Two distortion theorems in quaternionic hyperbolic geometry

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Abstract

We establish quaternionic analogues of two classical distortion theorems due to Beardon. Working in the ball model of quaternionic hyperbolic n-space with the chordal metric d on its boundary, we first prove that for every isometry g,

$$\sup_{z,w\in\partial\mathbf{H}^n_{\mathbb{H}}} \frac{d(g(z),g(w))}{d(z,w)} = \exp(\rho(o,g(o))),$$

where ρ is the hyperbolic distance and o the origin. Second, let Ω be a subdomain of the boundary and let ξ,ζ be two distinct boundary points not taken by g on Ω. Then for all $z,w\in\Omega$,

$$d(g(z),g(w)) \le \frac{8\,d(z,w)}{d(\xi,\zeta)\,d(z,\partial\Omega)^{1/2}\,d(w,\partial\Omega)^{1/2}},$$

and the constant 8 is best possible. The proofs combine the geometry of the ball model, a quaternionic cross-ratio, and careful estimates that reduce the problem to the real hyperbolic case when testing sharpness.

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1 Introduction

Let $\mathbb{R},\mathbb{C}$ or $\mathbb{H}$ be respectively the real numbers, the complex numbers or the quaternions. The sphere $S(a,r)$ in $\mathbb{R}^n$ is given by

$$S(a,r)=\{x\in\mathbb{R}^n : |x-a|=r\}.$$

The reflection in $S(a,r)$ is

$$\phi(x)=a+\frac{r^2(x-a)}{|x-a|^2}.$$

The plane in $\mathbb{R}^n$ is of the form

$$P(a,t)=\{x\in\mathbb{R}^n : (x,a)=t\}\cup\{\infty\}.$$

The reflection in $P(a,t)$ is

$$\phi(x)=x-\frac{2[(x,a)-t]a}{|a|^2}.$$

A Möbius transformation acting in $\overline{\mathbb{R}^n}=\mathbb{R}^n\cup\{\infty\}$ is a finite composition of reflections in spheres or planes. The group of Möbius transformation acting in $\overline{\mathbb{R}^n}=\mathbb{R}^n\cup\{\infty\}$ is called the general Möbius group and is denoted by $\mathrm{GM}(\overline{\mathbb{R}^n})$.

In $\mathbf{H}^{n+1}=\{(x_1,\cdots,x_n,x_{n+1}) : x_{n+1}>0\}$ with metric $ds=\frac{|dx|}{x_{n+1}}$, the hyperbolic distance ρ_H is defined by

$$\cosh\rho_H(x,y)=1+\frac{|x-y|^2}{2x_{n+1}y_{n+1}}.$$

In $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, the chordal metric $d_c(x, y)$ is defined by

$$d_c(x,y) = \begin{cases} \frac{2|x-y|}{(1+|x|^2)^{\frac{1}{2}}(1+|y|^2)^{\frac{1}{2}}}, & \text{if } x, y \neq \infty; \\ \\ \frac{2}{(1+|x|^2)^{\frac{1}{2}}}, & \text{if } y = \infty. \end{cases} \tag{1}$$

Beardon proved the following two distortion theorems.

Theorem B1 (See [1, Theorem 3.6.1]) Let g be a Möbius transformation acting in $\overline{\mathbb{R}^n}$ and ρ_H the hyperbolic metric in $\mathbf{H}^{n+1}$. Then

$$\sup_{z,w\in\overline{\mathbb{R}^n}} \frac{d_c(g(z), g(w))}{d_c(z, w)} = \exp(\rho_H(e_{n+1}, g(e_{n+1}))). \tag{2}$$

Theorem B2 (See [1, Theorem 3.6.2]) Let Ω be a subdomain of $\overline{\mathbb{R}^n}$ and suppose that ξ and ζ are distinct points in $\overline{\mathbb{R}^n}$. If $g \in \mathrm{GM}(\overline{\mathbb{R}^n})$ does not assume the values ξ and ζ in Ω, then for all z and w in Ω

$$d_c(g(z), g(w)) \leq \frac{8d_c(z, w)}{d_c(\xi, \zeta)d_c(z, \partial\Omega)^{\frac{1}{2}}d_c(w, \partial\Omega)^{\frac{1}{2}}}.$$

The constant 8 is best possible.

The group $\mathrm{M}(\overline{\mathbb{R}^n})$ is the subgroup of $\mathrm{GM}(\overline{\mathbb{R}^n})$ consisting of all orientation-preserving Möbius transformation in $\mathrm{GM}(\overline{\mathbb{R}^n})$. By Poincare extension[1], $\mathrm{GM}(\overline{\mathbb{R}^n})$ can be thought of as the isometry group $\mathrm{Isom}(\mathbf{H}^{n+1})$ of $\mathbf{H}^{n+1}$. We embed $x \in \overline{\mathbb{R}^n}$ in $\overline{\mathbb{R}^{n+1}}$ as $\tilde{x} = (x, 0)$. Let $\phi_0(x)$ be the reflection in $S(e_{n+1}, \sqrt{2})$, that is

$$\phi_0(x) = e_{n+1} + \frac{2(x - e_{n+1})}{|x - e_{n+1}|^2}.$$

Let $\Phi = \phi_0\sigma$, where σ is the reflection in the plane $x_{n+1} = 0$. Then Φ maps $x_{n+1} = 0$ onto $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$ and $\mathbf{H}^{n+1}$ onto $\mathbb{B}^{n+1} = \{x \in \mathbb{R}^{n+1} : |x| < 1\}$. If one view $(x, y) \in \mathbb{R}^2$ as $z = x + y\mathbf{i} \in \mathbb{C}$, then in the case of $n = 1$,

$$\Phi(z) = \frac{z\mathbf{i} + 1}{z + \mathbf{i}} = \mathbf{i}\frac{z - \mathbf{i}}{z + \mathbf{i}}.$$

If $\psi \in \mathrm{Isom}(\mathbf{H}^{n+1})$, then $\Phi\psi\Phi^{-1} \in \mathrm{Isom}(\mathbb{B}^{n+1})$. This shows that $\mathrm{GM}(\overline{\mathbb{R}^n})$ is conjugate in $\mathrm{GM}(\overline{\mathbb{R}^{n+1}})$ to the subgroup of $\mathrm{GM}(\overline{\mathbb{R}^{n+1}})$ consisting of those elements which leave $\mathbb{B}^{n+1}$ invariant.

We want to reformulate Beardon's theorems in ball model. It is obvious that

$$\Phi(\tilde{x}) = (\frac{2x_1}{1 + |x|^2}, \cdots, \frac{2x_n}{1 + |x|^2}, \frac{|x|^2 - 1}{|x|^2 + 1}),$$

$$d_c(x, y) = |\Phi(\tilde{x}) - \Phi(\tilde{y})| \tag{3}$$

and

$$\Phi(e_{n+1}) = (0, \cdots, 0)^T := o_{n+1} \in \mathbb{R}^{n+1}.$$

Let g be a Möbius transformation with $g(\mathbb{B}^{n+1}) = \mathbb{B}^{n+1}$. Then $\Phi^{-1}g\Phi$ is a Möbius transformation acting in $\overline{\mathbb{R}^n}$. Let $x = \Phi(z), y = \Phi(w)$. By using (2), we have

$$\sup_{z,w\in\overline{\mathbb{R}^n}} \frac{d_c(\Phi^{-1}g\Phi(z), \Phi^{-1}g\Phi(w))}{d_c(z, w)} = \exp(\rho_H(e_{n+1}, \Phi^{-1}g\Phi(e_{n+1}))).$$

Since $\Phi^{-1}g(x), \Phi^{-1}g(y) \in \overline{\mathbb{R}^n}$, by (3), we have

$$\sup_{x,y\in\mathbb{S}^n} \frac{d_c(\Phi^{-1}g(x), \Phi^{-1}g(y))}{d_c(\Phi^{-1}(x), \Phi^{-1}(y))} = \sup_{x,y\in\mathbb{S}^n} \frac{|g(x) - g(y)|}{|x - y|} = \exp(\rho_H(\Phi^{-1}(o_{n+1}), \Phi^{-1}g(o_{n+1}))).$$

Let $J_b = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}$ and

$$\langle x,\, y\rangle = y^T J_b x = y_1x_1 + \cdots y_nx_n - y_{n+1}x_{n+1}.$$

The following is the key observation for reformulation of Beardon's theorem.

Proposition 1.1. *Let $z, w \in \mathbb{S}^n$, $\mathbf{z} = \begin{pmatrix} z \\ 1 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} w \\ 1 \end{pmatrix}$. Then the angle θ between Oz and Ow satisfies*

$$\cos\theta = w^T z$$

and

$$|\langle \mathbf{z},\, \mathbf{w}\rangle| = |1 - w^T z| = |1 - \cos\theta| = 2\sin^2\frac{\theta}{2}.$$

Since

$$|z-w|^2 = |z|^2 + |w|^2 - 2z\cdot w = 1 + 1 - 2\cos\theta = 2(1-\cos\theta) = 4\sin^2\frac{\theta}{2},$$

we have

$$|z-w| = 2\sin\frac{\theta}{2} = |2\langle \mathbf{z},\, \mathbf{w}\rangle|^{\frac{1}{2}}. \tag{4}$$

We define

$$d_e(z,w) = |z-w| = |2\langle \mathbf{z},\, \mathbf{w}\rangle|^{\frac{1}{2}} = \frac{|2\langle \lambda\mathbf{z},\, \mu\mathbf{w}\rangle|^{\frac{1}{2}}}{|\lambda|^{\frac{1}{2}}|\mu|^{\frac{1}{2}}}. \tag{5}$$

Here d_e is the Euclidean distance on the sphere $\mathbb{S}^n$; it is exactly the chordal metric after the Cayley transform (see (3)). We will keep the notation d_e for this real metric and reserve d for the quaternionic metric introduced later.

Let ρ_B the hyperbolic metric in $\mathbb{B}^{n+1}$ given by $ds = \frac{2|dx|}{1-|x|^2}$.

Proposition 1.2. *(See [1, page 37]) The Cayley transformation $\Phi : \mathbf{H}^{n+1} \to \mathbb{B}^{n+1}$ is an isometry from $\mathbf{H}^{n+1}$ to $\mathbb{B}^{n+1}$.*

By Proposition 1.2, we have

$$\rho_H(\Phi^{-1}(o_{n+1}), \Phi^{-1}g(o_{n+1})) = \rho_B(o_{n+1}, g(o_{n+1})). \tag{6}$$

Beardon's theorems can be reformulated as the following theorems with $n \geq 2$.

Theorem B1a (See [1, Theorem 3.6.1]) Let g be a Möbius transformation with $g(\mathbb{B}^n) = \mathbb{B}^n$ and ρ_B the hyperbolic metric in $\mathbb{B}^n$. Then

$$\sup_{z,w\in\mathbb{S}^{n-1}} \frac{d_e(g(z), g(w))}{d_e(z,w)} = \exp(\rho_B(o_n, g(o_n)). \tag{7}$$

Theorem B2a (See [1, Theorem 3.6.2]) Let Ω be a subdomain of $\mathbb{S}^{n-1}$ and suppose that ξ and ζ are distinct points in $\mathbb{S}^{n-1}$. If $g \in \mathrm{GM}(\mathbb{B}^n)$ does not assume the values ξ and ζ in Ω, then for all z and w in Ω

$$d_e(g(z), g(w)) \leq \frac{8d_e(z,w)}{d_e(\xi,\zeta)d_e(z,\partial\Omega)^{\frac{1}{2}}d_e(w,\partial\Omega)^{\frac{1}{2}}}.$$

The constant 8 is best possible.

In what follows, we want to reformulate Beardon's theorems in hyperboloid model.

Proposition 1.3. (See [1, Theorem 3.7.5]) *Let Q be hyperboloid model defined by*

$$Q = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \langle x,\, x\rangle = -1, x_{n+1} > 0\},$$

Let $ds^2 = dx_1^2 + \cdots dx_n^2 - dx_{n+1}^2$ be the metric on Q. Then the map

$$F : (x_1, \ldots, x_{n+1})^T \to \big(\frac{x_1}{1+x_{n+1}}, \ldots, \frac{x_n}{1+x_{n+1}}\big)^T$$

is an isometry of Q onto $\mathbb{B}^n$ with metric $ds = \frac{2|dx|}{1-|x|^2}$.

The group $G(Q)$ of isometries of Q and $GM(\mathbb{B}^n)$ are isomorphic by virtue of the relation

$$GM(\mathbb{B}^n) = FG(Q)F^{-1}.$$

Let

$$O(n,1) = \{A \in GL(n+1, \mathbb{R}) : A^T J_b A = J_b\}.$$

In fact, $G(Q) = O^+(n,1)$, which is the component of $O(n,1)$ containing the identity.

Let

$$V_- = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \langle x, x\rangle < 0\}$$

and

$$V_0 = \{(x_1, \ldots, x_{n+1})^T \in \mathbb{R}^{n+1} : \langle x, x\rangle = 0\}.$$

Let $\mathbb{P} : V_- \cup V_0 \longrightarrow \mathbb{R}^n$ be the projection map given by

$$\mathbb{P} : (x_1, \ldots, x_{n+1})^T \longmapsto (x_1 x_{n+1}^{-1}, \ldots, x_n x_{n+1}^{-1})^T.$$

It is obvious that $Q \subset V_-$. We will extend F from Q to $V_- \cup V_0$.

If $x = (x_1, \ldots, x_{n+1})^T \in V_-$ with $\langle x, x\rangle = -\frac{1}{u^2}$, then $ux \in Q$, where we choose $u \in \mathbb{R}$ such that $ux_{n+1} > 0$. Thus $\mathbb{P}(Q) = \mathbb{P}(V_-)$ and we can extend F from Q to V_- as

$$F : (x_1, \ldots, x_{n+1})^T \in V_- \to \big(\frac{ux_1}{1+ux_{n+1}}, \ldots, \frac{ux_n}{1+ux_{n+1}}\big)^T = \big(\frac{x_1}{\frac{1}{u}+x_{n+1}}, \ldots, \frac{x_n}{\frac{1}{u}+x_{n+1}}\big)^T.$$

When $u \to \infty$, we get $F(V_0) = \mathbb{S}^{n-1} = \mathbb{P}(V_0)$, that is

$$F : (x_1, \ldots, x_{n+1})^T \in V_0 \to (x_1 x_{n+1}^{-1}, \ldots, x_n x_{n+1}^{-1})^T.$$

Let $\mathbf{H}^n_{\mathbb{R}} = \mathbb{P}(V_-)$ with boundary $\partial\mathbf{H}^n_{\mathbb{R}} = \mathbb{P}(V_0) = \mathbb{S}^{n-1}$. The element $g \in O^+(n,1)$ acts on $x \in \mathbf{H}^n_{\mathbb{R}} = \mathbb{B}^n$ as $g(x) = \mathbb{P}g\mathbb{P}^{-1}(x)$. Note that

$$F^{-1}(o_n) = \begin{pmatrix} o_n \\ 1 \end{pmatrix} = o_Q, F(o_Q) = o_n$$

and

$$\rho_B(o_n, FgF^{-1}(o_n)) = \rho_B(F(o_Q), Fg(o_Q)) = \rho_Q(o_Q, g(o_Q)).$$

Let $z, w \in \mathbb{S}^{n-1}$ and $\mathbf{x} = F^{-1}(z) = \begin{pmatrix} z \\ 1 \end{pmatrix}, \mathbf{y} = F^{-1}(w) = \begin{pmatrix} w \\ 1 \end{pmatrix}$. By using (7), we have

$$\sup_{\mathbf{x},\mathbf{y}\in V_0} \frac{d_e(Fg\mathbf{x}, Fg\mathbf{y})}{d_e(F(\mathbf{x}), F(\mathbf{y}))} = \sup_{z,w\in\mathbb{S}^{n-1}} \frac{d_e(FgF^{-1}(z), FgF^{-1}(w))}{d_e(z,w)} = \exp(\rho_B(o, FgF^{-1}(o)) = \exp(\rho_Q(o_Q, g(o_Q))).$$

The Bergman metric on $\mathbf{H}^n_{\mathbb{R}} = \mathbb{B}^n$ is given by the distance formula

$$\cosh^2(\rho(z,w)) = \frac{\langle \mathbf{z}, \mathbf{w}\rangle\langle \mathbf{w}, \mathbf{z}\rangle}{\langle \mathbf{z}, \mathbf{z}\rangle\langle \mathbf{w}, \mathbf{w}\rangle}, \ \mathbf{z} \in \mathbb{P}^{-1}(z), \mathbf{w} \in \mathbb{P}^{-1}(w). \tag{8}$$

Thus $\cosh^2(\rho(o, g(o))) = |\langle o_Q, g(o_Q)\rangle|^2$.

Proposition 1.4. *Let $g \in O^+(n,1)$. Then*

$$\rho(o, g(o)) = \rho_Q(o_Q, g(o_Q)).$$

From the above description, $d_e(F(\mathbf{x}), F(\mathbf{y})) = d_e(z, w)$ and $d_e(Fg\mathbf{x}), Fg\mathbf{y}) = d_e(g(z), g(w))$. Hence Beardon's theorems can be reformulated as the following theorems.

Theorem B1b (See [1, Theorem 3.6.1]) Let $o = (0, \cdots, 0)^t \in \mathbf{H}^n_{\mathbb{R}}$. If $g \in O^+(n, 1)$, then

$$\sup_{z,w \in \partial \mathbf{H}^n_{\mathbb{R}}} \frac{d_e(g(z), g(w))}{d_e(z, w)} = \exp(\rho(o, g(o))).$$

Theorem B2b (See [1, Theorem 3.6.2]) Let Ω be a subdomain of $\partial \mathbf{H}^n_{\mathbb{R}}$ and suppose that ξ and ζ are distinct points in $\partial \mathbf{H}^n_{\mathbb{R}}$. If $g \in O^+(n, 1)$ does not assume the values ξ and ζ in Ω, then for all z and w in Ω

$$d_e(g(z), g(w)) \leq \frac{8 d_e(z, w)}{d_e(\xi, \zeta) d_e(z, \partial\Omega)^{\frac{1}{2}} d_e(w, \partial\Omega)^{\frac{1}{2}}}.$$

The constant 8 is the best possible.

Kamiya [6] had generalized Theorem B1b to complex hyperbolic geometry.

Theorem K1 (See [6, Theorem 1.3]) Let $o = (0, \cdots, 0)^t \in \mathbf{H}^n_{\mathbb{C}}$. If $g \in U(n, 1; \mathbb{C})$, then

$$\sup_{z,w \in \mathbf{H}^n_{\mathbb{C}}} \frac{d(g(z), g(w))}{d(z, w)} = \exp(\rho(o, g(o))).$$

As interest in quaternionic hyperbolic geometry has grown, many of real and complex hyperbolic problems have been translated into the quaternionic arena [2, 3, 7, 9].

Our purpose of this paper is to establish analogues of Theorem B1b and B2b in quaternionic hyperbolic geometry. We will use the ball model of quaternionic hyperbolic space $\mathbf{H}^n_{\mathbb{H}}$, which is the open unit ball

$$\mathbb{B}^n_{\mathbb{H}} = \{\zeta = (\zeta_1, \cdots, \zeta_n)^T \in \mathbb{H}^n : \|\zeta\|^2 = \sum_{i=1}^n |\zeta_i|^2 < 1\}.$$

Its boundary is the sphere

$$\partial \mathbf{H}^n_{\mathbb{H}} = \{\zeta \in \mathbb{H}^n : \|\zeta\| = 1\}.$$

The following conception is the key tool for our generalization.

Definition 1.1. *For $z, w \in \overline{\mathbb{B}^n_{\mathbb{H}}}$, we define*

$$d(z, w) = \sqrt{2} \left| 1 - \sum_{i=1}^n \overline{w_i} z_i \right|^{\frac{1}{2}} = |2(1 - w^* z)|^{\frac{1}{2}}. \tag{9}$$

Remark 1.1. *When $z, w \in \partial \mathbb{B}^n_{\mathbb{R}}$, d is the Euclidean chordal metric and $d(z, w) = |z - w|$(see Proposition 1.1). When restricted to the boundary $\partial \mathbf{H}^n_{\mathbb{H}}$, d becomes the standard chordal metric of quaternionic hyperbolic space (see Proposition 3.1 (iii)); it is the quaternionic analogue of the Euclidean chordal metric from the real case, but note that in general $d(z, w) \neq |z - w|$.*

Let ρ be the Bergman metric on $\mathbb{B}^n_{\mathbb{H}}$ and $\mathrm{Sp}(n, 1)$ the group of orientation-preserving isometries of $\mathbb{B}^n_{\mathbb{H}}$.

Our main results are the following two theorems.

Theorem 1.1. *Let $o = (0, \cdots, 0)^t \in \mathbf{H}^n_{\mathbb{H}}$. If $g \in \mathrm{Sp}(n, 1)$, then*

$$\sup_{z,w \in \partial \mathbf{H}^n_{\mathbb{H}}} \frac{d(g(z), g(w))}{d(z, w)} = \exp(\rho(o, g(o))).$$

Theorem 1.2. *Let Ω be a subdomain of $\partial \mathbf{H}^n_{\mathbb{H}}$ and suppose that ξ and ζ are distinct points in $\partial \mathbf{H}^n_{\mathbb{H}}$. If $g \in \mathrm{Sp}(n, 1)$ does not assume the values ξ and ζ in Ω, then for all z and w in Ω*

$$d(g(z), g(w)) \leq \frac{8\, d(z, w)}{d(\xi, \zeta)\, d(z, \partial\Omega)^{\frac{1}{2}}\, d(w, \partial\Omega)^{\frac{1}{2}}}.$$

The paper is organized as follows. In Section 2 we shall collect some basic facts in quaternionic hyperbolic geometry. Section 3 is devoted to obtaining some properties of the metric d on $\overline{\mathbb{B}^n_{\mathbb{H}}}$ and quaternionic cross-ratio. Section 4 contains the proofs of Theorems 1.1 and 1.2.

2 Ball model of quaternionic hyperbolic space

We briefly recall some necessary material on quaternionic hyperbolic geometry here and we refer to [3, 4, 7] for further details.

We recall that a real quaternion is of the form $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \in \mathbb{H}$ where $q_i \in \mathbb{R}$ and $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. Let $\overline{q} = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}$ and $|q| = \sqrt{\overline{q}q} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$ be the conjugate and the modulus of q, respectively. For a matrix $A = (a_{ij}) \in \mathbb{H}^{m\times n}$, let $\|A\|^2 = \sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2$. If $m = n$, we define $tr(A) = \sum_{i=1}^m a_{ii}$.

Let $\mathbb{H}^{n,1}$ be the vector space of dimension n+1 over $\mathbb{H}$ with the unitary structure defined by the Hermitian form

$$\langle \mathbf{z},\, \mathbf{w}\rangle = \mathbf{w}^* J_b \mathbf{z} = \overline{w_1} z_1 + \cdots + \overline{w_n} z_n - \overline{w_{n+1}} z_{n+1},$$

where $\mathbf{z}$ and $\mathbf{w}$ are the column vectors in V with entries $(z_1, \cdots, z_{n+1})$ and $(w_1, \cdots, w_{n+1})$ respectively, $\cdot^*$ denotes the conjugate transpose and J_b is the Hermitian matrix

$$J_b = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}.$$

Let

$$\begin{aligned} V_0 &= \Big\{\mathbf{z} \in V\backslash\{0\} : \langle \mathbf{z},\, \mathbf{z}\rangle = 0\Big\}, \\ V_- &= \Big\{\mathbf{z} \in V : \langle \mathbf{z},\, \mathbf{z}\rangle < 0\Big\}, \\ V_+ &= \Big\{\mathbf{z} \in V : \langle \mathbf{z},\, \mathbf{z}\rangle > 0\Big\}. \end{aligned}$$

We define an equivalence relation $\sim$ on $\mathbb{H}^{n,1}$ by $\mathbf{z} \sim \mathbf{w}$ if and only if there exists a non-zero quaternion λ so that $\mathbf{w} = \mathbf{z}\lambda$. Let $[\mathbf{z}]$ denote the equivalence class of $\mathbf{z}$. Let $\mathbb{P} : \mathbb{H}^{n,1} - \{0\} \longrightarrow \mathbb{HP}^n$ be the *right projection* map given by $\mathbb{P} : \mathbf{z} \longmapsto [\mathbf{z}]$. The ball model of *quaternionic hyperbolic n space* is defined to be $\mathbf{H}^n_{\mathbb{H}} = \mathbb{P}(V_-)$ with boundary $\partial\mathbf{H}^n_{\mathbb{H}} = \mathbb{P}(V_0)$.

For $\mathbf{z} = (z_1, \cdots, z_{n+1})^T \in V_0 \cup V_-$, since $\mathbf{z}^* J_b \mathbf{z} = \sum_{i=1}^n |z_i|^2 - |z_{n+1}|^2 \le 0$, we know that $z_{n+1} \neq 0$. This fact is useful to parameterize $\mathbb{P}(V_0 \cup V_-)$. In fact, for $\mathbf{z} = (z_1, \cdots, z_{n+1})^T \in V_0 \cup V_-$, we can define coordinates for $[\mathbf{z}]$ as

$$\zeta = (\zeta_1, \cdots, \zeta_n)^T = (z_1 z_{n+1}^{-1}, \cdots, z_n z_{n+1}^{-1})^T \in \overline{\mathbb{B}^n_{\mathbb{H}}} = \{\zeta \in \mathbb{H}^n : \|\zeta\| \le 1\}.$$

The Bergman metric on $\mathbf{H}^n_{\mathbb{H}}$ is given by the distance formula

$$\cosh^2(\rho(z,w)) = \frac{\langle \mathbf{z},\, \mathbf{w}\rangle\langle \mathbf{w},\, \mathbf{z}\rangle}{\langle \mathbf{z},\, \mathbf{z}\rangle\langle \mathbf{w},\, \mathbf{w}\rangle},\ \mathbf{z} \in \mathbb{P}^{-1}(z), \mathbf{w} \in \mathbb{P}^{-1}(w). \tag{10}$$

We define a *unitary transformation* g to be an automorphism of $\mathbb{H}^{n,1}$, that is, a linear bijection such that $\langle g(\mathbf{z}),\, g(\mathbf{w})\rangle = \langle \mathbf{z},\, \mathbf{w}\rangle$ for all $\mathbf{z}$ and $\mathbf{w}$ in V. We denote the group of all unitary transformations by $\mathrm{Sp}(n,1)$.

If $g \in \mathrm{Sp}(n,1)$, by definition g preserves the Hermitian form. Hence

$$\mathbf{w}^* J_b \mathbf{z} = \langle \mathbf{z},\, \mathbf{w}\rangle = \langle g\mathbf{z},\, g\mathbf{w}\rangle = \mathbf{w}^* g^* J_b g \mathbf{z}$$

for all $\mathbf{z}$ and $\mathbf{w}$ in V. Thus

$$\mathrm{Sp}(n,1) = \{g \in \mathrm{GL}(n+1, \mathbb{H}) : g^* J_b g = g\}.$$

It follows from $J_b = g^* J_b g$ that $g^{-1} = J_{\mathbb{B}}^{-1} g^* J_b$. That is:

$$g^{-1} = \begin{pmatrix} A^* & -\beta^* \\ -\alpha^* & \overline{a_{n+1,n+1}} \end{pmatrix} \quad \text{for} \quad g = (a_{ij})_{i,j=1,\cdots,n+1} = \begin{pmatrix} A & \alpha \\ \beta & a_{n+1,n+1} \end{pmatrix}.$$

Using the identities $gg^{-1} = g^{-1}g = I$ we obtain:

$$AA^* - \alpha\alpha^* = I_n, \tag{11}$$
$$-A\beta^* + \alpha\overline{a_{n+1,n+1}} = 0, \tag{12}$$
$$-|\beta|^2 + |a_{n+1,n+1}|^2 = 1, \tag{13}$$
$$A^*A - \beta^*\beta = I_n, \tag{14}$$
$$A^*\alpha - \beta^* a_{n+1,n+1} = 0, \tag{15}$$
$$-|\alpha|^2 + |a_{n+1,n+1}|^2 = 1. \tag{16}$$

We mention that $g \in \mathrm{Sp}(n,1)$ acts on $\mathbf{H}^n_{\mathbb{H}} \cup \partial\mathbf{H}^n_{\mathbb{H}}$ as $g(z) = \mathbb{P}g\mathbb{P}^{-1}(z)$. This implies that $g(z) = (-g)(z)$, and hence we define

$$\mathrm{PSp}(n,1) = \mathrm{Sp}(n,1)/\{\pm I_{n+1}\},$$

which is the group of orientation-preserving isometries of $\mathbf{H}^n_{\mathbb{H}}$.

Proposition 2.1. *(See [6, Proposition 1.1]) Let $o = (0, \cdots, 0)^t \in \mathbf{H}^n_{\mathbb{H}}$. If $g \in \mathrm{Sp}(n,1)$ then*

$$\|g\|^2 = \|I_{n+1}\|^2 + 4\sinh^2(\rho(o, g(o))).$$

Proof. Let

$$g = \begin{pmatrix} A & \alpha \\ \beta & a_{n+1,n+1} \end{pmatrix} \in \mathrm{Sp}(n,1).$$

Note that a lift of $g(o)$ is $(\alpha^T, a_{n+1,n+1})^T$ and so we have

$$\cosh^2(\rho(o, g(o))) = |a_{n+1,n+1}|^2.$$

Thus

$$\sinh^2(\rho(o, g(o))) = |a_{n+1,n+1}|^2 - 1.$$

By the equalities (11)-(16) and $tr(\alpha\alpha^*) = \alpha^*\alpha$, we have

$$\begin{aligned} \|g\|^2 &= tr(AA^*) + \alpha^*\alpha + \beta\beta^* + |a_{n+1,n+1}|^2 \\ &= tr(I_n + \alpha\alpha^*) + \alpha^*\alpha + \beta\beta^* + |a_{n+1,n+1}|^2 \\ &= tr(I_n) + 2\alpha^*\alpha + \beta\beta^* + |a_{n+1,n+1}|^2 \\ &= 4(|a_{n+1,n+1}|^2 - 1) + n + 1 \\ &= \|I_{n+1}\|^2 + 4\sinh^2(\rho(o, g(o))). \end{aligned}$$

□

If g has fixed point o, then g is of the form

$$g = \begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix},\ A \in \mathrm{Sp}(n),\ a \in \mathrm{Sp}(1).$$

This implies

$$\mathrm{Stab}(o) \cong \mathrm{Sp}(n) \times \mathrm{Sp}(1).$$

We record these facts as the following proposition.

Proposition 2.2. *(See [6, Proposition 1.5]) If $g \in \mathrm{Sp}(n,1)$ then the following statements are equivalent.*

(i) $\|g\|^2 = \|I_{n+1}\|^2$;

(ii) $g \in \mathrm{Sp}(n) \times \mathrm{Sp}(1)$;

(iii) $g(o) = o$.

It is easy to verify the following proposition.

Proposition 2.3. *Let* $o \neq p \in \mathbb{B}^n_{\mathbb{H}}$ *and* $t = \frac{1}{\sqrt{1-\|p\|^2}}$. *Then*

$$f_p = \begin{pmatrix} I_n - \frac{t^2}{t-1} pp^* & tp \\ -tp^* & t \end{pmatrix} \in \mathrm{Sp}(n,1) \tag{17}$$

and

$$f_p(p) = o, f_p(o) = p, f_p^2 = id. \tag{18}$$

The action of f_p *on* $\mathbb{H}^n$ *is given by*

$$f_p(z) = \frac{p}{1-\sqrt{1-\|p\|^2}} + \sqrt{1-\|p\|^2}\left(z - \frac{p}{1-\sqrt{1-\|p\|^2}}\right)(1-p^*z)^{-1}. \tag{19}$$

Proposition 2.4. *(See [6, Theorem 1.2]) If* $g \in \mathrm{Sp}(n,1)$ *and* $g(p) = o$. *Then* g *can be decomposed as*

$$g = uf_p,$$

where $f_p(p) = o, f_p(o) = p, f_p^2 = I_{n+1}$ *and* $u \in \mathrm{Sp}(n) \times \mathrm{Sp}(1)$.

Proof. Let f_p be given by Proposition 2.3. It follows from $gf_p^{-1}(o) = o$ and Proposition 2.2 that $gf_p^{-1} \in \mathrm{Sp}(n) \times \mathrm{Sp}(1)$. Hence there exists a $u \in \mathrm{Sp}(n) \times \mathrm{Sp}(1)$ such that $g = uf_p$. □

3 The metric d and quaternion cross-ratio

In this section, we collect properties of the function d defined on the closed ball and the quaternionic cross-ratio.

Recall from (9) that for any $z, w \in \overline{\mathbb{B}^n_{\mathbb{H}}}$,

$$d(z,w) = \sqrt{2}\,|1 - w^*z|^{\frac{1}{2}}. \tag{20}$$

Let $\mathbf{z}$ and $\mathbf{w}$ be the standard lifts of z, w, that is, $\mathbf{z} = \begin{pmatrix} z \\ 1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} w \\ 1 \end{pmatrix}$. Since $\langle \mathbf{z}\lambda, \mathbf{w}\mu\rangle = \bar{\mu}\langle \mathbf{z}, \mathbf{w}\rangle\lambda$, we have for all $z, w \in \overline{\mathbb{B}^n_{\mathbb{H}}}$

$$|\langle \mathbf{z}, \mathbf{w}\rangle| = \frac{1}{2}\, d(z,w)^2. \tag{21}$$

Hence we have

$$\cosh^2(\rho(z,w)) = \frac{\langle \mathbf{z},\, \mathbf{w}\rangle\langle \mathbf{w},\, \mathbf{z}\rangle}{\langle \mathbf{z},\, \mathbf{z}\rangle\langle \mathbf{w},\, \mathbf{w}\rangle} = \frac{d(z,w)^4}{d(z,z)^2 d(w,w)^2}, \forall z, w \in \mathbb{B}^n_{\mathbb{H}}. \tag{22}$$

Proposition 3.1. *(See [5, Proposition 3.2])*

(i) $d(h(z), h(w)) = d(z,w), \forall h \in \mathrm{Sp}(n) \times \mathrm{Sp}(1)$;

(ii) $d(z,w) \leq d(z,x) + d(x,w), \forall z, x, w \in \overline{\mathbb{B}^n_{\mathbb{H}}}$;

(iii) d *is a metric on* $\partial\mathbb{B}^n_{\mathbb{H}}$;

(iv) $d(z,w) = \sqrt{2}|1 - w^*z|^{\frac{1}{2}} \leq 2$.

Proof. Let $h = diag(U, v) \in \mathrm{Sp}(n) \times \mathrm{Sp}(1)$. Then the standard lifts of $h(z)$ and $h(w)$ are $\begin{pmatrix} Uzv^{-1} \\ 1 \end{pmatrix}$ and $\begin{pmatrix} Uwv^{-1} \\ 1 \end{pmatrix}$. Therefore we have

$$d(h(z), h(w))^2 = 2|(Uwv^{-1})^* Uzv^{-1} - 1| = 2|v(w^*z - 1)v^{-1}| = 2|1 - w^*z| = d(z, w)^2. \tag{23}$$

This proves (i).

By using (i), we may assume that $x = (a, 0, \cdots, 0)$ with $|a| \leq 1$ in (ii). Note that

$$|a - w_1| \leq |1 - \overline{a}w_1|, 1 - |\overline{a}z_1|^2 = (1 + |\overline{a}z_1|)(1 - |\overline{a}z_1|) \leq 2(1 - |\overline{a}z_1|)$$

and

$$1 - |\bar{a}w_1| \leq |1 - \bar{a}w_1|.$$

Hence

$$\begin{aligned}
|1 - w^*z| &= |1 - \overline{w_1}z_1 - \sum_{j=2}^{n} \overline{w_j}z_j| = |1 - \overline{a}z_1 + (\overline{a}z_1 - \overline{w_1}z_1) - \sum_{j=2}^{n} \overline{w_j}z_j| \\
&\leq |1 - \overline{a}z_1| + |a - w_1| + (\sum_{j=2}^{n} |w_j|^2 \sum_{j=2}^{n} |z_j|^2)^{\frac{1}{2}} \\
&\leq |1 - \overline{a}z_1| + |1 - \overline{a}w_1| + [(1 - |w_1|^2)(1 - |z_1|^2)]^{\frac{1}{2}} \\
&\leq |1 - \overline{a}z_1| + |1 - \overline{a}w_1| + [(1 - |\overline{a}w_1|^2)(1 - |\overline{a}z_1|^2)]^{\frac{1}{2}} \\
&\leq |1 - \overline{a}z_1| + |1 - \overline{a}w_1| + 2[(1 - |\overline{a}w_1|)(1 - |\overline{a}z_1|)]^{\frac{1}{2}} \\
&\leq (|1 - \overline{a}z_1|^{\frac{1}{2}} + |1 - \overline{a}w_1|^{\frac{1}{2}})^2.
\end{aligned}$$

This implies that $d(z, w) \leq d(z, x) + d(x, w)$, which proves (ii).

In $\partial\mathbb{B}^n_{\mathbb{H}}$, it is obvious that $d(z, w) = 0$ if and only if $z = w$. So (iii) and (iv) are immediate.

□

Proposition 3.2. *If $g \in \mathrm{Sp}(n, 1)$ with $g(p) = o$ then*

$$d(g(z), g(w)) = \frac{\sqrt{2}\, d(p, p)}{d(p, z)\, d(p, w)}\, d(z, w).$$

Proof. By Proposition 2.4, we can write $g = uf_p$. By Proposition 3.1, we only need to prove

$$d(f_p(z), f_p(w)) = \frac{\sqrt{2}\, d(p, p)}{d(p, z)\, d(p, w)}\, d(z, w).$$

Let $A = I_n - \frac{t^2}{t-1}pp^*$. Then

$$f_p = \begin{pmatrix} I_n - \frac{t^2}{t-1}pp^* & tp \\ -tp^* & t \end{pmatrix} = \begin{pmatrix} A & tp \\ -tp^* & t \end{pmatrix}$$

and

$$Ap = -tp,\ A^*A = I_n + t^2pp^*.$$

Choose two lifts of $f_p(z)$ and $f_p(w)$ as

$$z_p = \begin{pmatrix} Az + tp \\ t(1 - p^*z) \end{pmatrix},\ w_p = \begin{pmatrix} Aw + tp \\ t(1 - p^*w) \end{pmatrix}.$$

Then we have

$$|\langle z_p, w_p \rangle| = |(Aw + tp)^*(Az + tp) - t^2(1 - w^*p)(1 - p^*z)| = |w^*z - 1| = \frac{d(z, w)^2}{2}.$$

Hence

$$\begin{aligned} d(f_p(z), f_p(w))^2 &= \frac{2|\langle z_p, w_p\rangle|}{t^2|(1-p^*z)(1-p^*w)|} \\ &= 2\frac{d(z,w)^2\, d(p,p)^2}{d(p,z)^2\, d(p,w)^2}. \end{aligned}$$

□

Cross-ratio is an important tool in real hyperbolic geometry [1]. It has been generalized to complex and quaternionic hyperbolic geometry in [2, 8, 9]. There are two definitions of quaternionic cross ratios in Siegel domain [2]. Let C be the Cayley transformation mapping the ball model to the Siegel domain. Then the relation of the two models can be mainly expressed by the following two equations:

$$\mathbf{w}^* J_b \mathbf{z} = (C\mathbf{w})^* J_s (C\mathbf{z}),\ g^* J_b g = J_b = C^{-1} J_s C = C^{-1}(CgC^{-1})^* J_s (CgC^{-1})C,$$

where

$$C = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & I_{n-1} & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}, J_s = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

By the above relations, we can define the quaternionic cross-ratio as in [2] on the ball model as follows.

Let $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4)$ be a quadruple of points in $V_0 \cup V_-$, we define

$$\mathbb{P}(\mathbf{p}) = \big(\mathbb{P}(\mathbf{p}_1), \mathbb{P}(\mathbf{p}_2), \mathbb{P}(\mathbf{p}_3), \mathbb{P}(\mathbf{p}_4)\big) \in (\overline{\mathbf{H}^n_{\mathbb{H}}})^4.$$

Definition 3.1. *Let $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4)$ be a quadruple of points in $V_0 \cup V_-$ such that $\mathbb{P}(\mathbf{p})$ are pairwise distinct points in $\overline{\mathbf{H}^n_{\mathbb{H}}}$. The quaternionic cross-ratio of $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4)$ in $V_0 \cup V_-$ is defined as*

$$\mathbb{X}(\mathbf{p}) = \mathbb{X}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) := \langle \mathbf{p}_3, \mathbf{p}_1\rangle \langle \mathbf{p}_3, \mathbf{p}_2\rangle^{-1} \langle \mathbf{p}_4, \mathbf{p}_2\rangle \langle \mathbf{p}_4, \mathbf{p}_1\rangle^{-1}. \tag{24}$$

Observe that, for nonzero quaternions $\lambda_i, i = 1, \cdots, 4$,

$$\mathbb{X}(\mathbf{p}_1\lambda_1, \mathbf{p}_2\lambda_2, \mathbf{p}_3\lambda_3, \mathbf{p}_4\lambda_4) = \overline{\lambda}_1 \mathbb{X}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4)\overline{\lambda}_1^{-1}. \tag{25}$$

The above equality is dominant among the properties of quaternionic cross-ratio in $V_0 \cup V_-$.

Definition 3.2. *Let $\mathfrak{p} = (p_1, p_2, p_3, p_4)$ be any quadruple of pairwise distinct points in $\overline{\mathbf{H}^n_{\mathbb{H}}}$. The quaternionic cross-ratio of $\mathfrak{p} = (p_1, p_2, p_3, p_4)$ in $\overline{\mathbf{H}^n_{\mathbb{H}}}$ is defined as*

$$\mathbb{X}(\mathfrak{p}) = \mathbb{X}(p_1, p_2, p_3, p_4) := \mathbb{X}(\hat{\mathbf{p}}_1, \hat{\mathbf{p}}_2, \hat{\mathbf{p}}_3, \hat{\mathbf{p}}_4), \tag{26}$$

where $\hat{\mathbf{p}}_i$ is the standard lift of p_i.

Proposition 3.3. *Let $\mathfrak{p} = (p_1, p_2, p_3, p_4)$ be any quadruple of pairwise distinct points in $\overline{\mathbf{H}^n_{\mathbb{H}}}$ with lifts $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4$ and $h \in \mathrm{Sp}(n,1)$. Then*

(i) $\Re\big(\mathbb{X}(\mathfrak{p})\big) = \Re\big(\mathbb{X}(\mathbf{p})\big) = \Re\big(\mathbb{X}(h(p_1), h(p_2), h(p_3), h(p_4))\big)$;

(ii) $|\mathbb{X}(\mathfrak{p})| = |\mathbb{X}(\mathbf{p})| = |\mathbb{X}(h(p_1), h(p_2), h(p_3), h(p_4))|$.

Proof. From (25) we see that changing lifts conjugates the cross-ratio by a non-zero quaternion, so its real part and absolute value remain unchanged. Moreover, for $h \in \mathrm{Sp}(n,1)$ we have $\langle h\mathbf{p}_i, h\mathbf{p}_j\rangle = \langle \mathbf{p}_i, \mathbf{p}_j\rangle$, hence $\mathbb{X}(h(\mathbf{p})) = \mathbb{X}(\mathbf{p})$. The statement follows. □

4 The proofs of the main theorems

We now are ready to give a proof of Theorems 1.1 and 1.2, which for convenience will be restated below.

Theorem 4.1. *If $g \in \mathrm{Sp}(n,1)$, then*

$$\sup_{z,w\in\partial\mathbf{H}^n_{\mathbb{H}}} \frac{d(g(z),g(w))}{d(z,w)} = \exp(\rho(o,g(o))).$$

Proof. Let $p = g^{-1}(o)$. Then $g(p) = o$ and by Proposition 2.4 we can write $g = uf_p$ with $u \in \mathrm{Sp}(n)\times\mathrm{Sp}(1)$. Since u preserves d, we only need to prove

$$\sup_{z,w\in\partial\mathbf{H}^n_{\mathbb{H}}} \frac{d(f_p(z),f_p(w))}{d(z,w)} = \exp(\rho(o,f_p(o))). \tag{27}$$

Noting that

$$\cosh^2(\rho(o,f_p(o))) = \frac{1}{1-\|p\|^2},$$

we have

$$\exp(\rho(o,f_p(o)))^2 = \frac{1+\|p\|}{1-\|p\|}.$$

Choose two lifts of $f_p(z)$ and $f_p(w)$ as

$$z_p = \begin{pmatrix} Az+tp \\ t(1-p^*z)\end{pmatrix}, w_p = \begin{pmatrix} Aw+tp \\ t(1-p^*w)\end{pmatrix}.$$

Note that

$$1-\|p\| \le |1-p^*z|, 1-\|p\| \le |1-p^*w|.$$

By Proposition 3.2, we have

$$\begin{aligned} d(f_p(z),f_p(w))^2 &= \frac{2|\langle z_p,w_p\rangle|}{t^2|(1-p^*z)(1-p^*w)|} \\ &= \frac{2\,d(z,w)^2\,d(p,p)^2}{d(p,z)^2\,d(p,w)^2} \\ &= \frac{d(z,w)^2\,(1-\|p\|^2)}{|1-p^*z|\,|1-p^*w|} \\ &\le d(z,w)^2\,\frac{1+\|p\|}{1-\|p\|} \\ &= d(z,w)^2\exp(\rho(o,f_p(o)))^2. \end{aligned}$$

Thus $\sup_{z,w}\frac{d(g(z),g(w))}{d(z,w)} \le \exp(\rho(o,g(o)))$.

To see that the supremum is exactly $\exp(\rho(o,g(o)))$, consider the point $q = p/\|p\| \in \partial\mathbf{H}^n_{\mathbb{H}}$ and any sequence of distinct points $z_n, w_n \in \partial\mathbf{H}^n_{\mathbb{H}}$ converging to q. By continuity, $d(p,z_n)\to d(p,q) = \sqrt{2}\,(1-\|p\|)^{1/2}$ and similarly for w_n. Using the identity from Proposition 3.2,

$$\frac{d(f_p(z_n),f_p(w_n))}{d(z_n,w_n)} = \frac{\sqrt{2}\,d(p,p)}{d(p,z_n)\,d(p,w_n)}.$$

The right-hand side tends to $\frac{2\,(1-\|p\|^2)^{1/2}}{2\,(1-\|p\|)} = \sqrt{\frac{1+\|p\|}{1-\|p\|}} = \exp(\rho(o,f_p(o)))$. Hence the supremum is attained in the limit, completing the proof. □

Theorem 4.2. *Let Ω be a subdomain of $\partial\mathbf{H}^n_{\mathbb{H}}$ and suppose that ξ and ζ are distinct points in $\partial\mathbf{H}^n_{\mathbb{H}}$. If $g \in \mathrm{Sp}(n,1)$ does not assume the values ξ and ζ in Ω, then for all z and w in Ω*

$$d(g(z), g(w)) \leq \frac{8\, d(z,w)}{d(\xi,\zeta)\, d(z,\partial\Omega)^{\frac{1}{2}}\, d(w,\partial\Omega)^{\frac{1}{2}}}.$$

Proof. Suppose that z, w are distinct points in Ω and α, β are distinct points outside of Ω. By our definitions of cross-ratios and (21), we have

$$|\mathbb{X}(z,\alpha,w,\beta)| = \frac{d(z,w)^2\, d(\alpha,\beta)^2}{d(z,\beta)^2\, d(\alpha,w)^2}$$

and

$$|\mathbb{X}(z,\beta,w,\alpha)| = \frac{d(z,w)^2\, d(\alpha,\beta)^2}{d(z,\alpha)^2\, d(\beta,w)^2}.$$

Also we have

$$|\mathbb{X}(g(z),g(\alpha),g(w),g(\beta))| = \frac{d(g(z),g(w))^2\, d(g(\alpha),g(\beta))^2}{d(g(z),g(\beta))^2\, d(g(\alpha),g(w))^2}$$

and

$$|\mathbb{X}(g(z),g(\beta),g(w),g(\alpha))| = \frac{d(g(z),g(w))^2\, d(g(\alpha),g(\beta))^2}{d(g(z),g(\alpha))^2\, d(g(\beta),g(w))^2}.$$

By Proposition 3.3, we have

$$|\mathbb{X}(z,\alpha,w,\beta)\mathbb{X}(z,\beta,w,\alpha)| = |\mathbb{X}(g(z),g(\alpha),g(w),g(\beta))\mathbb{X}(g(z),g(\beta),g(w),g(\alpha))|. \tag{28}$$

Note that

$$d(u,v) \leq 2, \forall u, v \in \partial\mathbf{H}^n_{\mathbb{H}}.$$

By $d(z,\alpha) \geq d(z,\partial\Omega)$, $d(z,\beta) \geq d(z,\partial\Omega)$ and

$$d(\alpha,\beta) \leq d(\alpha,z) + d(z,\beta),\ \ d(\alpha,\beta) \leq d(\alpha,w) + d(w,\beta),$$

we have

$$\begin{aligned}
\frac{d(\alpha,\beta)^2}{d(z,\alpha)d(\beta,w)d(z,\beta)d(\alpha,w)} &\leq \big(\frac{d(\alpha,\beta)}{d(\alpha,z)d(z,\beta)}\big)\big(\frac{d(\alpha,\beta)}{d(\alpha,w)d(w,\beta)}\big) \\
&\leq \big(\frac{1}{d(\alpha,z)} + \frac{1}{d(\beta,z)}\big)\big(\frac{1}{d(\alpha,w)} + \frac{1}{d(\beta,w)}\big) \\
&\leq \frac{4}{d(z,\partial\Omega)d(w,\partial\Omega)}.
\end{aligned}$$

Note that

$$d(g(z),g(\beta))\, d(g(\alpha),g(w))\, d(g(z),g(\alpha))\, d(g(\beta),g(w)) \leq 16.$$

It follows from (28) that

$$\begin{aligned}
\Big[\frac{d(g(z),g(w))}{d(z,w)}\Big]^2 &= \Big[\frac{d(\alpha,\beta)}{d(g(\alpha),g(\beta))}\Big]^2 \frac{d(g(z),g(\beta))\, d(g(\alpha),g(w))\, d(g(z),g(\alpha))\, d(g(\beta),g(w))}{d(z,\alpha)\, d(\beta,w)\, d(z,\beta)\, d(\alpha,w)} \\
&\leq \Big[\frac{4}{d(g(\alpha),g(\beta))}\Big]^2 \frac{d(\alpha,\beta)^2}{d(z,\alpha)\, d(\beta,w)\, d(z,\beta)\, d(\alpha,w)} \\
&\leq \frac{64}{d(g(\alpha),g(\beta))^2\, d(z,\partial\Omega)\, d(w,\partial\Omega)}.
\end{aligned}$$

The inequality follows by taking $\alpha = g^{-1}(\xi)$ and $\beta = g^{-1}(\zeta)$. □

Remark 4.1. *The constant* 8 *in Theorem 1.2 is best possible. To see this, embed the real hyperbolic* 4*-space* $\mathbf{H}^4_{\mathbb{R}}$ *isometrically into the quaternionic hyperbolic line* $\mathbf{H}^1_{\mathbb{H}}$ *via the totally real subspace* $\mathbb{R}^4 \subset \mathbb{H}$*. On the boundary* $\mathbb{S}^3$ *of this subspace, the quaternionic chordal metric* d *restricts to the Euclidean chordal metric. Beardon's original extremal example for* $\mathbf{H}^4_{\mathbb{R}}$ *(see [1, Theorem 3.6.2]) therefore provides an explicit sequence of points in* $\mathbb{S}^3$ *for which the ratio in Theorem 1.2 can be made arbitrarily close to* 8*, showing that* 8 *cannot be lowered. For* $n > 1$*, the same sequence can be placed inside a totally real quaternionic line (setting all but the first coordinate to zero) and yields the same conclusion. Hence the constant* 8 *is sharp in all dimensions.*

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