CHAPTER II

BASIC ESTIMATES

The quasilinear partial differential equations studied in this thesis are essentially first-order hyperbolic equations, second-order hyperbolic equations and second order parabolic systems. In this chapter,we establish the basic energy estimates and existence theorems for the corresponding linear equations of these types.

§ 2.1 Basic Lemmas

First we note two basic lemmas playing an important role throughout this thesis.

Lemma 2.1 (Sobolev's Lemma) Set $N = [n/2] + 1$ and let m be a nonnegative integer.

i) Suppose $f \in H^{N+m}$. Then we have

$$f \in \mathcal{B}^{m+\sigma} \text{ and } \| f \|_{\mathcal{B}^{m+\sigma}} \leq C\| f \|_{N+m}$$

where $\sigma \in [0,N - (n/2))$.

ii) Suppose $f \in H^m$ $(m \leq N)$. Then we have

$$f \in L^p \ (\frac{1}{p} = \frac{1}{2} - \frac{m}{n}) \text{ and } \| f \|_{L^p} \leq C\| f \|_m$$

iii) Suppose $f \in H^{N+m}$. Then we have

$$D_x^\alpha f \in L^p \quad \text{for } m + 1 \leq |\alpha| \leq m + N$$

and $\| D_x^\alpha f \|_{L^p} \leq C \| f \|_{N+m}$

where

$$\begin{cases} \frac{1}{p} \in [\frac{|\alpha|}{n} - \frac{m+1}{n}, \frac{1}{2}] - \{0\} & \text{for } n = \text{even} \\ \frac{1}{p} \in [\frac{|\alpha|}{n} - \frac{2m+1}{2n}, \frac{1}{2}] & \text{for } n = \text{odd.} \end{cases}$$

Lemma 2.2 Suppose $a \in \mathcal{B}^1$ and $f \in L^2$. Set

$$C_\delta^i \equiv \phi_\delta * (a f_{x_i}) - a(\phi_\delta * f_{x_i}) \qquad (1 \leq i \leq n)$$

where $\phi_\delta *$ denotes the Friedrichs' mollifier with respect to x. Then we have

i) $\| C_\delta^i \| \leq C \| a \|_{\mathcal{B}^1} \| f \|$

ii) $\| C_\delta^i \| \to 0$ as $\delta \to 0$.

For a proof of Lemmas 2.1 and 2.2, see [42] for example. Next we mention only the results of the usual theory of linear evolution equation in Banach space (See [22], for example). Now consider

$$(2.1) \quad \begin{cases} \frac{d}{dt} U(t) = A(t)U(t) + F(t), \\ U(0) = U_0, \ 0 \leq t \leq T, \end{cases}$$

in some Banach space X with the norm $\| \cdot \|_X$.

Lemma 2.3 Suppose the following:

i) For every fixed $t \in [0,T]$,

a) $A(t)$ is closed linear operator with the domain $D(A(t))$,

b) $D(A(t))$ is dense in X,

c) $\| (I - A(t))^{-1} V \|_X \le (Re\ \lambda - \beta)^{-1} \| V \|_X$

for some constant β (independent of t), all $V \in X$ and all λ such that $Re\ \lambda > \beta$.

ii) There exists a ~~closed~~ dense linear subspace Y in X such that

d) Y is regarded as a Banach space with the norm $\| \cdot \|_Y$,

e) $\| V \|_X \le C \| V \|_Y$ for all $V \in Y$,

f) $S(t)A(t)S(t)^{-1} = A(t) + B(t)$

for some $S(t) \in C^1(0,T;B(Y,X))$ and some $B(t) \in C^0(0,T;B(X,X))$ where $B(X_1,X_2)$ denotes the usual Banach space of linear bounded operators from X_1 to X_2 with strong topology.

iii) For every fixed $t \in [0,T]$,

g) $Y \subset D(A(t))$ and $A(t) \in C^0(0,T;B(Y,X))$.

Then, the problem (2.1) has a unique solution

$$U(t) \in C^0(0,T;Y) \cap C^1(0,T;X)$$

for $U_0 \in Y$ and $F(t) \in C^0(0,T;Y)$.

§ 2.2 First Order Hyperbolic Equations

We consider the initial value problem to the first order hyperbolic equations;

$$(2.2)\quad \begin{cases} L(u) = u_t + \sum_{i=1}^{n} a_i(x,t)u_{x_i} = f(x,t), \\ u(x,0) = u_0(x),\ x \in R^n,\ 0 \leq t \leq T. \end{cases}$$

Then the following energy estimates (Proposition 2.4) and existence theorem (Proposition 2.5) can be proved.

Proposition 2.4 Let s and l be nonnegative integers satisfying $s \geq [n/2] + 2$ and $0 \leq l \leq s$. Suppose that

$$a_i \in L^0_\infty(0,T;H^s) \qquad (1\leq i\leq n),$$

$$f \in L_2(0,T;H^l) \cap L^0_\infty(0,T;H^{l-1}),$$

$$u_0 \in H^l,$$

and that the problem (2.2) has a solution

$$u \in L^0_\infty(0,T;H^l) \cap L^1_\infty(0,T;H^{l-1}).$$

Then we have the energy estimates

$$(2.3)\qquad \| u(t) \|_l \leq e^{CMt} \{ \| u_0 \|_l + \sqrt{t} \left(\int_0^t \| f(s) \|_l^2\, ds \right)^{\frac{1}{2}} \}$$

where $M = \sup_{0\leq\tau\leq T} \left(\sum_{i=1}^{n} \| a_i(\tau) \|_s \right)$, and moreover have

$$(2.4)\qquad u \in C^0(0,T;H^l).$$

Proposition 2.5 Let s and l be positive integers satisfying $s \geq [n/2] + 2$ and $1 \leq l \leq s$. Suppose that

$$a_i \in C^0(0,T;H^s) \qquad (1 \leq i \leq n),$$

$$f \in L_2(0,T;H^l) \cap C^0(0,T;H^{l-1}),$$

$$u_0 \in H^l.$$

Then the initial value problem (2.2) has a unique solution

$$u \in C^0(0,T;H^l) \cap C^1(0,T;H^{l-1})$$

and the energy estimates (2.3) hold.

In order to prove Propositions 2.4 and 2.5,we prepair

Lemma 2.6 Let s and l be nonnegative integers satisfying $s \geq [n/2] + 2$ and $0 \leq l \leq s$. For $a \in H^s$ and $u \in H^l$,we have

i) $\sum_{k=1}^{l} \| D_x^k(au_{x_i}) - aD_x^k u_{x_i} \| \leq C\| a \|_s \| u \|_l \qquad (1 \leq i \leq n)$,

ii) $\| \phi_\delta * (aD_x u) - a(\phi_\delta * D_x u) \|_l \leq C\| a \|_s \| u \|_l$,

iii) $\| \phi_\delta * (aD_x u) - a(\phi_\delta * D_x u) \|_l \to 0$ as $\delta \to 0$.

Proof of Lemma 2.6

i) We show the estimate only for

$$(2.5) \qquad D_x^l(au_{x_i}) - aD_x^l u_{x_i}$$

because the other terms $(1 \leq k \leq l-1)$ are estimated more easily.
By Leibniz's formula,

$$(2.5) = \sum_{|\alpha|=l} \sum_{\substack{\beta \leq \alpha \\ 1 \leq |\beta|}} \binom{\alpha}{\beta} \left(\frac{\partial}{\partial x}\right)^{\beta} a \left(\frac{\partial}{\partial x}\right)^{\alpha-\beta} u_{x_i}$$

Now suppose $l \geq [n/2] + 1$. Then, since i) of Lemma 2.1 asserts

$$\| (\frac{\partial}{\partial x})^{\beta} a \|_{\mathcal{B}^0} \leq C \| a \|_s \quad \text{for } |\beta| \leq s - [n/2] - 1,$$

$$\| (\frac{\partial}{\partial x})^{\alpha-\beta} u_{x_i} \|_{\mathcal{B}^0} \leq C \| u \|_l \quad \text{for } |\beta| \geq [n/2] + 2,$$

we have easily the desired estimate for the terms $|\beta| \leq s-[n/2]-1$ or $|\beta| \geq [n/2]+2$ in (2.5). So it suffices to estimate (2.5) only for the terms $s-[n/2] \leq |\beta| \leq [n/2]+1$. Now, in addition, suppose n = odd. Then iii) of Lemma 2.1 asserts

$$(2.6) \quad \begin{cases} (\frac{\partial}{\partial x})^{\beta} a \in L^{p_1} \\ \frac{1}{p_1} \in [\frac{|\beta|}{n} - \frac{2s-n}{2n}, \frac{1}{2}], \end{cases}$$

$$(2.7) \quad \begin{cases} (\frac{\partial}{\partial x})^{\alpha-\beta} u_{x_i} \in L^{p_2} \\ \frac{1}{p_2} \in [\frac{l-|\beta|+1}{n} - \frac{2l-n}{2n}, \frac{1}{2}]. \end{cases}$$

In (2.6) and (2.7), since

$$\left(\frac{|\beta|}{n} - \frac{2s-n}{2n} \right) + \left(\frac{l-|\beta|+1}{n} - \frac{2l-n}{2n} \right)$$

$$= \frac{1}{2} - \frac{1}{n} \left(s - (\frac{n}{2} + 1) \right) < \frac{1}{2},$$

we can choose p_1 and p_2 satisfying (2.6) and (2.7) as

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2}.$$

Therefore we have

$$(2.8) \qquad \Big\| \sum_{|\alpha|=l} \sum_{\substack{s-[n/2]\le|\beta|\le[n/2]+1 \\ \beta\le\alpha}} \binom{\alpha}{\beta} \Big(\frac{\partial}{\partial x}\Big)^{\beta} a \, \Big(\frac{\partial}{\partial x}\Big)^{\alpha-\beta} u_{x_i} \Big\|$$

$$\le C \sum \Big(\int \Big| \Big(\frac{\partial}{\partial x}\Big)^{\beta} a \Big|^{p_1} dx \Big)^{\frac{1}{p_1}} \Big(\int \Big| \Big(\frac{\partial}{\partial x}\Big)^{\alpha-\beta} u_{x_i} \Big|^{p_2} dx \Big)^{\frac{1}{p_2}}$$

$$\le C \| a \|_s \| u \|_l .$$

For n = even,we can prove (2.8) similarly. Thus i) of Lemma 2.6 is proved for $l \ge [n/2] + 1$. We can also give a proof for the case $l \le [n/2]$ by Lemma 2.1 in the same way as above.

ii) and iii) It suffices to show ii) and iii) only for

$$(2.9) \qquad D_x^l \{ \phi_\delta * (a u_{x_i}) - a(\phi_\delta * u_{x_i}) \}.$$

By Leibniz's formula,

$$(2.9) = \sum_{|\alpha|=l} \Big\{ \phi_\delta * \Big(a \Big(\frac{\partial}{\partial x}\Big)^{\alpha} u_{x_i}\Big) - a\Big(\phi_\delta * \Big(\frac{\partial}{\partial x}\Big)^{\alpha} u_{x_i}\Big)$$

$$+ \sum_{|\alpha|=l} \sum_{\substack{\beta\le\alpha \\ 1\le|\beta|}} \binom{\alpha}{\beta} \Big\{ \phi_\delta * \Big(\Big(\frac{\partial}{\partial x}\Big)^{\beta} a \Big(\frac{\partial}{\partial x}\Big)^{\alpha-\beta} u_{x_i} \Big) - \Big(\frac{\partial}{\partial x}\Big)^{\beta} a \Big(\frac{\partial}{\partial x}\Big)^{\alpha-\beta} u_{x_i} \Big\}$$

$$+ \sum_{|\alpha|=l} \sum_{\substack{\beta\le\alpha \\ 1\le|\beta|}} \binom{\alpha}{\beta} \Big\{ \Big(\frac{\partial}{\partial x}\Big)^{\beta} a \Big(\frac{\partial}{\partial x}\Big)^{\alpha-\beta} (\phi_\delta * u - u)_{x_i} \Big\}$$

$$= I_1 + I_2 + I_3 .$$

Since

$$a \in H^s \subset \mathcal{B}^1 ,$$

Lemma 2.2 implies

$$\| I_1 \| \leq C\| a \|_{\mathcal{B}^1}\| u \|_l,$$

$$\| I_1 \| \to 0 \quad \text{as } \delta \to 0.$$

By fundumental properties for the Friedrichs' mollifier and the same arguments as in i),we have

$$\| I_2 \| \leq C\| u \|_l\| a \|_s,$$

$$\| I_2 \| \to 0 \quad \text{as } \delta \to 0,$$

$$\| I_3 \| \leq C\| u - \phi_\delta * u \|_l\| a \|_s,$$

$$\leq C\| u \|_l\| a \|_s,$$

$$\| I_3 \| \to 0 \quad \text{as } \delta \to 0.$$

This completes the proof of Lemma 2.6.

Proof of Proposition 2.4 First,suppose that

$$\text{(2.10)} \qquad \begin{aligned} &f \in L^0_\infty(0,T;H^l), \\ &u \in L^0_\infty(0,T;H^\infty) \cap L^1_\infty(0,T;H^\infty). \end{aligned}$$

Then we estimate the equality

$$\text{(2.11)} \qquad \sum_{k=0}^{l}\int D^k_x(L(u))\cdot D^k_x u \; dx = \sum_{k=0}^{l}\int D^k_x f\cdot D^k_x u \; dx$$

by virtue of Lemma 2.6 as follows;

$$\sum_{k=0}^{l}\int D^k_x(L(u))\cdot D^k_x u \; dx = \sum_{k=0}^{l}\int L(D^k_x u)\cdot D^k_x u + (D^k_x L(u) - L(D^k_x u))\cdot D^k_x u \; dx$$

$$= \sum_{k=0}^{l} \{ \int D^k_x u_t \cdot D^k_x u + \sum_{i=1}^{n} a_i D^k_x u_{x_i} \cdot D^k_x u \; dx +$$

$$+ \int \sum_{i=1}^{n} (D_x^k(a_i u_{x_i}) - a_i D_x^k u_{x_i}) \cdot D_x^k u \, dx \}$$

$$\geq \sum_k \frac{d}{dt} \int \frac{1}{2} |D_x^k u|^2 \, dx - \sum_{k,i} \int a_{i,x_i} |D_x^k u|^2 \, dx - C \| a \|_s \| u \|_l$$

$$\geq \frac{1}{2} \frac{dt}{dt} \| u \|_l^2 - C \| a \|_s \| u \|_l ,$$

and

$$\sum_{k=0}^{l} \int D_x^k f \cdot D_x^k u \, dx \leq \| f \|_l \| u \|_l .$$

Therefore (2.11) implies

$$\frac{1}{2} \frac{d}{dt} \| u \|_l^2 - C \| a \|_s \| u \|_l \leq \| f \|_l \| u \|_l ,$$

that is

$$\frac{d}{dt} \| u \|_l - CM \| u \|_l \leq \| f \|_l ,$$

which implies

(2.12) $$\| u(t) \|_l \leq e^{CMt} (\| u_0 \|_l + \int_0^t e^{-CM\tau} \| f(\tau) \|_l \, d\tau)$$

$$\leq e^{CMt} (\| u_0 \|_l + \sqrt{t} (\int_0^t \| f(\tau) \|_l^2 \, d\tau)^{\frac{1}{2}}).$$

Let us show that (2.12) holds even for $f \in L_2(0,T;H^l) \cap L_\infty^0(0,T;H^{l-1})$ and $u \in L_\infty^0(0,T;H^l) \cap L_\infty^1(0,T;H^{l-1})$. Denote $\phi_\delta * f$ and $\phi_\delta * u$ by f_δ and u_δ respectively. Then we note

$$f_\delta \in L_\infty^0(0,T;H^\infty),$$

$$u_\delta \in L_\infty^0(0,T;H^\infty) \cap L_\infty^1(0,T;H^\infty)$$

$$\subset C^0(0,T;H^\infty).$$

Applying $\phi_\delta *$ to $L(u) = f$,

$$L(u_\delta) = f_\delta + C_\delta$$

where

$$C_\delta = L(u_\delta) - \phi_\delta * L(u)$$
$$= \sum_{i=1}^{n} \{ a_i(\phi_\delta * u_{x_i}) - \phi_\delta * (a_i u_{x_i}) \}.$$

By (2.12), we have

$$\text{(2.13)} \qquad \| u_\delta(t) \|_l \le e^{CMt}\{ \| u_{0\delta} \|_l + \sqrt{t} \left(\int_0^t (\| f_\delta(\tau) \|_l + \| C_\delta(\tau) \|_l)^2 d\tau \right)^{\frac{1}{2}} \}.$$

Since

$$\| u_\delta(t) \|_l \to \| u(t) \|_l, \quad \sup_{0 \le \tau \le t} \| C_\delta(\tau) \|_l \to 0,$$

$$\int_0^t \| f_\delta(\tau) \|_l^2 \, d\tau \to \int_0^t \| f(\tau) \|_l^2 \, d\tau \text{ as } \delta \to 0.$$

for every fixed t, we have (2.3) by taking $\delta \to 0$ in (2.13). Finally let us show $u \in C^0(0,T;H^l)$. Similarly as above,

$$L(u_\delta - u_{\delta'}) = f_\delta - f_{\delta'} + C_\delta - C_{\delta'},$$

which gives

$$\| (u_\delta - u_{\delta'})(t) \|_l \le C(T)\{ \| u_{0\delta} - u_{0\delta'} \|_l +$$
$$+ \left(\int_0^t \| f_\delta - f_{\delta'} \|_l^2 + \| C_\delta \|_l^2 + \| C_{\delta'} \|_l^2 \, d\tau \right)^{\frac{1}{2}} \},$$

which implies

$$\sup_{0 \le t \le T} \| (u_\delta - u_{\delta'})(t) \|_l \to 0 \text{ as } \delta, \delta' \to 0.$$

Therefore by noting that

$$u \in C^0(0,T;H^l),$$

we reach

$$u \in C^0(0,T;H^l).$$

This completes the proof of Proposition 2.4.

Proof of Proposition 2.5 First let us show the statements for $l = 1$. For the moment, suppose $f \in C^0(0,T;H^1)$. Then since Lemma 2.1 asserts

$$a_i \in C^0(0,T;\mathcal{B}^{1+\sigma}),$$

we immediately have a unique solution

$$u \in C^0(0,T;H^1) \cap C^1(0,T;L^2)$$

from the arguments of Chapter 6 in [42]. In fact, Lemma 2.3 can be applied to (2.2) by setting

$$\begin{cases} X = L^2, \ Y = H^1, \\ U = u, \ F = f, \\ AU = -\sum_{i=1}^{n} a_i u_{x_i}, \ D(A) = \{U \in L^2 \mid AU \in L^2\}, \\ S = 1 + \Lambda, \ B = \sum_{i=1}^{n} (a_i \Lambda - \Lambda a_i)\Lambda^{-1}\partial_{x_i}, \end{cases}$$

where

$$\Lambda U \equiv (2\pi)^{-\frac{n}{2}} \int e^{ix\cdot\xi} |\xi| \hat{U}(\xi)\, d\xi.$$

For $f \in L_2(o,T;H^1) \cap C^0(o,T;L^2)$, we may consider the following initial value problem;

$$(2.14)\quad \begin{cases} u_t^\delta + \sum_{i=1}^{n} a_i u_{x_i}^\delta = f_\delta \\ u^\delta(0) = u_0 \in H^1, \end{cases}$$

where $f_\delta = \phi_\delta * f$. Since

$$f_\delta \in C^0(0,T;H^1),$$

we have from the above arguments that

$$u^\delta \in C^0(0,T;H^1) \cap C^1(0,T;L^2),$$

and

$$\| (u^\delta - u^{\delta'})(t) \|_1 \leq \sqrt{t} e^{CMt} \left(\int_0^t \| (f_\delta - f_{\delta'})(\tau) \|_1^2 \, d\tau \right)^{\frac{1}{2}}$$

which implies

$$\sup_{0 \leq t \leq T} \| (u^\delta - u^{\delta'})(t) \|_1 \to 0 \quad \text{as} \quad \delta, \delta' \to 0.$$

Thus we have a solution of (2.2)

$$u \in C^0(0,T;H^1) \cap C^1(0,T;L^2)$$

even for $f \in L_2(0,T;H^1) \cap C^0(0,T;L^2)$. The uniqueness follows from the energy inequality such as (2.3) immediately.

Next, suppose that the statements of Proposition 2.5 hold for $l = k$ $(1 \leq k \leq s-1)$. Then, if we show the statements for $l = k + 1$, we complete the proof of Proposition 2.5. Differentiating (2.2) with respect to x,

$$(2.15)\quad \begin{cases} L(u_{x_j}) = f_{x_j} + \sum_{i=1}^{n} (a_i)_{x_j} u_{x_i}, \\ u_{x_j}(0) = u_{0 x_j} \quad (1 \leq j \leq n). \end{cases}$$

Since we can obtain

$$\| \sum_{i=1}^{n} (a_i)_{x_j} u_{x_i} \|_k \leq C \| a \|_s \| Du \|_k,$$

in the same way as i) of Lemma 2.6, we note that if $\{u_{x_i}\}_{i=1}^{n} \in C^0(0,T;H^k)$,

$$f_{x_j} + \sum_{i=1}^{n} (a_i)_{x_j} u_{x_i} \in L_2(0,T;H^k) \cap C^0(0,T;H^{k-1}).$$

Therefore, by constructing the approximate sequence $(u_{x_i})^{(m)}$ $(m \geq 0)$ as

$$(u_{x_j})^{(0)} \equiv u_{0x_j} \qquad (m = 0),$$

$$(2.16) \qquad \begin{cases} L((u_{x_j})^{(m)}) = f_{x_j} + \sum_{i=1}^{n} (a_i)_{x_j} (u_{x_i})^{(m-1)} \\ (u_{x_j})^{(m)}(0) = u_{0x_j} \qquad (m \geq 1), \end{cases}$$

and by applying Proposition 2.5 with $l = k$ to (2.16), we have for all $m \geq 2$ that

$$(u_{x_j})^{(m)} \in C^0(0,T;H^k) \cap C^1(0,T;H^{k-1}),$$

$$\sum_{j=1}^{n} \| (u_{x_j})^{(m)}(t) - (u_{x_j})^{(m-1)}(t) \|_k^2$$

$$\leq CTe^{CMT} \sum_{j=1}^{n} \int_0^t \| \sum_{i=1}^{n} (a_i)_{x_j} ((u_{x_j})^{(m-1)} - (u_{x_j})^{(m-2)})(\tau) \|_k^2 \, d\tau$$

$$\leq C(T) \int_0^t \sum_{j=1}^{n} \| (u_{x_j})^{(m-1)}(\tau) - (u_{x_j})^{(m-2)}(\tau) \|_k^2 \, d\tau.$$

Thus we have

$$u_{x_j} \in C^0(0,T;H^k) \cap C^1(0,T;H^{k-1}) \qquad (1 \leq j \leq n),$$

which consequently imply

$$u \in C^0(0,T;H^{k+1}) \cap C^1(0,T;H^k).$$

This completes the proof of Proposition 2.5.

§ 2.3 Second Order Hyperbolic Equations

We consider the initial value problem to the second order hyperbolic equations;

$$
(2.17)\qquad \begin{cases} L(u) = u_{tt} - \sum_{i,j=1}^{n} a_{ij}(x,t) u_{x_i x_j} = f(x,t), \\ \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in R^n,\ 0 \le t \le T, \end{cases}
$$

where the functions a_{ij} satisfy

$$
(2.18)\qquad \begin{aligned} & a_{ij} \in L_\infty^0(0,T;\mathcal{B}^1) && (1 \le i,j \le n), \\ & a_{ij} = a_{ji} && (1 \le i,j \le n), \\ & \nu|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \le \nu^{-1}|\xi|^2, \end{aligned}
$$

for some positive constant ν (≤ 1) and all $\xi \in R^n, x \in R^n$ and $0 \le t \le T$. Then the following energy estimates (Proposition 2.7) and existence theorem (Proposition 2.8) can be proved.

Proposition 2.7 Let s and l be nonnegative integers satisfying $s \ge [n/2] + 2$ and $0 \le l \le s$. Suppose that

$$
\begin{aligned} & D_{x,t} a_{ij} \in L_\infty^0(0,T;H^{s-1}) \qquad (1 \le i,j \le n), \\ & f \in L_\infty^0(0,T;H^l), \\ & u_0 \in H^{l+1},\ u_1 \in H^l, \end{aligned}
$$

and moreover suppose that the problem (2.17) has a solution

$$
u \in L_\infty^i(0,T;H^{l+1-i}) \qquad (0 \le i \le 2).
$$

Then we have the energy estimate

$$(2.19) \qquad \| u(t) \|_{l+1}^2 + \| u_t(t) \|_l^2$$

$$\leq \nu^{-2} e^{C(1+M)t} (\| u_0 \|_{l+1}^2 + \| u_1 \|_l^2 + \int_0^t \| f(\tau) \|_l^2 \, d\tau),$$

where $M = \sup_{0 \leq t \leq T} (\sum_{i,j=1}^{n} \| D_{x,t} a_{ij}(t) \|_{s-1})$, and moreover have

$$(2.20) \qquad u \in C^0(0,T;H^{l+1}) \cap C^1(0,T;H^l).$$

Proposition 2,8 Let s and l be positive integers satisfying $s \geq [n/2] + 2$ and $1 \leq l \leq s$. Suppose that

$$a_{ij} \in C^0(0,T;\mathcal{B}^0) \quad (1 \leq i,j \leq n),$$

$$D_{x,t} a_{ij} \in C^0(0,T;H^{s-1}) \quad (1 \leq i,j \leq n),$$

$$f \in C^0(0,T;H^l),$$

$$u_0 \in H^{l+1}, \; u_1 \in H^l.$$

Then the initial value problem (2.17) has a unique solution

$$u \in C^i(0,T;H^{l+1-i}) \quad (0 \leq i \leq 2),$$

and the energy estimate (2.19) holds.

In the same way as Lemma 2.6, we can obtain

Lemma 2.9 Let s and l be nonnegative integers satisfying $s \geq [n/2] + 2$ and $0 \leq l \leq s$. For every fixed t and $u(t) \in H^{l+1}$, $u\ (t) \in H^l$, $a(t) \in \mathcal{B}^0$ and $D_{x,t} a(t) \in H^{s-1}$, we have

i) $\sum_{k=1}^{l} \| D_x^k(au_{x_ix_j}) - aD_x^k u_{x_ix_j} \|$

$\leq C\| D_x a \|_{s-1} \| D_x u \|_l \qquad (1\leq i,j\leq n),$

ii) $\sum_{k=0}^{l} \| a_{x_j} D_x^k u_{x_i} \cdot D_x^k u_t \|_{L^1} + \sum_{k=0}^{l} \| a_t D_x^k u_{x_i} \cdot D_x^k u_{x_j} \|_{L^1}$

$\leq C\| D_{x,t} a \|_{s-1} \| D_{x,t} u \|_l^2 \qquad (1\leq i,j\leq n),$

iii) $\| \phi_\delta * (au_{x_ix_j}) - a(\phi_\delta * u_{x_ix_j}) \|_l$

$\leq C(\| a \|_{\mathcal{B}^1} + \| D_x a \|_{s-1}) \| u \|_{l+1} \qquad (1\leq i,j\leq n),$

iv) $\|\phi_\delta * (au_{x_ix_j}) - a(\phi_\delta * u_{x_ix_j}) \|_l \to 0$ as $\delta \to 0 \qquad (1\leq i,j\leq n).$

Proof of Proposition 2.7 We show (2.19) only for $u \in L^2_\infty(0,T; H^\infty)$ because ,for $u \in L^i_\infty(0,T;H^{l+1-i})$ $(0\leq i\leq 2)$,we may use the similar arguments on the mollifier $\phi_\delta *$ as in the proof of Proposition 2.4. In this case,using Lemma 2.9,we estimate the equality

$$\sum_{k=0}^{l} \int D_x^k(L(u)) \cdot D_x^k u_t \, dx = \sum_{k=0}^{l} \int D_x^k f \cdot D_x^k u_t \, dx$$

as follows;

$$\sum_k \int D_x^k f \cdot D_x^k u_t \, dx \leq \| f \|_l \| u_t \|_l$$
$$\leq \frac{1}{2}\| f \|_l^2 + \frac{1}{2}\| u_t \|_l^2 ,$$

and

$$\sum_k \int D_x^k(L(u)) \cdot D_x^k u_t \, dx =$$

$$= \sum_k \int D_x^k u_{tt} \cdot D_x^k u_t - \sum_{i,j} D_x^k (a_{ij} u_{x_i x_j}) \cdot D_x^k u_t \, dx$$

$$= \sum_k \frac{1}{2} \frac{d}{dt} \int |D_x^k u_t|^2 + \sum_{i,j} a_{ij} D_x^k u_{x_i} \cdot D_x^k u_{x_j} \, dx$$

$$+ \sum_{i,j,k} \int (a_{ij} D_x^k u_{x_i x_j} - D_x^k (a_{ij} u_{x_i x_j})) \cdot D_x^k u_t \, dx$$

$$+ \sum_{i,j,k} \int (a_{ij})_{x_j} D_x^k u_{x_i} \cdot D_x^k u_t - \frac{1}{2} (a_{ij})_t D_x^k u_{x_j} \cdot D_x^k u_{x_i} \, dx$$

$$\geq \frac{1}{2} \frac{d}{dt} E_l(t) - CM \| D_{x,t} u \|_l^2$$

where $E_l(t)$ is defined by

$$E_l(t) = \sum_{k=0}^{l} \int |D_x^k u_t(t)|^2 + \sum_{i,j=1}^{n} a_{i,j}(t) D_x^k u_{x_i}(t) \cdot D_x^k u_{x_j}(t) \, dx.$$

Therefore we have

$$\frac{d}{dt} E_l(t) - C(1+M) \| D_{x,t} u(t) \|_l^2 \leq \| f(t) \|_l^2.$$

Noting that (2.18) gives

(2.21) $$\nu \| D_{x,t} u(t) \|_l^2 \leq E_l(t) \leq \nu^{-1} \| D_{x,t} u(t) \|_l^2,$$

we have

$$\frac{d}{dt} (E_l(t) + \| u(t) \|^2) - C(1+M) (E_l(t) + \| u(t) \|^2)$$

$$\leq \| f(t) \|_l^2$$

which implies

(2.22) $$E_l(t) + \| u(t) \|^2 \leq e^{C(1+M)t} (E_l(0) + \| u_0 \|^2 + \int_0^t e^{-C(1+M)\tau} \| f(\tau) \|_l^2 \, d\tau).$$

Thus the desired estimate (2.19) follows from (2.21) and (2.22) immediately. Finally we can show the regularity (2.20) in the same way as in the proof of (2.4).

This completes the proof of Proposition 2.7.

Proof of Proposition 2.8 Set

$$U = \begin{pmatrix} u \\ u_t \end{pmatrix}, \quad U_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ f \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & 1 \\ \sum_{i,j} a_{ij} \partial^2_{x_i x_j} & 0 \end{pmatrix} .$$

Then we can rewrite (2.17) in the form

$$\text{(2.23)} \qquad \begin{cases} \dfrac{d}{dt} U(t) = A(t)U(t) + F(t), \\ U(0) = U_0 \ , \ 0 \leq t \leq T. \end{cases}$$

Although we omit the details (cf [ ]),Lemma 2.3 is applied to (2.23) by setting

$$X = \begin{pmatrix} H^1 \\ L^2 \end{pmatrix} \ , \ Y = D(A) = \begin{pmatrix} H^2 \\ H^1 \end{pmatrix} \ ,$$

$$S = \lambda I - A \ , \ B = 0$$

where λ is some positive constant determined later,and consequently we obtain a unique solution

$$U \in C^0(0,T;Y) \cap C^1(0,T;X),$$

that is

$$u \in C^i(0,T;H^{2-i}) \quad (0 \leq i \leq 2).$$

Thus we can prove Proposition 2.8 for $l = 1$. For $l > 1$, we can give a proof in the same way as in the proof of Proposition 2.5.

This completes the proof of Proposition 2.8.

2.4 Second Order Parabolic System

We consider the initial value problem to the following prabolic system for $u = \{u^i\}_{i=1}^p$;

$$(2.24)\qquad \begin{cases} L^i(u) = u_t^i - \sum\limits_{\substack{1\le k\le p\\ 1\le l,m\le n}} a_{l,m}^{k,i}(x,t)u_{x_l x_m}^k = f^i(x,t), \ (1\le i\le p) \\ u^i(x,0) = u_0^i(x), 1\le i\le p, \ x \in R^n, 0 \le t \le T, \end{cases}$$

where the functions $a_{l,m}^{k,i}$ satisfy for $1\le i,k\le p, 1\le l,m\le n$

$$(2.25)\qquad \begin{aligned} &a_{j,m}^{k,i} \in L_\infty^0(0,T;\mathcal{B}^1), \\ &a_{l,m}^{k,i} = a_{m,l}^{k,i}, \\ &a_{l,m}^{k,i} = a_{l,m}^{i,k}, \end{aligned}$$

and the system (2.24) is strongly parabolic system in the following sence;

$$(2.26)\qquad \inf_{\substack{x\in R^n\\ 0\le t\le T}} \sum_{i,k,l,m} a_{l,m}^{k,i}(x,t)\xi_l\xi_m V^k V^i \ge \nu|\xi|^2|V|^2$$

for some positive constant ν,all $\xi \in R^n$ and $V \in R^p$. By virtue of (2.25) and (2.26),we have the Gårding inequality (cf.[42][54])

$$(2.27)\qquad \begin{aligned} &- \int \Sigma\, a_{l,m}^{k,i} u_{x_l x_m}^k u^i \, dx \\ &\ge \nu' \| u \|_1^2 - C(\Sigma \| a_{l,m}^{k,i} \|_{\mathcal{B}^1} + 1) \| u \|^2, \end{aligned}$$

$$(2.28)\qquad \begin{aligned} &\int \Sigma\, a_{l,m}^{k,i} u_{x_l x_m}^k u_{x_j x_j}^i \, dx \\ &\ge \nu' \| u \|_2^2 - C(\Sigma \| a_{l,m}^{k,i} \|_{\mathcal{B}^1} + 1) \| u \|_1^2 \end{aligned}$$

for some positive constant ν' and all $u \in H^2$. Then the following energy estimates (Proposition 2.10) and existence theorem (Proposition 2.11) can be proved.

Proposition 2.10 Let s and l be nonnegative integers satisfying $s \geq [n/2] + 2$ and $0 \leq l \leq s$. Suppose that

$$D_x a^{k,i}_{l,m} \in L^0_\infty(0,T;H^{s-1}) \qquad (1 \leq i,k \leq p, 1 \leq l,m \leq n),$$

$$f \in L^0_\infty(0,T;H^l),$$

$$u_0 \in H^{l+1},$$

and moreover suppose that the problem (2.24) has a solution

$$u \in L^0_\infty(0,T;H^{l+1}) \cap L^1_\infty(0,T;H^{l-1}).$$

Then we have

$$u \in C^0(0,T;H^{l+1}) \cap L_2(0,T;H^{l+2}), \tag{2.29}$$

and the energy estimate

$$\| u(t) \|^2_{l+1} + \nu' \int_0^t \| u(\tau) \|^2_{l+2} \, d\tau \tag{2.30}$$

$$\leq e^{C(1+M)t}(\| u_0 \|^2_{l+1} + \int_0^t C\| f(\tau) \|^2_l \, d\tau),$$

where $M = \sup\limits_{0 \leq t \leq T} (\Sigma\| D_x a^{k,i}_{l,m}(t) \|_{s-1} + \Sigma\| a^{k,i}_{l,m}(t) \|_{\mathcal{B}^0})$.

Proposition 2.11 Let s and l be positive integers satisfying $s \geq [n/2] + 2$ and $1 \leq l \leq s$. Suppose that

$$a^{k,i}_{l,m} \in C^0(0,T; \mathcal{B}^1) \qquad (1 \leq i,k \leq p, 1 \leq l,m \leq n),$$

$$D_x a_{l,m}^{k,i} \in C^0(0,T;H^{s-1}) \qquad (1 \le k,i \le p, 1 \le l,m \le n),$$

$$f \in C^0(0,T;H^l),$$

$$u_0 \in H^{l+1}.$$

Then the initial value problem (2.24) has a unique solution

$$u \in C^0(0,T;H^{l+1}) \cap C^1(0,T;H^{l-1}) \cap L_2(0,T;H^{l+2})$$

and the energy estimates (2.30) hold.

Proof of Proposition 2.10 It suffices to show (2.30) only for $u \in L^1_\infty(0,T;H^\infty)$. First, by using (2.27), we estimate the equality

$$\int L^i(u)u^i \, dx = \int f^i u^i \, dx$$

as follows;

$$\int f^i u^i \, dx \le \frac{1}{2}(\| f \|^2 + \| u \|^2) \quad ,$$

and

$$\int L^i(u)u^i \, dx = \int u_t^i u^i - \Sigma a_{l,m}^{k,i} u^k_{x_l x_m} u^i \, dx$$

$$\ge \frac{1}{2}\frac{d}{dt}\int |u|^2 \, dx + \nu' \| u \|_1^2 - C(1+M)\| u \|^2.$$

Therefore we have

$$(2.31) \qquad \frac{d}{dt}\| u \|^2 + 2\nu' \| u \|_1^2 - C(1+M)\| u \|^2 \le \| f \|^2.$$

Next, using i) in Lemma 2.9 and (2.28), we estimate the equality

$$\sum_{h=0}^{l} \int D_x^h(L^i(u)) \cdot D_x^h(-\Delta u^i) \, dx = \sum_{h=0}^{l} \int D_x^h f^i \cdot D_x^h(-\Delta u^i) \, dx$$

as follows;

$$\sum_{i,h} \int D_x^h f^i \cdot D_x^h(-\Delta u^i)\ dx \leq \| f \|_l \| u \|_{l+2}$$

$$\leq \varepsilon \| u \|_{l+2}^2 + \varepsilon^{-1} \| f \|_l^2 \quad \text{for any } \varepsilon > 0 ,$$

and

$$\sum_{i,h} \int D_x^h(L^i(u)) \cdot D_x^h(-\Delta u^i)\ dx$$

$$= \sum \int - D_x^h u_t^i \cdot D_x^h(\Delta u^i) + \sum a_{l,m}^{k,i} D_x^h u_{x_l x_m}^k \cdot D_x^h(\Delta u^i) +$$

$$+ \sum \{ D_x^h(a_{l,m}^{k,i} u_{x_l x_m}^k) - a_{l,m}^{k,i} D_x^h u_{x_l x_m}^k \} \cdot D_x^h(\Delta u^i)\ dx$$

$$\geq \frac{1}{2} \frac{d}{dt} \int \sum_{h=1}^{l+1} |D_x^h u|^2\ dx + \nu' \| u \|_{l+2}^2 - C(1+M) \| u \|_{l+1}^2 .$$

Therefore we have

$$\frac{d}{dt} (\| u \|_{l+1}^2 - \| u \|^2) + (2\nu' - \varepsilon) \| u \|_{l+2}^2 - C(1+M) \| u \|_{l+1}^2 \tag{2.32}$$

$$\leq 2\varepsilon^{-1} \| f \|_l^2 .$$

Taking ε small as $\varepsilon \leq \nu'$ in (2.32) and combining (2.31) with (2.32), we reach

$$\frac{d}{dt} \| u \|_{l+1}^2 + \nu' \| u \|_{l+2}^2 - C(1+M) \| u \|_{l+1}^2 \leq C \| f \|_l^2$$

from which the desired energy estimate (2.30) follows immediately. This completes the proof of Proposition 2.10.

Proof of Proposition 2.11 We show the statements only for $l = 1$ because for $l > 1$ we give a proof in the same way as in the proof of Proposition 2.5.

For the moment, we more suppose $f^i \in C^0(0,T;H^\infty)$ and $a^{k,i}_{l,m} \in C^\infty(0,T;\mathcal{B}^\infty)$.
Then, setting

$$X = L^2,\ U = u = \{u^i\}, U_0 = \{u_0^i\},$$

$$A = \{A^{i,k}\} = \{\sum_{l,m} a^{k,i}_{l,m} \partial^2_{x_l x_m}\},$$

$$F = \{f^i\},\ D(A) = Y = H^2,$$

$$S = \lambda I - A,\ B = 0,$$

where λ is some positive constant determined later, we can write (2.24) as

$$(2.33)\qquad \begin{cases} U_t = AU + F, \\ U(0) = U_0,\ 0 \le t \le T, \end{cases}$$

and consequently we can apply Lemma 2.3 to (2.33). Thus we can obtain a solution

$$(2.34)\qquad u \in C^0(0,T;H^2) \cap C^1(0,T;L^2).$$

By Proposition 2.10, (2.34) implies that $u \in L_2(0,T;H^3)$ and the energy estimate (2.30) with $l = 1$ holds. Next, we are going to obtain a solution for $f^i \in C^0(0,T;H^1), a^{k,i}_{l,m} \in C^0(o,T;\mathcal{B}^1)$ and $D_x a^{k,i}_{l,m} \in C^0(0,T;H^{s-1})$.
Now extend $a^{k,i}_{l,m}$ as

$$\begin{cases} a^{k,i}_{l,m}(x,t) \equiv a^{k,i}_{l,m}(x,T) & t > T, \\ \qquad\qquad\quad \equiv a^{k,i}_{l,m}(x,0) & t < 0. \end{cases}$$

Then define f^i_δ and $a_{\delta,}{}^{k,i}_{l,m}$ by

$$f^i_\delta = \phi_\delta * f^i,\ a_{\delta,}{}^{k,i}_{l,m} = \phi^t_\delta * \phi_\delta * a^{k,i}_{l,m},$$

where $\phi^t_\delta *$ is the Friedrichs' mollofier with respect to t. We note that

$$f_\delta^i \in C^0(0,T;H^\infty), a_{\delta,l,m}^{k,i} \in C^\infty(0,T;\mathcal{B}^\infty),$$

$$\sup_{0\le t\le T} \Sigma \| f_\delta^i(t) - f^i(t) \|_1 \to 0 \text{ as } \delta \to 0,$$

$$\sup_{0\le t\le T} \Sigma(\| a_{\delta,l,m}^{k,i}(t) - a_{l,m}^{k,i}(t) \|_{\mathcal{B}^1} +$$

$$+ \| D_x(a_{\delta,l,m}^{k,i}(t) - a_{l,m}^{k,i}(t)) \|_{s-1}) \to 0 \text{ as } \delta \to 0,$$

and (2.26) holds also for $a_{\delta,l,m}^{k,i}$. By $u^\delta = \{u^{\delta,i}\}$, denote the solution to the initial value problem ;

$$(2.34) \qquad \begin{cases} u_t^{\delta,i} - \Sigma\, a_{\delta,l,m}^{k,i}\, u_{x_l x_m}^{\delta,i} = f_\delta^i, \\ u^\delta(0) = u_0, x \in R^n, 0\le t\le T. \end{cases}$$

From the previous arguments, we have a solution of (2.34) such that

$$u^\delta \in C^0(0,T;H^2) \cap C^1(0,T;L^2) \cap L_2(0,T;H^3),$$

$$(2.35) \qquad \sup_{0\le t\le T} \| u^\delta(t) \|_2^2 + \nu \int_0^T \| u^\delta(\tau) \|_3^2\, d\tau$$

$$\le e^{C(1+M)T}(\| u_0 \|_2^2 + \int_0^T C\| f(\tau) \|_1^2\, d\tau\ ).$$

Furthermore, for any δ and $\delta' > 0$

$$\begin{cases} (u^\delta - u^{\delta'})_t^i - \Sigma a_{\delta,l,m}^{k,i}\, (u^\delta - u^{\delta'})_{x_l x_m}^k = \\ = f_\delta^i - f_{\delta'}^i + \Sigma(a_{\delta,l,m}^{k,i} - a_{\delta',l,m}^{k,i}) u_{x_l x_m}^{\delta',k}, \\ (u^\delta - u^{\delta'})(0) = 0, \end{cases}$$

which implies, noting (2.35),

$$\sup_{0\le t\le T} \| (u^{\delta} - u^{\delta'})(t) \|_2^2 + \nu \int_0^T \| (u^{\delta} - u^{\delta'})(\tau) \|_3^2 \, d\tau$$

$$\le e^{C(1+M)T} \int_0^T \| (f_\delta - f_{\delta'})(\tau) \|_1^2 + \Sigma \| (a_{\delta, l,m}^{k,i} - a_{\delta', l,m}^{k,i}) u_{x_l x_m}^{\delta',k}(\tau) \|_1^2 \, d\tau$$

$$\le e^{C(1+M)T} \int_0^T \| (f_\delta - f_{\delta'})(\tau) \|_1^2 + \| (a_\delta - a_{\delta'})(\tau) \|_{\mathcal{B}^1}^2 \| u^{\delta'}(\tau) \|_3^2 \, d\tau$$

$$\le C(T) \sup_{0\le t\le T} (\| f_\delta(t) - f_{\delta'}(t) \|_1^2 + \| a_\delta(t) - a_{\delta'}(t) \|^2_1)$$

$$\to 0 \text{ as } \delta, \delta' \to 0.$$

Thus we have a solution u as a limit of u^{δ} such that

$$u \in C^0(0,T;H^2) \cap C^1(0,T;L^2) \cap L_2(0,T;H^3).$$

The uniqueness follows from the energy inequalities immediately. This completes the proof of Proposition 2.11.