ON QUASI-STRONGLY KEEN HEEGAARD SPLITTINGS

KANAKO OIE

Abstract. In this paper, we define a Heegaard splitting to be quasi-strongly keen if the pair of vertices in the curve complex realizing the Hempel distance is unique and the set of all geodesics connecting them is finite. Focusing on the case where the Hempel distance is 2, we prove that every quasi-strongly keen genus-g (≥ 3) Heegaard splitting is in fact strongly keen, meaning that there exists exactly one geodesic connecting the pair.

1. Introduction

The curve complex $\mathcal{C}(S)$, introduced by Harvey [1], plays a fundamental and central role in low-dimensional topology, particularly in the study of the Heegaard splittings of 3-manifolds and their Goeritz groups. A Heegaard splitting is a classical and powerful method for describing 3-manifolds by decomposing them into two handlebodies. To measure the complexity of such a splitting, Hempel introduced the notion of *Hempel distance* [2], which is defined as the distance in the curve complex $\mathcal{C}(S)$ between the disk complexes associated with the handlebodies.

In order to refine the notion of Hempel distance, Ido–Jang–Kobayashi [3] introduced the concept of *keen* Heegaard splittings, where the pair of vertices in the respective disk complexes realizing the Hempel distance is unique. They further defined *strongly keen* Heegaard splittings, where there exists exactly one geodesic connecting the pair. They proved the existence of both strongly keen Heegaard splittings and keen Heegaard splittings with infinitely many geodesics realizing the Hempel distance (hence, they are not strongly keen).

In this paper, we consider an intermediate refinement, namely, Heegaard splittings for which the pair of vertices realizing the Hempel distance is unique, but the set of all geodesics connecting the pair is finite. We refer to such splittings as *quasi-strongly keen.* This property

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is closely related to the finiteness of geodesics in the curve complex, which can be described in terms of vertex pairs of type F and strongly type F, the definitions of which are given in Subsection 2.2.

Using this terminology, the central question we address in this paper is the following:

Question : Does there exist a Heegaard splitting that is quasi-strongly keen but not strongly keen?

In a previous work [6], the author showed that there exist distance-2 vertex pairs in the curve complex which are of type F but not strongly type F. (We note that in [5], Matsuda, Shiga and the author gave a vast refinement of the results in [6]. In fact they gave an explicit upper bounds of the number of the geodesics joining a type F pair of vertices with distance 2, and gave concrete constructions realizing various finite counts.)

This result suggests that a distance 2 Heegaard splitting that is quasi-strongly keen but not strongly keen might exist. However, in this paper, we show that such an expectation does not hold in the category of distance 2 and genus at least 2 Heegaard splittings. Our main result is the following:

Theorem 1.1. *Every quasi-strongly keen genus-*g (≥ 3) *Heegaard splitting with distance 2 is strongly keen.*

REMARK 1.2. Theorem 1.1 does not hold for low-genus cases. When $g = 1$, for example, the genus-1 Heegaard splitting of the lens space $L(5, 2)$ admits exactly two geodesics between the disk complexes, since the curve complex of the torus is the Farey graph (for details, see Appendix B in [4]). When $g = 2$, it is known that any Heegaard splitting is not keen (see Remark 3.2 in [3]). Thus, the assumption $g \geq 3$ is essential.

2. Preliminaries.

2.1. **Curve complex.** Let S be an orientable surface of genus $g \geq 0$ with $c \geq 0$ boundary components. In this subsection, we recall the definition of the curve complex associated to S.

A *simple closed curve* in S means a closed connected 1-manifold embedded in the interior of S. A simple closed curve α is said to be *inessential* if either α bounds a disk in S, or α is parallel to a component of ∂S. Otherwise, α is called *essential*. Two simple closed curves are said to be *isotopic* if there exists an ambient isotopy of S carrying one to the other. The surface S is said to be *simple* if it does not admit any essential simple closed curve. Otherwise, we say S is *non-simple*. We say that S is *sporadic* if it does not contain any pair of non-isotopic disjoint essential simple closed curves. Otherwise, S is said to be *non-sporadic*. It is elementary to show that S is a sporadic, non-simple surface if and only if S is homeomorphic to either a torus with at most one boundary component, or a sphere with 4 boundary components.

DEFINITION 2.1. Suppose that S is non-sporadic. The *curve complex* $\mathcal{C}(S)$ is the simplicial complex defined as follows: its vertices are the isotopy classes of essential simple closed curves in S. A collection of $k+1$ mutually distinct vertices spans a k-simplex if they can be represented by pairwise disjoint curves in S.

DEFINITION 2.2. Suppose that S is a sporadic, non-simple surface. Then the *curve complex* $\mathcal{C}(S)$ is defined as follows: its vertices are the isotopy classes of essential simple closed curves in S. When S is a torus with at most one boundary component, a collection of $k+1$ distinct vertices spans a k-simplex if each pair can be realized to intersect exactly once. When S is a 4-punctured sphere, a collection of $k+1$ distinct vertices spans a k-simplex if each pair can be realized to intersect twice.

We denote by $\mathcal{C}^0(S)$ the 0-skeleton of $\mathcal{C}(S)$. Throughout this paper, for a vertex $\ell \in \mathcal{C}^0(S)$, we often identify ℓ with a geometric representative in its isotopy class. Given a collection $\{\ell_0, \ell_1, \ldots, \ell_n\}$ of vertices in $\mathcal{C}(S)$, we always assume that the geometric intersection number $|\ell_i \cap \ell_j|$ is minimal in their isotopy classes for any $i \neq j$.

The *distance* $d_S(\alpha, \beta)$ between two vertices α and β in $\mathcal{C}(S)$ is defined as the minimal number of 1-simplices in any simplicial path connecting them. A path $[\ell_0, \ell_1, \ldots, \ell_n]$ in $\mathcal{C}(S)$ is called a *geodesic* if $n = d_S(\ell_0, \ell_n)$.

2.2. **Vertex pairs in $\mathcal{C}(S)$ with finite geodesic set.** Let S be a non-sporadic surface, and let $\mathcal{C}(S)$ be defined as in Section 2.1.

DEFINITION 2.3. A pair of vertices (α, β) in the curve complex $\mathcal{C}(S)$ is said to be of *type F* if the set of all geodesics connecting α and β consists of finite number of elements. Moreover, (α, β) is said to be of *strongly type F* if there exists exactly one geodesic connecting α and β.

REMARK 2.4. In [3], the condition that (α, β) is strongly type F is described as the unique geodesic from α to β.

Throughout this paper, for a submanifold Y of a manifold X, we denote by $N_X(Y)$ a regular neighborhood of Y in X.

Suppose that a pair $(\ell_0, \ell_2) \in \mathcal{C}^0(S) \times \mathcal{C}^0(S)$ satisfies the following two conditions:

- $d_S(\ell_0, \ell_2) = 2$,
- (ℓ_0, ℓ_2) is of type F but not strongly type F.

That is, there exist only finitely many (≥ 2) geodesics of distance 2 connecting ℓ_0 and ℓ_2. We denote them by

$$[\ell_0, \ell_1^{(i)}, \ell_2] \quad (i = 1, \ldots, n).$$

Lemma 2.5. *Under the above notations, we have the following: For any $1 \leq i < j \leq n$, the curves $\ell_1^{(i)}$ and $\ell_1^{(j)}$ are disjoint; that is,*

$$\ell_1^{(i)} \cap \ell_1^{(j)} = \emptyset.$$

Proof. Suppose, for a contradiction, that there exist $\ell_1^{(i)}, \ell_1^{(j)}$ such that $\ell_1^{(i)} \cap \ell_1^{(j)} \neq \emptyset$. Let $\partial(N_S(\ell_1^{(i)} \cup \ell_1^{(j)}))$ be the boundary of a regular neighborhood of $\ell_1^{(i)} \cup \ell_1^{(j)}$ in S. If some components of the boundary are inessential in S, denote them by $e_1, \ldots, e_p$. For each such component e_k, define a region E_k bounded by e_k as follows:

- If e_k bounds a disk in S, let E_k be that disk.
- If e_k is parallel to a boundary component of S, let E_k be the annulus it cobounds with the boundary component.

Let $\mathcal{E} = \bigcup_{k=1}^{p} E_k$, and define $X = N_S(\ell_1^{(i)} \cup \ell_1^{(j)}) \cup \mathcal{E}$. Then X is a non-simple subsurface of S such that each component of ∂X is essential in S. In fact, if X is simple, then every essential simple closed curve in X is isotopic to a boundary component of X, implying $\ell_1^{(i)}$ and $\ell_1^{(j)}$ are disjoint, which contradicts our assumption.

Since X is non-simple, it is easy to show that there exist infinitely many pairwise distinct essential simple closed curves $m_1^{(1)}, m_1^{(2)}, \ldots$ in X. Then, for each k, the path $[\ell_0, m_1^{(k)}, \ell_2]$ is a geodesic of distance 2 connecting ℓ_0 and ℓ_2, contradicting the assumption that there are only finitely many such geodesics.

This completes the proof of the lemma. □

2.3. **Heegaard splitting.** In this subsection, we recall the notion of Heegaard splittings and introduce several related concepts concerning geodesics in the curve complex.

A *handlebody* is a compact 3-manifold that is homeomorphic to a 3-ball with a finite number of 1-handles attached. Let M be a closed orientable 3-manifold. A *Heegaard splitting* of M is a decomposition

$M = V_1 \cup_S V_2$, where V_1 and V_2 are handlebodies such that $V_1 \cap V_2 = \partial V_1 = \partial V_2 = S$. The surface S is called the *Heegaard surface*, and its genus is called the genus of the Heegaard splitting.

Let V be a handlebody. The *disk complex* $\mathcal{D}(V)$ is the full subcomplex of the curve complex $\mathcal{C}(\partial V)$ whose vertices correspond to the isotopy classes of essential simple closed curves in ∂V that bound properly embedded disks in V. A collection of $k+1$ distinct such vertices forms a k-simplex in $\mathcal{D}(V)$ if they can be represented by pairwise disjoint curves.

DEFINITION 2.6 (Hempel [2]). Let $V_1 \cup_S V_2$ be a Heegaard splitting of a closed orientable 3-manifold. The *Hempel distance* of the splitting is defined by

$$d(V_1 \cup_S V_2) := d_S(\mathcal{D}(V_1), \mathcal{D}(V_2)) = \min\{d_S(\alpha, \beta) \mid \alpha \in \mathcal{D}(V_1),\ \beta \in \mathcal{D}(V_2)\}.$$

DEFINITION 2.7. A Heegaard splitting $V_1 \cup_S V_2$ is said to be *keen* if there exists a unique pair $(\alpha, \beta) \in \mathcal{D}(V_1) \times \mathcal{D}(V_2)$ that realizes the Hempel distance.

Note that even if a Heegaard splitting is keen, the geodesics in $\mathcal{C}(S)$ connecting the unique pair (α, β) may not be unique.

DEFINITION 2.8. Let $V_1 \cup_S V_2$ be a keen Heegaard splitting. We say that the splitting is *quasi-strongly keen* if the unique pair (α, β) realizing the Hempel distance is of type F in $\mathcal{C}(S)$. If the pair (α, β) is of strongly type F, then we say that the splitting is *strongly keen.*

3. Proof of main result

In this section, we give a proof of the main result of this paper (Theorem 1.1).

Assume, for contradiction, that there exists a quasi-strongly keen Heegaard splitting $V_1 \cup_S V_2$ of distance 2 and genus at least 3 that is not strongly keen. Then, there exist two geodesics $[\ell_0, \ell_1, \ell_2], [\ell_0, \ell_1', \ell_2]$ in $\mathcal{C}(S)$ realizing the Hempel distance.

Lemma 3.1. *Under the above notation, the following holds:*

1. *ℓ_0, ℓ_1, and ℓ_1' are all non-separating in S.*
2. *The union $\ell_0 \cup \ell_1$ (resp. $\ell_0 \cup \ell_1'$) separates S.*

Proof. (1) Suppose that ℓ_0 is separating in S. Let D_0 be a disk properly embedded in V_1 bounded by ℓ_0. Then D_0 separates V_1 into two components, say V_1' and V_1''. We suppose that $\ell_1 \subset \partial V_1''$. Then we can find a disk D_0' properly embedded in V_1' such that $D_0 \cap D_0' = \emptyset$

and D_0' is not isotopic to D_0. Then the path $[\partial D_0', \ell_1, \ell_2]$ is a geodesic, contradicting the keenness of the splitting.

Now suppose ℓ_1 is separating in S. Then ℓ_1 separates S into two components, say S_1 and S_2. Since $\ell_0 \cap \ell_2 \neq \emptyset$, we may assume both curves lie in S_1. Since S_2 has genus at least one, it is non-simple. Thus, we can find infinitely many pairwise distinct essential simple closed curves $m_1^{(1)}, m_1^{(2)}, \ldots$ in S_2. Then, for each k, the path $[\ell_0, m_1^{(k)}, \ell_2]$ is a geodesic of distance 2 connecting ℓ_0 and ℓ_2, contradicting the assumption that there are only finitely many such geodesics.

The same argument applies symmetrically to ℓ_1'.

(2) Suppose that $\ell_0 \cup \ell_1$ is non-separating in S. Then there exists an essential simple closed curve ℓ^* intersecting $\ell_0 \cup \ell_1$ transversely in one point contained in ℓ_0. Let D_0 be a disk properly embedded in V_1 bounded by ℓ_0. Let D_0^* be a disk properly embedded in V_1, obtained by a band sum of two parallel copies of the compressing disk $D_0 \subset V_1$ along ℓ^*. Then the path $[\partial D_0^*, \ell_1, \ell_2]$ is a geodesic, contradicting the keenness of the splitting.

The case for $\ell_0 \cup \ell_1'$ follows similarly. □

By Lemma 3.1(2), the union $\ell_0 \cup \ell_1$ separates S into two components, say P_1 and P_*. By Lemma 2.5, we have $\ell_1 \cap \ell_1' = \emptyset$, and we also recall that $\ell_0 \cap \ell_1' = \emptyset$. These show that ℓ_1' is contained in either P_1 or P_*; assume it lies in P_*.

Since $\ell_0 \cup \ell_1'$ also separates S, and ℓ_1' is non-separating, it follows that ℓ_1' must separate P_* into two components, say P_2 and P_3, where $\ell_0 \subset \partial P_2$ and $\ell_1 \subset \partial P_3$. Thus, the surface S is decomposed into three components as $S = P_1 \cup P_2 \cup P_3$, with $\partial P_1 = \ell_0 \cup \ell_1, \partial P_2 = \ell_0 \cup \ell_1', \partial P_3 = \ell_1 \cup \ell_1'$.

At this point, note that if the genus of S is 3, then both P_1 and P_* have genus 1. In this case, there is no way to choose a non-separating curve ℓ_1' in P_* that further separates it into two components as required. This contradicts the assumption that such ℓ_1' exists. Therefore, we may assume the genus of S is at least 4 from now on.

Since ℓ_1 and ℓ_1' are not isotopic, the genus of P_3 is at least one. Therefore, there exist infinitely many pairwise non-isotopic essential simple closed curves $m_1^{(1)}, m_1^{(2)}, \ldots$ in P_3. Furthermore, since $\ell_0 \cap \ell_2 \neq \emptyset$ and $\ell_2 \cap (\ell_1 \cup \ell_1') = \emptyset$, it follows that $\ell_2 \cap P_3 = \emptyset$. Therefore, for each k, the path $[\ell_0, m_1^{(k)}, \ell_2]$ is a geodesic of distance 2 connecting ℓ_0 and ℓ_2, contradicting the assumption that there are only finitely many such geodesics.

This completes the proof of Theorem 1.1.

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Graduate School of Humanities and Sciences, Nara Women's University, Kitauoya-Nishimachi, Nara 630-8506, Japan

Email address: yak_oie@cc.nara-wu.ac.jp