熱伝導性理想気体の空間一次元モデルに対する時間大域解の漸近挙動

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平成27年1月9日

System of non-viscous and heat-conductive ideal gas

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p)_x = 0, \\ (\rho(e + \frac{u^2}{2}))_t + (\rho u(e + \frac{u^2}{2}) + pu)_x = \kappa\theta_{xx}, \end{cases}$$

$$p = R\rho\theta, \quad e = \frac{R}{\gamma - 1}\theta.$$

$\rho\,(> 0)$: density,

u : fluid velocity,

p : pressure,

e : internal energy per unit mass,

$\gamma\,(\geq 1)$: adiabatic constant,

R : gas constant,

κ : coefficient of heat-conductivity (a positive constant).

Energy method for nonlinear PDEs (around 1980)

- Global solutions in time for multi-dimensional Quasi-linear dissipative wave equations around the zero solution (Publ. RIMS Kyoto Univ., 1977):

$$u_{tt} - \sum_{i,j=1}^{n} a_{ij}(u, u_t, \nabla u) u_{x_i x_j} + \kappa u_t = 0$$

- Global solutions in time for three dimensional System of viscous and heat-conductive ideal gas around the constant state (J. Math. Kyoto Univ., 1980, with T. Nishida).

- 1981, M. Slemlod : System of nonlinear thermo-elasticity

Global Existence, Uniqueness, and Asymptotic Stability of Classical Smooth Solutions in One-Dimensional Non-linear Thermoelasticity

M. SLEMROD

(Arch. Rat. Mech. Ana. 76, 97-133 (1981))

Linearized system of non-linear thermo-elasticity

$$\begin{cases} v_t - u_x = 0, \\ u_t - v_x + \theta_x = 0, \\ \theta_t + u_x = \kappa\theta_{xx}. \end{cases}$$

Remark.

If we set $v = 1/\rho$ (specific volume), the system of non-viscous and heat-conductive gas is rewritten in the form

$$\begin{cases} v_t + uv_x - vu_x = 0, \\ u_t + uu_x - pv_x + R\theta_x = 0, \\ \theta_t + u\theta_x + (\gamma - 1)\theta u_x = \frac{(\gamma-1)\kappa v}{R}\theta_{xx}. \end{cases}$$

- Energy estimates for the linearized system

$$\begin{cases} v_t - u_x = 0, & \times v \\ u_t - v_x + \theta_x = 0, & \times u \\ \theta_t + u_x = \kappa\theta_{xx}. & \times \theta \end{cases}$$

$$\frac{d}{dt}\int \frac{1}{2}(v^2 + u^2 + \theta^2)\, dx + \kappa \int |\theta_x|^2\, dx = 0.$$

$$\frac{d}{dt}\int \frac{1}{2}(|v_x|^2 + |u_x|^2 + |\theta_x|^2)\, dx + \kappa \int |\theta_{xx}|^2\, dx = 0.$$

To restore the dissipativity for v and u,

$$\begin{cases} v_t - u_x = 0, & \times\frac{\alpha}{2}u_x \\ u_t - v_x + \theta_x = 0, & \times\alpha(-\frac{1}{2}v_x - \theta_x) \\ \theta_t + u_x = \kappa\theta_{xx}. & \times\alpha u_x \end{cases}$$

$$\frac{d}{dt}\int \alpha(\theta u_x - \frac{1}{2}uv_x)\,dx + \frac{\alpha}{2}\int (|v_x|^2 + |u_x|^2)\,dx$$

$$= \alpha\int (\frac{3}{2}v_x\theta_x + |\theta_x|^2 + \kappa\theta_{xx}u_x)\,dx,$$

$$\|(v,u,\theta)(t)\|_{H^1}^2 + \int_0^t (\|(v_x,u_x)(\tau)\|_{L^2}^2 + \|\theta_x(\tau)\|_{H^1}^2)\,d\tau$$

$$\leq C\|(v,u,\theta)(0)\|_{H^1}^2.$$

Cauchy problem

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p)_x = 0, \qquad t \geq 0,\ x \in \mathrm{R}, \\ (\rho(e + \frac{u^2}{2}))_t + (\rho u(e + \frac{u^2}{2}) + pu)_x = \kappa \theta_{xx}, \end{cases} \tag{1}$$

$$p = R\rho\theta, \quad e = \frac{R}{\gamma - 1}\theta,$$

with the initial data

$$(\rho, u, \theta)(0, x) = (\rho_0, u_0, \theta_0)(x), \quad x \in \mathrm{R}, \tag{2}$$

and the far field condition

$$\lim_{x \to \pm\infty} (\rho, u, \theta)(t, x) = (\rho_\pm, u_\pm, \theta_\pm), \quad t \geq 0, \tag{3}$$

where $\rho_\pm > 0$, $\theta_\pm > 0$ and $u_\pm \in \mathrm{R}$ are given constants.

Viscous and Heat-conductive case

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p)_x = \mu u_{xx}, \\ (\rho(e + \frac{u^2}{2}))_t + (\rho u(e + \frac{u^2}{2}) + pu)_x = \kappa\theta_{xx} + (\mu u u_x)_x. \end{cases}$$

Riemann Problem for the Euler system

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p)_x = 0, \qquad t > 0,\ x \in \mathrm{R}, \\ (\rho(\frac{R}{\gamma-1}\theta + \frac{u^2}{2}))_t + (\rho u(\frac{R}{\gamma-1}\theta + \frac{u^2}{2}) + pu)_x = 0, \\ (\rho, u, \theta)(0, x) = \begin{cases} (\rho_-, u_-, \theta_-), & x < 0, \\ (\rho_+, u_+, \theta_+), & x > 0. \end{cases} \end{cases}$$

Charactaristic speeds :

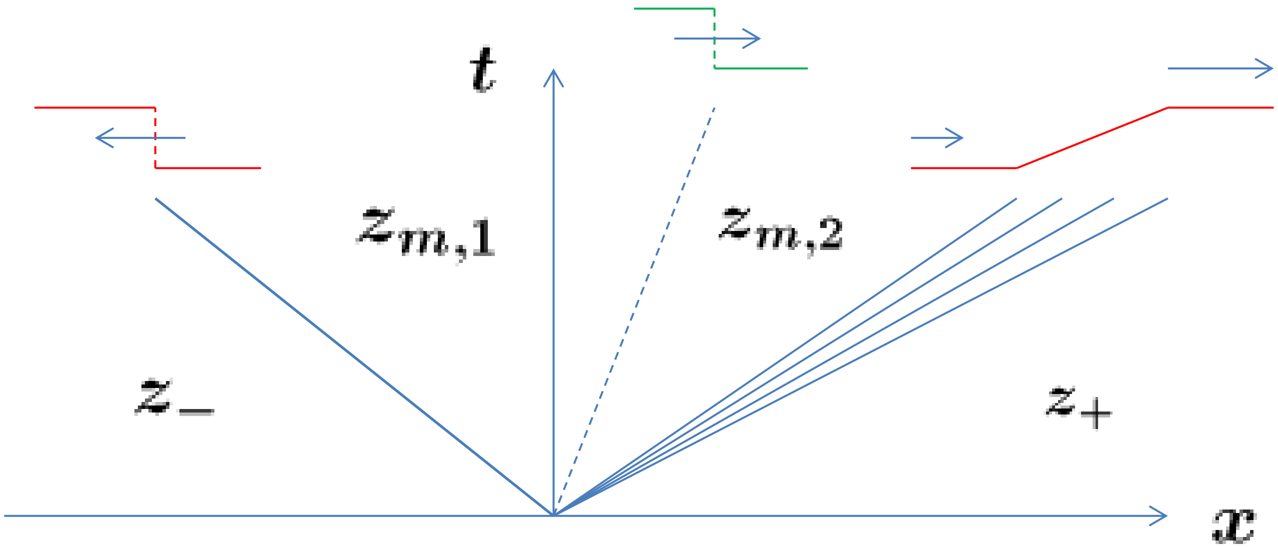
$$\lambda_1(z) = u - c_s, \quad \lambda_2(z) = u, \quad \lambda_3(z) = u + c_s,$$

where $z = {}^t(\rho, u, \theta)$.

Sound speed : $$c_s = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\gamma R\theta}$$

i-characteristic field ($i = 1, 3$) is genuinely nonlinear,

2-characteristic field is linearly degenerate.



Riemann (1860)

$$\begin{cases} \rho_t + (\rho w)_x = 0, \\ (\rho w)_t + (\rho w^2 + p)_x = 0, \\ p = p(\rho) = a\rho^\gamma. \end{cases}$$

Ueber die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite.

(Aus dem achten Bande der Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen. 1860.)

Georg Friedrich Bernhard Riemann (1826 – 1866)

Peter David Lax (1926--)

Lax (1957)

$$u_t + f(u)_x = 0.$$

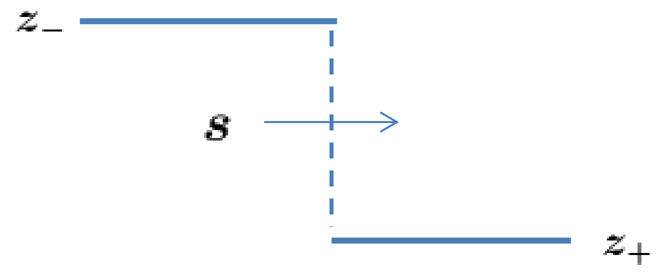
Mathematical Theory of Conservation Laws

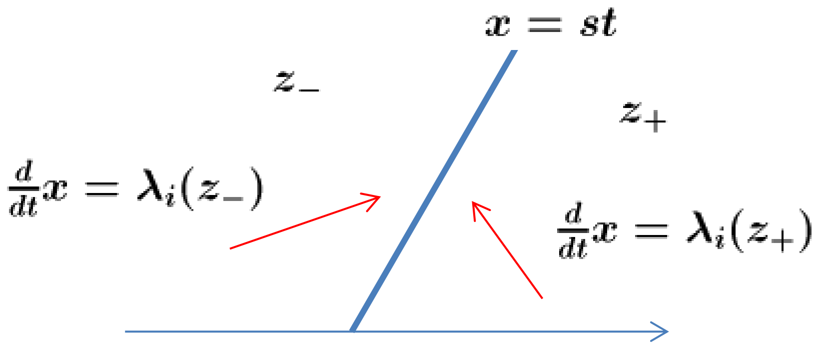
Commun. Pure Appl. Math., 10 (1957)

Simple Waves

In the case the characteristic field is **genuinely nonlinear**

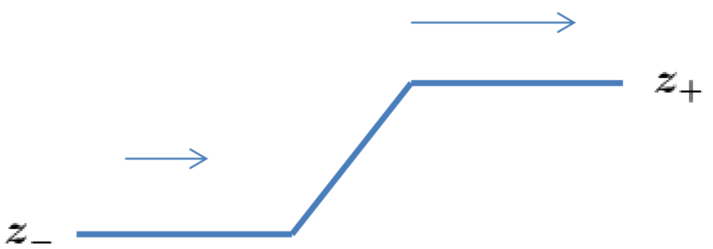
Shock wave : $z_i^s(x - st; z_-, z_+)$

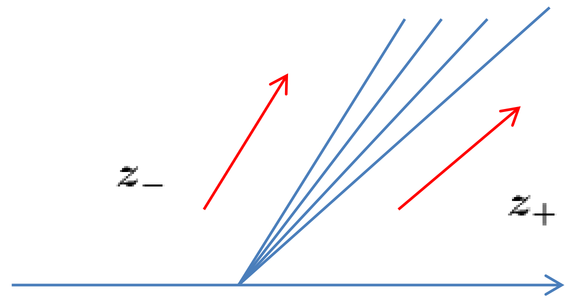




Rankine-Hugoniot Condition + Entropy Condition

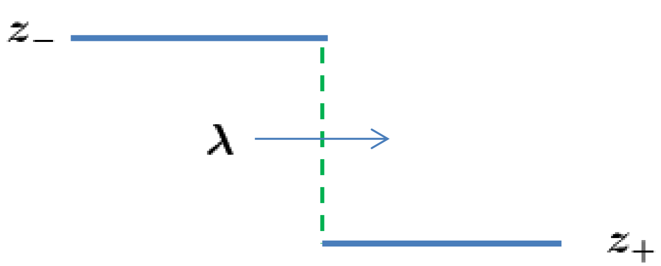
Rarefaction Wave : $z_i^r(x/t; z_-, z_+)$

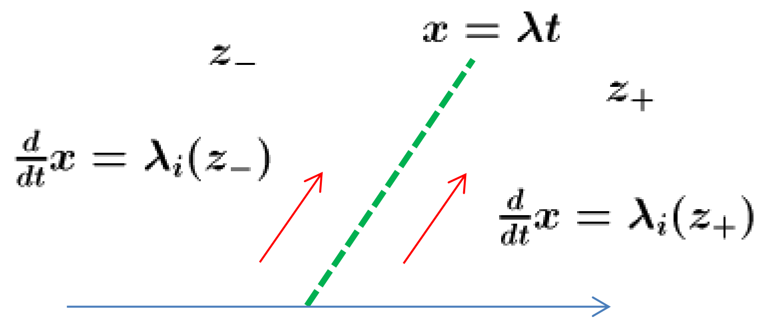




Simple Waves : In the case the characteristic field is **linearly degenerate**

Contact discontinuity : $z_i^c(x - \lambda t; z_-, z_+)$



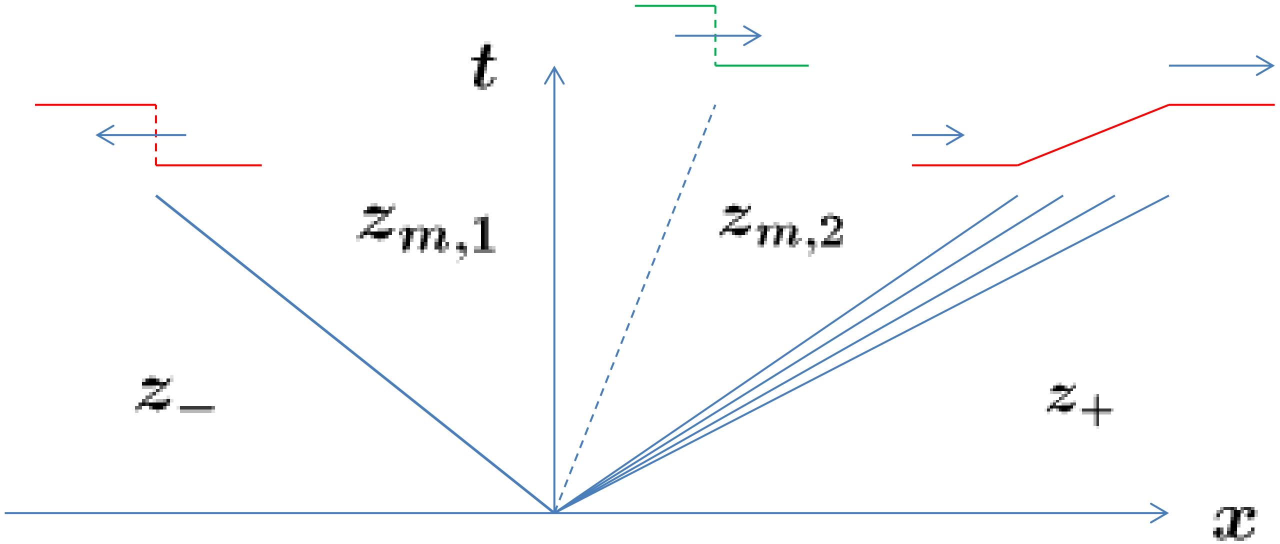


$$\lambda = \lambda_i(z_-) = \lambda_i(z_+)$$

General Riemann Solution

A linear superposition of simple waves

An example



$$z^R(t,x) = z_1^s(x - s_1 t; z_-, z_{m,1})$$

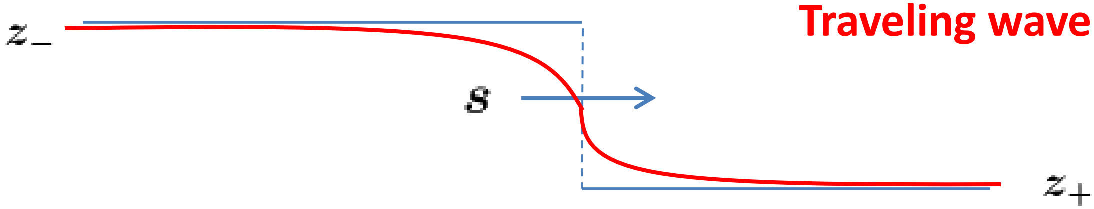
$$+ z_2^c(x - \lambda_2 t; z_{m,1}, z_{m,2}) - z_{m,1}$$

$$+ z_3^r(x/t; z_{m,2}, z_+) - z_{m,2}$$

Asymptotic state under viscous effect

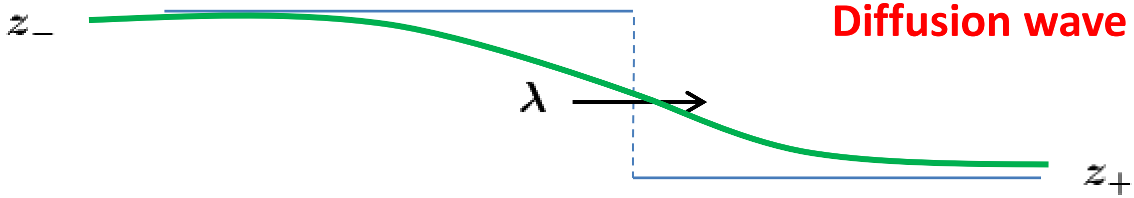
- Rarefaction wave ⟹ Rarefaction wave
- Shock wave ⟹ Viscous shock wave
- Contact discontinuity ⟹ Viscous contact wave

Viscous shock wave : $U_i^{vs}(x - st; u_-, u_+)$



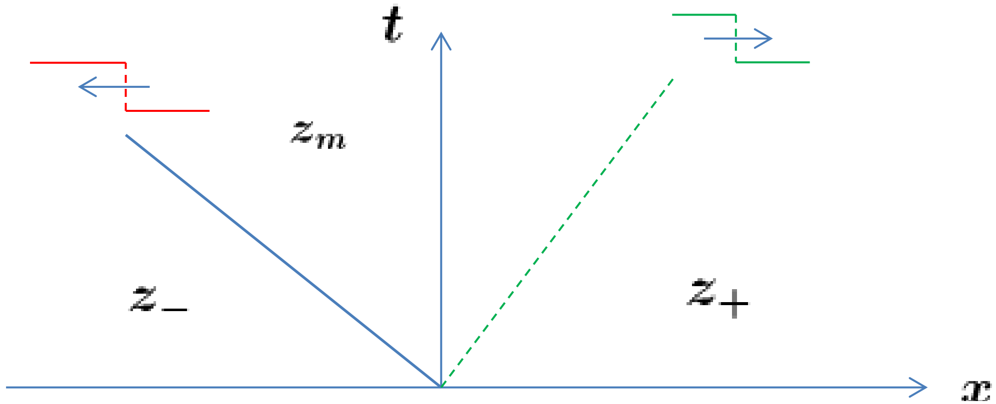
Traveling wave relaxed by viscosity

Viscous contact wave: $U_i^{vc}(\frac{x-\lambda t}{\sqrt{t}}; u_-, u_+)$



Diffusion wave relaxed by viscosity

An example



Consider the case where the Riemann solution consists of two shocks:

Riemann Solution :

$$z^R(t,x) = z_1^s(x - st; z_-, z_m) + z_2^c(x - \lambda t; z_m, z_+) - z_m$$

Asymptotic Solution :

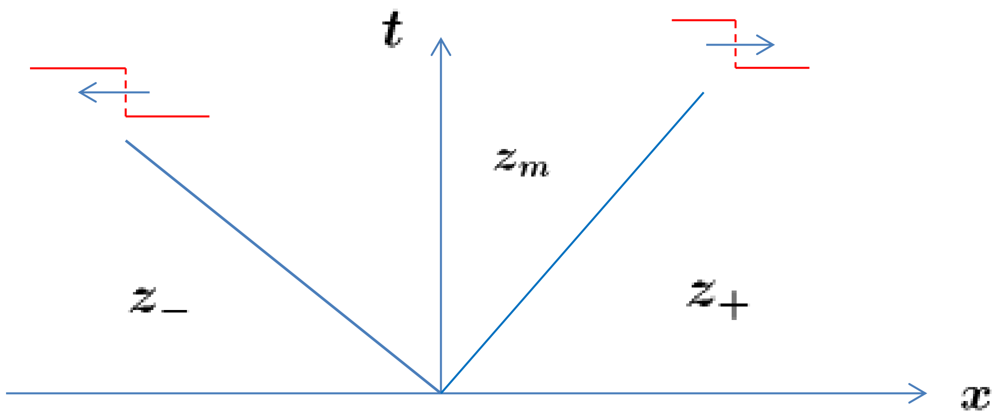
$$Z(t,x) = Z_1^{vs}(x - st + \alpha; z_-, z_m) + Z_2^{vc}\left(\frac{x-\lambda t}{\sqrt{t}}; z_m, z_+\right) - z_m$$

Known results on Viscous and Heat-conductive case :

- Single Shock → **Kawashima-M** (1985) (zero mass initial perturbations); **Liu** (1997); **Zumbrum** (2004); **Liu-Zen** (2009)
- Shock + Shock → **Huang-M** (2009)
- Rarefaction + (Rarefaction) → **Kawashima-Nishihara-M** (1986)
- Single Contact discontinuity → **Huanag-Xin-M** (2006) (zero mass initial perturbation); **Huang-Xin-Yang** (2008)
- Contact discontinuity + Rarefactions → **Huang-Li-M** (2010)
- Rarefaction + Shock → **Open**
- Contact discontinuity + Shocks → **Open**
- Rarefaction + Contact discontinuity + Shock → **Open**

Non-viscous and Heat-conductive case

Consider the case where the Riemann solution consists of two shocks:



$$z^R(t,x) = z_1^s(x - s_1 t; z_-, z_m) + z_3^s(x - s_3 t; z_m, z_+) - z_m.$$

The corresponding asymptotic solution for the system (1) :

$$Z_{\alpha_1,\alpha_3}(t,x) = Z_1^{vs}(x - s_1 t + \alpha_1; z_-, z_m) + Z_3^{vs}(x - s_3 t + \alpha_3; z_m, z_+) - z_m.$$

Lili Fan -M, J. Differential Equations, 2015

If the strengthens of shock waves $|\boldsymbol{z_m} - \boldsymbol{z_-}|$ *and* $|\boldsymbol{z_+} - \boldsymbol{z_m}|$ *are suitably small with same order, and the initial perturbation from* $\boldsymbol{Z_{0,0}}$ *is suitably small in* $\boldsymbol{H^2}$ *and further satisfies some technical smallness conditions, there exist a unique time-global solution* $\boldsymbol{z} = {}^t(\boldsymbol{\rho}, \boldsymbol{u}, \boldsymbol{\theta})$ *of the Cauchy problem* (1)-(3) *satisfying* $\boldsymbol{z} - \boldsymbol{Z_{\alpha_1,\alpha_3}} \in \boldsymbol{C}([0,\infty);\ \boldsymbol{H^2})$ *and the asymptotic behavior*

$$\sup_{x\in\mathrm{R}} |(z - Z_{\alpha_1,\alpha_3})(t,x)| \to 0, \quad t \to \infty,$$

where the spatial shifts $\boldsymbol{\alpha_1}$ *and* $\boldsymbol{\alpha_3}$ *are uniquely determined by the initial perturbation.*

Remarks:

- As for the proof, basically follow the arguments in Kawashima-M and Huang-Li-M, except careful manipulation to a dissipative structure which is weaker than one for the viscous and heat-conductive case.

- Other cases are also expected to be similar as in the viscous and heat-conductive case.

Murakami, preprint : rarefaction wave

Lili, preprint : viscous contact wave

Remarks on the initial boundary value problems on the half space

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p)_x = \mu u_{xx}, \qquad t \geq 0, \ x \geq 0, \\ (\rho(e + \frac{u^2}{2}))_t + (\rho u(e + \frac{u^2}{2}) + pu)_x = \kappa\theta_{xx} + (\mu u u_x)_x. \end{cases}$$

with the initial data

$$(\rho, u, \theta)(0, x) = (\rho_0, u_0, \theta_0)(x), \quad x \geq 0,$$

and the far field condition

$$\lim_{x \to \infty} (\rho, u, \theta)(t, x) = (\rho_+, u_+, \theta_+), \quad t \geq 0,$$

and the boundary condition of Dirichet's type.

The boundary condition has to be imposed for the initial boundary value problem to be well-posed as for hyperbolic-parabolic system .

Boundary condition for $\mu > 0$ (Viscous and Heat-conductive case)

$$\begin{cases} \rho_t + u\rho_x = l.o.t, \\ u_t - \dfrac{\mu}{\rho} u_{xx} = l.o.t., \\ \dfrac{R}{\gamma - 1}\theta_t - \dfrac{\kappa}{\rho}\theta_{xx} = l.o.t. \end{cases}$$

The boundary condition is depending on the sign of u.

$$u_- \leq 0 \quad \Rightarrow \quad \text{B.C.} \begin{cases} u(t,0) = u_-, \\ \theta(t,0) = \theta_-. \end{cases} \quad t \geq 0.$$

$$u_- > 0 \quad \Rightarrow \quad \text{B.C.} \begin{cases} \rho(t,0) = \rho_-, \\ u(t,0) = u_-, \\ \theta(t,0) = \theta_-, \end{cases} \quad t \geq 0.$$

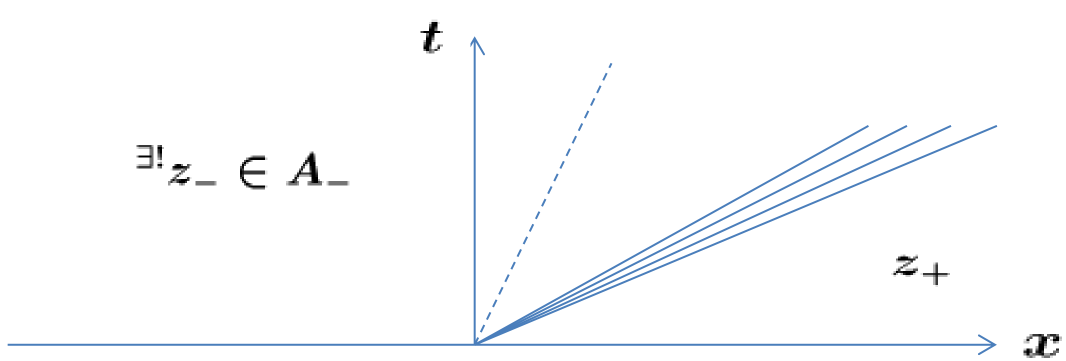
General principle to predict asymptotic behavior

Admissible set for the boundary condition :

$$A_- := \{z \in \mathrm{R}^3{}_{\rho>0,\theta>0} \;|z : \text{ consistent with } B.C. \}$$

- For the fixed far field state z_+, consider the Riemann problem for any $z_- \in A_-$.
- In the case where ${}^{\exists!}z_- \in A_-$ such that the Riemann solution includes no incoming wave:

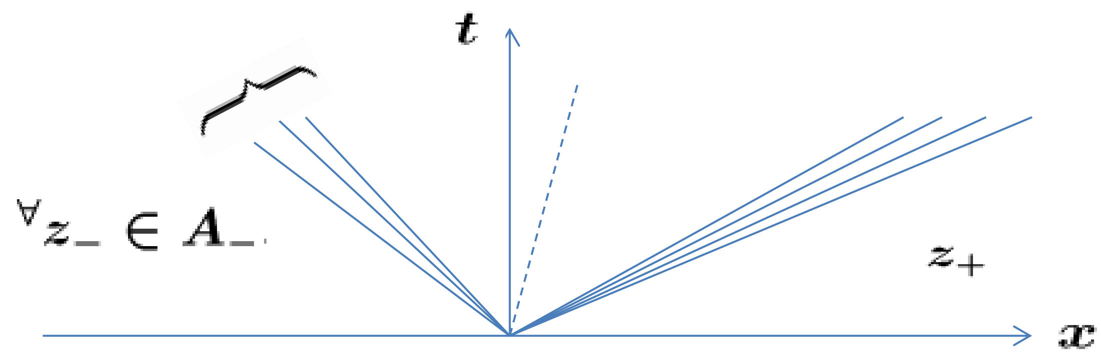
$$z \longrightarrow \text{outgoing asymptotic waves}, \quad t \to \infty.$$



- In the case where ${}^{\forall} z_- \in A_-$, the Riemann solution includes an incoming wave :

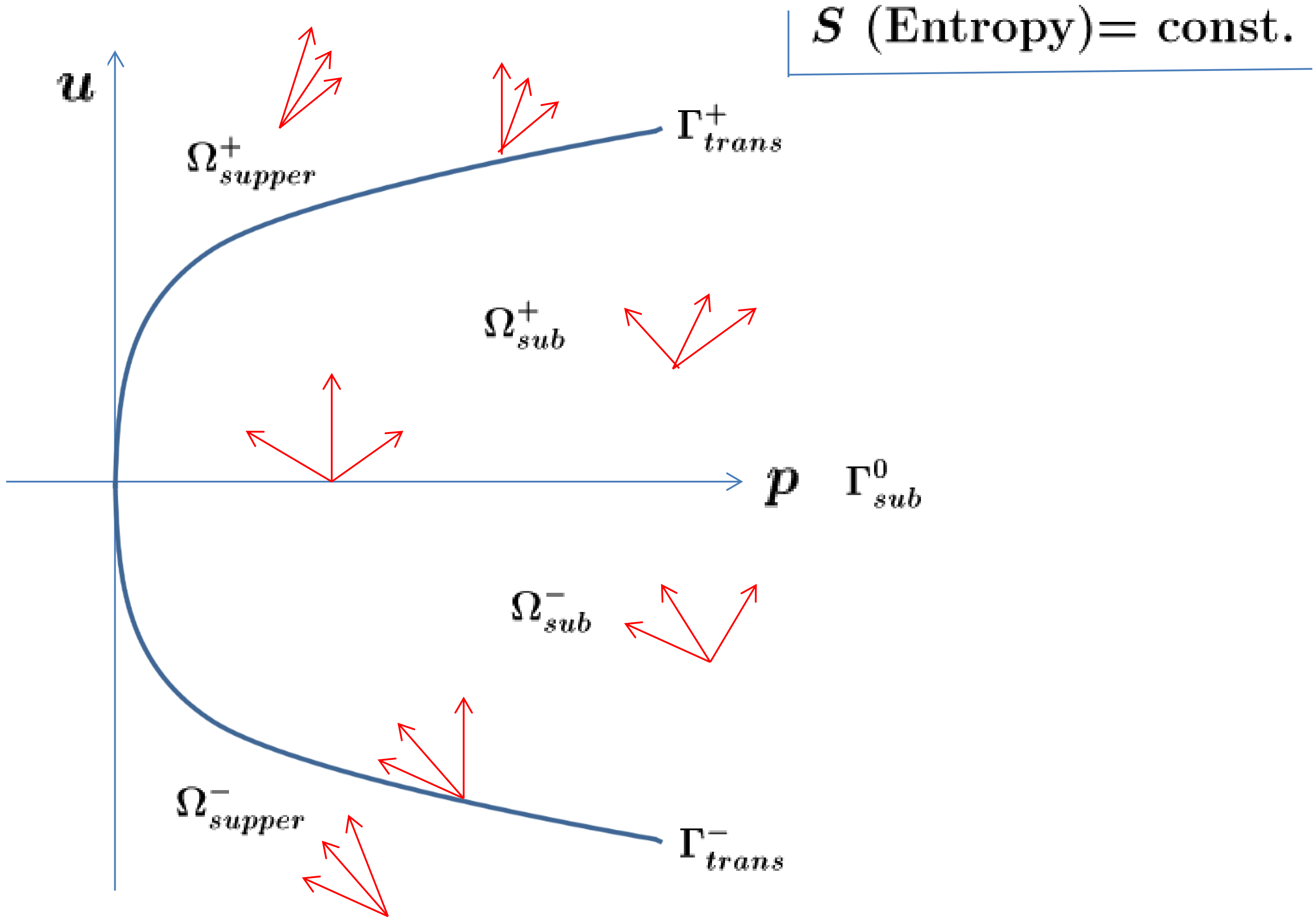
$$z \longrightarrow \text{stationary solution} + \text{outgoing asymptotic waves}, \quad t \to \infty.$$

(boundary layer solution)



Important are the signs of the characteristic speeds

$$\lambda_1(z) = u - c_s, \quad \lambda_2(z) = u, \quad \lambda_3(z) = u + c_s.$$


S (Entropy)= const.
u
p
Ω^+_{supper}
Γ^+_{trans}
Ω^+_{sub}
Γ^0_{sub}
Ω^-_{sub}
Ω^-_{supper}
Γ^-_{trans}

$$\Omega^+_{sub} := \{z \mid 0 < u < c_s\}, \quad \Omega^-_{sub} := \{z \mid -c_s < u < 0\},$$

$$\Omega^+_{supper} := \{z \mid u > c_s\}, \quad \Omega^-_{supper} := \{z \mid u < -c_s\},$$

$$\Gamma^+_{trans} := \{z \mid u = c_s\}, \quad \Gamma^0_{sub} := \{z \mid u = 0\},$$

$$\Gamma^-_{trans} := \{z \mid u = -c_s\}.$$

- For any fixed far field state z_+, consider the situation where the initial data $z_0(\cdot)$ belongs a sufficiently small neighborhood ω_{z_+} of z_+.

For all cases $z_+ \in \Omega^\pm_{sub}, \Omega^\pm_{supper}, \Gamma^\pm_{trans}$, and Γ^0_{sub}, we can classify all the asymptotic behaviors depending on the boundary conditions in ω_{z_+} by the general principle .

Boundary condition for $\mu = 0$ (Non-viscous and heat-conductive case)

$$\begin{cases} \rho_t + u\rho_x + \rho u_x = 0, \\ u_t + uu_x + \dfrac{p}{\rho^2}\rho_x = l.o.t., \\ \dfrac{R}{\gamma - 1}\theta_t - \dfrac{\kappa}{\rho}\theta_{xx} = l.o.t. \end{cases}$$

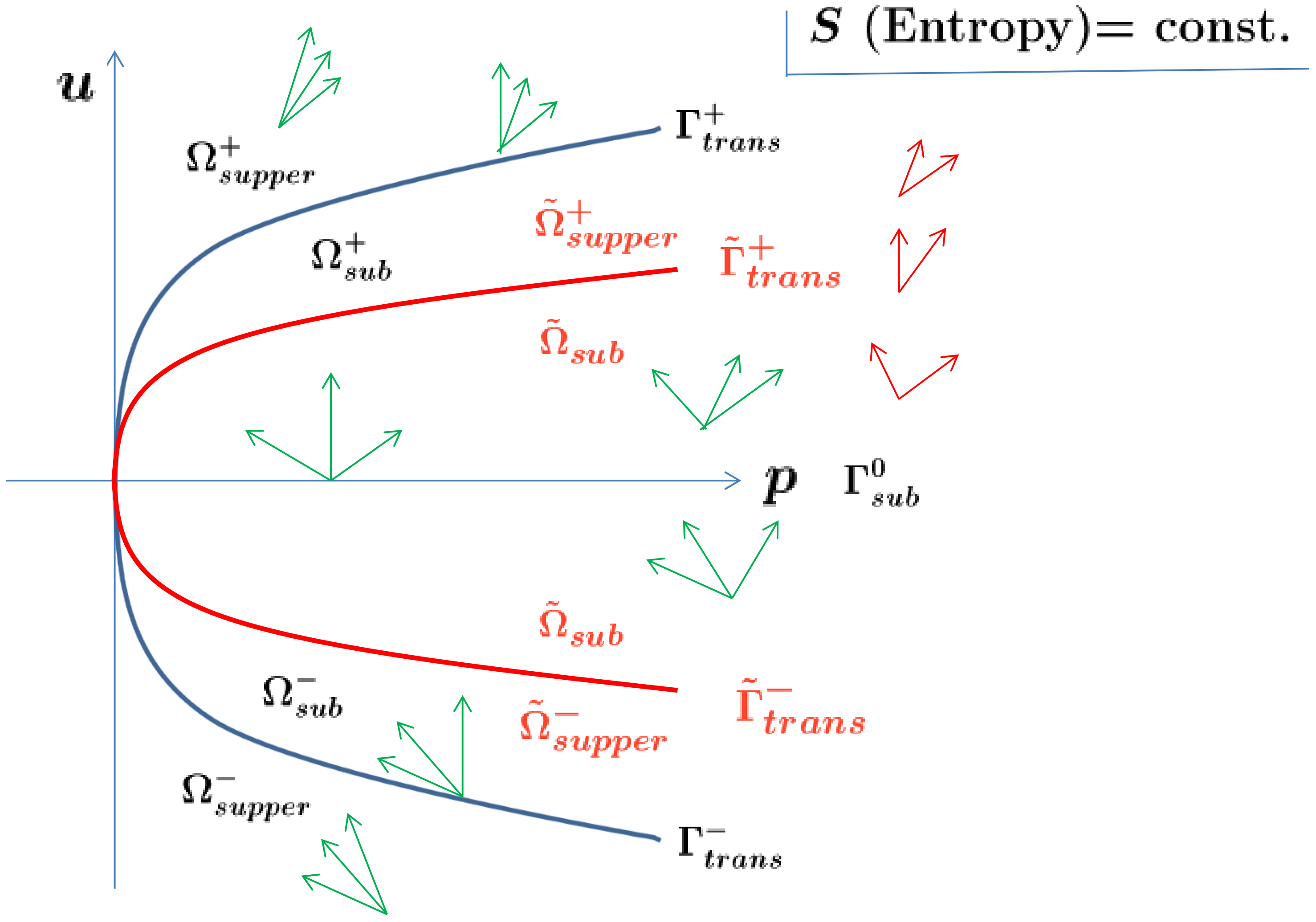
The boundary condition is depending on the signs of

$$u \pm \tilde{c_s}$$

where

$$\tilde{c_s} = \sqrt{\frac{p}{\rho}} = \frac{c_s}{\sqrt{\gamma}} < c_s.$$

$$\tilde{\Omega}_{sub} := \{ z \mid |u| < \tilde{c_s} \}, \quad \tilde{\Gamma}^{\pm}_{trans} := \{ z \mid u = \pm\tilde{c_s} \},$$
$$\tilde{\Omega}^{+}_{supper} := \{ z \mid u > \tilde{c_s} \}, \quad \tilde{\Omega}^{-}_{supper} := \{ z \mid u < -\tilde{c_s} \}.$$



S (Entropy)= const.
u
p
Ω^{+}_{supper}
Γ^{+}_{trans}
$\tilde{\Omega}^{+}_{supper}$
Ω^{+}_{sub}
$\tilde{\Gamma}^{+}_{trans}$
$\tilde{\Omega}_{sub}$
Γ^{0}_{sub}
$\tilde{\Omega}_{sub}$
Ω^{-}_{sub}
$\tilde{\Gamma}^{-}_{trans}$
$\tilde{\Omega}^{-}_{supper}$
Ω^{-}_{supper}
Γ^{-}_{trans}

$z_+ \in \tilde{\Omega}^+_{supper}$

In a neighborhood ω_{z_+},

$$\text{B.C.} \begin{cases} \rho(t,0) = \rho_-, \\ u(t,0) = u_-, \qquad t \geq 0. \\ \theta(t,0) = \theta_-, \end{cases}$$

- For $z_+ \in \Omega^+_{supper} \subset \tilde{\Omega}^+_{supper}$, the asymptotic behavior is expected to be the same as that for the Cauchy problem.

- For $z_+ \in \Omega^+_{sub} \cap \tilde{\Omega}^+_{supper}$, the asymptotic behavior is expected to be

$$z \longrightarrow \text{stationary solution} + \text{outgoing asymptotic waves}, \quad t \to \infty.$$

$z_+ \in \tilde{\Omega}^-_{supper}$

In a neighborhood ω_{z_+},

$$\text{B.C.} \quad \theta(t, 0) = \theta_-.$$

- For $z_+ \in \Omega^-_{supper} \subset \tilde{\Omega}^-_{supper}$, the asymptotic behavior is expected to be

$$z \longrightarrow \text{stationary solution}, \quad t \to \infty.$$

Nakamura-Nishibata, preprint : Existence and asymptotic stability of stationary solutions for the Kawashima-Shizuta system.

- For $z_+ \in \Omega^-_{sub} \cap \tilde{\Omega}^-_{supper}$, the asymptotic behavior is expected to be

$$z \longrightarrow \text{an outgoing asymptotic wave}, \quad t \to \infty.$$

$z_+ \in \tilde{\Omega}_{sub}$

In a neighborhood ω_{z_+},

$$\text{B.C.} \begin{cases} u(t,0) = u_-, \\ \theta(t,0) = \theta_-, \end{cases} \quad t \geq 0.$$

- For $z_+ \in \Omega_{sub}^+ \cap \tilde{\Omega}_{sub}$, the asymptotic behavior is expected to be

$$z \longrightarrow \text{outgoing asymptotic waves}, \quad t \to \infty.$$

- For $z_+ \in \Omega_{sub}^- \cap \tilde{\Omega}_{sub}$, the asymptotic behavior is expected to be

$$z \longrightarrow \text{stationary solution} + \text{an outgoing asymptotic wave}, \quad t \to \infty.$$

- For $z_+ \in \Gamma_{sub}^0$ and $u_- = 0$, the asymptotic behavior is expected to be

$$z \longrightarrow \text{a viscous contact wave with zero convection}, \quad t \to \infty.$$

(one of basic open problems)

$z_+ \in \tilde{\Gamma}^{\pm}_{trans}$

In a neighborhood ω_{z_+},

B.C. : Subtle!

In this case, we may have to consider the well-posedness of the problem without separating the system to the hyperbolic part and parabolic part.

These situations above for the half space problem shows much differences from the viscous case $\mu > 0$, and so would be more interesting as the next topics.

A Toy Model

$$\begin{cases} u_t + uu_x + \theta_x = 0, \\ \theta_t + u_x = \kappa\theta_{xx}, \end{cases} \qquad t \geq 0, x \geq 0,$$

with the initial data

$$(u, \theta)(0, x) = (u_0, \theta_0)(x), \quad x \geq 0,$$

the far field condition

$$\lim_{x \to \infty} (u, \theta)(t, x) = (u_+, \theta_+), \quad t \geq 0,$$

and the boundary conditions

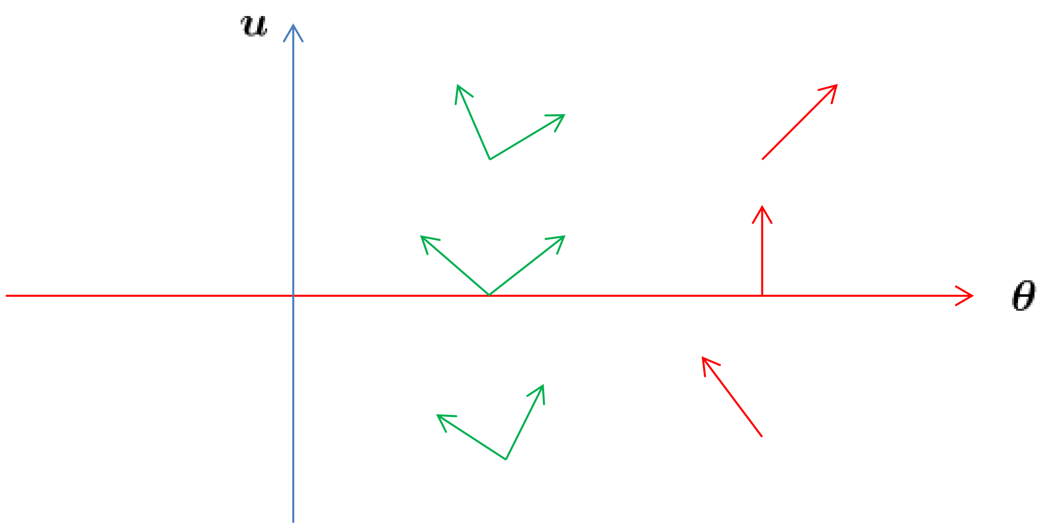
$$\theta(t, 0) = 0, \quad u(t, 0) = \text{depends !}.$$

Hyperbolic part

$$\begin{pmatrix} u \\ \theta \end{pmatrix}_t + \begin{pmatrix} u & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ \theta \end{pmatrix}_x = 0.$$

Characteristic speeds

$$\lambda_1(u) = \frac{u - \sqrt{u^2+4}}{2} < 0 < \lambda_2(u) = \frac{u + \sqrt{u^2+4}}{2}$$



Boundary conditions and asymptotic behaviors for $u \neq 0$

- For $u_+ > 0$ and z_- is in a neighborhood of z_+, the boundary condition should be

$$\text{B.C.} \quad \begin{cases} u(t,0) = u_-, \\ \theta(t,0) = 0, \end{cases} \qquad t \geq 0,$$

and the asymptotic behavior is expected to be

$$z \longrightarrow \text{stationary solution} + \text{an outgoing asymptotic wave}, \qquad t \to \infty.$$

- For $u_+ < 0$ and z is in a neighborhood of z_+, the boundary condition should be only

$$\text{B.C.} \quad \theta(t,0) = 0,$$

and the asymptotic behavior is expected to be

$$z \longrightarrow \text{outgoing asymptotic wave}, \qquad t \to \infty.$$

The case $u_- = 0$ or $u_+ = 0$ is a subtle problem!

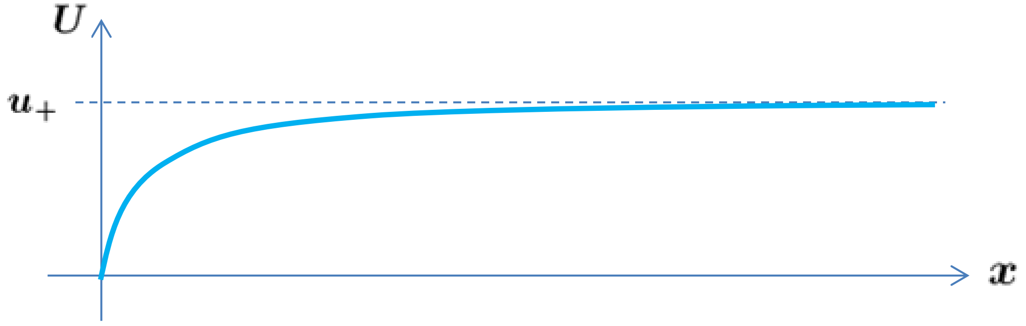
Stationary solution

Assume $\frac{1}{2}u_-^2 = \frac{1}{2}u_+^2 + \theta_+$.

$$\begin{cases} \frac{1}{2}U^2 + \Theta = \frac{1}{2}u_+^2 + \theta_+, \\ \kappa\Theta_x = U - u_+, \end{cases} \qquad x > 0.$$

- In the case $u_+ > 0$ and $u_- = 0$

$$\begin{cases} \kappa U_x = \frac{u_+ - U}{U}, \quad x > 0, \\ U(0) = 0, \ U(\infty) = u_+. \end{cases}$$



$$U(x) \sim \sqrt{x}, \quad x \to 0.$$

- **In the case $u_+ = 0$ and $u_- > 0$**

$$\begin{cases} \kappa U_x = -1 \quad \text{or} \quad U = 0, & x > 0, \\ U(0) = u_-, \ U(\infty) = 0. \end{cases}$$

Problem: Investigate the asymptotic behaviors of the solution around these stationary solutions.

Thank You!