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EDITED BY

MITIO NAGUMO

Volume 1. Number 1

MARCH , 1950

OSAKA

SUGAKU SIZYO DANWAKAI

is expected to be issued monthly generally hereafter substituting for the ZENKOKU SIZYO SUGAKU DANWAKAISI which has been published in Japanese up until today for the purpose of promotion of the study of mathematics.

All the pages of the SUGAKU SIZYO DANWAKAI are being offered to those who are inclined to make a debate, a review, a critique, or an announcement of study.

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A REMARK ON THE COVERING HOMOTOPY THEOREM

By Tatsuji Kudo

One of the most important theorems in the theory of fibre bundles is the covering homotopy theorem of Steenrod [1]. In this note it will be shown that this theorem is an immediate consequence of Theorem I, [2] of the author.

Let A be a fibre bundle over B with fibre F, then it is equivalent to the fibre bundle $\mathfrak{G} = \mathfrak{G}(f,B)$ generated by the F-continuous function f of B into A, where $f = \pi^{-1}$, π being the projection (mapping) of the fibre bundle A.

Let $k(x) = (h(x), \omega(x))$, $x \in X$, $h(x) \in B$, $\omega(x) \in f(h(x)) \subset A$, be a continuous mapping of a compact space X into $\mathfrak{G}(f,B)$, and $H(x,t)$, $0 \leq t \leq 1$, be a homotopy of $h(x)$ in B. Then $f(H(x,t))$ is an F-continuous function of $X \times I$ into A. Since $g(x,t:s) = f(H(x,ts))$ gives an F-homotopy of $g(x,t;0) = f(H(x,t\,0)) = f(h(x))$ to $g(x,t:1) = f(H(x,t))$, the fibre bundles $\mathfrak{G}(f(h(x));X \times I)$ and $\mathfrak{G}(f(H(x,t));X \times I)$ are equivalent to each other by Theorem I, [2]. This means that there exists a continuous mapping $\omega' = \omega'(x,t,\omega)$ of $\mathfrak{G}(f(h(x));X \times I)$ into A such that $\omega'(x,t,\omega) \in f(H(x,t))$.

Put $\omega(x,t) = \omega'(x,t,\omega(x))$, then $K(x,t) = (H(x,t), \omega(x,t))$ is a homotopy of $k(x)$ in $\mathfrak{G}(f,B)$, with $\pi(K(x,t)) = H(x,t)$.

For the practical purpose it is more powerful to repeat this argument in each case than to use the resulting theorem.

[1] . N.E.Steenrod: The Classification of Sphere Bundles, Annals of Math. vol.45, No.2, 1944.

[2] . T.Kudo: Classification of Topological Fibre Bundles, Osaka Math. Journ. vol.1, No2, 1949.

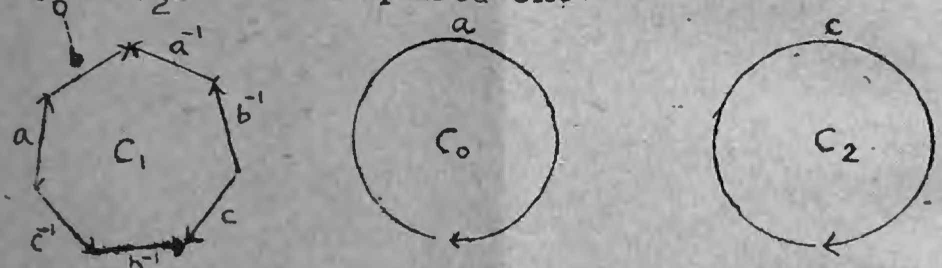
AN EXAMPLE OF A 2-DIMENSIONAL COMPLEX WITH SOME SINGULAR PROPERTY

By Takeshi Yajima

Recently Mr. Tatsuji Kudo proposed the following interesting question: "If C is a 2-dimensional simply connected complex and C_0 one of its 2-cells, is it true that the fundamental group of $C - C_0$ is generated by the boundary of C_0?"

In the following I shall show by an example that the answer is in the negative.

Let C_1 be a complex represented by the schema $aba^{-1}b^{-1}cb^{-1}c^{-1}$ and let C_0 and C_2 be the 2-cells with boundaries a and c respectively. Then the complex $C = C_1 + C_0 + C_2$ is the required one.



The simple connectedness of C is obvious, since we have from the relations $a = 1$ and $c = 1$ the relation $b = 1$. To see that the fundamental group of $C - C_0$ can not be generated by a, or to see that the relation $aba^{-1}b^{-1}$ does not infer $b = a^n$ and therefore not $b = 1$, put

$$a = \begin{pmatrix} 1/\sqrt{2} & 0 \\ 1 & \sqrt{2} \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix};$$

the free group generated by a and b has the relation $aba^{-1}b^{-1} = b$, but we have $b \neq 1$.

I am thankful to my colleagues for their suggestions and aids in preparing this note.

REMARK ON THE STRUCTURE OF LIE AND JORDAN RINGS

By Masahiko Atsuchi (Hokkaido University)

The following theory is the abstract of the researches which will be published in the next volume of the Journal of the Faculty of Science, The Hokkaido University.

Our purpose is to investigate the form of multiplication of Lie and Jordan rings when we seek for their model in a non-commutative ring.

Suppose L is the module with operator in which a multiplication is defined and the following conditions are satisfied:

(1) $\quad a \times (b + c) = a \times b + a \times c$

$\quad (b + c) \times a = b \times a + c \times a$

(2) $\quad \lambda(a \times b) = (\lambda a) \times b = a \times (\lambda b)$

$\quad a, b, c \in L, \quad \lambda \in P,$

where P is a operator domain.

Suppose that P is the field whose characteristic is m $(0 < m \leq \infty)$ (in our case we say the characteristic of a field is ∞ when it is 0 in usual sense), and S is the non-commutative ring in which L is mapped as the module by an operator-homomorphism φ.

Now we assume that $a \times b$ is mapped on the polynomial with coefficients in P of degree n $(< m)$ with respect to x, y which are the images of a, b by the homomorphism φ respectively:

$$\varphi(a \times b) = \sum \xi^{i_1, \ldots, i_r}_{j_1, \ldots, j_r} x^{i_1} y^{j_1} \ldots x^{i_r} y^{j_r} + C$$

$$\xi \in P \qquad x, y, C \in S$$

$$i_1 + \ldots + i_r \leq n \qquad i_k \geq 0$$

$$j_1 + \ldots + j_r \leq n \qquad j_k \geq 0$$

$$i_1 + \dots + i_r + j_1 + \dots + j_r \neq 0$$

where $\sum$ covers $i_1, \dots, i_r, j_1, \dots, j_r$ satisfying the above conditions.

Being $0 \times b = a \times 0 = 0 \times 0 = 0$, it follows immediately that C and the sum of all terms with $i_1 + \dots i_r = 0$ or $j_1 + \dots j_r = 0$ vanish. Therefore we can assume from the beginning that $C = 0$ and

$$i_1 + \dots + i_r > 0$$
$$j_1 + \dots + j_r > 0 .$$

Now for an arbitrary element λ of P it holds from (2)

$$\sum \lambda^{j_1 + \dots + j_r} \xi^{i_1, \dots, i_r}_{j_1, \dots, j_r} x^{i_1} y^{j_1} \dots x^{i_r} y^{j_r}$$
$$= \sum \lambda \xi^{i_1, \dots, i_r}_{j_1, \dots, j_r} x^{i_1} y^{j_1} \dots x^{i_r} y^{j_r} .$$

Arranging the expantion with respect to λ, we have

$$(3) \qquad \sum_{i=1}^{n} \lambda^i A_i = 0$$

where A_i is a polynomial in x, y not containing λ.

Since there exist at least n elements different from 0 in P, we have from (3):

$$\begin{cases} A_1 + \lambda_1 A_2 + \dots + \lambda_1^{n-1} A_n = 0 \\ \quad \dots \\ \quad \dots \\ A_1 + \lambda_n A_2 + \dots + \lambda_n^{n-1} A_n = 0 \end{cases}$$
$$\lambda_i \neq 0 \qquad (i = 1, 2, \dots, n).$$

Since

$$\begin{vmatrix} 1, & \lambda_1, & \dots, & \lambda_1^{n-1} \\ & \dots & & \\ & \dots & & \\ 1, & \lambda_n, & \dots, & \lambda_n^{n-1} \end{vmatrix} \neq 0$$

$$A_i = 0 \quad (i = 1, \dots, n).$$

Hence

$$j_1 + \dots + j_r = 1.$$

Similarly we have

$$i_1 + \dots + i_r = 1.$$

Therefore changing the coefficients, we can put

$$(4) \qquad \varphi(a \times b) = \xi x y + \xi' y x .$$

Now if we add to L the next condition (5) besides (1), (2)

$$(5) \qquad a \times b = b \times a,$$

it follows evidently that $\xi' = \xi$ as S is con-commutative, i.e. (4) will be written as

$$\varphi(a \times b) = \xi (x y + y x),$$

and the module L is represented as a Jordan ring.

When there exists an element x in S such that $x^2 = 0$ and if we adopt besides (1), (2) the next condition (6) instead of (5):

$$(6) \qquad a \times a = 0,$$

we have from (4) $\xi' = -\xi$ i.e. (4) will be written as

$$\varphi(a \times b) = \xi (xy - yx),$$

and in this case the polynomial

$$a \times (b \times c) + b \times (c \times a) + c \times (a \times b)$$

is mapped on 0, i.e. the module L is represented as a Lie ring.

When for all elements x of S $x^2 = 0$, then

$$yx = -xy$$

therefore we can describe (4) as

$$\varphi(a \times b) = \alpha xy$$

and the module L is represented as an associative ring.

ON THE LOGARITHMIC FUNCTIONS OF MATRICES.

By Kakutarō Morinaga and Takayuki Nōno.

In this paper we shall consider the logarithmic functions of matrices over the field of complex numbers. Let $\mathfrak{A}$ be the totality of matrices of order n whose characteristic values λ_i have the imaginary parts $I(\lambda_i)$ such that $0 \leq I(\lambda_i) < 2\pi$, and $\mathfrak{M}$ the totality of matrices of order n whose determinant $\det M$ are not zero. We write $\mathfrak{A}(n)$ and $\mathfrak{M}(n)$ for $\mathfrak{A}$ and $\mathfrak{M}$ respectively, when the order of these matrices must be shown. The exponential function of A is defined by the series

$$\exp A = E + \sum_{r=1}^{\infty} \frac{A^r}{r!}$$

THEOREM 1. If $M \in \mathfrak{M}$, then A such as $\exp A = M$ exists in $\mathfrak{A}$ uniquely.

Proof. The existency of such A in $\mathfrak{A}$ is well-known.(1) Our purpose is to prove the uniqueness. But it is convenient for us to begin with the proof of the existency.

Let $\mathring{M}$ the canonical form of M, and M_i the block for a characteristic value λ_i, i.e.,

$$(1) \quad \mathring{M} = P^{-1}MP = \begin{pmatrix} M_1 & & & 0 \\ & M_2 & & \\ & & \ddots & \\ 0 & & & M_p \end{pmatrix}, \quad M_i = \begin{pmatrix} \lambda_i & \epsilon_{i,1} & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \epsilon_{i,\alpha_i-1} \\ 0 & & & \lambda_i \end{pmatrix} = \lambda_i E + N_i$$

where $\lambda_i \neq \lambda_j$ for $i \neq j$: $\epsilon_{ij} = 0$ or 1 $(j = 1, \cdots, \alpha_i - 1)$, and $N_i^{\alpha_i} = 0$ and $\lambda_i \neq 0$, since $\det M \neq 0$.

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(1) cf. J.H.M. WEDDERBURN: Lectures on matrices. (1934). p.p.122-123.
K. YOSHIDA. DANWA 309. ZENKOKU-SHIJO-SUGAKU-DANWAKAI. No.72. (1935)
[illegible] 310. " " " No.72. (1935)
[illegible] 320. " " " No.74. (1935)

Let now A_i be

$$(2) \quad A_i = \operatorname{Log}\lambda_i \cdot E + \sum_{r=1}^{\alpha_i - 1} (-1)^{r-1} \frac{N_i^r}{r\lambda_i^r} \qquad (0 \leq I(\operatorname{Log}\lambda_i) < 2\pi)$$

Then, since $N_i^{\alpha_i} = 0$ and $EN_i = N_iE$; these expressions have the same forms as we shall obtain by substituting E, N, and λ_i for 1, x, and λ respectively in the next expression for complex numbers.

$$\operatorname{Log}(\lambda + x) = \operatorname{Log}\lambda + \sum_{r=1}^{\infty} (-1)^{r-1} \frac{x^r}{r\lambda^r} \qquad (\lambda \neq 0,\ |\tfrac{x}{\lambda}| < 1)$$

Therefore, by the same calculation as we obtained

$$\exp \cdot \operatorname{Log}(\lambda + x) = \lambda + x,$$

we have

$$(3) \quad \exp \cdot A_i = \lambda_i E + N_i = M_i$$

where A_i is the polynomial of E and N_i, i.e., E and M_i with the coefficients of complex numbers.

Let now

$$(4) \quad \mathring{A} = \begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_p \end{pmatrix}$$

then

$$(5) \quad \exp \mathring{A} = \mathring{M} = P^{-1}MP,$$

and therefore

$$\exp(P\mathring{A}P^{-1}) = M \qquad P\mathring{A}P^{-1} \in \mathfrak{A}$$

That is, $P\mathring{A}P^{-1}$ which satisfies the equation $\exp A = M$ exists in $\mathfrak{A}$.

Next we shall prove the uniqueness.

Let us suppose that

$$A = P\mathring{A}P^{-1}, \quad \exp B = M, \quad B \in \mathfrak{A}$$

$$(6) \quad \text{i.e.,} \quad \exp A = \exp B = M.$$

Since $\quad M = \exp B$,

(7) $\quad BM = MB$.

Putting now $P^{-1}BP = \mathring{B}$, we get from (6) and (7)

(8) $\quad \exp \mathring{A} = \exp \mathring{B} = \mathring{M}$

and

(9) $\quad \mathring{B}\mathring{M} = \mathring{M}B$,

respectively.

Therefore, we see from (9) that $\mathring{B}$ has the form

(10) $$\mathring{B} = \begin{pmatrix} B_1 & & & 0 \\ & B_2 & & \\ & & \ddots & \\ 0 & & & B_p \end{pmatrix}$$

where B_i is the matrix of the same oder as M_i , and

(11) $\quad B_i M_i = M_i B_i$

Since, we have shown above, A_i is the polynomial of E and M_i with the coefficients of complex numbers, we get from (11)

(12) $\quad A_i B_i = B_i A_i$

Moreover we have from (1), (4), (8) and (10)

(13) $\quad \exp A_i = \exp B_i = M_i$

Let now $\mu_{ij}\ (j = 1, \cdots, \alpha_i)$ be the characteristic values of B_i , then

(14) $\quad \mu_{ij} = \operatorname{Log} \lambda_i \qquad (j = 1, \cdots, \alpha_i)$

since $B \in \mathfrak{U}$ i.e., $0 \leq I(\mu_{ij}) < 2\pi$ and λ_i is the characteristic values of M_i . Hence we can conclude that all characteristic values of A_i and B_i are $\operatorname{Log} \lambda_i$. From this result and (12), by Frobenius' Theorem for the characteristic values, we can obtain that all characteristic values of $A_i - B_i$ are equal to zero. Therefore

we can choose R such that

(15) $$R^{-1}(A_i - B_i)R = \begin{pmatrix} 0 & \eta_{i1} & & 0 \\ & 0 & \ddots & \\ & & \ddots & \eta_{i\alpha_i-1} \\ 0 & & & 0 \end{pmatrix} . \ \eta_{ij} = 0 \text{ or } 1 \quad (j = 1, \cdots, \alpha_i - 1)$$

On the other hand, from (12) and (13) we get

$$\exp \cdot (A_i - B_i) = E \quad \text{i.e.,} \quad \exp \cdot (R^{-1}(A_i - B_i)R) = E ,$$

Hence we can conclude that $\eta_{ij}\ (j = 1, \cdots, \alpha_i - 1)$ in the above expression (15) must be zero. Therefore from (15) we have $A_i = B_i$ i.e., $A = B$. This completes the proof of the uniqueness.

<u>THEOREM 2. Let $A, B \in \mathfrak{U}$, then the relation $\exp A \exp B = \exp B \exp A$ holds if and only if $AB = BA$</u> .

Proof. It is obvious that if $AB = BA$, then $\exp A \exp B = \exp B \exp A$. Next we shall prove the converse. Suppose that

(16) $\quad \exp A \exp B = \exp B \exp A$.

Let now $\exp B = D$, then $\det D \neq 0$ and from (16) we have

$$\exp A \cdot D = D \cdot \exp A \ .$$

Therefore

(17) $\quad \exp A = D \cdot \exp A \cdot D^{-1} = \exp(DAD^{-1})$

Since $A \in \mathfrak{U}$, $DAD^{-1} \in \mathfrak{U}$. Using the theorem 1, we have from (17)

$$A = DAD^{-1} ,$$

(18) $\quad$ thus $\quad AD = DA$ i.e. $\quad A \cdot \exp B = \exp B \cdot A$

Here if $\det A \neq 0$; we put $A' = A$; if $\det A = 0$ $A' = A + \epsilon E$, where we choose a number ϵ such ...

Then we have from (18)

(19) $\quad A' \cdot \exp B = \exp B \cdot A'$.

And, since A^{-1} exists, from (19) we get

$$(20) \qquad \exp B = \exp(A^{-1} B A)$$

Here, since $B \in \mathfrak{U}$, $A^{-1}BA \in \mathfrak{U}$. Again using the theorem 1, we have

$$B = A^{-1} B A \quad \text{i.e.,} \quad AB = BA .$$

Hence always we get $AB = BA$. q.e.d.

By theorem 1 and 2 we obtain the following corollary.

COROLLARY. Let $M, N \in \mathfrak{U}$, then the relation $\operatorname{Log} M \operatorname{Log} N = \operatorname{Log} N \operatorname{Log} M$ if and only if $MN = NM$.

THEOREM 3. $M \in \mathfrak{U}$: when and only when M has the form

$$M = \begin{pmatrix} U & W \\ 0 & V \end{pmatrix}$$

where U and V are the matrices of order n_1 and n_2 respectively, then $\operatorname{Log} M$ has also the form

$$\operatorname{Log} M = \begin{pmatrix} H & X \\ 0 & K \end{pmatrix}$$

where H and K are the matrices of order n_1 and n_2 respectively.

Proof. As theorem 1 shows that the correspondence from $\mathfrak{U}$ to $\mathfrak{M}$ is one to one, we have only to prove that there exist H, K and X such that

$$(21) \quad \exp\begin{pmatrix} H & X \\ 0 & K \end{pmatrix} = \begin{pmatrix} U & W \\ 0 & V \end{pmatrix}, \qquad \begin{pmatrix} H & X \\ 0 & K \end{pmatrix} \in \mathfrak{U}$$

for given U, V and W. (where $\det U \neq 0$ and $\det V \neq 0$). Further, this relation (21) is equivalent to

$$(22) \qquad \exp H = U \quad , \quad \exp K = V$$

and

$$(23) \quad \sum_{r=0}^{\infty} \frac{1}{(r+1)!} \sum_{m=0}^{r} H^{r-m} X K^{m} = W , \text{ where } H^{0} = E_{n_1} , \ K^{0} = E_{n_2}$$

(It is obvious that the left-hand member of (23) is absolutely and uniformly convergent for all matrices.)

By theorem 1 there exists uniquely

$$(24) \qquad H = \operatorname{Log} U \qquad \text{and} \qquad K = \operatorname{Log} V$$

in $\mathfrak{U}(n_1)$ and $\mathfrak{U}(n_2)$ respectively. We see that if there exists X such that satisfies (23) together with (24), this $\begin{pmatrix} H & X \\ 0 & K \end{pmatrix}$ satisfies the conditions (21).

To prove the existency of such X, we shall transform H and K by S and T respectively:

$$\overset{\circ}{H} = SHS^{-1} = \begin{pmatrix} H_1 & & & 0 \\ & H_2 & & \\ & & \ddots & \\ 0 & & & H_s \end{pmatrix} , \quad H_i = \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix} , \ (i = 1, \cdots, s)$$

(H_i is the matrix of order)

$$(25) \quad \overset{\circ}{K} = TKT^{-1} = \begin{pmatrix} K_1 & & & 0 \\ & K_2 & & \\ & & \ddots & \\ 0 & & & K_t \end{pmatrix} , \quad K_j = \begin{pmatrix} \mu_j & & & 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & \mu_j \end{pmatrix} , \ (j = 1, \cdots, t)$$

(K_j is the matrix of order)

and we shall put

$$\overset{\circ}{X} = SXT^{-1} = \begin{pmatrix} x_{11} & \cdots & x_{1n_2} \\ \vdots & & \vdots \\ x_{n_1 1} & \cdots & x_{n_1 n_2} \end{pmatrix} , \quad \overset{\circ}{W} = SWT^{-1} = \begin{pmatrix} w_{11} & \cdots & w_{1n_2} \\ \vdots & & \vdots \\ w_{n_1 1} & \cdots & w_{n_1 n_2} \end{pmatrix} .$$

Then the equation (23) is reduced to the following equation

$$(26) \quad \sum_{r=0}^{\infty} \frac{1}{(r+1)!} \sum_{m=0}^{r} \overset{\circ}{H}{}^{r-m} \overset{\circ}{X} \overset{\circ}{K}{}^{m} = \overset{\circ}{W} , \text{ where } \overset{\circ}{H}{}^{0} = E_{n_1} , \ \overset{\circ}{K}{}^{0} = E_{n_2}$$

The matric equation (26) is the system of linear equations consisting of $n_1 n_2$ equations of $n_1 n_2$ unknown variables $x_{uv} (u = 1, \cdots, n_1 ; v = 1, \cdots, n_2)$. Let then this system of equations be expressed in the following

(27) $$\sum_{k,l} g_{uv,kl}\, x_{kl} = w_{uv} \qquad \left(\begin{matrix} k,u = 1, \cdots\cdots, n_1 \\ l, v = 1, \cdots\cdots, n_2 \end{matrix}\right)$$

If the x_{uv} are arranged in the proper order, the matrix $G \equiv \| g_{uv,kl} \|$ is

(28) $$G = \sum_{r=0}^{\infty} \frac{1}{(r+1)!} \sum_{m=0}^{r} \mathring{H}^{r-m} \times (\mathring{K}^*)^m \qquad \text{where } \mathring{\mathring{H}} = E_{n_1}, \mathring{\mathring{K}} = E_{n_2}^{(2)}$$

From (25) and (28), we get

(29) $$G = \begin{pmatrix} \times & & \# \\ & \times \ddots & \\ 0 & & \ddots \times \end{pmatrix} .$$

and further, the diagonal elements of G are calculated as the following

(30) $$\sum_{r=0}^{\infty} \frac{1}{(r+1)!} \sum_{m=0}^{r} \lambda_i^{r-m} \mu_j^m \quad \begin{aligned} &= \frac{e^{\lambda_i} - e^{\mu_j}}{\lambda_i - \mu_j}, \text{ for } \lambda_i \neq \mu_j \\ &= e^{\lambda_i}, \text{ for } \lambda_i = \mu_j \end{aligned}$$

so that all these values are not zero. Therefore we obtain that

(31) $$\det G \neq 0 .$$

Thus there exists the matrix $\mathring{X}$ such that satisfies (27) i.e., (26). Hence we see that there exists the matrix X such that satisfies (23). This completes the proof of theorem 3.

From these results, we will proceed to investigate for the properties of the finite continuous transformations groups.

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(2) Let now $H \times K = Z$ i.e., $h_{uk} x_{kl} k_{lv} = Z_{uv}$. If the x_{uv} are arranged in the order. $x_{11}, x_{21}, \cdots x_{n_1 1}; x_{12}, x_{22}, \cdots x_{n_1 2}; \cdots\cdots; x_{1 n_2}, \cdots, x_{n_1 n_2}$ the matrix of the system of equations is $H \times K^*$ where K^* denote the transposed matrix of K and $H \times K^*$ shows the direct product of H and K^*;

$$H \times K^* = \begin{pmatrix} H k_{11} & & H k_{n_2 1} \\ & \ddots & \\ H k_{1 n_2} & & H k_{n_2 n_2} \end{pmatrix}$$

AN EXAMPLE OF SOME MAXIMALLY ALMOST PERIODIC GROUP

By Shingo Murakami.

1. In his paper "Topologische Gruppen mit genügend vielen fastperiodischen Funktionen" (Ann. of Math. vol 37 (1936) pp.57-75), H. Freudenthal proved, using his results on the structure of maximally almost periodic groups, the following theorem: for a connected locally compact group G, the following conditions are equivalent,

a) G is maximally almost periodic (m.a.p. for short),

b) G has a two-sided invariant uniform structure.

Evidently, if G satisfies the first countability axiom, the latter condition is equivalent to the condition that G admits a two-sided invariant metric.

Even if G is not connected, the equivalency of these conditions a) and b) are still valid under a weaker condition: In fact, we can show it for such a locally compact group as its factor group by the component of identity is compact. This can be proved by means of its structural characterization, which may be obtained by the results of M. Kuranishi's paper "On non-connected maximally almost periodic groups" (forthcoming).

In more general cases, b) does not necessarily imply a) (e.g. non-m.a.p. discrete group). With regard to the converse, it is mentioned in H. Freudenthal's paper (cited above that it should always hold without any restriction for the group in question. However, we shall illustrate below that, in general, b) does not follow from a), constructing an example of such a group. This is the purpose of the present note.

2. <u>LEMMA. Any extension of a m.a.p. group by a finite group is also m.a.p.</u>

Proof. Let G be a topological group and N its closed normal subgroup such that itself is m.a.p. and that G/N is finite. Then, in order to prove for G to be m.a.p., it is sufficient to show that any continuous almost periodic function f(x) ($x \in N$) has an extension onto whole group G. This extension of f(x) can be given by the following function:-

$$F(g) = \begin{cases} f(g), & \text{if } g \in N, \\ 0, & \text{if } g \notin N. \end{cases}$$

Since G/N is discrete, the continuity of F(g) is obvious. We shall prove that F(g) is an almost periodic function on G, i.e. that every sequence of functions of the form

$F(h_n g)$, $n = 1, 2, \ldots$

where h_n are fixed elements of G,

g is the variable runing over G,

contains a subsequence which uniformly converges on G. Since the number of the cosets modulo N is finite, there exists one coset to which infinitely mant number of h_n belong. Therefore, if necessary passing to the sub-sequence composed of all the functions that are defined by such h_n, we may assume that all elements h_n are expressed in the form

$$h_n = a x_n \quad (x_n \in N),$$

with a fixed representative element a of a certain coset modulo N. Then, as is easily seen from the definition of F(g), all the functions of this sequence vanish on every coset modulo N, except one of the form $a^{-1}N$. and on the latter one they are represented as a sequnce $\{f(a x_n a^{-1} x)\}$ (x varies on N). Thus, referring to the almost periodicity of f(x) on N, we can extract

from the original sequence a subsequence which uniformly converges on this coset and so on the whole group G. q.e.d.

Remark. This lemma is applicable to give a simpler proof for a lemma in M.Kuranishi's paper (cited in 1.).

3.. Let G be a totally disconnected locally compact group which has two-sided invariant uniform structure. Then, there exists in G an open compact normal subgroup. In fact, since G is totally disconnected and locally compact, G has an open compact subgroup. The intersection of all its conjugate subgroups is the required normal subgroup, because its openness can be seen by means of the special structure of G.

Thus we can state as follows: <u>an example of a locally compact group which is m.a.p., but has no two-sided invariant uniform structure, can be given by a locally compact totally disconnected m.a.p. group which admits no open compact normal subgroup.</u>

In the next section, we shall construct such a group concretely.

4. CONSTRUCTION OF EXAMPLE.

At first, we shall define two kinds of groups.

Let D be the weak direct product of countable number of groups, each of which is isomorphic to the additive group of integers. Namely, an element of D is represented by $\{x_n\}$, where x_n (n = 1,2,...) are integers. = 0 except for finite number of n, and the multiplication in D is defined as $\{x_n\} \cdot \{y_n\} = \{x_n + y_n\}$ for $\{x_n\}, \{y_n\} \in D$. Denote by D(r) and $\bar{D}(r)$ two subgroups of D which consist of those elements that satisfies the condition $x_m = 0$ for $m > r$, and the codition $x_m = 0$ for $m \leq r$ respectively.

Secondly, let K denote the direct product of countable number of groups, each of which is isomorphic to the multiplicative group of two numbers ± 1. That is, an element of K is expressed in the form $\{\varepsilon_n\}$, where $\varepsilon_n = \pm 1$, and the product of two elements $\{\varepsilon_n\} \cdot \{\delta_n\}$ of K are defined by $\{\varepsilon_n \delta_n\}$. In K, $K(r)$ and $\overline{K}(r)$ are defined analogous to $D(r)$ and $\overline{D}(r)$ in D.

These groups D and K are topologized as usual, and then, the former is discrete, the latter is compact and totally disconnected.

Now, the desired group G is the set of all pairs $(\{\varepsilon_n\}, \{x_n\})$ of elements from K and D. The multiplication in G is defined as follows;

$$(\{\varepsilon_n\},\{x_n\})\ (\{\delta_n\},\{y_n\}) = (\{\varepsilon_n\delta_n\},\{\varepsilon_n y_n + x_n\})$$
$$\text{for}\quad (\{\varepsilon_n\},\{x_n\}),\ (\{\delta_n\},\{y_n\}) \in G.$$

It is easily verified that then G forms an algebraic group, and that the set of all the elements of the form $(\{\varepsilon_n\},\{0\})$, and the set of those of the form $(\{1\},\{x_n\})$ consistute respectively a subgroup isomorphic to K and a normal subgroup isomorphic to D. (By this reason, we shall designate corresponding groups by the same letters). We note further that $D(r)$ and $\overline{D}(r)$ are nomal subgroups in G and that $D(r)$ and $\overline{K}(r)$ are elementwise commutative.

Moreover, the following relations come in G,

$G = KD$, $K \cap D = 1$, where 1 denotes the identity of G.

Hence we can introduce in G the direct product topology of K and D. Clearly, by this topologization G is a totally disconnected locally compact topological group, in which $K(r), \overline{K}(r), D(r)$ and $\overline{D}(r)$ are closed.

We are to show for G to be m.a.p. Since G/D is compact, we must only to prove that any element $(\neq 1)$ of

D can be separated from 1 by an almost periodic function of $\cdot$. Further, for this purpose it is enough to show that $G/\overline{D}(r)$ is m.a.p., because each element $(\neq 1)$ is contained in some $D(r)$ and then has its natural homomorphic image different from the identity in $G/\overline{D}(r)$. In $G/\overline{D}(r)$, its normal subgroup $D\overline{K}(r)/\overline{D}(r)$ decomposes into the direct product of two subgroups $D/\overline{D}(r)$ and $(\overline{D}(r)\overline{K}(r))/\overline{D}(r)$, for $D(r)$ and $\overline{K}(r)$ are elementwise commutative. Therefore $D\overline{K}(r)/\overline{D}(r)$ is m.a.p., as a direct product of a discrete abelian group and a compact group. While the factor group $(G/\overline{D}(r))/(D\overline{K}(R)/\overline{D}(r))$ is isomorphic to the finite group $K(r)$, we can conclude by the previous lemma, that G is m.a.p.

At last we shall prove that G admits no open compact normal subgroup. Suppose there exist such a normal subgroup N in G. Then, by the definition of the topology in G, there should exist a suitable r such as

$$N \supset \overline{K}(r),$$

and therefore an element $k = (\{\varepsilon_n\}, \{0\})$ in which $\varepsilon_m = 1$ for $m \leq r$, and $\varepsilon_m = -1$ for $m > r$, must belong to N. Let $d = (\{1\}, \{x_n\})$ be an element of D in which $x_{r+1} \neq 0$. In virtue of the normality of N and D, the element

$$dkd^{-1}k^{-1} = (\{1\}, \{x_n - \varepsilon_n x_n\})$$

must be contained both in N and in D. This means, as this element is different from 1 because of $x_{r+1} - \varepsilon_{r+1}x_{r+1} = 2x_{r+1} \neq 0$, that D should have non-trivial compact (that is finite) subgroup, which contradicts to the algebraic structure of D.

Thus G possesses all the properties mentioned in 3. and gives the example which we required at the end of 1.

FIKSPUNKTA TEOREMO EN FUNKCIALA SPACO.

De Tokui Sato (Kyusyu Univ.)

Mi povas ĝeneraligi teoremon 3 en mia noto "Pri fikspunkta teoremo" (1).

D estu domajno en la n-dimensia Euklid-a spaco kaj $\mathfrak{F}$ familio(2) de f(x) funkcioj kontinuaj en D. Tiam $\mathfrak{F}$ faciĝas L-spaco rilate al la topologio difinata per la konverĝo unuforma je larĝa senco en D.

Se $f(x) + g(x)$ kaj $\lambda f(x)$ (λ konstanto) signifas la ordinarajn adicion kaj produtigon, ĉi tiu spaco (nomiĝas simple L-spaco) estas linia spaco kie $f(x) + g(x)$ kaj $f(x)$ estas kontinua respektive rilate al $f(x)$ kaj $g(x)$, kaj rilate al λ kaj $f(x)$.

Por la domajno D. ni povas preni sekvajon $\{D_\nu\}$ de limita kaj fermitaj domajnon tiaj ke

$$D_\nu \subset D_{\nu+1} \qquad (\nu = 1,2,\dots),$$
$$\lim_{\nu\to\infty} D_\nu = D.$$

Signu $U(f(x);\varepsilon,n)$ $(f(x)\in\mathfrak{F})$ familion de g(x) tiaj ke

$$\max_{x\in D_n} |g(x) - f(x)| < \varepsilon, \qquad g(x)\in\mathfrak{F},$$

kie ε estas pozitiva nombro kaj n ajna entjero. Tiam $U(f(x);\varepsilon,n)$ plenumas la aksiomon de Hausdorff kaj la unuan aksiomon de komputebleco.

Se $g(x), h(x) \in U(f(x);\varepsilon,n)$, $\lambda, \mu \geq 0$, $\lambda+\mu = 1$, ni ricevas $\lambda g(x) + \mu h(x) \in U(f(x);\varepsilon,n)$, t.e. $U(f(x);\varepsilon,n)$ estas konveksa.

Tiu ĉi spaco (nomiĝas V-spaco) estas homeomorfa al la L-spaco.

Efektive estu D_0 ajna limita kaj fermita domajno enhavita en D, tiam ekzistas D_n tia ke $D_0 \subseteq D_n$. Se $f_\nu(x) \in U(f(x);\varepsilon,n)$ $(\nu=1,2,\dots)$ por $\varepsilon\,(>0)$ donita

antaŭe, ni ricevas $\max_{x\in D_n} |f_\nu(x) - f(x)| < \varepsilon$ $(\nu=1,2,\dots)$, t.e. la funkcisekvajo $\{f_\nu(x)\}$ konverĝas al f(x) unuforme je larĝa senco en la domajno D. Kontraŭe se la funkcisekvajo $\{f_\nu(x)\}$ konverĝas al f(x) unuforme je larĝa senco en la domajno D, por $\varepsilon\,(>0)$ kaj n antaŭe donitaj, ni povas preni entjeron N tia ke $\max_{x\in D_n} |f_\nu(x) - f(x)| < \varepsilon$, $\nu\geq N$, t.e. $f_\nu(x) \in U(f(x);\varepsilon,n)$ $\nu\geq N$.

V-spaco estas do linia topologia lokale konveksa spaco(3). Per la teoremo(4) de G. Birkhoff kaj S. Kakutani V-spaco estas metricebla. Sekve per la fikspunkta teoremo(5) en linia metrica spaco ni alvenas al jena

Teoremo. Estu $\mathfrak{F}$, $\mathfrak{F}_K$, $\mathfrak{F}_F$ kaj $\mathfrak{F}_N$ familioj de funkcioj kontinuaj en domajno D kaj posedantaj iun propreco P, tiaj ke

$$\mathfrak{F}_N \subseteq \mathfrak{F}_F \subseteq \mathfrak{F}_K \subseteq \mathfrak{F}.$$

Supozu ke

i) $\mathfrak{F}$ estas ne malplena:

ii) estu $f(x), g(x)\in\mathfrak{F}_K$, $\lambda, \mu \geq 0$, $\lambda+\mu=1$, tiam $\lambda f(x) + \mu g(x) \in \mathfrak{F}_K$:

iii) se funkcisekvajo $\{f_\nu(x)\}$ $(f_\nu(x)\in\mathfrak{F}_F)$ konverĝas al f(x) unuforme je larĝa senco en D, la limfunkcio f(x) apartenas ankaŭ al $\mathfrak{F}_F$;

iv) $\mathfrak{F}_N$ estas normala familio de funkcioj kontinuaj en D. Respondu $f^*(x)$ funkcion kontinuan en D al f(x) $(f(x)\in\mathfrak{F})$, kaj estu $\mathfrak{F}^*$ la familio de la funkcioj $f^*(x)$.

Se ni havas $\mathfrak{F}^* \subseteq \mathfrak{F}_N$ kaj por funkcisekvajo $\{f_n(x)\}$ konverĝanta al f(x) unuforme je larĝa senco en D, $f_n^*(x)$ konverĝas al $f^*(x)$ unuforme je larĝa senco en D, kie $f_n^*(x)$ kaj $f^*(x)$ respondas respektive al $f_n(x)$ kaj f(x), ekzistas almenaŭ unu funkcio f(x) $(\in\mathfrak{F}^*)$ kiu respondas al ĝi mem.

Rimarko. Se familio $\mathfrak{F}$ de funkcioj kontinuaj en konsajno D estas egale kontinua kaj limita ĉe punktoj kiuj apartenas al densa aro da punktoj en D, $\mathfrak{F}$ fariĝas normala familio.

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(1) T. Sato, Pri fikspunkta teoremo, Mem. Fac. Kyusyu Univ. 4(1949), 33-44.

(2) Mi riceves similajn rezultatojn ĉe tiu ĉi noto rilate al sistemo de kontinuaj funkcioj.

(3) Vidu (1).

(4) S. Kakutani, Über die Metrisation der topologischen Gruppen, Proc. Imp. Akad. Tokyo, 12(1936), 82-84.

(5) Vidu (1).

ON THE NON-EXISTENCE OF SOLUTION OF SCHRÖDINGER EQUATION.

By Osamu Miyatake.

(received VIII 25)

The operator, representing the mutual action of a nucleon and a scalar meson field, has no domain in the space in which the operator is defined other than the null vector. Accordingly, the Schrödinger equation $i\,d\Psi/dt = H\Psi$ has no solution. In the present paper, we put the light velocity c as 1 and Planck constant h as 2π.

1. Determination of the space. As already has been known (Yukawa & Sakata 1), a scalar meson field can be described by a scalar function $U(x,y,z,t)$ and its conjugate $\tilde{U}(x,y,z,t)$. In the presence of a nucleon, the Hamiltonian for the total system is given by

(1) $$H = H(M)+H(U)+H',$$

where
$$H(M) = \alpha p + \beta\left(\frac{1+\tau_3}{2}M_N + \frac{1-\tau_3}{2}M_p\right),$$
$$H(U) = \sum_k E_k(N^+(k) + N^-(k)),$$

$$H' = -ig\sum_k \sqrt{\frac{2\pi}{E_k}}\left\{(a_k^* - b_k)e^{-ikr}Q^* - (a_k - b_k^*)e^{ikr}Q\right\}\beta .$$

In these formulae, $H(M)$, $H(U)$ and H' are the Hamiltonians of the nucleon, the meson field and the mutual action respectively. (α, β) are Dirac's spin matrices. (τ_1, τ_2, τ_3) are the isotopic spin matrices. In the present paper, we use the representations

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

M_p and M_N are the masses of proton and neutron respectively and they are nearly equal to each other. So that, let us put as $M_p = M_N = M$ for simplicity. The zero point energy $\sum_k E_k$ that is contained in $H(U)$ has been omitted. $E_k = \sqrt{k^2 + m_U^2}$, where m_U and k being the mass and the momentum of a meson respectively. The k's in the series of $H(U)$ and H' stand for the momenta k's. Hereafter, we use the letter k in this sense. The operator $N^+(k) \equiv a^* a$ has eigenvalues 0,1,2,..., and represents the number of mesons whose charge, momentum and energy are $+e$, k and E_k respectively. The operator $N^-(k) \ell^* \ell$ represents the number of mesons whose charge, momentum and energy are $-e$, $-k$ and E_k respectively. For brevity, let us call the meson belonging to the operators $N^+(k)$ and $N^-(k)$ as $N^+(k)$-meson and $N^-(k)$-meson respectively. In the above formulae, $p = -i\,\mathrm{grad}$, $Q = \frac{1}{2}(\tau_1 + i\tau_2)$ and $Q^* = \frac{1}{2}(\tau_1 - i\tau_2)$. Accordingly, we obtain the representations

$$Q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Q^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The components of k are either zero or integers multiplied by 2π.

In the Schrödinger picture,

$$a_k = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & \cdots \\ \cdots & & & & \\ \cdots & & & & \end{pmatrix}, \quad a_k^* = \begin{pmatrix} 0 & 0 & \cdots \\ \sqrt{1} & 0 & \cdots \\ 0 & \sqrt{2} & \cdots \\ \cdot & 0 & \cdots \end{pmatrix}$$

b_k and b_k^* have the matrix representations similar to those of a_k and a_k^* respectively. a_k is an operator which absorbs a $N^+(k)$ -meson, and a_k^* is an operator which produces a $N^-(k)$ -meson.

We represent a state in which there are $\beta^+(k) N^+(k)$-mesons with a vector

$$\varphi(\beta^+(k)) \equiv \begin{pmatrix} \delta(\beta^+(k), 0) \\ \delta(\beta^+(k), 1) \\ \vdots \\ \vdots \end{pmatrix} \exp(i\beta^+(k) E_k t),$$

where $\beta^+(k) = 0, 1, 2, \ldots$, and $\delta(\beta^+(k), i)$ is equal to 1, when $\beta^+(k) = i$ and is equal to zero when $\beta^+(k) \neq i$. The total $\{\varphi(\beta^+(k))\}$ of $\varphi(\beta^+(k))$'s determines a Hilbert space. Let us represent this space with $\mathfrak{H}^+(k)$. In like manner, we define the vectors $\varphi(\beta^-(k))$'s, where $\beta^-(k) = 0, 1, 2, \ldots$ The total $\{\varphi(\beta^-(k))\}$ determines a Hilbert space $\mathfrak{H}^-(k)$. It can easily be shown that the operators $N^+(k)$ and $N^-(k)$ are self-adjoint in the respective spaces $\mathfrak{H}^+(k)$ and $\mathfrak{H}^-(k)$.

The incomplete direct product

$$\mathfrak{H}(U) \equiv \prod_k{}^{c} \otimes \mathfrak{H}^+(k) \otimes \prod_k{}^{c} \otimes \mathfrak{H}^-(k)$$

is the space whose vectors represent states of a free meson field. As to the notation used here, see v. Neumann 2.

The space $\mathfrak{H}(U)$ is determined by a c.n.o.s. (complete normalized orthogonal set) $\{\varphi_\beta\}$, where

(2) $$\varphi_\beta \equiv \prod_k \otimes \varphi(\beta^+(k)) \otimes \prod_k \otimes \varphi(\beta^-(k))$$

and $\beta^+(k)$ and $\beta^-(k)$ are positive integers for only a

finite number of k's, and zero for all other k's (v. Neumann 2, Lemma 4.1.4). Prof. v. Neumann has used a notation $\beta \in F$ in order to show that both $\beta^+(k)$ and $\beta^-(k)$ have this character. In order that the energy of the meson field may be finte, this condition $\beta \in F$ is necessary. It can easily be shown that the operator $H(U)$ is self-adjoint in $\mathfrak{H}(U)$.

In the next place, let us consider the eigenvectors of the operator $H(M)$. These are given as

(3) $$\Psi_N \equiv \begin{pmatrix} \psi(N) \\ 0 \end{pmatrix}, \qquad \Psi_P \equiv \begin{pmatrix} 0 \\ \psi(P) \end{pmatrix}$$

where Ψ_N and Ψ_P are the neutron and proton-state respectively, and

(4) $$\psi(N) = u(N)\exp(i(pv - Wt)), \; \psi(P) = u(P)\exp(i(pv - wt)).$$

$u(N)$ and $u(P)$ are both four-components vectors dependent on the momentum p of the nucleon. And $W^2 = p^2 + M^2$.

As has been well known, two values of W are determined by one value of p, and two vectors $u(N)$'s by each value of W. That is, four vectors $u(N)$'s are determined by one value of p. These vectors are orthogonal with one another. As to $u(p)$'s, the similar holds valid. Thus, eight vectors, representing the states of a nucleon, are given once a p is fixed, and they are mutually orthogonal. When we impose a condition that the wave functions of the nucleon are periodical with unit period in spatial coordinates, the components of the momentum p contained in Ψ_N and Ψ_P are zero or integers multiplied by 2π.

The Hilbert space determined by such eigenvectors of $H(M)$ is denoted by $\mathfrak{H}(M)$. Then the direct product

(5) $$\mathfrak{H} \equiv \mathfrak{H}(M) \otimes \mathfrak{H}(U)$$

is the space whose vectors represent states of the system, consisting of a nucleon and a meson field. This space is determined by a c.n.o.s.

$$(6) \qquad \{\phi_\beta\} = \{\phi_{\beta(N)}\} + \{\phi_{\beta(P)}\},$$

where

$$\phi_{\beta(N)} = \psi_N \otimes \varphi_\beta, \qquad \phi_{\beta(P)} = \psi_P \otimes \varphi_\beta.$$

The total k's are numbered suitably. Let the order number of k be K.

2. On the domain of H'.

Theorem 1. The vectors $\phi_{\beta(N)}$ and $\phi_{\beta(P)}$ do not belong to the domain of H'.

Proof. For brevity, let us put as

$$-ig\sqrt{\frac{2\pi}{E_k}}\{(a_k^* - b_k)e^{-ikr}Q^* - (a_k - b_k^*)e^{ikr}Q\}\beta \equiv H_k.$$

Then

$$H' = \sum_k H_k \quad \text{and} \quad H'\phi_{\beta(N)} = \lim_{K\to\infty}\sum^{K} H_k\phi_{\beta(N)}$$

As

$$Q^*\psi_N = \begin{pmatrix} 0 \\ \psi(N) \end{pmatrix}, \quad Q^*\psi_P = 0, \quad Q\psi_N = 0, \quad Q\psi_P = \begin{pmatrix} \psi(P) \\ 0 \end{pmatrix}.$$

we obtain

$$(7) \quad H_k\phi_{\beta(N)} = -ig\sqrt{\frac{2\pi}{E_k}}\Big\{\sqrt{\beta^+(k)+1}\,\varphi(\beta^+(k)+1)\otimes\prod_{k'\neq k}\otimes\varphi(\beta^+(k'))$$
$$\otimes\prod\otimes\varphi(\beta^-(k))e^{iE_kt} - \sqrt{\beta^-(k)}\,\varphi(\beta^-(k)-1)\otimes\prod\otimes\varphi(\beta^+(k))$$
$$\otimes\prod_{k'\neq k}\otimes\varphi(\beta^-(k'))e^{-iE_kt}\Big\}e^{-ikr}\beta Q^*\psi_N$$

The two vectors in the brackets are orthogonal with each other. And $H_{k_1}\phi_{\beta(N)}$ and $H_{k_2}\phi_{\beta(N)}$ are orthogonal with each other when $k_1 \neq k_2$. So that, we obtain

$$\|H'\phi_{\beta(N)}\|^2 = \lim_{K\to\infty}\Big(\sum^{K} H_k\phi_{\beta(N)}, \sum^{K} H_k\phi_{\beta(N)}\Big)$$
$$= g^2\lim_{K\to\infty}\sum^{K}\frac{2\pi}{E_k}(\beta^+(k)+\beta^-(k)+1) \geq 2\pi g^2\sum_{l,m,n=0}^{\infty}\frac{1}{\sqrt{4\pi^2(l^2+m^2+n^2)+m_0^2}} = \infty$$

accordingly, we obtain

$$\|H'\phi_{\beta'}\| = \infty$$

In like manner,

$$\|H'\phi_{\beta(P)}\| = \infty$$

Corollary. $\phi_{\beta^{(i)}(N)} = \psi_N^{(i)}\otimes\varphi_{\beta^{(i)}}$, $i = 1.2.3....m$, and $\psi_{\beta^{(1)}(N)}, \psi_{\beta^{(2)}(N)}, \dots \psi_{\beta^{(m)}(N)}$ are orthogonal with one another. Then $\|\sum_{i=1}^{m} c_i\phi_{\beta^{(i)}(N)}\|^2 = \sum_{i=1}^{m}|c_i|^2\|H'\phi_{\beta^{(i)}(N)}\|^2 = \infty$

Proof. When $\psi_N^{(i)}$ and $\psi_N^{(j)}$ are orthogonal with each other, obtain

$$(\beta Q^*\psi_N^{(i)}, \beta Q^*\psi_N^{(j)}) = 0.$$

so that, from (7), it can be proved that $H'\phi_{\beta^{(i)}(N)}$ and $H'\phi_{\beta^{(j)}(N)}$ are mutually orthogonal. Accordingly we obtain the result.

Let the momenta belonging to $\phi_{\beta(P)}$ and $\phi_{\beta'(N)}$ be p and p' respectively. When those two vectors are in the relation

$$\beta^{+'}(k_1)+1 = \beta^+(k_1), \qquad \beta^{-'}(k_1) = \beta^-(k_1),$$
$$\beta^{\pm'}(k) = \beta^{\pm}(k) \quad \text{for all } k\text{'s unequal to } k_1,$$
$$p' = p + k_1,$$

we use the notation

$$(8) \qquad \begin{pmatrix} \beta'(N) \\ \beta(P) \end{pmatrix} = \left(\begin{matrix} \beta^{+'}(k_1) \\ \beta^{+'}(k_1)+1 \end{matrix}\,\middle|\qquad\middle|\, p' = p + k_1\right)$$

In this case, we obtain

$$\lim_{K\to\infty}\Big(\sum^{K} H_k\phi_{\beta'(N)}, \phi_{\beta(P)}\Big) = (H_{k_1}\phi_{\beta'(N)}, \phi_{\beta(P)})$$

$$(9) \qquad = -ig\sqrt{\frac{2\pi}{E_{k_1}}}\sqrt{\beta^+(k_1)+1}\,(\beta u'(N), u(P))e^{-i(W'-E_{k_1}-W)t}$$

In like manner, we use the relation

$$(10) \qquad \begin{pmatrix} \beta'(N) \\ \beta(P) \end{pmatrix} = \left(\qquad\middle|\begin{matrix} \beta^{-'}(k_1) \\ \beta^{-'}(k_1)-1 \end{matrix}\middle|\, p' = p + k_1\right)$$

when two vectors $\phi_{\beta'(N)}$ and $\phi_{\beta(P)}$ are in the relation

$$\beta^{-\prime}(k_1) - 1 = \beta^{-}(k_1) \qquad \beta^{+\prime}(k_1) = \beta^{+}(k_1)$$

$$\beta^{\pm\prime}(k) = \beta^{\pm}(k) \quad \text{for all } k\text{'s unequal to } k_1,$$

$$p' = p + k_1.$$

In this case, we obtain

$$\lim_{K\to\infty}\Big(\sum^{K} H_k \phi_{\beta'(N)}, \phi_{\beta(P)}\Big) = \big(H_{k_1}\phi_{\beta'(N)}, \phi_{\beta(P)}\big)$$

(11) $$= -ig\sqrt{\frac{2\pi}{E_{k_1}}}\sqrt{\beta_-^{\prime}(k_1)}\,(\beta u'(N), u(P))\, e^{-i(w'+E_{k_1}-w)t}$$

Theorem 2. H' has no domain other than the null vector.

Proof. We assume that the domain of H' contains a non null vector. Let this vector be ϕ. And we expand ϕ into a series of c.n.o.s. $\{\phi_\beta\}$:

$$\phi = \sum c_{\beta'}\phi_{\beta'} = \sum c_{\beta'(N)}\phi_{\beta'(N)} + \sum c_{\beta'(P)}\phi_{\beta'(P)}.$$

As ϕ is non null, at least one of the two series in the right hand side is non null. So that, let us assume that $\sum c_{\beta'(N)}\phi_{\beta'(N)}$ is non null. When, in the above series, $\varphi_{\beta^{(i)}} = \varphi_{\beta^{(j)}} = \cdots$, and $\psi_N^{(i)}, \psi_N^{(j)}, \cdots$ belong to the same momentum. we rewrite the partial sum $c_{\beta^{(i)}(N)}\phi_{\beta^{(i)}(N)} + c_{\beta^{(j)}(N)}\phi_{\beta^{(j)}(N)}$ as $\Psi_N^{(i)} \otimes \varphi_{\beta^{(i)}}$, i.e.,

$$\Psi_N^{(i)} = c_{\beta^{(i)}(N)}\psi_N^{(i)} + c_{\beta^{(j)}(N)}\psi_N^{(j)} + \cdots\cdots$$

and let this vector $\Psi_N^{(i)} \otimes \varphi_{\beta^{(i)}}$ be denoted as $\phi_{\beta^{(i)}(N)}$. As has already been stated, the number of i, j .... is at most four. In the same way, we define the vectors $\phi_{\beta'(P)}$'s. Thus we can rewrite the series (12) as

(13) $$\phi = \sum \phi_{\beta'(N)} + \sum \phi_{\beta'(P)}$$

Accordingly, if we put as

$$\phi_{\beta^{(i)}(N)} = \Psi_N^{(i)} \otimes \varphi_{\beta^{(i)}}, \quad \phi_{\beta^{(j)}(N)} = \Psi_N^{(j)} \otimes \varphi_{\beta^{(j)}} \quad (i \neq j)$$

then $\Psi_N^{(i)}$ and $\Psi_N^{(j)}$ must belong to the different

momenta when $\varphi_{\beta^{(i)}} = \varphi_{\beta^{(j)}}$ and $\varphi_{\beta^{(i)}} \neq \varphi_{\beta^{(j)}}$ when $\Psi_N^{(i)}$ and $\Psi_N^{(j)}$ belong to the same momentum. Accordingly, any two of the terms of the first series of (13) are mutually orthogonal. In the same way, we can prove that any two of the terms of the second series of (13) are mutually orthogonal. Moreover, the first and second series of (13) are orthogonal with each other. So that we obtain

(14) $$\|\phi\|^2 = \sum \|\phi_{\beta'(N)}\|^2 + \sum \|\phi_{\beta'(P)}\|^2 < \infty$$

And, as ϕ belongs to the domain of H' we have

(15) $$\|H'\phi\| < \infty$$

Here, we interpret the vector $H'\phi$ as $H'\phi = \lim_{K\to\infty}\sum^{K} H_k\phi$ and $H_k\phi = \lim_{B\to\infty}\sum^{B} H_k\phi_{\beta'}$, where B being the order number of β'.

According to (14), the number of the terms of the series (13) which are non zero is at most (enumerably infinite) (v. Neumann [2], Lemma 2.3.4.). From the corollary of the Theorem 1, we obtain

(16) $$\|H'\phi_{\beta'(N)}\| = \infty.$$

The condition (14), (15) and (16) can not, however, hold valid simultaneously, as can be shown below. Rewriting the inequality (15), we obtain

$$\infty > \|H'\phi\|^2 = \sum_{\beta\in F}|(H'\phi, \phi_\beta)|^2 = \sum_{\beta\in F}\Big|\lim_{K\to\infty}\Big(\sum^{K} H_k\phi, \phi_\beta\Big)\Big|^2$$

$$= \sum_{\beta\in F}\Big|\lim_{K\to\infty}\Big(\phi, \sum^{K} H_k\phi_\beta\Big)\Big|^2 = \sum_{\beta\in F}\Big|\lim_{K\to\infty}\lim_{B\to\infty}\Big(\sum^{B}\phi_{\beta'}, \sum^{K} H_k\phi_\beta\Big)\Big|^2$$

$$= \sum_{\beta\in F}\Big|\lim_{K\to\infty}\lim_{B\to\infty}\Big\{\sum^{(B)}\Big(\phi_{\beta'(N)}, \sum H_k\phi_\beta\Big) + \sum^{(B)}\Big(\phi_{\beta'(P)}, \sum^{K} H_k\phi_\beta\Big)\Big|^2$$

$$= \sum_{\beta\in F}\Big|\lim_{K\to\infty}\lim_{B\to\infty}\Big\{\sum^{(B)}\Big(\sum^{K}\phi_{\beta(N)}, \phi_\beta\Big) + \sum^{(B)}\Big(\sum^{K}\phi_{\beta'(P)}, \phi_\beta\Big)\Big|^2$$

$$\geq \sum_{\beta(P)\in F}\Big|\lim_{K\to\infty}\sum\Big(\sum^{K} H_k\phi_{\beta'(N)}, \phi_{\beta(P)}\Big)\Big|^2$$

For fixed $\psi_{\beta'(N)}$ and k_1, the number of $\phi_{\beta(P)}$'s, either in the relation (8) or (10), is four in each case.

In the next place, we consider vectors $\psi_{\beta''(N)}$'s which are in the relation

$$(18)\qquad \begin{pmatrix}\beta''(N)\\ \beta(P)\end{pmatrix}=\left(\begin{array}{c|c|c}\beta^{+''}(k_2) & & \\ & & p''=p+k_2\\ \beta^{+''}(k_2)+1 & & \end{array}\right)$$

or

$$(19)\qquad \begin{pmatrix}\beta''(N)\\ \beta(P)\end{pmatrix}=\left(\begin{array}{c|c|c} & \beta^{-''}(k_2) & \\ & & p''=p+k_2\\ & \beta^{-''}(k_2)-1 & \end{array}\right)$$

where $\phi_{\beta(P)}$'s are the same vectors that we considered above and k_2 is a certain fixed momentum and it may be equal to k_1. Then the vectors $\psi_{\beta'(N)}$ and $\psi_{\beta''(N)}$'s are in any one of the following relations:

From (8) and (18),

$$(20)\qquad \begin{pmatrix}\beta'(N)\\ \beta''(N)\end{pmatrix}=\left(\begin{array}{cc|c|c}\beta^{+'}(k_1) & \beta^{+'}(k_2) & & \\ & & & p'=p'-k_1+k_2\\ \beta^{+'}(k_2)+1 & \beta^{+'}(k_2)-1 & & \end{array}\right)$$

From (8) and (19),

$$(21)\qquad \begin{pmatrix}\beta'(N)\\ \beta''(N)\end{pmatrix}=\left(\begin{array}{c|c|c}\beta^{+'}(k_1) & \beta^{-'}(k_2) & \\ & & p''=p'-k_1+k_2\\ \beta^{+'}(k_1)+1 & \beta^{-'}(k_2)+1 & \end{array}\right)$$

From (10) and (18),

$$(22)\qquad \begin{pmatrix}\beta'(N)\\ \beta''(N)\end{pmatrix}=\left(\begin{array}{c|c|c}\beta^{+'}(k_2) & \beta^{-'}(-k_1) & \\ & & p''=p'-k_1+k_2\\ \beta^{+'}(k_2)-1 & \beta^{-'}(k_1)+1 & \end{array}\right)$$

From (10) and (19),

$$(23)\qquad \begin{pmatrix}\beta'(N)\\ \beta''(N)\end{pmatrix}=\left(\begin{array}{c|c|c} & \beta^{-'}(k_1),\ \beta^{-'}(k_2) & \\ & & p''=p'-k_1+k_2\\ & \beta^{-'}(k_1)-1,\ \beta^{-'}(k_2)+1 & \end{array}\right)$$

In these relations, p' is determined uniquely, when p', k_1 and k_2 are given. Even though there are four $\phi_{\beta(P)}$'s which are in the relation (8) or (10), there are at most one $\psi_{\beta''(N)}$ in the series (13) which is in any one of the relations (20), (21), (22) and (23). When $\beta'(N)$ and $\beta''(N)$'s are in the relation (20), the number of $\beta(P)$'s is ∞ and those of $\beta''(N)$'s which correspond to each of these $\beta(P)$'s are uniformly bounded. In such a case, we say that the ensemble of $\beta''(N)$'s is (∞,n)-type. Hereafter, an ensemble of $\beta''(N)$'s is denoted as $\{\beta''(N)\}$. When $\beta'(N)$ and $\beta''(N)$'s are in the relation (22), the number of $\beta(P)$'s is finite and those of $\beta''(N)$'s which correspond to each of these $\beta(P)$'s are all finite. In such a case, we say that $\{\beta''(N)\}$ is (n,n)-type. When $\beta'(N)$ and $\beta''(N)$'s are in the relation (23), the number of $\beta(P)$'s is finite and those of $\beta''(N)$'s which correspond to each of these $\beta(P)$'s are ∞. In such a case, we say that $\{\beta''(N)\}$ is (n,∞)-type. When $\beta'(N)$ and $\beta''(N)$'s are in the relation (21), the number of $\beta(P)$'s is ∞, and those of $\beta''(N)$'s which correspond to each of these $\beta(P)$'s may also be ∞. In such a case, we say that $\{\beta''(N)\}$ is (∞,∞)-type. When $\{\beta''(N)\}$ is a finite sum of ensembles of (n,n)-, (n,∞)- and (∞,∞)-type, we say that $\{\beta''(N)\}$ is (n,n,∞)-type.

Up to the present, the vectors $\psi_{\beta'(N)}$ and $\psi_{\beta''(N)}$ have been arbitrary ones. From now on, let $\psi_{\beta'(N)}$ and $\psi_{\beta''(N)}$ be vectors which are contained in the series (13).

Let us now prove that if we pick up $\psi_{\beta'(N)}$ suitably, the ensemble of $\beta''(N)$'s which are in the relation (21) with $\beta'(N)$ can also be (n,n,∞)-type. Let us say that a $\beta''(N)$ which is in the relation (21) with $\beta'(N)$ belongs to $(\beta'(N)+2)$-class.

First, we pick up a $\beta'(N)$ in the series (13) arbitrarily. If the $(\beta'(N)+2)$-class is (n,n,∞)-type from the beginning, we have nothing to prove. So that, we assume that the $(\beta(N)+2)$-class is not (n,n,α)-type. Then the number of its members must be α. Let these members be $\beta(N)^{(l)}, \beta(N)^{(m)}, \cdots$ and we construct $(\beta(N)^{(l)}+2)$-class, $(\beta(N)^{(m)}+2)$-class, ...: their members are, of course, picked up in the series (13). If one of these classes — say $(\beta(N)^{(l)}+2)$-class is (n,n,α)-type, we may regard $\beta(N)^{(l)}$ as desired $\beta'(N)$. So that, we assume that the above classes are all not (n,n,α)-type. Any two of these classes may have common members. For example, we assume that the $(\beta(N)^{(l)}+2)$- and $(\beta(N)^{(m)}+2)$-class have a common member $\beta(N)$. $\beta(N)$ is in the like relation (21) with $\beta(N)^{(l)}$ and $\beta(N)^{(m)}$. Let the crossing points of $\beta(N)^{(l)}$ and $\beta(N)$ be k_1, k_2 and those of $\beta(N)^{(m)}$ and $\beta(N)$ be k_1', k_2'. Then

$$p = p^{(l)} - k_1 + k_2 , \qquad p = p^{(m)} - k_1' + k_2' ,$$

where p is the momentum contained in $\beta(N)$. As $\beta(N)^{(l)}, \beta(N)^{(m)} \in$ all the order numbers of k_1, k_2, k_1', k_2' can not be sufficiently large. For, as $\beta(N), \beta(N)^{(l)}, \beta(N)^{(m)} \in F$, $\beta^+(k)$ must be zero for all k's which are unequal to k_1 and k_2 and whose order numbers are sufficiently large, and both $\beta^+(k_1)$ and $\beta^+(k_1')$ must be 1 when the order numbers of k_1, k_1' are both sufficiently large, so that it is concluded that $k_1 = k_1'$. In the same way, k_2 must be equal to k_2'. Accordingly, $p^{(l)} = p^{(m)}$. Then the vectors $\psi_{\beta(N)^{(l)}}$ and $\psi_{\beta(N)^{(m)}}$ must coincide. This is contradictory to the fact that $\psi_{\beta(N)^{(l)}}$ and $\psi_{\beta(N)^{(m)}}$ are different vectors. So that, the order numbers of k_1, k_1' or those of k_2, k_2' must be finite. If the order numbers of k_1, k_1' are finite, the ensemble of $\beta(N)$'s is at most (n,α)-type. If those of k_2, k_2' are finite, the ensemble of $\beta(N)$'s is at most (α, n)-type. So that, even though the two classes $(\beta(N)^{(l)}+2)$- and $(\beta(N)^{(m)}+2)$-class

have common members, their set is at most (n,n,α)-type.

In the next place, we number $(\beta(N)^{(l)}+2)$-class, $(\beta(N)^{(m)}+2)$-class, etc., and erase members in a class which are, at the same time, contained in a class with smaller order number. In this way, members erased in a class, are at most (n,n,α)-type, and, after this adjustment is conducted, there is no more common member in these classes. If the number of members which have not been erased in a class — say $(\beta(N)^{(l)}+2)$-class — is finite, we may regard the $\beta(N)^{(l)}$ as the desired $\beta'(N)$. If the number of members which have not been erased in each of the classes be α, we construct the $(\beta(N)+2)$-class corresponding to each of these members $\beta(N)$'s. If any two of these classes have same common members, we eliminate these multiplicities by using the same method as above. Then the set of members which are erased in a class is at most (n,n,α)-type. If the number of members which have not been erased in any one class be finite, we can set the desired $\beta'(N)$. If each of these classes has α member which have not been erased, we proceed on to the next step. By and by, we continue these processes. But these processes can not be continued infinitely. Because, if otherwise, the number of all the members differing with one another must be α^{α} which is greater than α, while the number of terms of the series (13) is at most α. This is contradictory. Accordingly, at least one of the classes which are constructed in the above processes must contain a finite number of members which have not been erased. Let this class be $(\beta(N)^{(8)}+2)$-class. Then this class is at most (n,n,α)-type. So we can regard $\beta(N)^{(8)}$ as the desired $\beta'(N)$. That is, the ensemble of $\beta''(N)$'s which are in the relation (21) with this $\beta(N)$ is a finite sum of ensembles of (n,n)-, (n,σ)-and (σ,n)-type. When it is (σ,n)-type, the number of $\beta''(N)$'s which are in the relation (18) or (19) is σ and the number of $\beta''(N)$'s corresponding to each of these $\beta(P)$'s is finite. Moreover, these numbers are bounded with respect to all $\beta(P)$'s.

In conclusion, there is at least one such $\beta'(N)$ in the series (13) as the set of $\beta''(N)$'s which are in any one of the relations (20), (21), (22) and (23) is at most (n,n,σ)-type.

Accordingly, the number of terms of the series in (17)

$$(24)\qquad \lim_{K\to\infty}\sum_{\beta''(N)}\Big(\sum^{K} H_k\phi_{\beta''(N)},\ \phi_{\beta(P)}\Big)\qquad \beta(P)\in F$$

may be infinite for only finite number of $\beta(P)$'s, and are finite for the other $\beta(P)$'s. Moreover, the number of terms of these finite series are bounded for all $\beta(P)$'s.

Let us denote the set of $\beta(P)$'s which are in either the relation (8) or (10) with $\beta'(N)$ as $F_{\beta'(N)}$, then we have, from (17),

$$(25)\qquad \|H'\phi\|\geq\sum_{\beta(P)\in F_{\beta'(N)}}\Big|\lim_{K\to\infty}\sum_{\beta''(N)}\Big(\sum^{K} H_k\phi_{\beta''(N)},\phi_{\beta(P)}\Big)\Big|^2,$$

As the number of $\phi_{\beta(P)}$'s which are in either the relation (18) or (19) with fixed $\beta''(N)$ and k_2 is at most eight, the inner product

$$(26)\qquad \lim_{K\to\infty}\Big(\sum^{K} H_k\phi_{\beta''(N)},\phi_{\beta(P)}\Big)$$

does not vanish at most eight times when $\beta(P)$ runs over $F_{\beta'(N)}$. So that we have

$$(27)\qquad \begin{aligned}\|H'\phi\| &\geq \sum_{\beta(P)\in F_{\beta'(N)}}\Big|\lim_{K\to\infty}\Big\{\Big(\sum^{K} H_k\phi_{\beta'(N)},\phi_{\beta(P)}\Big) + \sum_{\beta''\neq\beta'}\Big(\sum^{K} H_k\phi_{\beta''(N)},\phi_{\beta(P)}\Big)\Big\}\Big|^2 = \sum_{\beta(P)\in F_{\beta'(N)}}|\text{infinite series}|^2\\ &+ \sum_{\beta(P)\in F_{\beta'(N)}}\Big|\lim_{K\to\infty}\Big(\sum^{K} H_k\phi_{\beta'(N)},\phi_{\beta(P)}\Big)+\text{finite series}\Big|^2\end{aligned}$$

(32)

As $\phi_{\beta'(N)}$ is fixed, $\beta^{+}(k)$ and $\beta^{-}(k)$ contained in $\phi_{\beta(P)}$ are bounded for all values of k, so that also those contained in $\phi_{\beta''(N)}$'s which make the inner product (26) non null are bounded. Accordingly, using the results (9) and (11), it can be proved that the series

$$\sum_{\beta(P)\in F_{\beta'(N)}}\Big|\lim_{K\to\infty}\sum_{\beta''\neq\beta'}\Big(\sum^{K} H_k\phi_{\beta''(N)},\phi_{\beta(P)}\Big)\Big|^2$$

converges. While, $\sum_{\beta(P)\in F_{\beta'(N)}}\Big|\lim_{K\to\infty}\Big(\sum^{K} H_k\phi_{\beta'(N)},\phi_{\beta(P)}\Big)\Big|^2=\infty$. So that, we obtain, from (27), $\|H'\phi\|^2=\infty$. This is contradictory to (15). It has just been proved that the domain of H' does not contain any non null vector.

§3. Conclusion. In the Schrödinger equation,

$$(28)\qquad i\frac{d\psi}{dt}=H\psi\quad,\qquad H=H_0+H',$$

we put as $\psi=\exp(-iH_0t)\varphi$. Then we have

$$(29)\qquad i\frac{d\varphi}{dt}=e^{iH_0t}H'e^{-iH_0t}\varphi.$$

As has been proved in the preceding section, H' has no domain other than the null vector, so that also the operator $\exp(iH_0t)H'\exp(-iH_0t)$ has no domain other than the null vector. The equation (29), therefore, has no solution. Accordingly, the original equation (28) has no solution.

References.

1 Yukawa & Sakata: On the interaction of Elementary Particle II. Proc. Phys.-Math. Soc. 19 1084 (1937).

2 J.v.Neumann On Infinite Direct Product, Com. Math., 46, 1937.

おわび

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発行所　大阪市北区中之島
大阪大学理学部数学教室

振替　大阪　一七七四三番

大阪市北区大阪大学理学部数学教室
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