(m-2)})_t(\theta)\|_{s-1}^2 \, d\theta ,$$

$$\|(u^{(m)} - u^{(m-1)})_{tt}(t)\|_{s-2}$$

$$\leq C(\|(u^{(m)} - u^{(m-1)})(t)\|_s + \|(u^{(m)} - u^{(m-1)})_t(t)\|_{s-1} +$$

$$+ \|(u^{(m-1)} - u^{(m-2)})(t)\|_s + \|(u^{(m-1)} - u^{(m-2)})_t(t)\|_{s-1}),$$

which implies that there exists a $u \in C^i(0,\tau;H^{s-i}) \quad (0\leq i\leq 2)$ such that

$$\text{(3.14)} \qquad u^{(m)} \to u \quad \text{strongly in } C^i(0,\tau;H^{s-i}) \quad (0\leq i\leq 2).$$

On the other hand, it follows from (3.11) that for every fixed $t \in [0,\tau]$ there exist a subsequence $\{m_j(t)\}$ and $v_1(t) \in H^{s+1}$, $v_2(t) \in H^s$, $v_3(t) \in H^{s-1}$ satisfying

$$\text{(3.15)} \qquad \begin{aligned} u^{(m_j(t))} &\to v_1(t) \text{ weakly in } H^{s+1}, \\ u_t^{(m_j(t))} &\to v_2(t) \text{ weakly in } H^s , \\ u_{tt}^{(m_j(t))} &\to v_3(t) \text{ weakly in } H^{s-1}. \end{aligned}$$

Then it follows from (3.14) that $v_1(t) = u(t)$, $v_2(t) = u_t(t)$ and $v_3(t) = u_{tt}(t)$. Thus we obtain a solution of (3.1)-(3.2) satisfying

$$\text{(3.16)} \qquad \begin{aligned} &u \in L^i(0,\tau;H^{s+1-i}) \quad (0\leq i\leq 2) , \\ &\sup_{0\leq t\leq\tau} (\|u(t)\|_{s+1}^2 + \|u_t(t)\|_s^2) \leq E^2 . \end{aligned}$$

Finally, by applying Proposition 2.7 to (3.16), we obtain a solution

$$u \in X^s(0,\tau;E),$$

and the uniqueness easily follows from the energy estimates such as (3.13). Thus we arrive at

Theorem 3.3 (Local Existence Theorem) Suppose Assumption 3.1 and the initial data $u_0 \in H^{s+1}$, $u_1 \in H^s$ $(s \geq [n/2]+2)$. Then there exists a positive constant τ such that , if

$$\| u_0 \|_{s+1}^2 + \| u_1 \|_s^2 \leq (\frac{\nu}{2} E)^2 \qquad \text{for some } E \leq E_0 ,$$

then the initial value problem (3.1)-(3.2) has a unique solution

$$u \in X^s(0,\tau;E).$$

§ 3.3 Global Existence I, Quasilinear Dissipative Waves

Let us consider the following initial value problem to the quasi-linear dissipative wave equations :

$$(3.1)' \qquad L(u) \equiv u_{tt} - \sum_{i,j=1}^{n} a_{ij}(u,D_{x,t}u)u_{x_i x_j} + u_t + b(u,D_{x,t}u) = 0 ,$$
$$x \in R^n, t \geq 0 ,$$

with

$$(3.2) \qquad u(x,o) = u_0(x) , \ u_t(x,0) = u_1(x) ,$$

where we suppose Assumption 3.1 and in addition

Assumption 3.2

i) $(D_y b)(0) = 0.$

ii) $b(u,D_{x,t}u) = b_1(u) + b_2(D_{x,t})$,

$$b_1(u)u \geq 0 \ .$$

Let s be a positive integer not less than $[n/2] + 2$ and $X^s(0,+\infty;E)$ be the set of solutions defined in § 3.2. Then we have

Theorem 3.4 (Global Existence Theorem) Suppose Assumptions 3.1,3.2 and the initial data $u_0 \in H^{s+1}$, $u_1 \in H^s$ $(s \geq [n/2]+2)$. Then there exist positive constants ε_0 and C_0 such that , if

$$\| u_0 \|_{s+1} + \| u_1 \|_s \leq \varepsilon_0,$$

then the initial value problem (3.1)'-(3.2) has a unique solution

$$u \in X^s(0,+\infty;C_0(\| u_0 \|_{s+1} + \| u_1 \|_s) \ ,$$

satisfying

$$(3.17) \qquad \| u(t) \|_{\mathcal{B}^0} , \ \| D_x u(t) \|_{s-1} \rightarrow 0 \quad \text{as} \quad t \rightarrow +\infty.$$

Since the local existence is established in Theorem 3.3,it suffices to show the following a priori estimate:

Proposition 3.5 Suppose Assumptions 3.1,3.2 and the initial data $u_0 \in H^{s+1}, u_1 \in H^s$. Moreover suppose that (3.1)'-(3.2) has a solution $u \in X^s(0,h;E)$ for some h and $E \leq E_0$. Then there exist positive constants ε_1 and C_0 not depending on h such that , if $E \leq \varepsilon_1$, we have

$$u \in X^s(0,h;C_0(\| u_0 \|_{s+1} + \| u_1 \|_s) \ ,$$

and when $h = +\infty$,

$$\| u(t) \|_{\mathcal{B}^0} + \| D_x u(t) \|_{s-1} \to 0 \quad \text{as } t \to 0 .$$

By the arguments in Chapter I, the local existence theorem (Theorem 3.3) and a priori estimate (Proposition 3.5) complete the proof of Theorem3.4. Before proving Proposition 3.5, let us make some preparations.

Now define $E_\lambda\{u(t)\}$ for $0 < \lambda < 1$ by

$$E_\lambda\{u(t)\} = \sum_{k=0}^{s} \int \frac{\lambda}{2} |D_x^k u|^2 + \lambda D_x^k u \cdot D_x^k u_t + \frac{1}{2} |D_x^k u_t|^2 +$$

$$+ \frac{1}{2} \sum_{i,j} a_{ij}(u, D_{x,t} u) D_x^k u_{x_i} \cdot D_x^k u_{x_j} \, dx .$$

We have the following lemma in the same way as Lemmas 2.9 and 3.1.

Lemma 3.6 Suppose Assumptions 3.1 and 3.2. Then , for $u \in X^s$ $(0,h;E)$, we have the followings :

i) There exists a positive constant ν_1 not depending on h such that

$$\nu_1(\| u(t) \|_{s+1}^2 + \| u_t(t) \|_s^2)$$

$$\leq E_\lambda\{u(t)\} \leq \nu_1^{-1}(\| u(t) \|_{s+1}^2 + \| u_t(t) \|_s^2) \quad \text{for } 0 \leq t \leq h.$$

ii) $\sum_{k=0}^{s} \| (a_{ij})_{x_j} D_x^k u_{x_j} \cdot D_x^k u \|_{L^1} \leq C(E_0) E \| D_{x,t} u \|_s^2$.

iii) $\sum_{k=1}^{s} \| D_x^k b(u, D_{x,t} u) \| \leq C(E_0) E \| D_{x,t} u \|_s$.

iv) $\| b_2(D_{x,t} u) \| \leq C(E_0) \| D_{x,t} u \|^2$.

v) $\| u_{tt} \|_{s-1} \leq C(E_0) E$.

vi) $\| B(u) \|_{L^1} \leq C(E_0) \| u \|^2$ where $B(u) = \int_0^t b_1(v)\, dv$.

Proof of Proposition 3.5 We show the statements only for $u \in X^s(0,h:E) \cap C^i(0,h;H^\infty)$ $(0 \le i \le 2)$ because for $u \in X^s(0,h;E)$, we may use the arguments for the mollifier $\phi_\delta *$ as in the proof of Proposition 2.4. Then, we first estimate

$$\sum_{k=0}^{s} \int D_x^k(L(u)) \cdot D_x^k u_t \, dx$$

as follows :

(3.18)
$$\sum_{k=0}^{s} \int D_x^k(L(u)) \cdot D_x^k u_t \, dx$$

$$= \sum_k \int D_x^k u_{tt} \cdot D_x^k u_t - \sum_{i,j} D_x^k(a_{ij} u_{x_i x_j}) \cdot D_x^k u_t +$$

$$+ |D_x^k u_t|^2 + D_x^k b \cdot D_x^k u_t \quad dx$$

$$= \frac{1}{2}\frac{d}{dt}\Big(\sum_k \int |D_x^k u_t|^2 + \sum_{i,j} a_{ij} D_x^k u_{x_i} \cdot D_x^k u_{x_j} \, dx\Big) +$$

$$+ \sum_{i,j,k} \int (a_{ij} D_x^k u_{x_i x_j} - D_x^k(a_{ij} u_{x_i x_j})) \cdot D_x^k u_t \quad dx +$$

$$+ \sum_{i,j,k} \int (a_{ij})_{x_j} D_x^k u_{x_i} \cdot D_x^k u_t - \frac{1}{2}(a_{ij})_t D_x^k u_{x_i} \cdot D_x^k u_{x_j} \quad dx +$$

$$+ \sum_k \int |D_x^k u_t|^2 \, dx + \int b_1 u_t + b_2 u_t + \sum_{k=1}^{s} D_x^k b \cdot D_x^k u_t \quad dx$$

$$\le \frac{1}{2}\frac{d}{dt}\Big(\sum_k \int |D_x^k u_t|^2 + \sum_{i,j} a_{ij} D_x^k u_{x_i} \cdot D_x^k u_{x_j} + B(u) \, dx\Big) +$$

$$+ \| u_t \|_s^2 - C(E_0)E \| D_{x,t} u \|_s^2.$$

Next we estimate

$$\sum_{k=0}^{s} \int D_x^k(L(u)) \cdot D_x^k u \quad dx$$

as follows :

(3.19)
$$\sum_{k=0}^{s} \int D_x^k(L(u)) \cdot D_x^k u \quad dx$$

$$= \sum_k \int D_x^k u_{tt} \cdot D_x^k u \ - \sum_{i,j} D_x^k(a_{ij} u_{x_i x_j}) \cdot D_x^k u \ +$$

$$+ D_x^k u_t \cdot D_x^k u + D_x^k b \cdot D_x^k u \quad dx$$

$$= \frac{d}{dt} \left(\sum_k \int D_x^k u_t \cdot D_x^k u + \frac{1}{2} |D_x^k u|^2 \ dx \right) - \| u_t \|_s^2 +$$

$$+ \sum_{i,j,k} \int a_{ij} D_x^k u_{x_i} \cdot D_x^k u_{x_j} + (a_{ij} D_x^k u_{x_i x_j} - D_x^k(a_{ij} u_{x_i x_j})) \cdot D_x^k u \ +$$

$$+ (a_{ij})_{x_j} D_x^k u_{x_j} \cdot D_x^k u \ dx + \int b_1 u + b_2 u + \sum_{k=1}^{s} D_x^k b \cdot D_x^k u \ dx$$

$$\geq \frac{d}{dt} \left(\sum_k \int D_x^k u_t \cdot D_x^k u + \frac{1}{2} |D_x^k u|^2 \ dx \right) - \| u_t \|_s^2 +$$

$$+ \nu \| D_x u \|_s^2 - C(E_0) E \| D_{x,t} u \|_s^2 \ .$$

Therefore, taking some positive number λ $(\lambda<1)$, it follows from (3.18) and (3.19) that

$$\left[E_\lambda\{u(\tau)\} + \int B(u(\tau)) \ dx \ \right] \Big|_{\tau=0}^{\tau=t} + \int_0^t (1-\lambda) \| u_t(\tau) \|_s^2 +$$

$$+ \lambda\nu \| D_x u(\tau) \|_s^2 - C \ E \| D_{x,t} u(\tau) \|_s^2 \ d\tau \leq 0.$$

Taking E so small as

$$C_9 E \leq \frac{1}{2} \min(1-\lambda, \nu\lambda),$$

we obtain

$$(3.20) \qquad \left[E_\lambda\{u(\tau)\} + \int B(u(\tau))\, dx \right]\Big|_{\tau=0}^{\tau=t} + \frac{1}{2}\min(1-\lambda,\nu\lambda)\int_0^t \| D_{x,t} u(\tau) \|_s^2 \, d\tau \leq 0 .$$

Thus we arrive at

$$\begin{aligned} \| u(t) \|_{s+1}^2 + \| u_t(t) \|_s^2 &\leq \nu_1^{-1} E_\lambda\{u(t)\} \\ &\leq \nu_1^{-1} \left(E_\lambda\{u(0)\} + \int B(u(0))\, dx \right) \\ &\leq C^2 (\| u_0 \|_{s+1} + \| u_1 \|_s)^2 . \end{aligned}$$

Finally the inequality (3.20) directly implies the decay of the L^2-norm of the solution :

$$\| D_x u(t) \|_{s-1} \to 0 \quad \text{as} \quad t \to +\infty ,$$

and the Nirenberg's inequality ([ ])

$$\| u \|_{\mathcal{B}^0} \leq C \| D_x^{[n/2]+1} u \|^\alpha \| u \|^{1-\alpha} , \quad \alpha = n/2([n/2]+1),$$

gives the decay of the maximum norm of the solution :

$$\| u(t) \|_{\mathcal{B}^0} \to 0 \quad \text{as} \quad t \to +\infty .$$

This completes the proof of Proposition 3.5.

3.4 Global Existence II, Quasilinear Wave Equations

Let us consider the following initial value problem to the quasi-linear wave equations :

$$(3.1) \qquad L(u) \equiv u_{tt} - \sum_{i,j=1}^{n} a_{ij}(u,D_{x,t}u)u_{x_ix_j} + b(u,D_{x,t}u) = 0 ,$$

$$x \in R^n \ (n = 2 \text{ or } 3) , \ t \geq 0 ,$$

with

$$(3.2) \qquad u(x,0) = u_0(x) , \ u_t(x,0) = u_1(x) ,$$

where we suppose Assumption 3.1 and in addition

Assumption 3.3 There exist a positive constant p such that for $|y| \leq \gamma_0$ and $k = 0,1,2,..,$

$$|D_y^k(a_{ij}(y) - a_{ij}(0))| \leq C|y|^{max(p-k,0)} ,$$

$$|D_y^k b(y)| \leq C|y|^{max(p+1-k,0)}.$$

Let B_1^m be the Banach space with the norm $\| \cdot \|_{1,m}$ of the L^1-functions having all the m-th derivatives of L^1-functions. Define Φ_5 by

$$\Phi_5 = \| u_0 \|_5 + \| u_0 \|_{1,4} + \| u_1 \|_4 + \| u_1 \|_{1,3} .$$

Then we have

Theorem 3.7 (Global Existence Theorem) Consider the initial value problem (3.1)-(3.2) for $n = 3$ (resp. $n = 2$) and suppose Assumptions 3.1, 3.3, $u_0 \in H^5 \cap B_1^4$ and $u_1 \in H^4 \cap B_1^3$. Moreover suppose $p > 2$ (resp. $p > 3$).

Then there exist positive constants ε_0 and C_0 such that, if $\Phi_5 \leq \varepsilon_0$, then the initial value problem (3.1)-(3.2) has a unique solution

$$u \in X^4(0,+\infty; C_0\Phi_5),$$

which has the decay rate estimate

$$\| u(t) \|_{\mathfrak{B}^2} + \| u_t(t) \|_{\mathfrak{B}^1} \leq C_0(1+t)^{-1}$$

$$(\text{resp. } \leq C_0(1+t)^{-1/2}).$$

Remark. We can get the analogous results for $n \geq 4$ and $p \geq 2$, but we omit the arguments here.

We show the statements only for $n = 3$. Since the local existence theorem (Theorem 3.3) is established, it suffices to show the following a priori estimate.

Propsition 3.8 Suppose Assumptions 3.1 and 3.3 with $p > 2$, and that there is a solution of (3.1)-(3.2) satisfying

$$u \in X^4(0,h;E_0) \quad \text{for some } h > 0.$$

Then there exist positive constants ε_0' and C_0' not depending on h such that, if $\Phi_5 \leq \varepsilon_0'$, then the solution has the estimate

$$u \in X^4(0,h;C_0'\Phi_5)$$

and

$$\| u(t) \|_{\mathfrak{B}^2} + \| u_t(t) \|_{\mathfrak{B}^1} \leq C_0'(1+t)^{-1}.$$

By the arguments in Chapter I, the local existence theorem (Theorem 3.3) and a priori estimate (Proposition 3,8) complete the proof of Theorem 3.7.

Proof of Proposition 3.8 Proposition 3.8 is proved by a combination of the decay estimates for the solutions of the linearized equations and the L^2-energy estimates. Without loss of generality , by a linear change of coordinates we may consider the following instead of (3.1) :

$$L(u) \equiv u_{tt} - \Delta u - \sum_{i,j=1}^{n} \tilde{a}_{ij}(u, D_{x,t}u)u_{x_i x_j} + b(u, D_{x,t}u) = 0 , \tag{3.21}$$

where $\tilde{a}_{ij}$ satisfies for $|y| \leq \gamma_0$ and $k = 0, 1, 2, \ldots,$

$$|D_y^k \, a_{ij}(y)| \leq C|y|^{max(p-k,0)}.$$

Now we can write (3.21) in the form

$$u(t) = cos\ t\Lambda * u_0 + \frac{sin\ t\Lambda}{\Lambda} * u_1 + \tag{3.22}$$

$$+ \int_0^t \frac{sin(t-s)\Lambda}{\Lambda} * (\sum_{i,j} \tilde{a}_{ij} - b)(s)\, ds ,$$

$$u_t(t) = - \Lambda\ sin\ t\Lambda * u_0 + cos\ t\Lambda * u_1 + \tag{3.23}$$

$$+ \int_0^t cos(t-s)\Lambda * (\sum_{i,j} \tilde{a}_{ij} - b)(s)\, ds ,$$

where

$$\Lambda u \equiv (2\pi)^{-3} \int e^{ix \cdot \xi} |\xi| \hat{u}(\xi)\, d\xi ,$$

$$\cos t\Lambda * f \equiv (2\pi)^{-3}\int e^{ix\cdot\xi}(\cos t|\xi|)\hat{f}(\xi)\, d\xi ,$$

and so on.

Let us prepair the following :

Lemma 3.9 For $f \in L^2 \cap L^1$,

$$\| \cos t\Lambda * f \| \leq C\| f \| ,$$

$$\| \frac{\sin t\Lambda}{\Lambda} * f \| \leq C(\| f \| + \| f \|_{L^1}).$$

For $f \in H^3 \cap B^3_1$,

$$\| \cos t\Lambda * f \|_{\mathcal{B}^1} + \| \frac{\sin t\Lambda}{\Lambda} * f \|_{\mathcal{B}^2}$$

$$\leq C(\| f \|_3 + \| f \|_{1,3}).$$

For $f \in H^4 \cap B^4_1$,

$$\| \Lambda \sin t\Lambda * f \|_{\mathcal{B}^1} + \| \cos t\Lambda * f \|_{\mathcal{B}^2}$$

$$\leq C(\| f \|_4 + \| f \|_{1,4}).$$

Lemma 3.10 For $u \in X^4(0,h;E_0)$ and $1 \leq i,j \leq 3$,

$$\| \tilde{a}_{ij}(u,D_{x,t}u)u_{x_ix_j} \|_3 + \| \tilde{a}_{ij}(u,D_{x,t}u)u_{x_ix_j} \|_{1,3} ,$$

$$\| b(u,D_{x,t}u) \|_3 + \| b(u,D_{x,t}u) \|_{1,3}$$

$$\leq C(\| u \|_{\mathcal{B}^2} + \| u_t \|_{\mathcal{B}^1})^{p-1}(\| u \|_5 + \| u_t \|_4)^2.$$

For the proof of Lemma 3.9, refer to [74] for example, and the proof of Lemma 3.10 follows from Lemma 2.1 in the same way as in the previous sections. Applying Lemma 3.9 and 3.10 to (3.22) and (3.23), we have

$$(3.24)\qquad \| u(t) \|_{\mathcal{B}^2} + \| u_t(t) \|_{\mathcal{B}^1} \le C(1+t)^{-1}\Phi_5 + $$

$$+ C(E_0)\int_0^t (1+t-s)^{-1}(\| u(s) \|_{\mathcal{B}^2} + \| u_t(s) \|_{\mathcal{B}^1})^{p-1}\, ds ,$$

$$(3.25)\qquad \| u(t) \| \le C\Phi_5 + C(E_0)\int_0^t (\| u(s) \|_{\mathcal{B}^2} + \| u_t(s) \|_{\mathcal{B}^1})^{p-1}\, ds.$$

If $p > 2$, the inequality (3.24) gives

$$(3.26)\qquad M(t) \le C \quad \Phi_5 + C \quad (M(t))^{p-1} ,$$

where

$$M(t) \equiv \sup_{0\le s\le t} (1+s)(\| u(s) \|_{\mathcal{B}^2} + \| u_t(s) \|_{\mathcal{B}^1}).$$

Hence, if Φ_5 is suitably small, say $\Phi_5 \le \varepsilon_1$, we have $M(t) \le C\Phi_5$, i.e.,

$$(3.27)\qquad \| u(t) \|_{\mathcal{B}^2} + \| u_t(t) \|_{\mathcal{B}^1} \le C(1+t)^{-1}\Phi_5 .$$

Substituting (3.27) into (3.25), we obtain

$$\| u(t) \| \le C\Phi_5 \pm \int_0^t C(1+s)^{-(p-1)}\Phi_5^{p-1}\, ds$$

$$\le C\varepsilon_1^{p-2}\Phi_5 .$$

Thus we obtain

Proposition 3.11 There exist positive constants ε_1 and C' such that, if $\Phi_5 \leq \varepsilon_1$, then the solution $u \in X^4(0,h;E_0)$ has the estimate

$$
(3.28) \qquad
\begin{aligned}
&\| u(t) \|_{\mathcal{B}^2} + \| u_t(t) \|_{\mathcal{B}^1} \leq C' (1+t)^{-1} \Phi_5 , \\
&\| u(t) \| \leq C' \Phi_5 ,
\end{aligned}
$$

where ε_1 and C' do not depend on h.

Next the L^2-energy estimates are given by

Proposition 3.12 Suppose that the solution $u \in X^4(0,h;E_0)$ has the estimate (3.28). Then there exists a positive constant C'' not depending on h such that

$$
E_5(t) - E_5(0) \leq C'' \Phi_5^{p-1} ,
$$

where $E_5(t)$ is defined by

$$
E_5(t) \equiv \sum_{m=0}^{4} \int \frac{1}{2} \left| D_x^m u_t(t) \right|^2 + \frac{1}{2} \sum_{i,j=1}^{3} (\delta^{ij} + \tilde{a}_{ij})(t) D_x^m u_{x_i}(t) \cdot D_x^m u_{x_j}(t) \, dx .
$$

Noting Assumptions 3.1, we get Proposition 3.8 immediately by Propositions 3.11 and 3.12. To show Proposition 3.12, we need the following lemma that is proved in the same way as Lemmas 2.6 and 3.1.

Lemma 3.13 For the solution $u \in X^4(0,h;E_0)$ and $0 \leq m \leq 4$,

$$
\| (D_x^m (\tilde{a}_{ij} u_{x_i x_j}) - \tilde{a}_{ij} D_x^m u_{x_i x_j}) \cdot D_x^m u_t \|_{L^1} +
$$

$$+ \| (\tilde{a}_{ij})_{x_j} D_x^m u_{x_i} \cdot D_x^m u_{x_j} \|_{L^1} + \| (\tilde{a}_{ij})_t D_x^m u_{x_i} \cdot D_x^m u_{x_j} \|_{L^1}$$

$$\leq C(\| u \|_{\mathcal{B}^2} + \| u_t \|_{\mathcal{B}^1})^{p-1}(\| u \|_5^3 + \| u_t \|_4^3 + \| u_{tt} \|_{\mathcal{B}^0} \| u \|_5^2),$$

$$\| u_{tt} \|_{\mathcal{B}^0} \leq C(E_0)(\| u \|_{\mathcal{B}^2} + \| u_t \|_{\mathcal{B}^1}).$$

Then estimating the iquality

$$\sum_{m=0}^{4} \int_0^t \int \{D_x^m(u_{tt} - \Delta u - \sum_{i,j} \tilde{a}_{ij} u_{x_i x_j} + b)\} \cdot D_x^m u_t \ \ dx\, dt = 0 ,$$

by using Lemma 3.13 in the same way as in the proof of Proposition 3.5 , we consequently have

$$E_5(t) \leq E_5(0) + C(E_0) \int_0^t (\| u(s) \|_{\mathcal{B}^2} + \| u_t(s) \|_{\mathcal{B}^1})^{p-1} \, ds$$

$$\leq E_5(0) + C'' \ \Phi_5^{p-1} ,$$

which implies Proposition 3.12.

This completes the proof of Proposition 3.8.