Large time behaviors of solutions for the equations of one-dimensional motion of viscous and heat-conductive gas

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In this short note, we make a survey on the recent works on large time behaviors of solutions to the Cauchy problem for the equations of one-dimensional motion of viscous and heat-conductive gas.

The one-dimensional motion of the viscous and heat-conductive ideal gas is described in the Lagrangian mass coordinates by the system

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = \mu(\frac{u_x}{v})_x, \\ (e + \frac{u^2}{2})_t + (pu)_x = (\kappa\frac{\theta_x}{v} + \mu\frac{uu_x}{v})_x, \quad x \in R^1,\ t > 0 \end{cases} \tag{1}$$

where the unknown functions $v > 0$, u, $\theta > 0$, $e > 0$ and p are the specific volume, fluid velocity, internal energy, absolute temperature, and pressure respectively, while the constants $\mu > 0$ and $\kappa > 0$ denote the viscosity and heat conduction coefficients respectively. Here we study the ideal and polytoropic gas, that is, p and e are given by the state equations

$$p = \frac{R\theta}{v}, \quad e = \frac{R}{\gamma - 1}\theta$$

where $\gamma > 1$ is the adiabatic exponent and $R > 0$ is the gas constant. We consider the Cauchy problem for the system (1) with the initial data

$$(v, u, \theta)(x, 0) = (v_0, u_0, \theta_0)(x), \quad x \in R^1 \tag{2}$$

which satisfy

$$\lim_{x\to\pm\infty}(v_0, u_0, \theta_0)(x) = (v_\pm, u_\pm, \theta_\pm), \quad \inf_{x\in R^1} v_0(x) > 0, \quad \inf_{x\in R^1} \theta_0(x) > 0$$

where $v_\pm(> 0), u_\pm, \theta_\pm(> 0)$ are given constants.

We are interested in the global solutions in time of the Cauchy problem (1)-(2) and their large time behaviors in the relations with the spatial asymptotic states $(v_\pm, u_\pm, \theta_\pm)$. It has been known that these asymptotic behaviors are well characterized by those of the solutions of the corresponding Riemann problem for the hyperbolic part of (1) (Euler equation)

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \\ (e + \frac{u^2}{2})_t + (pu)_x = 0, \quad x \in R^1,\ t > 0 \\ (v, u, \theta)(x, 0) = (v_0^R, u_0^R, \theta_0^R)(x) := \begin{cases} (v_-, u_-, \theta_-),\ x < 0, \\ (v_+, u_+, \theta_+),\ x > 0. \end{cases} \end{cases} \tag{3}$$

The system of conservation laws (3) has three distinct real eigenvalues for positive v and θ

$$\lambda_1 = -\sqrt{\gamma p/v} < 0, \quad \lambda_2 = 0, \quad \lambda_3 = -\lambda_1 > 0$$

which implies the first and third characteristic fields are genuinely nonlinear and the second field is linearly degenerate. Then it is known that the solutions of (3) (Riemann solutions) consist of the various combinations of the three elementary nonlinear waves, that is, shock wave, rarefaction wave and contact discontinuity (in total, 17 cases). In what follows, we use the abbreviations $z = (v, u, \theta), z_\pm = (v_\pm, u_\pm, \theta_\pm), , ,$ and assume that for any fixed left state z_-, the right state z_+ is in a suitably small neighborhood of z_- in R^3.

In the case the Riemann solution of (3) consists of a single rarefaction wave $z_i^r(x/t)(i = 1, 3)$ corresponding to the i-characteristic field which connects the left constant state z_- to the right z_+, Kawashima-Matsumura-Nishihara [6] showed that around this rarefaction wave, the global solution in time of (1)-(2) exists and asymptotically tends toward the rarefaction wave $z_i^r(x/t)$ of the hyperbolic part. The case the Riemann solution consists of two rarefaction waves $z_1^r(x/t)$ and $z_3^r(x/t)$ can be treated similarly and the global solution in time is proved to tend toward the linear combination $z_1^r(x/t) + z_3^r(x/t) - z_m$. Here z_m is the uniquely determined intermediate constant state so that $z_1^r(x/t)$ connects z_- to z_m and $z_3^r(x/t)$ connects z_m to z_+.

In the case the Riemann solution of (3) consists of a single contact discontinuity corresponding to the 2-characteristic field, Huang-Matsumura-Xin [3] recently showed that the system (1) approximately has a corresponding "viscous contact wave" $z_2^{vc}(x/\sqrt{t})$ which connects z_- to z_+ and around this viscous contact wave the global solution in time of (1)-(2) exists and tends toward the viscous contact wave $z_2^{vc}(x/\sqrt{t})$ provided the integral of the initial perturbation is zero. Furthermore, Huang-Xin-Yang [4] extended the result for more general initial perturbation whose integral is not necessarily zero. The cases the Riemann solution consists of the contact discontinuity and shock waves (or rarefaction waves) are interesting open problems we should challenge as next targets (just recently, Huang-Li-Matsumura [1] solved the case where the Riemann solution consists of the contact discontinuity and rarefaction waves).

In the case the Riemann solution of (3) consists of a single shock wave $z_i^s(x - s_i t)(i = 1, 3)$ with the shock speed $s_i\,(s_1 < 0 < s_3)$ corresponding to the i-characteristic field which connects z_- to z_+, it is known that the system (1) has the corresponding traveling wave solution $z_i^{vs}(x - s_i t)$ which we call "viscous shock wave", and we expect that around $z_i^{vs}(x)$ the global solution in time of (1)-(2) exists and tends toward the $z_i^{vs}(x - s_i t + \alpha_i)$ with a suitable shift α_i. Kawashima-Matsumura [5] first showed this asymptotic stability provided the integral of the initial perturbation is zero. For more general initial perturbation whose integral is not necessarily zero, Szepessy-Xin [11] replaced the viscous terms by some artificial ones and showed the asymptotic stability. For the original physical system (1) it is still not clear (for the 2×2 viscous p-system, Mascia-Zumbrun [8] proved it). The case the Riemann solution consists of both shock and rarefaction waves is entirely open even for the 2×2 viscous p-system. The case the Riemann solution consists of two shock waves $z_1^s(x - s_1 t)$ and $z_3^s(x - s_3 t)$ has been another open problem. In this case, the global solution in time of (1)-(2) is expected to tend toward a linear combination of the corresponding combination of viscous shock waves

$$z_{\alpha_1,\alpha_3}(x, t) := z_1^{vs}(x - s_1 t + \alpha_1) + z_3^{vs}(x - s_3 t + \alpha_3) - z_m$$

with suitable shifts α_1 and α_3. Here z_m is the uniquely determined intermediate constant state so that $z_1^s(x - s_1 t)$ connects z_- to z_m and $z_3^s(x - s_3 t)$ connects z_m to z_+. Recently, Huang-Matsumura [2] showed that this asymptotic stability does hold in a small neighborhood of $z_{0,0}(x, 0)$. The proof is technically given by constructing a good approximation of the linear diffusion wave around the constant state z_m and combining the arguments by Liu [7] on how the shifts α_1, α_3 and the strength of the diffusion wave are determined, together with the elementary L^2-energy method by Kawashima-Matsumura [5].

Finally we show our arguments on the asymptotic stability of z_{α_1,α_3} is applicable to the study of the following initial boundary value problem on the half line

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = \mu(\frac{u_x}{v})_x, & x > 0, \ t > 0, \\ (e + \frac{u^2}{2})_t + (pu)_x = (\kappa\frac{\theta_x}{v} + \mu\frac{uu_x}{v})_x, \end{cases} \tag{4}$$

with the initial data

$$(v, u, \theta)(x, 0) = (v_0, u_0, \theta_0)(x), \quad x > 0 \tag{5}$$

satisfying

$$\lim_{x\to\infty}(v_0, u_0, \theta_0)(x) = (v_+, u_+, \theta_+), \quad \inf_{x>0} v_0(x) > 0, \quad \inf_{x>0} \theta_0(x) > 0$$

and the boundary conditions

$$u(0, t) = 0, \quad \theta_x(0, t) = 0, \quad t > 0. \tag{6}$$

For this initial boundary value problem, if we extend the unknown v and θ as even functions and u as odd function on whole R^1, we can reduce the initial boundary value problem (4)-(6) to the Cauchy problem (1)-(2) with $z_+ = (v_+, u_+, \theta_+)$ and $z_- = (v_+, -u_+, \theta_+)$. Then it turns out that if u_+ is negative, the corresponding Reimann solution of (3) consists of two shock waves with $s_3 = -s_1$ and the intermediate constant state z_m is uniquely determined in the form $z_m = (v_m, 0, \theta_m)$. Thus we can show that if u_+ is negative, the global solution in time of (4)-(6) exists and asymptotically tends toward the outgoing viscous shock wave $z_3^{vs}(x - s_3t + \alpha_3)$ with a shift α_3 which connects the left state z_m to the right z_+, under several smallness conditions. For the case u_+ is positive, we can similarly show the solution tends toward the outgoing rarefaction wave. These results are natural extensions of those for the initial boundary value problems on the half line to the 2×2 viscous p-system discussed in [9] and [10].

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