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Stability of stationary flow of compressible fluids subject to large external potential forces

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1. INTRODUCTION

This paper is a brief note on some mathematical questions concerning a rest state $\mathscr{S}_0$ for a general polytropic moving in a bounded rigid vessel under the action of an external potential force. Precisely, we prove the uniqueness of $\mathscr{S}_0$, the existence of global (in time) motions starting (at the initial time) from small perturbations to $\mathscr{S}_0$, and their behavior at large time (stability of $\mathscr{S}_0$).

As well known, the problem of the global (in time) existence and of the asymptotic behavior of a non-steady compressible flow has attracted the attention of several authors (cf. Matsumura and Nishida (1980, 1982, 1983), Valli (1983), Valli and Zajaczkowski (1986), Padula (1986), Coscia and Padula (1990)). Moreover, also the existence and the uniqueness of a solution of steady compressible fluids, recently, received several contributions (cf., e.g., Padula (1981, 1983a, 1983b, 1987), Fujita Yashima (1986), Valli (1987), Beirao da Veiga (1987), Matsumura and Nishida (1989), Farwig (1989)). However, several important physical questions still remain unsolved.

In particular, there have been no results on the global existence and large-time behavior of the solutions for «large external forces» except some one-dimensional results. Here, we intend to furnish an answer to the following one:

In a bounded rigid fixed vessel, let a viscous heat-conducting fluid be subject only to a large potential force. Does any perturbation to the rest state eventually vanish?

Such a question for compressible fluids meets, in general, a crucial difficulty: the rest state may include a vaccum part. In this relation we remark that the constitutive form prescribed on the pressure appears to play a fundamental role. In fact, da Veiga (1987) provides necessary and sufficient conditions on the constitutive equations of the pressure for the existence of the rest state, with a positive density, for general barotropic fluid. The objective pursued here is just to prove, in the same class of constitutive assumptions on the pressure, that the rest state, with a positive density, is unique and is asymptotically stable. To this end, throughout this paper, we technically manipulate the thermodynamical relations and energy forms. Then, we derive the well-posedness of the steady problem, starting from showing the uniqueness of the rest state without smallness of the external forces. Next, the well-posedness and asymptotic stability for the unsteady problem around the rest state is achieved by a slight modification of the classical energy method, in the form used first by Matsumura and Nishida (1980) (also cf. Galdi and Padula (1990)).

An existence and regularization theorem of steady flows of heat-conducting gases in presence of large potential forces and small non-potential forces, is given in Novotny and Padula, in preparation.

The plan of the paper is the following. In section 2 we state the problem, and in section 3 we prove existence and uniqueness of compressible heat-conducting stationary flows satisfying the suitable constitutive assumptions, subject to the action of an external potential force. In section 4 we prove a cricial energy identity which enables us to state an universal stability criterion (cf. Serrin (1959)). Precisely, the exponential stability of the rest state holds for *any* large external force and for *any* regular flow which is uniformly bounded in time. Next, in section 5 we prove a global existence theorem of regular flows subject to large external potential forces for «*sufficiently*» small initial perturbations.

2. STATEMENT OF THE PROBLEM

Let Ω be a bounded three dimensional domain of $\mathbb{R}^3$ having smooth boundary $\partial\Omega$. The basic equation governing the flow of a viscous heat-conducting compressible fluid are

$$\begin{aligned}
&\rho_t + \nabla\cdot(\rho u) = 0,\\
&(\rho u)_t + \nabla\cdot(\rho u\otimes u + T) = f,\\
(2.1)\qquad &\left(\rho\left(1/2\,|u|^2 + e\right)\right)_t + \nabla\cdot\\
&\cdot\left(\rho u\left(1/2\,|u|^2 + e\right) - \kappa\nabla\theta + T\cdot u\right) = \rho u\cdot f,\\
&\rho(t,x)\geq 0,\quad T = pI - \mu\Theta - \lambda(\nabla\cdot u)I.
\end{aligned}$$

Here, ρ,θ and e denote the unsteady density, temperature and the internal energy per unit mass, respectively, furthermore, u is the velocity, $\times$ represents the diadic symbol, f the external force, p the pressure, $\Theta_{ij} \equiv (\partial u_i/\partial x_j) + (\partial u_j/\partial x_i)$. Moreover, the constant coefficients λ, μ are the shear and bulk viscosity and κ the thermal condictivity. From thermodynamical consideration we assume

(i) $$3\lambda + 2\mu \geq 0, \quad \mu > 0, \quad \kappa > 0,$$

(ii) $\exists$ smooth entropy $S(\rho, \theta)$ such that

$$\mathrm{d}e = \theta \mathrm{d}S - p\mathrm{d}(1/\rho), \quad e = e(\rho, \theta), \quad p = p(\rho, \theta);$$

(iii) $e = c_v\theta$, $c_v =$ positive constant $=$ specific heat at constant volume;
(iv) $p = p(\rho, \theta)$, $p > 0$ and $p_\rho > 0$ for $\rho > 0$ and $\theta > 0$.

Condition (iii) is the polytropy condition and it is usually made. Moreover, we assume,

$$f = -\nabla\phi, \quad \phi = \phi(x).$$

We have (cf. Courant and Friedrichs 1948).

LEMMA 2.1. The thermodynamical relations (ii), (iii), (iv) imply

$$(2.2) \qquad \begin{aligned} &p(\rho, \theta) = \theta G(\rho), \quad G > 0, \quad G_\rho(\rho) > 0, \quad \text{for} \quad \rho > 0; \\ &S = S(\rho, \theta) = c_v \log \theta - \int^\rho (G(s)/s^2)\mathrm{d}s. \end{aligned}$$

Proof. By (ii) it easily follows

$$\mathrm{d}S = (c_v/\theta)\,\mathrm{d}\theta - (p/\rho^2\theta)\,\mathrm{d}\rho,$$

from which

$$S_\theta = (c_v/\theta), \quad S_\theta = -(p/\rho^2\theta).$$

By $S_{\rho\theta} = 0$ it delivers $p = \theta p_\theta$, with $p_\theta \equiv G(\rho)$ independent of θ , and $(2.2)_1$ follows. Next, integrating (ii) we obtain $(2.2)_2$ up to a constant.

The Lemma 2.1, now proved, asserts that the constitutive equation of an ideal fluid $p = R\rho\theta$ is included as particular case.

In the sequel for any scalar function φ denoting a physical quantity as density, temperature, etc. we set by $\varphi = \varphi(t, x)$ the function computed for unsteady flows while $\widehat{\varphi} = \widehat{\varphi}(x)$ represents the same function computed for steady flows.

After Lemma 2.1 we can rewrite (2.1) as

$$
\begin{aligned}
&\rho_t + \nabla\cdot(\rho u) = 0\,,\\
(2.3)\qquad &(\rho u)_t + \nabla\cdot(\rho u\otimes u) - \nabla\cdot(\mu\Theta + \lambda(\nabla\cdot u)I) = -\nabla p - \rho\nabla\phi,\\
&c_v\rho\left(\theta_t + u\cdot\nabla\theta\right) - \kappa\Delta\theta + p\nabla\cdot u - \Psi = 0\,,
\end{aligned}
$$

where $\Psi \equiv (\mu/2)\Theta : \Theta + \lambda(\nabla\cdot u)^2$. For given positive constants $\overline{\theta}$ and $\overline{\rho}$, we have append the following initial and boundary conditions

$$
\begin{aligned}
&(\rho, u, \theta)(0, x) = \left(\rho_0, u_0, \theta_0\right)(x), \quad x\in\Omega,\\
(2.4)\qquad &\inf\rho_0 > 0, \quad \inf\theta_0 > 0, \quad \int_\Omega \rho_0\,\mathrm{d}\,x = |\Omega|\overline{\rho};
\end{aligned}
$$

and

$$
(2.5)\qquad u(t, x) = 0, \quad \theta(t, x) = \overline{\theta}, \quad t\geq 0, \quad x\in\partial\Omega.
$$

Since $\phi = \phi(x)$ and $\overline{\theta} =$ const., equation (2.3), (2.5) admit the triple $(\widehat{\rho}, \widehat{u}, \widehat{\theta})$ solution to the following boundary problem

$$
\begin{aligned}
&\nabla\cdot(\widehat{\rho}, \widehat{u}) = 0, \quad \widehat{\rho}(x) > 0,\\
(2.6)\qquad &\widehat{\rho}, \widehat{u}\cdot\nabla\widehat{u} + \nabla\widehat{p} - \nabla\cdot(\mu\widehat{\Theta} + \lambda(\nabla\cdot\widehat{u})I) = -\widehat{\rho}\nabla\phi,\\
&c_v\widehat{\rho}, \widehat{u}\cdot\nabla\theta + \widehat{p}\nabla\cdot\widehat{u} - \kappa\widehat{\theta} - \widehat{\Psi} = 0, \quad x\in\partial\Omega,
\end{aligned}
$$

with

$$
\begin{aligned}
&\widehat{u}|_{\partial\Omega} = 0, \quad \widehat{\theta}|_{\partial\Omega} = \overline{\theta},\\
(2.7)\qquad &\int_\Omega \widehat{\rho}(x)\,\mathrm{d}\,x = |\Omega|\overline{\rho}\left(= \int_\Omega \rho_0(x)\,\mathrm{d}\,x\right).
\end{aligned}
$$

3. EXISTENCE AND UNIQUENESS OF STATIONARY FLOWS FOR LARGE EXTERNAL FORCES

The aim of this section is to prove that under the thermodynamical assumptions (i),...,(iv), there exist a unique stationary solution to (2.6), (2.7) corresponding to any large potential ϕ . Since the existence has been already proven (cf. da Veiga (1987)), here we prove in the lemma 3.1 the uniqueness.

First note that a particular solution to (2.6), (2.7) is the triple $(\widehat{\rho}, 0, \overline{\theta})$ solution to

$$\begin{aligned} &\int_{\overline{\rho}}^{\widehat{\rho}} \frac{p_\rho(s,\overline{\theta})}{s} \mathrm{d}\, s + \phi = \text{ const.}, \\ &\int_\Omega \widehat{\rho}(x)\, \mathrm{d}\, x = |\Omega|\overline{\rho}. \end{aligned} \tag{3.1}$$

The lemma below proves that it is the unique solution to (2.6), (2.7).

LEMMA 3.1. Let $\widehat{\rho} \in C^1(\overline{\Omega})$, $(\widehat{u}, \widehat{\theta}) \in C^2(\overline{\Omega})$, $\inf \widehat{\rho} > 0$, $\inf \widehat{\theta} > 0$, then the triple $(\widehat{\rho}, 0, \overline{\theta})$ solving (3.1) is the unique solution to the boundary value problem (2.6), (2.7).

Proof. Multiplying $(2.6)_1$ by $\int_{\overline{\rho}}^{\widehat{\rho}} \frac{p_\rho(s,\overline{\theta})}{s} \mathrm{d}\, s$, $(2.6)_2$ by $\widehat{u}$ and $(2.6)_3$ by $(1 - (\overline{\theta}/\widehat{\theta}))$, next integrating over Ω it delivers

$$\begin{aligned} &\int_\Omega (\widehat{u} \cdot \nabla + \widehat{\Psi})\, \mathrm{d}\, x = 0, \\ &\int_\Omega \left(1 - (\overline{\theta}/\widehat{\theta})\right)\left(c_v \widehat{\rho}\widehat{u} \cdot \nabla\widehat{\theta} + \widehat{p}\nabla \cdot \widehat{u} - \kappa\Delta\widehat{\theta} - \widehat{\Psi}\right) \mathrm{d}\, x = 0, \\ &\int_\Omega \left(\int_{\overline{\rho}}^{\widehat{\rho}} \frac{p_\rho(s,\overline{\theta})}{s} \mathrm{d}\, s\right) \nabla \cdot (\widehat{\rho}\widehat{u})\, \mathrm{d}\, x = 0. \end{aligned} \tag{3.2}$$

Summing $(3.2)_1$ to $(3.2)_2$ it furnishes

$$\int_\Omega \left\{-(\overline{\theta}/\widehat{\theta})(\widehat{p}\nabla \cdot \widehat{u} - \widehat{\Psi}) + \kappa\left(\overline{\theta}/\widehat{\theta}^2\right)|\nabla\widehat{\theta}|^2\right\} \mathrm{d}\, x = 0. \tag{3.3}$$

Moreover, $(3.2)_3$ implies

$$0 = \int_\Omega p_\rho(\widehat{\rho}, \overline{\theta})\widehat{u} \cdot \nabla\widehat{\rho}\, \mathrm{d}\, x = \int_\Omega p(\widehat{\rho}, \overline{\theta})\nabla \cdot \widehat{u}\, \mathrm{d}\, x. \tag{3.4}$$

The particular functional dependence $(2.2)_1$ of p on θ and (3.4) allow to conclude from (2.10)

$$\int_\Omega \left\{(\overline{\theta}/\widehat{\theta})\Psi + k\left(\overline{\theta}/\widehat{\theta}^2\right)|\nabla\widehat{\theta}|^2\right\} \mathrm{d}\, x = 0. \tag{3.5}$$

From (3.5) we infer $\widehat{u} = 0$, $\widehat{\theta} = \overline{\theta}$ and $\widehat{\rho}$ solution to the equation $(3.1)_1$. The proof of the Lemma is so completed.

To solve (3.1) we assume $\phi \in C^1(\overline{\Omega})$,

$$\lim_{r\to\infty} \int_{\overline{\rho}}^{r} \frac{p_\rho(s,\overline{\theta})}{s} \mathrm{d}\, s = +\infty. \tag{3.6}$$

Relation (3.6) is a typical regularity assumption which infer regularity for the density, furthermore, it generalizes the hypothesis of ideal fluids, usually adopted. We also assume

$$\lim_{r\to 0} \int_{\overline{\rho}}^{r} \frac{p_\rho(s,\overline{\theta})}{s} \mathrm{d}\, s = -\infty, \tag{3.7}$$

or

$$\begin{gathered} \int_0^{\overline{\rho}} \frac{p_\rho(s,\overline{\theta})}{s} \mathrm{d}\, s < +\infty \\ (1/|\Omega|) \int_\Omega \widehat{\rho}_c(x) \mathrm{d}\, x < \overline{\rho}, \end{gathered} \tag{3.8}$$

where the function $\widehat{\rho}_c$ is defined by

$$\int_0^{\widehat{\rho}_c} \frac{p_\rho(s,\overline{\theta})}{s} \mathrm{d}\, s = -\phi(x) + \sup_x \phi(x). \tag{3.9}$$

From the results of da Veiga (1987) and Lemma 3.1 it easily follows

THEOREM 3.1. Let (3.6) be satisfied together with either (3.7) or (3.8). Then there exists a unique steady solution $(\widehat{\rho}, \widehat{u}, \widehat{\theta}) = (\widehat{\rho}, 0, \overline{\theta})$ to (2.6), (2.7) and a unique $\overline{\phi} \in \mathbf{R}$ such that

$$\widehat{\rho} \in C^1(\overline{\Omega}), \quad \inf \widehat{\rho} > 0, \quad \int_\Omega \widehat{\rho}(x) \mathrm{d}\, x = |\Omega| \overline{\rho},$$

$$\int_{\overline{\rho}}^{\widehat{\rho}} \frac{p_\rho(s,\overline{\theta})}{s} \mathrm{d}\, s + \phi = \overline{\phi}, \quad x \in \Omega.$$

The proof is an easy consequence of assumptions made on the constitutive equation for p and of Lemmas 3.1, 3.2, 3.3.

REMARK 3.1. It is worth of notice that the counter example of da Veiga (1987) applies equally to those fluids having constitutive equations for the pressure not obeying one of the restrictions (3.7), or (3.8). An obvious consequence of such result is that stationary flows of arbitrary polytropic gases, in a fixed rigid vessel, cannot exist, for arbitrary large external potential forces, unless the total prescribed mass of the gas is sufficiently large.

REMARK 3.2. Using the function $\widehat{\rho}_c(x)$ defined by (3.9) the condition $(3.8)_2$ can be interpreted as a (smallness) condition on the potential force field $\phi(x)$, once the total mass is let to be small.

4. A UNIVERSAL STABILITY CRITERION

In this section we shall prove a crucial energy inequality which allow us to state a universal stability criterion.

Let $(\widehat{\rho}, 0, \overline{\theta})$ be the steady flow whose existence and uniqueness are given in Theorem 3.1. As known (cf. Okada and Kawashima (1983)), the internal energy e when considered as function of $V = (1/\rho)$ and S, say $e = \varepsilon(V, S)$, has the Hessian positive definite for $\rho > 0$, $\theta > 0$. Therefore, by the Taylor polynomial, the form

$$(4.1) \qquad \rho\left[\varepsilon - \overline{\varepsilon} - \overline{\varepsilon}_V(V - \overline{V}) - \overline{\varepsilon}_S(S - \overline{S})\right]$$

results a positive definite quadratic form. Here we set $\overline{\varepsilon} \equiv \varepsilon(\overline{V}, \overline{S})$,..., etc. Moreover, our thermodynamical consideration infer

$$(4.2) \qquad \varepsilon_V = -p, \quad \varepsilon_S = \theta.$$

Inspired by (4.1), using $(2.2)_1$, we can manipulate suitably the total energy E as follows

$$\begin{aligned}
\rho E &= \rho\left\{e - e(\widehat{\rho}, \overline{\theta}) + p(\widehat{\rho}, \overline{\theta})[(1/\rho) - (1/\widehat{\rho})] + \right.\\
&\left. -\overline{\theta}(S - S(\widehat{\rho}, \overline{\theta})\right\} + (\rho/2)|u|^2 = \\
(4.3) \qquad &= (\rho/2)|u|^2 + c_v\rho[(\theta - \overline{\theta}) - \overline{\theta}\log(\theta/\overline{\theta})] + \\
&+ \left[\rho\int_{\widehat{\rho}}^{\rho} \frac{p(s, \overline{\theta})}{s^2} \mathrm{d}s - \frac{p(s, \overline{\theta})}{\widehat{\rho}}(\rho - \widehat{\rho})\right]
\end{aligned}$$

with

$$\begin{aligned}
&\left[\rho\int_{\widehat{\rho}}^{\rho} \frac{p(s, \overline{\theta})}{s^2} \mathrm{d}s - \frac{p(s, \overline{\theta})}{\widehat{\rho}}\sigma\right] = \\
&= \overline{\theta}\left[G_\rho(\xi)/2\xi\right]\sigma^2, && \exists\, \xi \text{ between } \widehat{\rho} \text{ and } \rho, \\
&\chi - \overline{\theta}\log(\theta/\overline{\theta}) = (\overline{\theta}/2\zeta^2)\,\chi^2, && \exists\, \zeta \text{ between } \overline{\theta} \text{ and } \theta,
\end{aligned}$$

and where $\sigma \equiv (\rho - \widehat{\rho}), u$, $\chi \equiv (\theta - \overline{\theta})$, denote the unsteady perturbation to the rest state $(\widehat{\rho}, 0, \overline{\theta})$. It becomes enormous the difference between the standard energy which is (regarded as) function of the total unsteady flow and the energy E deduced in (4.3) which is function of the perturbation only. As well known, for incompressible fluids there is no difference between the kinetic energy of the total flow and that of the perturbation only, because it is derived from a linear term. The energy E given by (4.3) will play an essential role in the sequel.

LEMMA 4.1. There exists a positive constant c depending on sup ρ, inf ρ, sup $\widehat{\rho}$, inf $\widehat{\rho}$, sup θ, inf θ and $\overline{\theta}$ such that

$$
\begin{aligned}
&c^{-1}\int_\Omega \{|u|^2+\sigma^2+\chi^2\}\,\mathrm{d}x \le \int_\Omega \rho E\,\mathrm{d}x \le \\
&\le c\int_\Omega \{|u|^2+\sigma^2+\chi^2\}\,\mathrm{d}x.
\end{aligned}
\tag{4.4}
$$

Keeping the form (4.3) in mind, we multiply $(2.3)_3$ by $[1-(\overline{\theta}/\theta)]$ and integrate over Ω. The use of the transport theorem implies after straightforward calculation

$$
\begin{aligned}
&\left(\int_\Omega c_v\rho[\chi-\overline{\theta}\log(\theta/\overline{\theta})]\right)_t + \\
&+\int_\Omega \{p[1-(\overline{\theta}/\theta)]\nabla\cdot u+\kappa(\overline{\theta}/\theta^2)|\nabla\theta|^2+[(\overline{\theta}/\theta)-1]\Psi\}=0.
\end{aligned}
\tag{4.5}
$$

Next, we multiply $(2.3)_2$ by u and integrate over Ω receiving

$$
\left(\int_\Omega \frac{1}{2}\rho|u|^2\mathrm{d}x\right)_t+\int_\Omega\{u\cdot\nabla p+\Psi+\rho u\cdot\nabla\phi\}\,\mathrm{d}x=0.
\tag{4.6}
$$

It also holds

$$
\int_{\widehat{\rho}}^{\rho}\frac{p(s,\overline{\theta})}{s^2}\mathrm{d}s+\frac{p(\rho,\overline{\theta})}{\rho}-\frac{p(\widehat{\rho},\overline{\theta})}{\widehat{\rho}}=\int_{\widehat{\rho}}^{\rho}\frac{p_\rho(s,\overline{\theta})}{s}\mathrm{d}s,
$$

this identity, together with the continuity equation $(2.3)_1$, provides an identity perfectly analogous to $(3.2)_3$

$$
\int_\Omega\left(\int_{\widehat{\rho}}^{\rho}\frac{p_\rho(s,\overline{\theta})}{s}\mathrm{d}s\right)(\rho_t+\nabla\cdot(\rho u))\,\mathrm{d}x=0.
$$

By this last relation we achieve the wanted identity for *sigma* (analogous to (3.4))

$$
\begin{aligned}
&\left(\int_\Omega\left\{\rho\int_{\widehat{\rho}}^{\rho}\frac{p(s,\overline{\theta})}{s^2}\mathrm{d}s-\frac{p(\widehat{\rho},\overline{\theta})}{\widehat{\rho}}\sigma\right\}\mathrm{d}x\right)_t+ \\
&-\int_\Omega\left\{\frac{p_\rho(\rho,\overline{\theta})}{\rho}(\rho u)\cdot\nabla\rho-\frac{p_\rho(\widehat{\rho},\overline{\theta})}{\widehat{\rho}}(\rho u)\cdot\nabla\widehat{\rho}\right\}\mathrm{d}x=0.
\end{aligned}
\tag{4.7}
$$

By noticing that

$$
\int_\Omega p_\rho(\rho,\overline{\theta})u\cdot\nabla\rho\,\mathrm{d}x=-\int_\Omega p(\rho,\overline{\theta})\nabla\cdot u\,\mathrm{d}x,
$$

$$
\int_\Omega\left(p_\rho(\widehat{\rho},\overline{\theta})/\widehat{\rho}\right)(\rho u)\cdot\nabla\widehat{\rho}\,\mathrm{d}x=\int_\Omega-(\rho u)\cdot\nabla\phi\,\mathrm{d}x,
$$

adding (4.5), (4.6) and (4.7) we receive

$$\left(\int_\Omega \rho E \,\mathrm{d}\,x\right)_t + \int_\Omega \left\{ \left(\overline{\theta}/\theta^2\right) \kappa |\nabla\theta|^2 + (\overline{\theta}/\theta)\Psi \right\} \mathrm{d}\,x = 0\,. \tag{4.8}$$

By Poincaré inequality, there exists a γ (depending on $\sup_{t,x}\theta, \overline{\theta}$ and $|\Omega|$) such that

$$\left(\int_\Omega \rho E \,\mathrm{d}\,x\right)_t + \gamma \,\|\,(u,\chi)\,\|_1^2 \le 0 \tag{4.9}$$

where $\|\cdot\|_k$ denotes $H^k(\Omega)$-norm, the L^2-norm will be indicated by $\|\cdot\|$, the L^p-norm will be indicated by $\|\cdot\|_{L^p}$. Inequality (4.9) is sufficient to ensure continuous dependence on the data and the asymptotic decay to zero for the perturbation to the kinetic and temperature fields along sequences of times, this is due to the parabolic character of the equation governing such fields. However, because of the conservative character of the continuity equation, an analogous dissipative term for the perturbation to the density field σ is not evident. In the remaining part of this section, we shall provide an algorithm, introduced in Valli (1983), which provides a dissipation for the L^2-norm of σ and requires few regularity properties on the perturbed flow. In the wake of the Valli (1983), we now consider the following two auxiliary problems.

Neumann problem

$$\begin{aligned} &\Delta\omega = \sigma, \\ &\left.\frac{\partial\omega}{\partial n}\right|_{\partial\Omega} = 0\,, \quad \int_\Omega \omega \mathrm{d}\,x = 0\,. \end{aligned} \tag{4.10}$$

Relation $(2.4)_4$ ensures that σ satisfies the compatibility condition, furthermore, $(4.10)_3$ fixes the constant up to wihch the Neumann solution exists in such a way that it results

$$\|\,\omega\,\|_2 \le C\,\|\,\sigma\,\|\,. \tag{4.11}$$

Stokes problem

$$\begin{aligned} &-\Delta w + \nabla q = 0\,, \\ &\nabla\cdot w = 0\,, \\ &\left. w\right|_{\partial\Omega} = \left.\nabla\omega\right|_{\partial\Omega}\,. \end{aligned} \tag{4.12}$$

The solution of the Stokes problem satisfies the compatibility condition $\int_{\partial\Omega} w \cdot n = 0$, because of $(4.10)_2$. Moreover, it satisfies the estimate,

$$\|w\|_1 \le C \|w\|_{1/2,\partial\Omega} \le C \|\omega\|_2 \le C \|\sigma\|, \tag{4.13}$$

where $\|\cdot\|_{1/2,\partial\Omega}$ denotes the fractional norm of order $1/2$ on the boundary (cf. Cattabriga (1961)).

The momentum equation $(2.3)_2$ can be rewritten as

$$\begin{aligned} &u_t + u \cdot \nabla u + \nabla \left(\int_{\widehat{\rho}}^{\rho} \frac{p_\rho(s, \overline{\theta})}{s} \mathrm{d}\, s \right) + \frac{p_\theta}{\rho} \nabla \theta - \frac{1}{\rho} \cdot \\ &\cdot \left[p_\rho(\rho, \overline{\theta}) - p_\rho(\rho, \theta) \right] \nabla \rho - \frac{1}{\rho} \nabla \cdot [\mu \Theta + \lambda(\nabla \cdot u) I] = 0 . \end{aligned} \tag{4.14}$$

Multiplying (4.14) by $w - \nabla\omega$, it delivers

$$\begin{aligned} &\left(\int_\Omega u \cdot (w - \nabla\omega) \mathrm{d}\, x \right)_t + \int_\Omega \{ (w - \nabla\omega) \cdot (u \cdot \nabla) u + \\ &- u \cdot (w - \nabla\omega))_t + \sigma \int_{\widehat{\rho}}^{\rho} \frac{p_\rho(s, \overline{\theta})}{s} \mathrm{d}\, s + \frac{1}{\rho} p_\theta (w - \nabla\omega) \cdot \nabla\theta + \\ &- \frac{1}{\rho} \left[p_\rho(\rho, \overline{\theta}) - p_\rho(\rho, \theta) \right] \cdot (w - \nabla\omega) \cdot \nabla\rho + \\ &+ \nabla \otimes (w - \nabla\omega) : (\mu\Theta + \lambda(\nabla \cdot u) I) - \frac{\nabla\rho}{\rho^2} \cdot \\ &\cdot [\mu\Theta + \lambda(\nabla \cdot u) I] \cdot (w - \nabla\omega) \} \mathrm{d}\, x . \end{aligned} \tag{4.15}$$

By taking the derivative with respect to the time of (4.10), (4.12) we obtain

$$\begin{aligned} &\Delta \omega_t = \sigma_t = -\nabla \cdot (\rho u) , \\ &\left. \frac{\partial \omega_t}{\partial n} \right|_{\partial\Omega} = 0 , \\ &- \Delta w_t + q_t = 0 , \\ &\nabla \cdot w_t = 0 , \\ &w_t|_{\partial\Omega} = \nabla \omega_t|_{\partial\Omega} . \end{aligned}$$

This, in turn, implies

$$\begin{aligned} &\|\nabla\omega_t\| \le C \|\rho u\| \le C \left(\sup_{t,x} \rho \right) \|u\|, \\ &\|w_t\| \le C \|\sigma_t\| \le C \|\nabla \cdot (\rho u)\|, \end{aligned} \tag{4.16}$$

and

$$\int_\Omega \sigma \left(\int_{\widehat{\rho}}^{\rho} \frac{p_\rho(s,\overline{\theta})}{s} \mathrm{d}\, s \right) \mathrm{d}\, x \geq \gamma \parallel \sigma \parallel^2,$$

with γ a positive constant depending on $\overline{\theta}$, sup $\widehat{\rho}$, inf $\widehat{\rho}$. After such considerations it is easy to deduce from (4.15) and (4.16) the

$$\begin{aligned}
&\left(\int_\Omega u \cdot (w - \nabla \omega) \mathrm{d}\, x \right)_t + \gamma \parallel \sigma \parallel^2 \leq \\
(4.17) \qquad &\leq C \int_\Omega (|\chi| + |\nabla u|) |\nabla \rho| (|w| + |\nabla \omega|) \mathrm{d}\, x + \\
&+ C \int_\Omega |u| |\nabla \cdot (\rho u)| \mathrm{d}\, x + C \left(\parallel u \parallel_1^2 + \parallel \theta \parallel_1^2 \right).
\end{aligned}$$

Assuming

$$(4.18) \qquad \nabla \rho \in L^\infty([0, j\infty), L^3(\Omega))$$

we deduce

$$\begin{aligned}
&\int_\Omega |\nabla u| |\nabla \rho| (|w| + |\nabla \omega|) \mathrm{d}\, x \leq \\
&\leq C \left(\parallel \nabla \omega \parallel_{L^6} + \parallel w \parallel_{L^6} \right) \parallel \nabla \rho \parallel_{L^3} \parallel \nabla u \parallel \leq \\
&\leq C \parallel \sigma \parallel \parallel u \parallel_1 \parallel \nabla \rho \parallel_{L^3} \leq \varepsilon \parallel \sigma \parallel^2 + \\
&+ \left(\frac{C}{\varepsilon} \right) \parallel \nabla \rho \parallel_{L^3} \parallel u \parallel_1^2, \\
&\int_\Omega |u| |\nabla \cdot (\rho u)| \mathrm{d}\, x \leq \parallel u \parallel \parallel \nabla \cdot (\rho u) \parallel \leq \\
&\leq C \parallel u \parallel_1^2 \left[(\sup \rho) + \parallel \nabla \rho \parallel_{L^3} \right]
\end{aligned}$$

and employing (4.17), we reach the following estimate

$$(4.19) \qquad \left(\int_\Omega u \cdot (w - \nabla \omega) \mathrm{d}\, x \right)_t + \gamma \parallel \sigma \parallel^2 \leq C \left(\parallel u \parallel_1^2 + \parallel \theta \parallel_1^2 \right),$$

here the constant C depends also on the L^3-norm of $\nabla \rho$. Next, we add to (4.9) relation (4.19) multiplied by a positive constant α. For α sufficiently small, we achieve the fundamental energy estimate

$$\begin{aligned}
(4.20) \qquad &\left(\int_\Omega \{ \rho E + \alpha u \cdot (w - \nabla \omega) \} \mathrm{d}\, x \right)_t + \\
&+ \gamma \left(\parallel \sigma \parallel^2 + \parallel u \parallel_1^2 + \parallel \theta \parallel_1^2 \right) \leq 0.
\end{aligned}$$

Since it is

$$\int_\Omega \{\rho E + \alpha(w - u \cdot \nabla\omega)\} \, dx \geq C \left(\| \sigma \|^2 + \| u \|^2 + \| \chi \|^2 \right),$$

employing Gronwall's Lemma in (4.17) we receive

$$\begin{aligned} \| (\sigma, u, \chi) \|^2 (t) &\leq C \int_\Omega \{\rho E + \alpha u \cdot (w - \nabla\omega)\} \, dx \leq \\ &\leq C e^{-\gamma t} \int_\Omega \{\rho E + \alpha u \cdot (w - \nabla\omega)\} \, dx \Big|_{t=0} \end{aligned} \tag{4.21}$$

The inequality (4.21) proves the any sufficiently regular perturbation to the rest decays exponentially in time *for every large potential force.* This is resumed in the main theorem of this section

THEOREM 4.1. Suppose the regular solution (ρ, u, θ) of (2.3), (2.4), (2.5) satisfies

$$\begin{aligned} &\sup_{t,x} \rho(t,x) < +\infty, \ \inf_{t,x} \rho(t,x) > 0, \\ &\sup_{t,x} \theta(t,x) < +\infty, \ \inf_{t,x} \theta(t,x) > 0, \\ &\sup_{t,x} |u(t,x)| < +\infty, \quad \text{and} \quad \nabla\rho \in L^\infty([0,\infty); L^3). \end{aligned} \tag{4.22}$$

Then the solution tends to the stationary solution $(\rho, 0, \overline{\theta})$ exponentially in the L^2- norm in the sense of (4.21).

Let us explicitly observe that the above theorem does not require smallness of external forces and of the initial data, as well. This latter statement has an important consequence because, in order to prove stability, it is enough to prove that the class of regular unsteady flow with the uniform (in time) estimates (4.22) is non empty. This will be the content of the next section.

5. ON THE EXISTENCE OF REGULAR GLOBAL FLOWS

In this section we shall prove the existence of global (in time) regular flows in a suitably small neighbourhood of the steady flow $(\widehat{\rho}, 0, \overline{\theta})$, and, consequently, prove the exponential asymptotic stability of the steady flow for any large external potential force. To do that, we further assume the followings:

$$\phi \in H^4(\Omega), \tag{5.1}$$

$$(\rho_0, u_0, \theta_0) \in H^3(\Omega) \tag{5.2}$$

and the compatibility conditions

$$\begin{aligned} &u_0|_{\partial\Omega} = 0, \qquad \theta_0|_{\partial\Omega} = \overline{\theta}, \\ &u_t(0)|_{\partial\Omega} = 0, \qquad \theta_0(0)|_{\partial\Omega} = \overline{\theta}, \end{aligned} \tag{5.3}$$

where $u_t(0, x)$ and $\theta_t(0, x)$ are determined by the equations (2.3) and the initial data (2.4). Our main theorem in this section reads as follows.

THEOREM 5.1. Suppose (5.1)-(5.3) hold, and let p satisfy (3.6) together with either (3.7) or (3.8). Then there exist positive constants ε_0, β, and C_0, depending on $\overline{\rho}$, $\overline{\theta}$ and $\| \phi \|_4$, such that if $\| \rho_0 - \widehat{\rho},\ u_0,\ \theta_0 - \overline{\theta} \|_3 \leq \varepsilon_0$, then the initial boundary value problem has a unique global in time solution (ρ, u, θ) satisfying

$$\begin{aligned} &(\rho, u, \theta) \in (C^0 \cap L^\infty)([0, +\infty); H^3(\Omega)), \\ &\inf_{t,x} \rho(t, x) > 0, \qquad \inf_{t,x} \theta(t, x) > 0, \end{aligned} \tag{5.4}$$

and as the asymptotic behavior

$$\sup_x |(\rho - \widehat{\rho}, u, \theta - \overline{\theta})(t, x)| \leq C_0 e^{-\beta t}. \tag{5.5}$$

Proof. First, we notice that once we get the global solution satisfying (5.4), the stability criterion in section 4 and the Sobolev type inequality

$$\sup_x |(\rho - \widehat{\rho}, u, \theta - \overline{\theta})| \leq$$

$$\leq C \| (\rho - \widehat{\rho}, u, \theta - \overline{\theta}) \|^{1/4} \| \rho - \widehat{\rho}, u, \theta - \overline{\theta} \|_2^{3/4},$$

easily imply the exponential stability (5.5). To obtain the global solution satisfying (5.4), we set

$$\rho = \widehat{\rho} + \sigma, \qquad \theta = \overline{\theta} + \chi,$$

and rewrite the original problem (2.3-6) in the following form

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\sigma + \nabla \cdot (\widehat{\rho} u) = h_0(\sigma, u), \\ &(u)_t - (\mu/\widehat{\rho})\Delta u - [(\mu + \lambda)/\widehat{\rho}]\nabla\nabla \cdot u + \nabla[(\overline{\theta} G_\rho(\widehat{\rho})/\widehat{\rho})\sigma] + \\ &+ (1/\widehat{\rho})\nabla(G(\widehat{\rho})\chi) = h(\sigma, u, \chi), \\ &\chi_t + [\overline{\theta} G(\widehat{\rho})/(c_v \widehat{\rho})]\nabla \cdot u - [\kappa/(c_v \widehat{\rho})]\Delta \chi = h_4(\sigma, u, \chi), \end{aligned} \tag{5.6}$$

with

$$(\sigma, u, \chi)(0, x) = (\sigma_0, u_0, \chi_0) = (\rho_0 - \widehat{\rho}, u_0, \theta_0 - \overline{\theta}) \in H^3, \tag{5.7}$$

and

$$u|_{\partial\Omega} = \chi|_{\partial\Omega} = 0, \tag{5.8}$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + u \cdot \nabla$ is the Lagrangian derivative, furthermore,

$$\begin{aligned}
&h_0(\sigma, u) \equiv -\sigma \nabla \cdot u, \\
&h(\sigma, u, \chi) \equiv -u \cdot \nabla u - [\sigma/(\widehat{\rho} + \sigma)\widehat{\rho}](\mu \Delta u + (\mu + \lambda)\nabla\nabla \cdot u) + \\
&- \nabla \left(\int_{\widehat{\rho}}^{\widehat{\rho}+\sigma} \left[\overline{\theta} G_\rho(s)/s \right] d s - \left[\overline{\theta} G_\rho(\widehat{\rho})/(\widehat{\rho}) \right] \sigma \right) + \\
&+ \left(\frac{G(\widehat{\rho})}{\widehat{\rho}} - \frac{G(\widehat{\rho} + \sigma)}{\widehat{\rho} + \sigma} \right) \nabla \chi + \\
&+ \left[\left(G_\rho(\widehat{\rho})/(\widehat{\rho}) \right) \nabla \widehat{\rho} - \left(G_\rho(\widehat{\rho} + \sigma)/(\widehat{\rho} + \sigma) \right) \nabla(\widehat{\rho} + \sigma) \right] \chi, \\
&h_4 \equiv (-u \cdot \nabla \chi) + \left[\left(\overline{\theta} G(\widehat{\rho})/c_v \widehat{\rho} \right) - \left(\overline{\theta} G(\widehat{\rho} + \sigma)/c_v(\widehat{\rho} + \sigma) \right) \right] \cdot \\
&\cdot (\nabla \cdot u) + \left[\Psi / c_v(\widehat{\rho} + \sigma) \right] - k \sigma \Delta \chi / c_v \widehat{\rho}(\widehat{\rho} + \sigma).
\end{aligned}$$

In the wake of the results by Matsumura and Nishida (1980, 1982, 1983), the existence of a regular global solution to (5.6-8) will be the consequence of the following suitable «*a priori*» estimates.

THEOREM 5.2. There exist positive constants ε_1 (small) and C_1 depending on $\overline{\rho}$, $\overline{\theta}$ and $\| \phi \|_4$ such that if $(\sigma, u, \chi) \in X(0, T)$ is a solution of (5.6-8) for some $T > 0$ and

$$\sup_{0 \le t \le T} \| (\sigma, u, \chi) \|_3^2 (t) < \varepsilon_1,$$

then it holds

$$\begin{aligned}
&\sup_{0 \le t \le T} \| (\sigma, u, \chi) \|_3^2 (t) + \int_0^T [\| \sigma \|_3^2 (t) + \| (u, \chi) \|_4^2 (t)] \, d t \le \\
&\le C_1 \{ \| (\sigma_0, u_0, \chi_0) \|_3^2 \},
\end{aligned}$$

where

$$X(0,T) = \{(\sigma,u,\chi) : \sigma \in C^0([0,T];H^3),$$
$$(u,\chi) \in C^0([0,T];H^3) \cap L^2([0,T];H^4)\}.$$

Proof. First by the same estimates (4.22) deduced in the stability criterion Theorem 4.1, for ε_1 properly small we receive

$$\sup_{0\le t\le T} \|(\sigma,u,\chi)\|^2(t) + \int_0^T \left[\|\sigma\|^2(t) + \|(u,\chi)\|_1^2(t)\right] dt \le \tag{5.9}$$
$$\le C_1 \|(\sigma_0,u_0,\chi_0)\|^2 .$$

Next, we add the nonlinear term $-u\cdot\nabla\sigma$ at the right hand sides of (5.6), in doing so we can deduce a very nice basic energy equality. In fact, multiplying by $(\overline{\theta}G_\rho(\widehat{\rho})/\widehat{\rho})\sigma$ relation $(5.6)_1$, by $\widehat{\rho}u$ relation $(5.6)_2$ and by $(\widehat{\rho}c_v/\overline{\theta})\chi$ relation $(5.6)_3$, integrating over Ω and summing the resulting equations it delivers

$$\begin{aligned}
&\left(\int_\Omega \left[\frac{\overline{\theta}G_\rho(\widehat{\rho})}{2\widehat{\rho}}|\sigma|^2 + \frac{\widehat{\rho}}{2}|u|^2 + \frac{\widehat{\rho}c_v}{2\overline{\theta}}|\chi|^2\right] dx\right)_t + \\
&+ \int_\Omega \left[\mu|\nabla u|^2 + (\lambda+\mu)|\nabla\cdot u|^2 + \frac{\kappa}{\overline{\theta}}|\nabla\chi|^2\right] dx = \\
&= \int_\Omega (h_0 - u\cdot\nabla\sigma)\frac{\overline{\theta}G_\rho(\widehat{\rho})}{\widehat{\rho}}\sigma\, dx + \\
&+ \int_\Omega \left[\widehat{\rho}h\cdot u + \frac{\widehat{\rho}c_v}{\overline{\theta}}h_4\chi\right] dx.
\end{aligned} \tag{5.10}$$

Here we emphasize that the equality (5.10) holds without the smallness condition of $\|\phi\|_4$, which is the crucial difference from the previous arguments. Then, keeping (5.9-10) in mind, we may follow the corresponding lines of Matsumura and Nishida (1982), to obtain the remaining estimates of higher derivatives. In what follows, let us show that we can simplify the arguments by using the auxiliary Neumann and Stokes problem (4.10), (4.12). Multiplying $w-\nabla\omega$ by $(5.6)_2$ and combining it together with (5.10), in the same way as in section 4, we have

$$\begin{aligned}
&E_\alpha(\sigma,u,\chi)(t) + \nu\int_0^t \left[\|\sigma\|^2(\tau) + \|(u,\chi)\|_1^2(\tau)\right] d\tau \le \\
&\le E_\alpha(\sigma_0,u_0,\chi_0) + C\int_0^t \left[\|u\|_3^2(\tau)\,\|\sigma\|^2(\tau) + \right. \\
&\left. + \|h_0\|^2(\tau) + \|(h,h_4)\|_{-1}^2(\tau)\right] d\tau,
\end{aligned} \tag{5.11}$$

for some positive constants α, ν and C depending on $\overline{\rho}$, $\overline{\theta}$ and $\widehat{\rho}$ (eventually on $\| \phi \|_4$), where the quadratic energy form $E_\alpha(\sigma, u, \chi)$ has the form

$$
E_\alpha(\sigma,u,\chi)(t) = \tag{5.12}
$$
$$
= \int_\Omega \left[\frac{\overline{\theta}G_\rho(\widehat{\rho})}{2\widehat{\rho}}|\sigma|^2 + \frac{\widehat{\rho}}{2}|u|^2 + \frac{\widehat{\rho}c_v}{2\overline{\theta}}|\chi|^2 + \alpha(w - \nabla\omega)\cdot u \right] \mathrm{d}\,x
$$

which is the just linearized counterpart to the linear form $\rho E + \alpha \int_\Omega (w - \nabla\omega)\cdot u \,\mathrm{d}\,x$ introduced in section 4. Here, we further note the term $\| u \|_3^2 \| \sigma \|^2$ in the right hand side of (5.11) comes from the only one non linear term $u \cdot \nabla\sigma$ in the left hand side of (5.6). Thus, taking ε_1 properly small, we have by (5.11) the basic estimate for the linear (except one term) system (5.6-8) with given h_0, h, h_4

$$
\begin{aligned}
&\sup_{0\le t\le T} \| (\sigma,u,\chi) \|^2 (t) + \int_0^T [\| \sigma \|^2 (t) + \\
&+ \| (u,\chi) \|_1^2 (t)]\,\mathrm{d}\,t \le C\{\| (\sigma_0,u_0,\chi_0) \|^2 + \\
&+ \int_0^T [\| h_0 \|^2 (t) + \| (h,h_4) \|_{-1}^2 (t)]\,\mathrm{d}\,t \Big\}.
\end{aligned} \tag{5.13}
$$

Let ∂ represent a linearly independent system of smooth first-order partial derivatives (vector fields) on Ω which are tangential to $\partial\Omega$. Applying ∂ to the equations (5.6-8) and following the same procedure as (5.11-13), for properly small ε_1, it is straightforward to obtain

$$
\begin{aligned}
&\sup_{0\le t\le T} \| \partial(\sigma,u,\chi) \|^2 (t) + \int_0^T [\| \partial\sigma \|^2 (t) + \\
&+ \| \partial(u,\chi) \|_1^2 (t)]\,\mathrm{d}\,t \le C\{\| (\sigma_0,u_0,\chi_0) \|_1^2 + \\
&+ \int_0^T [\| \sigma \|^2 (t) + \| (u,\chi) \|_1^2 (t) + \\
&+ \| (h_0,h,h_4) \|^2 + \| \partial h_0 \|^2]\,\mathrm{d}\,t\}.
\end{aligned} \tag{5.14}
$$

In fact, we may estimate the form $E_\alpha(\partial\sigma, \partial u, \partial\chi)$ which comes from the principal linear part of ∂ (5.6), and estimate the commutator of ∂ and the principal part of (5.6) which produces only lower order terms.

Next, we multiply $(5.6)_2$ by $\widehat{\rho}u_t$ and $(5.6)_3$ by $c_v\widehat{\rho}\chi_t$, integrate over Ω to obtain

$$\sup_{0\le t\le T}\int_\Omega [\mu|\nabla u|^2 + (\lambda+\mu)|\nabla\cdot u|^2]\,\mathrm{d}x +$$

$$+\int_0^T\int_\Omega \widehat{\rho}|u_t|^2\,\mathrm{d}x\,\mathrm{d}t \le C\{\|(\sigma_0,u_0,\chi_0)\|_1^2 +$$

$$+\sup_{0\le t\le T}\|(\sigma,u)\|^2(t) + \int_0^T [\|h\|^2 + \|\sigma\|^2 + \|(u,\chi)\|_1^2]\,\mathrm{d}t\Big\}, \tag{5.15}$$

$$\sup_{0\le t\le T}\int_\Omega \kappa|\nabla\chi|^2\mathrm{d}x + \int_0^T\int_\Omega \widehat{\rho}c_v|\chi_t|^2\mathrm{d}x\,\mathrm{d}t \le$$

$$\le C\left\{\|\chi_0\|_1^2 + \int_0^T [\|u\|_1^2 + \|h_4\|^2]\,\mathrm{d}t\right\},$$

where we used the fact

$$\int_\Omega \widehat{\rho}u_t\cdot\nabla\left(\frac{\overline{\theta}G_\rho(\widehat{\rho})}{\widehat{\rho}}\sigma\right)\mathrm{d}x = \int_\Omega\left(\frac{\overline{\theta}G_\rho(\widehat{\rho})}{\widehat{\rho}}\sigma\right)\nabla\cdot(\widehat{\rho}u_t)\mathrm{d}x =$$

$$= \left(\int_\Omega\left(\frac{\overline{\theta}G_\rho(\widehat{\rho})}{\widehat{\rho}}\sigma\right)\nabla\cdot \mathrm{u}\,\mathrm{d}x\right)_t - \int_\Omega\left(\frac{\overline{\theta}G_\rho(\widehat{\rho})}{\widehat{\rho}}\right)\nabla\cdot$$

$$\cdot(\widehat{\rho}u)\left(h_0 - u\cdot\nabla\sigma\right)\mathrm{d}x.$$

Let $\partial_r = n\cdot\nabla$ represent a smooth first-order partial derivative on Ω, where n is a smooth vector field on Ω which coincides with the unit outer normal to $\partial\Omega$ on the boundary. Operating ∂_r to $(5.6)_1$ and combining the form $\widehat{\rho}n\cdot(5.6)_2$ (refer to Matsumura and Nishida (1982)), we can have

$$[(\lambda+2\mu)/\widehat{\rho}]\partial_r\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right) + \overline{\theta}G_\rho(\widehat{\rho})\partial_r\sigma = -\widehat{\rho}n\cdot u_t + \text{ terms of } \partial^2u +$$

$$+ \text{ terms of } \partial\partial_r u + \text{ terms of } \sigma + \text{ first-order terms of } (u,\chi) + \tag{5.16}$$

$$+\rho n\cdot h + \text{ terms of } \left(h_0,\partial_r h_0\right).$$

Multiplying $\partial_r\sigma$ by (5.16) and taking ε_1 properly small, we deduce

$$\sup_{0\le t\le T}\|\partial_r\sigma\|^2(t) + \int_0^T\|\partial_r\sigma\|^2(t)\mathrm{d}t \le$$

$$\le C\left\{\|\sigma_0\|_1^2 + \int_0^T [\|u_t\|^2 + \|\partial u\|_1^2 + \|\sigma\|^2 + \right. \tag{5.17}$$

$$+\|(u,\chi)\|_1^2 + \|h_0\|_1^2 + \|h\|^2](t)\mathrm{d}t\}.$$

Thus, combining the estimates (5.13-17) together with the standard elliptic estimates, it yields

$$
\begin{aligned}
&\| \chi \|_2^2 \leq C \| \Delta \chi \|^2 \leq C (\| \nabla \cdot u \|^2 + \| \chi_t \|^2 + \| h_4 \|^2), \\
(5.18) \qquad &\| u \|_2^2 \leq C(\| [\mu \Delta u + (\lambda + \mu) \nabla \nabla \cdot u] \|^2 \leq \\
&\leq C (\| u_t \|^2 + \| (\sigma, u, \chi) \|_1^2 + \| h \|^2),
\end{aligned}
$$

we have, for properly small ε_1,

$$
\begin{aligned}
&\sup_{0 \leq t \leq T} \| (\sigma, u, \chi) \|_1^2 (t) + \int_0^T [\| \sigma \|_1^2 + \| (u, \chi) \|_2^2] (t) dt \leq \\
(5.19) \qquad &\leq C \left\{ \| (\sigma_0, u_0, \chi_0) \|_1^2 + \int_0^T [\| (h, h_4) \|^2 + \| h_0 \|_1^2] (t) dt \right\}.
\end{aligned}
$$

For remaining higher derivatives, after obtaining the corresponding estimates of $(\sigma, u, \chi)_t$, $(\partial\sigma, \partial u, \partial\chi)$ and $(\partial^2\sigma, \partial^2 u, \partial^2\chi)$ to the H^1-estimates (5.19), estimating ∂_τ (5.16), $\partial\partial_\tau$ (5.16), ∂_τ^2 (5.16), and the ellipticity in (5.18), we can finally have

$$
\begin{aligned}
&\sup_{0 \leq t \leq T} \| (\sigma, u, \chi) \|_3^2 (t) + \int_0^T [\| \sigma \|_3^2 (t) + \| (u, \chi) \|_4^2 (t)] dt \leq \\
(5.20) \qquad &\leq C \left\{ \| (\sigma_0, u_0, \chi_0) \|_3^2 + \int_0^T [\| (h, h_4) \|_2^2 + \| h_0 \|_3^2] (t) dt + \right. \\
&\left. + \int_0^T [\| (\partial_t h, \partial_t h_4) \|_3^2 + \| \partial_t h_0 \|_1^2] (t) dt \right\},
\end{aligned}
$$

for properly small ε_1. We omit the details (refer to Matsumura and Nishida (1982)). Last, it holds

$$
\begin{aligned}
&\int_0^T [\| (h, h_4) \|_2^2 + \| h_0 \|_3^2 + \| (\partial_t h, \partial_t h_4) \|_0^2 + \| \partial_t h_0 \|_1^2] (t) dt \leq \\
(5.21) \qquad &\leq C \sup_{0 \leq t \leq T} \| (\sigma, u, \chi) \|_3^2 (t) \cdot \int_0^T [\| \sigma \|_3^2 + \| (u, \chi) \|_4^2] (t) dt.
\end{aligned}
$$

Therefore, taking ε_1 small again in (5.20-21), the desired estimates in Theorem 5.1 are obtained. Thus, the proof of the Theorem is completed.

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