# 377. 有限個領域ニ於ケル函數ノ Erzeugung ニ就イテ

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角谷氏が第62号229ニ於イテ変域モ値域モ有限個領域 $\{0, 1, 2, \cdots\cdots, n\}$ デアル任意ノ函數 $f(x_1, x_2, \cdots\cdots x_p)$ が或ル一ツノ二変數ノ函数ヲ何回カ *iterieren* スルコトニヨリ *erzeugen* サレ得ルコトヲ証明サレタ。其ノ後、既ニ1935年5月ノ Am. National Acad. of Science 上デ米國ノ Webb がソレト異ナル函数ニヨツテ同様ノコトヲ論ジテキルノヲ見タ。然シ其ノ論文ハ、記号ソノ他ノ点デ多少理解シ難イ感ジがスルノデ其等ヲ書キ改ヘテ、次ニ御紹介シタイト思フ。

次ノ表ニヨツテ定義サレル函数ヲ $\varphi(x, y)$ トス。(空白ノ部分ノ数字ハ0トス)

$\longrightarrow y$

$\downarrow x$

| $\varphi$ | 0 | 1 | 2 | --- | --- | $n-1$ | $n$ |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | --- | --- | --- | --- | --- |
| 1 | 0 | 2 | 0 | --- | --- | --- | --- |
| 2 | ⋮ | 0 | 3 | ⋱ | | | |
| ⋮ | ⋮ | ⋮ | 0 | ⋱ | ⋱ | | |
| ⋮ | ⋮ | ⋮ | ⋮ | 0 | ⋱ | 0 | |
| $n-1$ | ⋮ | ⋮ | ⋮ | ⋮ | 0 | $n$ | 0 |
| $n$ | 0 | ⋮ | ⋮ | ⋮ | | 0 | 0 |

$g_1(x)=\varphi(x,x)\equiv x+1 \ (\mathrm{mod.}\ \nu=n+1)$ デアルカラ $g_1(x)$ ヲ *iterieren* スルコトニヨリ $g_h(x)=x+h$ ハ $\varphi(x,y)$ ニヨツテ *erzeugen* サレル。又 $g_{n-h}(x)=g_{-h}(x)$ トス。

函数 $R^i_{\lambda,\mu}(x,y)$ ヲ次ノ如ク定義シ、$R^i_{\lambda,\mu}$ ト略記ス。

$$R^i_{\lambda,\mu}(x,y)=i, \quad x=\lambda,\ y=\mu \text{ ナルトキ}$$

$$R^i_{\lambda,\mu}(x,y)=0, \quad x\neq\lambda \text{ or } y\neq\mu \text{ ナルトキ}$$

然ラバ

$$R^0_{\lambda,\mu}=\varphi(x,g_1(x))=0$$
$$R'_{\lambda,\mu}=\varphi\{g_{-1}[\varphi(g_{-\lambda}(x),g_{-\mu}(y))],\ \varphi(x,g_1(x))\}$$

何故ナラバ

$$\text{右辺}=\varphi\{g_{-1}[\varphi(x-\lambda,y-\mu)],0\}$$

$x=\lambda,\ y=\mu$ ノトキ $g_{-1}[\varphi(x-\lambda,y-\mu)]=g_{-1}(\varphi(0,0))=0$

$$\therefore\ \text{右辺}=\varphi\{0,0\}=1$$

$x\neq\lambda$ or $y\neq\mu$ ノトキ $\varphi(x-\lambda,y-\mu)\neq 1$ トナリ

$$g_{-1}[\varphi(x-\lambda,y-\mu)]\neq 0$$

$$\therefore\ \text{右辺}=0$$

次ニ

$$R^i_{\lambda,\mu}=\varphi\{g_{i-2}(R'_{\lambda,\mu}),\ g_{i-1}(R^0_{\lambda,\mu})\}$$

何故ナラバ

$$\text{右辺}=\varphi\{g_{i-2}(R'_{\lambda,\mu}),\ i-1\}$$

$x=\lambda,\ y=\mu$ ナルトキ 右辺$=\varphi\{g_{i-2}(1),\ i-1\}=i$

$x\neq\lambda$ or $y\neq\mu$ ナルトキ $R'_{\lambda,\mu}\neq 1$ トナリ

$$g_{i-2}(R'_{\lambda,\mu})\neq i-1$$

$\therefore$ 右辺$=0$

故ニ $R^{i}_{\lambda,\mu},\ (i=0,1,2,\text{------},n)$ ハ $\varphi(x,y)$ ヲ有限回 iterieren スルコトニヨッテ erzeugen スルコトガ出来ル。

$$\delta(x,y)=\varphi(x,\ g_n(y))$$

$$\alpha_i(x,y)=\delta(N_{i-1},\ g_1(M_{i-1}))$$

$$(i=1,2,\text{---------},n)$$

$$M_0=R^{0}_{0,0},\quad M_k=\alpha_k\left(M_{k-1},\ \alpha_k(R^{k}_{0,k+1},\ R^{k}_{k+1,0})\right)$$

$$N_0=R'_{0,0},\quad N_k=\alpha_k(M_k,\ N_0)$$

ト定義スレバ, $\delta(x,y)$ ハ次ノ表デ与ヘラレル。

$\longrightarrow y$ ; $x \downarrow$

| $\delta(x,y)$ | 0 | 1 | 2 | ------ | $n-1$ | $n$ |
|---|---|---|---|---|---|---|
| 0 | | 1 | 0 | | | |
| 1 | | 0 | 2 | | | |
| 2 | | | | | | |
| ⋮ | | | | | 0<br>$n-1$ | <br>0 |
| $n-1$ | | | | | | $n$ |
| $n$ | 0 | 0 | | | | |

次ニ函数 $\alpha_n(x, y)$ ガ次ノ表ニヨツテ表ハサレルコトヲ証明シヨウ。

$\longrightarrow y$

| $\alpha_n$ | 0 | 1 | 2 | ------------ | $n$ |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | ------------ | $n$ |
| 1 | 1 | 1 | 1 | | |
| 2 | 2 | 1 | | | |
| ⋮ | ⋮ | | | 1 | |
| $n$ | $n$ | | | | |

$\downarrow x$

即チ
$x=0$ or $y=0$
ナルトキ
$\alpha_n(x,y)=x+y$

（証明）　$i=1$ ナルトキ

$$\alpha_1(x, y) = \delta(N_0 \ g_1(M_0))$$

$$M_0 = R^0_{0,0} \qquad N_0 = R^1_{0,0}$$

ヨリ

$\longrightarrow y$

| $\alpha_1$ | 0 | 1 | 2 | ------------ $n$ |
|---|---|---|---|---|
| 0 | 0 | 1 | 1 | ------------ 1 |
| 1 | 1 | 1 | 1 | |
| 2 | 1 | 1 | 1 | |
| ⋮ | ⋮ | | | 1 |
| $n$ | 1 | | | |

$\downarrow x$

$i=2$ナルトキ

$$\alpha_2(x\,y)=\delta(N_1,\ g_1(M_1)),$$

$$M_0=R^0_{0,0} \qquad M_1=\alpha_1\,(M_0,\alpha_1\,(R'_{0,2},\ R'_{2,0}))$$

$$N_0=R'_{0,0} \qquad N_1=\alpha_1\,(M_1,N_0)$$

$\longrightarrow y$ ; $x \downarrow$

| $M_1$ | 0 | 1 | 2 | …… | $n$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 0 | |
| 1 | 0 | 0 | | | |
| 2 | 1 | | | | |
| ⋮ | 0 | | 0 | | |
| $n$ | | | | | |

$\longrightarrow y$ ; $x \downarrow$

| $N_1$ | 0 | 1 | 2 | …… | $n$ |
|---|---|---|---|---|---|
| 0 | 1 | 0 | 1 | 0 | |
| 1 | 0 | 0 | | | |
| 2 | 1 | | | | |
| ⋮ | 0 | | 0 | | |
| $n$ | | | | | |

| $g_1(M_1)$ | 0 | 1 | 2 | …… | $n$ |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 2 | 1 | |
| 1 | 1 | 1 | | | |
| 2 | 2 | | | | |
| ⋮ | 1 | | 1 | | |
| $n$ | | | | | |

$\longrightarrow y$ ; $x \downarrow$

| $\alpha_2$ | 0 | 1 | 2 | …… | $n$ |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 1 | |
| 1 | 1 | 1 | | | |
| 2 | 2 | | | | |
| ⋮ | 1 | | 1 | | |
| $n$ | | | | | |

$i=m$ナルトキ $\alpha_m(x,y)$, $M_{m-1}$, $N_{m-1}$ ガ夫々次表ノ如ク表ハサレルト假定スレバ

| $\alpha_m$ | 0 | 1 | 2 | ⋯ | $m$ | | ⋯ | $n$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | ⋯ | $m$ | 1 | ⋯ | |
| 1 | 1 | 1 | | | | | | |
| 2 | 2 | | | | | | | |
| ⋮ | ⋮ | | | | 1 | | | |
| $m$ | $m$ | | | | | | | |
| ⋮ | 1 | | | | | | | |
| ⋮ | ⋮ | | | | | | | |
| $n$ | ⋮ | | | | | | | |

| $M_{m+1}$ | 0 | 1 | 2 | | ⋯ | $m$ | | ⋯ | $n$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 2 | ⋯ | $m-1$ | 0 | ⋯ | |
| 1 | 0 | 0 | | | | | | | |
| 2 | 1 | | | | | | | | |
| ⋮ | 2 | | | | | | | | |
| ⋮ | ⋮ | | | | | 0 | | | |
| $m$ | $m-1$ | | | | | | | | |
| ⋮ | 0 | | | | | | | | |
| ⋮ | ⋮ | | | | | | | | |
| $n$ | ⋮ | | | | | | | | |

| $N_{m-1}$ | 0 | 1 | 2 | | ⋯ | $m$ | | ⋯ | $n$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 1 | 2 | ⋯ | $m-1$ | 0 | ⋯ | |
| 1 | 0 | 0 | | | | | | | |
| 2 | 1 | | | | | 0 | | | |
| ⋮ | 2 | | | | | | | | |
| ⋮ | ⋮ | | | | | | | | |
| $m$ | $m-1$ | | | | | | | | |
| ⋮ | 0 | | | | | | | | |
| ⋮ | ⋮ | | | | | | | | |
| $n$ | ⋮ | | | | | | | | |

$$\alpha_{m+1}(x, y) = \delta(N_m, g_1(M_m))$$

$$M_m = \alpha_m(M_{m-1}, \alpha_m(R^m_{0,m+1}, R^m_{m+1,0}))$$

$$N_m = \alpha_m(M_m, N_0)$$

ヨリ

$\alpha_m(R^m_{0,m+1}, R^m_{m+1,0})$

| | 0 | 1 | 2 | ⋯ | $m+1$ | ⋯ | $n$ |
|---|---|---|---|---|---|---|---|
| 0 | | | | | $m$ | | |
| 1 | | | | | | | |
| 2 | | | | | | | |
| ⋮ | | | | 0 | | | |
| $m+1$ | $m$ | | | | | | |
| ⋮ | | | | | | | |
| $n$ | | | | | | | |

| $M_m$ | 0 | 1 | 2 | | ⋯ | $m+1$ | | ⋯ | $n$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 2 | ⋯ | $m$ | 0 | ⋯ | |
| 1 | 0 | | | | | | | | |
| 2 | 1 | | | | | | | | |
| | 2 | | | | | | | | |
| ⋮ | ⋮ | | | | 0 | | | | |
| $m+1$ | $m$ | | | | | | | | |
| | 0 | | | | | | | | |
| ⋮ | ⋮ | | | | | | | | |
| $n$ | ⋮ | | | | | | | | |

| $N_m$ | 0 | 1 | 2 | 3 | ⋯ | $m+1$ | | ⋯ | $n$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 1 | 2 | ⋯ | $m$ | 0 | ⋯ | |
| 1 | 0 | | | | | | | | |
| 2 | 1 | | | | | | | | |
| 3 | 2 | | | | | | | | |
| ⋮ | ⋮ | | | | 0 | | | | |
| $m+1$ | $m$ | | | | | | | | |
| | 0 | | | | | | | | |
| ⋮ | ⋮ | | | | | | | | |
| $n$ | ⋮ | | | | | | | | |

| $g_1(M_m)$ | 0 | 1 | 2 | 3 | ⋯ | $m+1$ | | ⋯ | $n$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 2 | 3 | ⋯ | $m+1$ | 1 | ⋯ | |
| 1 | 1 | | | | | | | | |
| 2 | 2 | | | | | | | | |
| 3 | 3 | | | | | | | | |
| ⋮ | ⋮ | | | | 1 | | | | |
| $m+1$ | $m+1$ | | | | | | | | |
| | 1 | | | | | | | | |
| ⋮ | ⋮ | | | | | | | | |
| $n$ | ⋮ | | | | | | | | |

| $\alpha_{m+1}$ | 0 | 1 | 2 | 3 | ⋯ | $m+1$ | | ⋯ | $n$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | ⋯ | $m+1$ | 1 | ⋯ | |
| 1 | 1 | | | | | | | | |
| 2 | 2 | | | | | | | | |
| 3 | 3 | | | | | | | | |
| ⋮ | ⋮ | | | | 1 | | | | |
| $m+1$ | $m+1$ | | | | | | | | |
| | 1 | | | | | | | | |
| ⋮ | ⋮ | | | | | | | | |
| $n$ | ⋮ | | | | | | | | |

故ニ數學的歸納法ニヨッテ $\alpha_i(x, y)$ $(i=3, 4, \cdots\cdots, n)$ ヲ定

メル表ヲ知ルコトガ出來 $i=n$ トスレバ $\alpha_n$ ノ表ガ求メラレル。

サテ同ジ領域デ任意ニ與ヘラレタ函數 $f(x,y)$ ガ次ノ表ニヨツテ定義サレテヰルトキ

| $f(x,y)$ | 0 | 1 | 2 | ………… | $n$ |
|---|---|---|---|---|---|
| 0 | $t_{0,0}$ | $t_{0,1}$ | $t_{0,2}$ | ………… | $t_{0,n}$ |
| 1 | $t_{1,0}$ | $t_{1,1}$ | $t_{1,2}$ | ………… | $t_{1,n}$ |
| 2 | $t_{2,0}$ | $t_{2,1}$ | $t_{2,2}$ | ………… | $t_{2,n}$ |
| ⋮ | ⋮ | ⋮ | ⋮ | | ⋮ |
| $n$ | $t_{n,0}$ | $t_{n,1}$ | $t_{n,2}$ | ………… | $t_{n,n}$ |

$$
\begin{aligned}
f(x,y) = {} & R_{0,0}^{t_{0,0}} + R_{0,1}^{t_{0,1}} + R_{0,2}^{t_{0,2}} + \cdots\cdots + R_{0,n}^{t_{0,n}} \\
& + R_{1,0}^{t_{1,0}} + R_{1,1}^{t_{1,1}} + R_{1,2}^{t_{1,2}} + \cdots\cdots + R_{1,n}^{t_{1,n}} \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + R_{n,0}^{t_{n,0}} + R_{n,1}^{t_{n,1}} + R_{n,2}^{t_{n,2}} + \cdots\cdots + R_{n,n}^{t_{n,n}}
\end{aligned}
$$

トナリ $f(x,y)$ ハ $\alpha_n(x,y)$ ト $R_{0,0}^{t_{0,0}}$, $R_{0,1}^{t_{0,1}}$, ………等ニヨツテ表ハシ得ル。即チ $f(x,y)$ ハ $\varphi(x,y)$ ヲ有限回 *iterieren* スルコトニヨツテ表ハシ得ル。

次ニ第62号228ニ於テ定義シタ函數

$\psi_1(x, y)$　　$\psi_2(x, y)$

$f_k(x) \equiv k$　　$(k = 0, 1, \cdots\cdots, n)$

$\sigma(x, \lambda)$　　$(\lambda = 0, 1, \cdots\cdots, n)$

ハ総テ $\varphi(x, y)$ ヲ有限回 *iterieren* スルコトニヨツテ表ハサレ得ル。

（証明）　二変数函數 $\psi_1(x, y)$ ガ $\varphi(x, y)$ ニヨツテ表ハサレルコトハ既ニ証明シタ。$\psi_2(x, y) = \alpha_n(x, y)$。

$f_0(x) = \varphi(x, g(x))$，　$f_k(x) = g(f_{k-1}(x))$，$(k=1, \cdots\cdots, n)$。

$\sigma(x, \lambda) = \varphi(f_0(x), g_{-\lambda}(x))$，$(\lambda = 0, 1, \cdots\cdots, n)$

故ニ第62号228ノ定理ニヨツテ任意ノ函数 $f(x_1, x_2, \cdots\cdots, x_n)$ ガ $\varphi(x, y)$ ヲ何回カ *iterieren* スルコトニヨツテ表ハサレ得ルコトガ証明出来タ。