INITIAL VALUE PROBLEMS FOR SOME QUASILINEAR PARTIAL DIFFERENTIAL EQUATIONS IN MATHEMATICAL PHYSICS

by

Akitaka MATSUMURA

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ABSTRUCT

It is commonly known that *a priori* estimates play an important role in the study of global existence in time for the initial value problems to the quasilinear partial differential equations. In this thesis, two methods of obtaining *a priori* estimates for suitably "small" solutions are proposed ; one is composed of only energy estimates in the space of square summable functions, and the other involves a combination of the estimates for the decay rate of the linearized equations and the energy estimates. These methods are concretely applied to the pure initial value problems to quasilinear equations of wave propagation and ones of compressible fluid. In each problem, it is proved that a unique solution exists globally in time for suitably small initial data. Also, the asymptotic behavior of the solution as $t \to +\infty$ is investigated.

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INTRODUCTION

There are many interesting but not fully investigated quasilinear partial differential equations in mathematical physics. Among these equations, we will focus our attention on the quasilinear equations of wave propagation and ones of compressible viscous fluid. One of the fundamental mathematical problems associated with these equations is the pure initial value problem. Especially our most interesting theme is existence of "global solutions in time". Since the equations are highly nonlinear with respect to the unknowns, we abandon to seek global solutions in general, and restrict ourselves to seek global solutions which are suitably close to constant equilibrium states. Therefore we may say that we shall investigate stability of constant equilibrium states.

When we construct a global solution in time by continuation of local solutions in time, we need "a priori" estimates for the solution. There have been many methods to derive a priori estimates. Two main methods of them, especially in multi-space-dimensional cases, are the energy method to construct and estimate an energy form of the solution skillfully, and the method to estimate the decay rate of the solutions of the linearized equations and absorb nonlinearity by using these estimates of decay rate (cf.[14][33][44][57][62][63][64][67][68][73]). But they all have been applied to the "semilinear" equations. In this thesis, by using the fact that the solution is suitably close to the constant equilibrium state, we improve the energy method so that it may be applied to "quasilinear" equations. In fact it is applied to the quasilinear dissipative wave equations and the equations of ideal polytropic gase, and their global solutions in time are obtained for the initial data which are suitably close to the constant equilibrium states. Furthermore, by using a combination of this energy method and the estimates of decay rate for the solutions of the linearized equations, we obtain the global solutions in time of the equations of the generel isotropic Newtonian fluid and the quasilinear wave equations with suitably high nonlinearity for the initial data which are suitably close to the constant equilibrium states. Also, as a consequence of the a priori estimates, the solutions are proved to decay to the constant equilibrium states as time tends to infinity. This method to obtain the small but global solutions in time for these particular nonlinear problems is rather general and can be applied to many systems of nonlinear partial differential equations, if the solution of the linearized equation has an appropriate decay rate as time tends to infinity and if the equations are amenable to ordinary energy estimates.

This thesis is devided to four parts. In Chapter I, we will mention what kind of local existence theorem and a priori estimate guarantee existence of a global solution in time, and roughly mention our methods to derive a priori estimates. Prepairing the basic energy estimates and existence theorems for the linear equations in Chapter II, we will investigate the quasilinear wave equations in Chapter III and the equations of compressible viscous fluid in Chapter IV.

SOME NOTATIONS

(i) L^p $(1\leq p<\infty)$: the Lebesgue space of measurable functions on R^n $(n=1,2,..)$ whose p-th powers are integrable , with the norm

$$\| f \|_{L^p} \equiv \left(\int_{R^n} |f(x)|^p \, dx\right)^{1/p}.$$

For $p=2$, we simply write $\| \cdot \|$.

(ii) $\mathcal{B}^k$ $(k=0,1,2..)$: the Banach space of bounded continuous functions on R^n such that all their partial derivatives of order $\leq k$ exist and are moreover bounded continuous, with the norm

$$\| f \|_{\mathcal{B}^k} \equiv \sum_{|\alpha|\leq k} \sup_{R^n} \left| \left(\frac{\partial}{\partial x}\right)^\alpha f(x) \right| ,$$

where $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_n)$, $|\alpha| = \sum_{i=1}^{n} \alpha_n$ and

$$\left(\frac{\partial}{\partial x}\right)^\alpha f = \partial^{|\alpha|} f / \partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}.$$

(iii) $\mathcal{B}^{k+\sigma}$ $(k=0,1,2,..,0<\sigma<1)$: the Hölder space of $\mathcal{B}^k$-functions such that their partial derivatives of order k are uniformly Hölder continuous with an exponent σ, with the norm

$$\| f \|_{\mathcal{B}^{k+\sigma}} \equiv \| f \|_{\mathcal{B}^k} + \sum_{|\alpha|=k} \sup_{\substack{x\neq y \\ x,y\in R^n}} \frac{\left| \left(\frac{\partial}{\partial x}\right)^\alpha f(x) - \left(\frac{\partial}{\partial y}\right)^\alpha f(y) \right|}{|x - y|^\sigma} .$$

(iv) Let $z \in R^m$ and $f = (f_1(z),f_2(z),\ldots,f_r(z))$. Then $D_z^k f$ denotes a vector composed of all k-th partial derivatives with respect to z, i.e.,

$$D_z^k f \equiv \{ (\frac{\partial}{\partial z})^\alpha f_i \ : \ |\alpha| = k \ , \ 1 \le i \le r \}.$$

Especially we write $D_z f$ instead of $D_z^1 f$. $D_z^k f \cdot D_z^k g$ denotes the usual inner product of $D_z^k f$ and $D_z^k g$, and $|D_z^k f|$ is defined by

$$|D_z^k f| \equiv (D_z^k f \cdot D_z^k f)^{1/2}.$$

(v) H^l $(l=0,1,2\ldots)$: the Sobolev space on R^n of L^2-functions whose partial derivatives of order $\le l$ are also L^2-functions , with the norm

$$\| f \|_l \equiv (\sum_{0 \le k \le l} \int_{R^n} |D_x^k f|^2 \ dx)^{1/2}.$$

(vi) Let B be a Banach space, k be a nonnegative integer and T be some positive constant. Then,

$C^k(0,T;B)$ (respectively $L_\infty^k(0,T;B)$) : the Banach space of functions $f(t)$ on $[0,T]$ which have the values in B for every fixed $t \in [0,T]$ and are k-times continuously (resp. boundedly) differentiable with respect to t in B-topology.

$L_2(0,T;B)$: the Banach space of functions $f(t)$ on $[0,T]$ which are square summable on $[0,T]$ in B-topology.

(vii) C , C_i , $C(X)$, $C_i(X)$ $(i=0,1,2,..)$ denote some constants. We write C_i when we want to distinguish it from the others, and $C(X)$ when we must to emphasize its dependence on a quantity X. Also, $h_i(\tau)$ $(i=0,1,2,..)$ denote some continuous nonnegative and nondecreasing functions on $\tau \ge 0$.

CHAPTER I

LOCAL EXISTENCE AND GLOBAL EXISTENCE

In this chapter,we abstractly mention what kind of local existence theorem and a priori estimate are sufficient to guarantee a global solution in time for suitably small initial data and how to construct the global solution by a combination of them. Especially in § 1.2,we roughly mention how to derive the desired a priori estimate in this thesis.

§ 1.1 Local Existence and Global Existence

Let B_1, B_2 and B_3 be some Banach spaces with the norms $\| \cdot \|_{B_1}, \| \cdot \|_{B_2}$ and $\| \cdot \|_{B_3}$ such that

$$B_3 \subsetneq B_2 \subsetneq B_1, \quad \| U \|_{B_1} \leq \| U \|_{B_2} \leq \| U \|_{B_3} \text{ for } U \in B_3.$$

Let $N(\cdot)$ be a continuous operator from B_2 to B_1 such that $N(0) = 0$. Then we consider the following nonlinear evolution equation in B_1;

$$(1.1) \quad \begin{cases} \dfrac{d}{dt} U(t) = N(U(t)) \quad t > 0, \\ U(0) = U_0 \in B_3. \end{cases}$$

Our purpose is to obtain a global solution in time of (1,1) satisfying

$$U \in C^0(0,+\infty;B_2) \cap C^1(0,+\infty;B_1).$$

First we choose a positive constant E_0 and restrict ourselves to seek only a solution satisfying

$$\sup_{t \geq 0} \| U(t) \|_{B_2} \leq E_0$$

so that the equation (1.1) may keep to be a convenient type to be studied for all $t \geq 0$, for example, strictly hyperbolic type or uniformly parabolic type. Let E be a positive number not more than E_0. Then let us define the set of solution $X(t_1, t_2; E)$ for $0 \leq t_1 \leq t_2 \leq +\infty$ by

$$(1.2) \qquad X(t_1, t_2; E) = \{ U \mid U(t) \in C^0(t_1, t_2; B_2) \cap C^1(t_1, t_2; B_1) \text{ and }$$

$$\sup_{t_1 \leq t \leq t_2} \| U(t) \|_{B_2} \leq E \quad (\leq E_0) \}.$$

Let us consider a problem what kind of local existence theorem and a priori estimate are sufficient to guarantee a global solution $U \quad X(0, +\infty; E_0)$ of (1.1). One answer is in the followings :

<u>Local Existence</u> For every nonnegative number h, consider the initial value problem

$$(1.3) \qquad \begin{cases} \dfrac{d}{dt} U(t) = N(U(t)), \\ U(h) = U_h \in B_2, \quad t \geq h. \end{cases}$$

Then there exist positive constants τ and δ $(\delta < 1)$ which do not depend on h such that if

$$\| U_h \|_{B_2} \leq \delta E \quad \text{for some } E \ (\leq E_0),$$

then the problem (1.3) has a unique solution

$$U \in X(h, h+\tau; E).$$

A Priori Estimate   Suppose that the initial value problem (1.1) has a solution

$$U \in X(0,h;E)$$

for some h and E ($\leq E_0$). Then there exist positive constants C_0, ε_1 and ε_2 which do not depend on h such that if $E \leq \varepsilon_1$ and $\| U_0 \|_{B_3} \leq \varepsilon_2$, we have

$$U \in X(0,h;C_0 \| U_0 \|_{B_3}).$$

Remark 1   Since $N(U)$ dose not include t explicitly, it suffices to show Local Existence only for $h = 0$.

Now,once Local Existence and A Priori Estimate are proved, we have

Global Existence   There exist positive constants C_0 and ε_0 such that if $\| U_0 \|_{B_3} \leq \varepsilon_0$, then the problem (1.1) has a unique solution

$$U \in X(0,+\infty;C_0 \| U_0 \|_{B_3}).$$

In fact,we may take ε_0 as

$$\varepsilon_0 = min\ (\ \delta\varepsilon_1,\ \frac{\delta\varepsilon_1}{C_0}\ ,\ \varepsilon_2\ ).$$

Then, by Local Existence with $h = 0$ and $E = \varepsilon_1$, we have a local solution

$$U \in X(0,\tau;\varepsilon_1).$$

By $\| U_0 \|_{B_3} \leq \varepsilon_0 \leq \varepsilon_2$ and $\| U_0 \|_{B_3} \leq \delta\varepsilon_1/C_0$, A Priori Estimate implies

$$U \in X(0,\tau;C_0\| U_0\|_{B_3}) \subset X(0,\tau;\delta\varepsilon_1).$$

Therefore, by using Local Existence with $h = \tau$ and $E = \varepsilon_1$ again, we have

$$U \in X(0,2\tau;\varepsilon_1)$$

which implies

$$U \in X(0,2\tau;\delta\varepsilon_1)$$

by virtue of A Priori Estimate again. Repeating the same arguments, we have Global Existence.

Occasionally, we have a case that it is convenient to seek a solution in more restricted space than in $C^0(0,+\infty;B_2)$. For such cases, we may take some suitable space $X(t_1,t_2)$ with the norm $\| \cdot \|_{X(t_1,t_2)}$ satisfying

$$X(t_1,t_2) \subsetneq C^0(t_1,t_2;B_2),$$

$$\sup_{t_1\leq t\leq t_2} \| U(t) \|_{B_2} \leq \| U \|_{X(t_1,t_2)},$$

$$\| U \|_{X(t_1,t_2)} \leq \| U \|_{X(t_1,t)} + \| U \|_{X(t,t_2)} \text{ for } t_1 \leq t \leq t_2,$$

and modify (1.2) as

$$(1.4) \qquad X(t_1,t_2;E) = \{ U \mid U \in X(t_1,t_2) \cap C^1(t_1,t_2;B_1) \text{ and } \| U \|_{X(t_1,t_2)} \leq E \ (\leq E_0) \}.$$

Then, if we take ε_0 as

$$\varepsilon_0 = \min\left\{ \frac{\delta\varepsilon_1}{2}, \frac{\delta\varepsilon_1}{2C_0}, \varepsilon_2 \right\},$$

Local Existence and A Priori Estimate imply Global Existence also in the case (1.4). In fact, by Local Existence with $E = \varepsilon_1/2$ and $h = 0$, we have

$$U \in X(0,\tau;\varepsilon_1/2).$$

Then A Priori Estimate implies

$$\begin{aligned} U &\in X(0,\tau;C_0\| U_0 \|_{B_3}) \\ &\subset X(0,\tau;\delta\varepsilon_1/2). \end{aligned}$$

Using Local Existence with $E = \varepsilon_1/2$ and $h = \tau$ again, we have

$$U \in X(\tau,2\tau;\varepsilon_1/2)$$

Therefore, noting that

$$\begin{aligned} \| U \|_{X(0,2\tau)} &\leq \| U \|_{X(0,\tau)} + \| U \|_{X(\tau,2\tau)} \\ &\leq \delta\varepsilon_1/2 + \varepsilon_1/2 \\ &\leq \varepsilon_1, \end{aligned}$$

we have

$$U \in X(0,2\tau;\varepsilon_1).$$

Then A Priori Estimate implies

$$U \in X(0, 2\tau;\ \delta\varepsilon_1/2).$$

Thus, repeating the same arguments, we have Global Existence in the cace (1.4).

§ 1.2 A Priori Estimates

Although Local Existence is usually obtained along the linear theory, there are no general methods to derive an estimate as A Priori Estimate. In this thesis, we use an energy method and a combination of the estimates of decay rate for the linearized equation and the energy estimates. Our energy method is, roughly speaking, to find a suitable energy form $E_1(U)$ associated with a structure of (1.1) satisfying

$$(1.5) \qquad C_1^{-1} \| U \|_{B_2} \leq E_1(U) \leq C_1 \| U \|_{B_2} \text{ for } \| U \|_{B_2} \leq E_0$$

where C_1 is some positive constant, and to get the energy inequality for the solution $U \in X(0,h;E)$ of (1.1) such as

$$(1.6) \qquad E_1(U(t)) - E_1(U_0) + \int_0^t \nu(1 - O(E))E_2(U(s))\, ds \leq 0$$

where ν is some positive constant and $E_2(U)$ is another nonnegative energy form. If (1.6) is obtained, A Priori Estimate is easily proved by (1.5) and taking E small in (1.6).

Usually we may not expect to get such convenient inequality as (1.6). More generally, let us assume that (1.6) has the form

$$(1.7) \qquad E_1(U(t)) - E_1(U_0) + \int_0^t \nu(1 - O(E))E_2(U(s))\, ds$$

$$\leq \int_0^t h_0(|||\, U(s)\, |||)\, ds$$

where $|||\cdot|||$ is some seminorm. Then <u>A Priori Estimate</u> follows from (1.7) again, if the following estimates hold ; for $U \in X(0,h;E)$

$$(1.8) \quad \begin{cases} |||\, U(t)\, ||| \leq C_2(1+t)^{-k} \|\, U_0\, \|_{B_3}, \\ h_0(|||\, U(t)\, |||) \leq C_2 |||\, U(t)\, |||^p \qquad (\ k > 0,\ p \geq 1,\ kp > 1\ ). \end{cases}$$

In fact, (1.8) gives

$$\int_0^t h_0(|||\, U(s)\, |||)\, ds \leq C_2^{p+1} \|\, U_0 \|_{B_3}^p \int_0^t (1+s)^{-kp}\, ds$$

$$\leq \frac{C_2^{p+1} \varepsilon_2^{p-1}}{kp - 1} \|\, U_0 \|_{B_3}.$$

Now suppose that $N(U)$ is Frechet differentiable at $U = 0$ and denote the Frechet derivative at $U = 0$ by A. Then we rewrite (1.1) in the form

$$(1.9) \quad \begin{cases} U_t - AU = N(U) - AU \\ \\ U(0) = U_0 \end{cases}$$

or in the form

$$(1.10) \qquad U(t) = e^{tA} U_0 + \int_0^t e^{(t-s)A} \{\ N(U(s)) - AU(s)\ \}\, ds,$$

if A generate the semigroup e^{tA} in B_1. There are many various cases that we can obtain the estimates of decay rate such as (1.8) by using (1.10). So, we mention only the simplest case. Let us assume

$$(1.11)\qquad \begin{aligned} &|||e^{tA}F||| \leq C_3\| F \|_{B_1}(1+t)^{-l}, \qquad (l \geq k, l > 1) \\ &\| N(F) - AF \|_{B_1} \leq C_4|||F|||\cdot\| F \|_{B_2} \text{ for } \| F \|_{B_2} \leq E_0. \end{aligned}$$

By applying (1.11) to (1.10), we have for $U \in X(0,h;E)$

$$(1.12)\qquad \begin{aligned} |||U(t)||| &\leq C_3(1+t)^{-l}\| U_0 \|_{B_1}+\int_0^t C_4(1+t-s)^{-l}|||U(s)|||\cdot\| U(s) \|_{B_2}ds \\ &\leq C_3(1+t)^{-l}\| U_0 \|_{B_3}+ C_4E \int_0^t (1+t-s)^{-l}|||U(s)||| \, ds. \end{aligned}$$

Set $M(t) = \sup\limits_{0\leq\tau\leq t} (1+\tau)^k|||U(\tau)|||$. Then it follows from (1.12) that

$$\begin{aligned} M(t) &\leq C_3\| U_0 \|_{B_3}+ C_4EM(t)(1+t)^k \int_0^t (1+t-s)^{-l}(1+s)^{-k}\, ds \\ &\leq C_3\| U_0 \|_{B_3}+ C_5EM(t) \end{aligned}$$

which implies

$$\begin{aligned} M(t) &\leq \frac{C_3}{1-C_5E} \| U_0 \|_{B_3} \\ &\leq 2C_3\| U_0 \|_{B_3} \text{ for } E \leq 1/2C_5 \end{aligned}$$

which consequently gives

$$|||U(t)||| \leq 2C_3(1+t)^{-k}\| U_0 \|_{B_3} \text{ for } E \leq 1/2C_5.$$