STABLE PROPERTY OF DIMENSIONS OF SPHERES WITH EFFECTIVE SMOOTH ONE-FIXED-POINT ACTION

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In memory of late Professor Mikio Furuta

Abstract. Let G be a finite group. Let $D(G)$ denote the set of dimensions of spheres which possess effective smooth G-actions with exactly one G-fixed point and let A denote the set of non-negative integers. For a non-negative integer d, let $d\,A$ denote the set of non-negative integers which are multiples of d. It is interesting to study whether there is an integer $d(G)$ such that $D(G)$ is cofinal with $d(G)\,A$, namely $D(G) \smallsetminus F$ coincides with $d(G)\,A \smallsetminus F$ for some finite subset F of A. In this paper, we show that if G has $d(G)$ then $d(G)$ is equal to p^m or $2p^m$ for some prime p and a non-negative integer m. We also determine $d(G)$ for many finite groups G.

1. Introduction

Throughout the paper, G is a finite group, manifolds are smooth manifolds possibly with boundary, G-actions on manifolds are smooth actions. For a manifold X with G-action, we call the G-action a *one-fixed-point G-action* (resp. *fixed-point-free G-action*) if the G-fixed-point set of X is a singleton (resp. the empty set). For a G-set X, let X^G denote the set of G-fixed points in X. R. Oliver [20, 21] studied finite groups G which possess fixed-point-free G-actions on disks. We call a finite group G an *Oliver group* if there is a fixed-point-free G-action on a disk. For a disk X (resp. a sphere Y) with G-action, we call the G-action on X (resp. Y) *linear* if there is a real G-representation V with G-invariant inner-product such that X (resp. Y) is G-diffeomorphic to the unit disk $D(V)$ (resp. sphere $S(V)$) of V, and we call the G-action on X *∂-linear* if the G-action on the boundary ∂X of X is linear. For a disk X with ∂-linear fixed-point-free G-action, the G-action on the homotopy sphere $\Sigma = X \bigcup_{\partial} D(V)$ obtained by gluing X and $D(V)$ along the boundary is a one-fixed-point G-action, where V is a G-representation such that $S(V)$ is G-diffeomorphic to ∂X. Using equivariant connected sum of copies of Σ, we can obtain a one-fixed-point

2020 *Mathematics Subject Classification.* Primary 57S17, Secondary 20C15.
Key words and phrases. finite group, representation, smooth action, sphere, fixed point.

G-action on the sphere of same dimension as Σ ([12, Theorem 1.1]). Conversely, for a sphere Σ with one-fixed-point G-action, the G-action on the disk $X = \Sigma \smallsetminus \overset{\circ}{D}(x_0)$ is fixed-point-free and ∂-linear, where $\{x_0\} = \Sigma^G$, $D(x_0)$ is a G-invariant closed disk neighborhood of x_0 in Σ, and $\overset{\circ}{D}(x_0)$ is the interior of $D(x_0)$. Therefore we can say that the study of ∂-linear fixed-point-free G-actions on disks is equivalent to the study of one-fixed-point G-actions on spheres.

We have seen in [6, Theorem A] that there is an effective one-fixed-point G-action on a sphere (of some dimension) for an arbitrary Oliver group G. Note that if there is an effective fixed-point-free G-action on the disk of dimension n then there is an effective fixed-point-free G-action on the disk of dimension m for any $m \geq n$. However, it is not true for dimensions of spheres with effective one-fixed-point G-action. Let $\mathfrak{S}(G)_1$ denote the family of spheres with effective one-fixed-point G-action and let $\mathfrak{S}(G)_1^*$ be a complete set of representatives of all G-diffeomorphism classes of G-manifolds in $\mathfrak{S}(G)_1$. Let $\mathfrak{R}(G)$ denote the family of real G-representations and let $\mathfrak{R}(G)^*$ be a complete set of representatives of all isomorphism classes of real G-representations. Let $\mathfrak{R}(G)_1$ denote the family of tangential G-representations $T_x(X)$ at $x \in X^G$, where X ranges over $\mathfrak{S}(G)_1$, and let $\mathfrak{R}(G)_1^*$ be a complete set of representatives of all isomorphism classes of real G-representations in $\mathfrak{R}(G)_1$. We suppose $\mathfrak{R}(G)_1^* \subseteq \mathfrak{R}(G)^*$. Let $\mathrm{Dim}(G, 1)$ denote the set of dimensions of spheres with effective one-fixed-point G-action, i.e.

$$\mathrm{Dim}(G, 1) = \{\dim X \mid X \in \mathfrak{S}(G)_1\} = \{\dim V \mid V \in \mathfrak{R}(G)_1\}.$$

Let $\mathbb{N}$ and $\mathbb{Z}$ denote the sets of natural numbers and integers, respectively, and set $\overline{\mathbb{N}} = \{0\} \cup \mathbb{N}$. For a, $b \in \mathbb{Z}$, let $[a..b]$ denote the set of integers m satisfying $a \leq m \leq b$, and let $[a..\infty)$ denote the set of integers n satisfying $a \leq n$. For a subset S of $\mathbb{Z}$ and an integer k, let $k\,S$ and $S + k$ denote the sets of integers kx and of $x + k$, respectively, where x ranges over S. It is easy to prove the next remark.

Remark 1.1. If G is an Oliver group of odd order then $\mathrm{Dim}(G, 1) \subseteq 2\,\mathbb{N}$.

The remark is slightly generalized in Proposition 6.5.

By [14, Theorem 1.1] and [17, Theorem 1.1], it holds that

$$\mathrm{Dim}(S_5, 1) = \{6, 10, 11, 12\} \cup [14..\infty),$$
$$\mathrm{Dim}(\mathrm{SL}(2, \mathbb{F}_5), 1) = \{7\} \cup [10..\infty), \quad \text{and}$$
$$\mathrm{Dim}(\mathrm{TL}(2, \mathbb{F}_5), 1) = \{14, 18, 19, 20\} \cup [22..\infty).$$

For subsets S and T of $\overline{\mathbb{N}}$, we say that S is *cofinal with* T if $S \cap [a..\infty) = T \cap [a..\infty)$ holds for some $a \in \overline{\mathbb{N}}$. We write $S \approx T$ when S is cofinal with T. It would be likely that there exists a positive integer n such that $\mathrm{Dim}(G, 1) \approx n\,\mathbb{N}$. If the integer n exists, then we denote it by δ_G.

First, we state a result of the case where G is an extension of a nontrivial perfect group.

Theorem 1.1. *Let G be a finite group with a normal subgroup N such that G/N is a nontrivial perfect group. Then $\mathrm{Dim}(G, 1) \approx \mathbb{N}$, i.e. $\delta_G = 1$.*

Second, we state a result of a case in which $|G/[G, G]|$ is not a prime power, where $[G, G]$ is the commutator subgroup of G.

Theorem 1.2. *Let G be an Oliver group such that $|G/[G, G]|$ is divisible by two or more odd primes.*

(1) *If $|G|$ is even, then $\mathrm{Dim}(G, 1) \approx \mathbb{N}$ i.e. $\delta_G = 1$.*
(2) *If $|G|$ is odd, then $\mathrm{Dim}(G, 1) \approx 2\,\mathbb{N}$, i.e. $\delta_G = 2$.*

The next two results which describe the feature of δ_G are interesting.

Theorem 1.3. *Let G be an Oliver group. If $\mathrm{Dim}(G, 1)$ is cofinal with $\delta_G\,\mathbb{N}$, then $\delta_G = 2^m$, p^m or $2p^m$, where m is some non-negative integer and p is some odd prime.*

Theorem 1.4. *Let p be an odd prime and $m \in \overline{\mathbb{N}}$. For each $n \in \{2^m, p^m, 2p^m\}$, there exist infinitely many (with respect to orders) Oliver groups G such that $\mathrm{Dim}(G, 1) \approx n\,\mathbb{N}$, i.e. $\delta_G = n$.*

The proofs of Theorems 1.1–1.3 are given in Section 8. Theorem 1.4 follows from Theorem 1.2, Examples 7.3, 7.4, 7.7, Dirichlet's result (Corollary 5.6) and Proposition 5.8.

The following sections of the present paper are organized as follows. In Section 2 we describe supplementary explanations and results to Section 1. In Section 3, we define

a preferable G-representation and describe a realization lemma of one-fixed-point G-actions on spheres (Lemma 3.2) which is useful to study $\mathfrak{R}(G)_1$. In Section 4 we discuss algebraic elementary facts which are frequently used in the study of $\mathrm{Dim}(G,1)$. At the end of the section, we prove a key lemma (i.e. Lemma 2.2). In Section 5 we prepare known algebraic facts and give basic results helpful for the study of $\mathrm{Dim}(G,1)$. In Section 6 we give several results in the theory of transformation groups which will be used in Section 7. Particularly, Lemma 6.11 would be interesting. In Section 7, we study $\mathrm{Dim}(G,1)$ of various Oliver groups G and determine δ_G for those groups. In addition, we will show that $\delta_G = 1$ for all $G = S_n$ with $n \geq 5$ and $G = \mathrm{GL}(2,\mathbb{F}_q)$ with $q > 4$ in Examples 7.5 and 7.8. In Sections 8 and 9, we complete the proofs of theorems in Section 1 and the proofs of propositions in Section 2, respectively.

We remark that there is still left a problem asking whether $\delta_G = 1$ for any nonsolvable group G.

Notation

Throughout the paper, let E denote the trivial group and let $\mathbb{F}_q$, A_n, S_n, C_n, D_{2n}, $\mathrm{GL}(2,\mathbb{F}_q)$ (resp. $\mathrm{SL}(2,\mathbb{F}_q)$) stand for the field consisting of q elements, the alternating group on n letters, the symmetric group on n letters, a cyclic group of order n, a dihedral group of order $2n$ and the general (resp. special) linear group of 2×2-matrices on the field $\mathbb{F}_q$, respectively. In addition, $\mathrm{TL}(2,\mathbb{F}_5)$ stands for the group $\mathrm{SmallGroup}(240,89)$ in GAP [5] (a nontrivial double covering of S_5).

2. Supplements for the introduction

Firstly, we recall that a finite group G is an Oliver group if and only if there never exists a normal series $P \trianglelefteq H \trianglelefteq G$ such that P and G/H are of prime-power order and H/P is cyclic ([20]). It follows that G is an Oliver group if G is nonsolvable, and that a nilpotent group G is an Oliver group if and only if G possesses three or more non-cyclic Sylow subgroups. It is readily shown that the smallest (with respect to the order) Oliver groups are isomorphic to A_5.

To describe our results, we recall several definitions. Let $\mathscr{S}(G)$ denote the set of all subgroups of G. A subset $\mathscr{F}$ of $\mathscr{S}(G)$ has the (partial) order $\leq$ such that $H \leq K \Leftrightarrow H \subseteq K$, where H, $K \in \mathscr{F}$. Let $\max(\mathscr{F})$ (resp. $\min(\mathscr{F})$) denote the set of maximal (resp. minimal) elements of the ordered set $\mathscr{F}$. Let $\mathscr{S}(G)^*$ be a complete set of representatives of G-conjugacy classes of subgroups of G. For a subset $\mathscr{F}$ of $\mathscr{S}(G)$

closed under taking conjugations by elements of G, we set $\mathscr{F}^* = \mathscr{F} \cap \mathscr{S}(G)^*$. Let p be a prime. Let $\mathscr{P}_p(G)$ denote the set of all subgroups P with p-power order of G, where we allow the case $P = E$. Set $\mathscr{P}(G) = \bigcup_q \mathscr{P}_q(G)$, where q runs over the set of all primes. Let $G^{\cap p}$ denote the intersection of all normal subgroups of G with index p. Therefore $G^{\cap p}$ is normal in G and $G/G^{\cap p}$ is a (possibly trivial) elementary abelian p-group. Let $G^{\{p\}}$ denote the smallest normal subgroup N of G such that the order $|G/N|$ of G/N is a power of p (possibly $|G/N| = 1$). We call the subgroup $G^{\{p\}}$ of G the *Dress subgroup of type* p, cf. [2, Proposition 1 (d)]. Let $\mathscr{L}^p(G)$ denote the set of subgroups L such that $L \supseteq G^{\{p\}}$. Therefore, G belongs to $\mathscr{L}^p(G)$ for any prime p. We set $\mathscr{L}(G) = \bigcup_q \mathscr{L}^q(G)$, where q runs over the set of all primes, and set $\mathscr{M}(G) = \mathscr{S}(G) \smallsetminus \mathscr{L}(G)$. Let $\mathfrak{P}(G)$ denote the set of primes q such that $G^{\{q\}} \neq G$. Let G^{nil} (resp. G^{sol}) denote the smallest normal subgroup N such that G/N is nilpotent (resp. solvable). We remark that G^{nil} coincides with $\bigcap_q G^{\{q\}}$, where q ranges over the set of all primes. It is readily seen that for a prime p, the Sylow p-subgroup of G/G^{nil} is canonically isomorphic to $G/G^{\{p\}}$. It follows that $\mathfrak{P}(G)$ coincides with the set of primes q dividing $|G/[G,G]|$.

To obtain effective one-fixed-point G-actions on spheres, we use the equivariant surgery theory, for which we need 'gap condition' between dimensions of fixed-point sets of subgroups of G. Let V be a real (or complex) G-representation. For a subset $\mathscr{F} \subseteq \mathscr{S}(G)$, we say that V is *$\mathscr{F}$-free* if $V^H = \{0\}$ for all $H \in \mathscr{F}$. For $H \in \mathscr{S}(G)$, let $\mathscr{U}_G(H)$ denote the set of $K \in \mathscr{S}(G)$ satisfying $K > H$. We set $\overline{\mathscr{U}}_G(H) = \{H\} \cup \mathscr{U}_G(H)$. Let H be a subgroup of G. We say that V satisfies the *gap condition* (resp. *weak-gap condition*) at H if the inequalities

$$2 \dim V^K < \dim V^H \quad (\text{resp. } 2 \dim V^K \leq \dim V^H)$$

hold for all $K \in \mathscr{U}_G(H)$. We say that V is a *gap representation* (resp. *weak-gap representation*) if it satisfies the gap condition (resp. weak-gap condition) at all $H \in \mathscr{P}(G)$.

Definition 2.1. We call a real G-representation V a *stably weak-gap representation* if there is an $\mathscr{L}(G)$-free weak-gap G-representation W such that $V \oplus W$ is a weak-gap representation. The G-representation W is called a *weak-gap stabilizer* of V.

If W is a weak-gap stabilizer of V, then $W^{\oplus n}$, $n \in \mathbb{N}$, are also weak-gap stabilizers of V. We call a finite group G a *gap group* if there is an $\mathscr{L}(G)$-free gap G-representation.

If G is a gap group, then all $\mathscr{L}(G)$-free real G-representations are stably weak-gap representations. T. Sumi [26, Theorem 1.4] showed that if $|\mathfrak{P}(G)| = 2$ and G is nonsolvable, then G is a gap group. On the other hand, P. Mizerka [8, Theorem 7.3] showed that there are nongap solvable Oliver groups G satisfying $\mathfrak{P}(G) = \{2, p\}$ for odd primes p.

Let V be a real (or complex) G-representation. For a normal subgroup N of G, we have the orthogonal decomposition $V = V^N \oplus V_N$ as real (or complex) G-representations with respect to a G-invariant inner-product on V. For $\mathscr{F} \subseteq \mathscr{S}(G)$ such that $\min(\mathscr{F})$ consists of normal subgroups of G, let $V^{\mathscr{F}}$ denote the smallest G-subrepresentation of V containing all V^N for $N \in \min(\mathscr{F})$, let $V_{\mathscr{F}}$ denote the orthogonal complement of $V^{\mathscr{F}}$ in V. We define the real G-representation $V(G)$ as $\mathbb{R}[G]_{\mathscr{L}(G)}$, namely

$$V(G) = (\mathbb{R}[G] - \mathbb{R}) - \bigoplus_p (\mathbb{R}[G/G^{\{p\}}] - \mathbb{R}), \tag{2.1}$$

where p ranges over the set of all primes. The definition (2.1) gives

$$\dim V(G)^H = (|G/H| - 1) - \sum_p (|G/HG^{\{p\}}| - 1). \tag{2.2}$$

Since G/G^{nil} is nilpotent, for any $H < K \in \mathscr{S}(G)$, there is a sequence $H \le K_0 \lhd K_1 \lhd \cdots \lhd K_r = K$ such that $HG^{\mathrm{nil}} = K_0G^{\mathrm{nil}}$ and $|K_{i+1}/K_i| = |K_{i+1}G^{\mathrm{nil}} : K_iG^{\mathrm{nil}}| = p_i$, where p_i are primes, for all $i \in [0..(r-1)]$. Using (2.2), we can easily prove that $\dim V(G)^H \ge |K_0 : H| \dim V(G)^{K_0}$ and $\dim V(G)^{K_i} \ge |K_{i+1} : K_i| \dim V(G)^{K_{i+1}}$ for all $i \in [0..(r-1)]$ (cf. [6, Lemma 2.10]), and therefore we obtain $\dim V(G)^H \ge |K : H| \dim V(G)^K$ as is stated in [6, Theorem 2.3 (2.3.2)].

Let $\mathscr{PH}(G)$ denote the set of pairs (P, H) such that $P \in \mathscr{P}(G)$ and $H \in \mathscr{U}_G(P)$, and let $\mathscr{PH}_2(G)$ denote the set of $(P, H) \in \mathscr{PH}(G)$ satisfying

(i) $PG^{\{q\}} = G$ for all odd primes q and
(ii) $|H : P| = |HG^{\{2\}} : PG^{\{2\}}| = 2$.

Remark 2.1 (see [6, Theorem 2.3] and [16, Proposition 1.9]). For any Oliver group G, $V(G)$ has the following properties.

(1) $\dim V(G)^H = 0$ for all $H \in \mathscr{L}(G)$.
(2) $\dim V(G)^H \ge 2$ for all $H \in \mathscr{M}(G)$.
(3) $\dim V(G)^H \ge |K : H| \dim V(G)^K$ for all $H < K \in \mathscr{S}(G)$.

(4) $V(G)$ is an $\mathscr{L}(G)$-free weak-gap representation.
(5) If $2 \notin \mathfrak{P}(G)$ or $|\mathfrak{P}(G) \smallsetminus \{2\}| \geq 2$, then $V(G)$ is a gap representation.
(6) For $(P, H) \in \mathscr{PH}(G)$, the equality $\dim V(G)^P = 2 \dim V(G)^H$ holds if and only if $(P, H) \in \mathscr{PH}_2(G)$.

For further information of $V(G)$, see [19]. For a stably weak-gap real G-representation V, we can choose a weak-gap stabilizer W of V such that

$$\dim W \equiv 0 \mod \dim V(G).$$

Let $\mathscr{R}$ be a set or family of real G-representations and $\mathscr{F}$ a set of subgroups of G. Let $\mathrm{Dim}(\mathscr{R})$ denote the set of $\dim V$, where V ranges over $\mathscr{R}$. Let $\mathscr{R}_{\mathscr{F}}$ denote the set or family consisting of $V \in \mathscr{R}$ such that V is $\mathscr{F}$-free. Let $\mathscr{R}^{\mathrm{w.gap}}$ (resp. $\mathscr{R}^{\mathrm{s.w.gap}}$) denote the set or family consisting of $V \in \mathscr{R}$ such that V is a weak-gap (resp. stably weak-gap) representation. The next lemma is fundamental in the study of one-fixed-point actions on spheres.

Lemma 2.1. *Let G be an Oliver group. If W lies in $\mathfrak{R}(G)^{\mathrm{w.gap}}_{\mathscr{L}(G)}$ (possibly $W = \{0\}$), then $V(G) \oplus W$ lies in $\mathfrak{R}(G)_1 \cap \mathfrak{R}(G)^{\mathrm{w.gap}}_{\mathscr{L}(G)}$.*

This lemma is proven in Section 3.

For a subset A of $\overline{\mathbb{N}}$ with $A \neq \{0\}$, let $\gcd(A)$ denote the greatest common divisor of integers in $A \smallsetminus \{0\}$. For the sake of convenience, we set $\gcd(A) = \infty$ in the case where $A \subseteq \{0\}$. For simplicity of notations, we set

$$\mathrm{Dim}(G, 0) = \mathrm{Dim}(\mathfrak{R}(G)_{\{G\}}), \quad d(G, 0) = \gcd(\mathrm{Dim}(G, 0)),$$
$$\mathrm{Dim}(G, \mathrm{w}) = \mathrm{Dim}(\mathfrak{R}(G)^{\mathrm{w.gap}}_{\mathscr{L}(G)}), \quad d(G, \mathrm{w}) = \gcd(\mathrm{Dim}(G, \mathrm{w})),$$
$$\mathrm{Dim}(G, \mathrm{s}) = \mathrm{Dim}(\mathfrak{R}(G)^{\mathrm{s.w.gap}}_{\mathscr{L}(G)}) \text{ and } d(G, \mathrm{s}) = \gcd(\mathrm{Dim}(G, \mathrm{s})).$$

By definition, we see $\mathrm{Dim}(G, \mathrm{w}) \subseteq \mathrm{Dim}(G, \mathrm{s}) \subseteq \mathrm{Dim}(G, 0)$ and $d(G, \mathrm{w}) \geq d(G, \mathrm{s}) \geq d(G, 0)$.

If G is a nontrivial group, then there are $\{G\}$-free real G-representations of dimension $2|G|$ as well as of dimension $|G| - 1$. It is easy to see that if G is a nontrivial finite group of even (resp. odd) order then $d(G, 0) = 1$ (resp. 2). In the case where G is an Oliver group, $V(G)$ lies in $\mathfrak{R}(G)^{\mathrm{w.gap}}_{\mathscr{L}(G)}$ and therefore $d(G, \mathrm{w})$ divides $\dim V(G)$. The next lemma is obtained by elementary arguments but plays important roles in the study of $\mathrm{Dim}(G, 1)$.

Lemma 2.2. *Let G be an Oliver group and set $d = d(G, \mathrm{w})$. Then the following holds.*

(1) *The integer* $d(G, \mathrm{w})$ *coincides with* $d(G, \mathrm{s})$.
(2) $\mathrm{Dim}(G, 1) \cap d\,\mathbb{N}$ *is cofinal with* $d\,\mathbb{N}$.
(3) *There is an integer* N *such that*

$$\mathrm{Dim}(G, \mathrm{s}) + d\,N \subseteq \mathrm{Dim}(G, 1).$$

In particular, if there is an $\mathscr{L}(G)$*-free, stably weak-gap real* G*-representation of dimension relatively prime to* $\dim V(G)$*, then* $\mathrm{Dim}(G, 1)$ *is cofinal with* $\mathbb{N}$.

The proof is given in Section 4.

Let G be an Oliver group. We note that $\mathfrak{P}(G) = \emptyset$ if and only if G is perfect. In the case, Theorem 1.1 implies $\delta_G = 1$. In the case where $|\mathfrak{P}(G) \smallsetminus \{2\}| \geq 2$, Theorem 1.2 says $\delta_G = 1$ (resp. 2) if $|G|$ is even (resp. odd). Remark that if G is nilpotent then $|\mathfrak{P}(G)| \geq 3$ and therefore $\delta_G = 1$ (resp. 2) for G of even (resp. odd) order. We discuss other cases in the following three propositions.

Proposition 2.3 (Case 1)**.** *Let* G *be an Oliver group such that* $|\mathfrak{P}(G)| = 2$ *and* $2 \in \mathfrak{P}(G)$*. Then* $\mathrm{Dim}(G, 1) \cap 2\,\mathbb{N} \approx 2\,\mathbb{N}$*. If there is an* $\mathscr{L}(G)$*-free, stably weak-gap real* G*-representation of odd dimension, then* $\mathrm{Dim}(G, 1) \approx \mathbb{N}$.

Proposition 2.4 (Case 2)**.** *Let* G *be an Oliver group such that* $\mathfrak{P}(G) = \{p\}$ *for an odd prime* p*. Therefore* $V(G)$ *is an* $\mathscr{L}(G)$*-free gap real* G*-representation. Set* $d = |G/G^{\{p\}}|$.

(i) *Suppose* $|G|$ *is even. Then* $\mathrm{Dim}(G, 1) \cap d\,\mathbb{N} \approx d\,\mathbb{N}$.
(ii) *Suppose* $|G|$ *is odd. Then* $\mathrm{Dim}(G, 1) \cap 2d\,\mathbb{N} \approx 2d\,\mathbb{N}$.

Proposition 2.5 (Case 3)**.** *Let* G *be an Oliver group such that* $\mathfrak{P}(G) = \{2\}$*. Set* $d = |G/G^{\{2\}}|$.

(i) *Suppose* $|G^{\{2\}}|$ *is even. Then* $\mathrm{Dim}(G, 1) \cap d\,\mathbb{N} \approx d\,\mathbb{N}$*. If there is an* $\mathscr{L}(G)$*-free, stably weak-gap real* G*-representation of odd dimension, then* $\mathrm{Dim}(G, 1) \approx \mathbb{N}$.
(ii) *Suppose* $|G^{\{2\}}|$ *is odd. Then* $\mathrm{Dim}(G, 1) \subseteq 2\,\mathbb{N}$ *and* $\mathrm{Dim}(G, 1) \cap 2d\,\mathbb{N} \approx 2d\,\mathbb{N}$*. If there is an* $\mathscr{L}(G)$*-free, stably weak-gap complex* G*-representation of odd dimension, then* $\mathrm{Dim}(G, 1) \approx 2\,\mathbb{N}$.

We have a remark on Oliver groups G with $\delta_G = 1$.

Proposition 2.6. *Let* G *be an Oliver group.*

(i) *If* $\mathrm{Dim}(G,1)$ *is cofinal with* $\mathbb{N}$*, then there is an* $\mathscr{L}(G)$*-free real* G*-representation of odd dimension.*

(ii) *Suppose* G *is a gap group and* $\mathfrak{P}(G) \neq \{p\}$ *for any odd prime* p*. If there is a* $\{G^{\cap 2}\}$*-free real* G*-representation of odd dimension, then* $\mathrm{Dim}(G,1)$ *is cofinal with* $\mathbb{N}$*.*

(iii) *Suppose* $\mathfrak{P}(G) = \{p\}$ *for an odd prime* p*. If there is a* $\{G\}$*-free irreducible real* G*-representation of odd dimension relatively prime to* p*, then* $\mathrm{Dim}(G,1)$ *is cofinal with* $\mathbb{N}$*.*

We prove Propositions 2.3–2.6 in Section 9.

3. Preferable G-representations and G-actions on spheres

Let G be an Oliver group and let V be a $\{G\}$-free real G-representation. We invoke several properties on V below in order to obtain a G-map $f : (X, \partial X) \to (D(V), S(V))$ such that X is a disk with fixed-point-free G-action, $\partial f : \partial X \to S(V)$ is a G-diffeomorphism, and $f : X \to D(V)$ is a homotopy equivalence rel. ∂. Let $\mathscr{G}^1(G)$ denote the set of subgroups H of G such that H/P is a cyclic group for some $P \in \mathscr{P}(H)$ such that $P \trianglelefteq H$. Let $\Omega(G)$ denote the Burnside ring. For each $H \in \mathscr{S}(G)$, we have the homomorphism $\chi_H : \Omega(G) \to \mathbb{Z}$ defined by $\chi_H([X]) = \chi(X^H)$, where X is a finite G-CW complex and $\chi(X^H)$ is the Euler characteristic of X^H. In the case where G is an Oliver group, then there is $\beta \in \Omega(G)$ satisfying

$$\chi_G(\beta) = 0 \text{ and } \chi_H(\beta) = 1 \text{ for all } H \in \mathscr{M}(G). \tag{3.1}$$

We can write β in the form

$$\beta = \sum\nolimits_{H \in \mathscr{H}^*} b_H[G/H] \quad (b_H \in \mathbb{Z} \smallsetminus \{0\})$$

for some $\mathscr{H}^* \subseteq \mathscr{S}(G)^* \smallsetminus \{G\}$. Define $\mathrm{Iso}(G, \beta)$ by

$$\mathrm{Iso}(G,\beta) = \{gHg^{-1} \mid H \in \mathscr{H}^* \text{ and } g \in G\}.$$

Set

$$\mathscr{F}(G,1) = \{H \in \max(\mathscr{M}(G)) \mid N_G(H) = H\}.$$

For a G-set X and $H \in \mathscr{S}(G)$, let $X^{>H}$ denote the set $\bigcup_{K \in \mathscr{U}_G(H)} X^K$.

Definition 3.1. Let G be an Oliver group and V a real G-representation. We call V *preferable* if the conditions (A1)–(A4) below are satisfied.

(A1) $\dim V \geq 5$.

(A2) For any $P \in \mathscr{P}(G) \smallsetminus \mathscr{F}(G,1)$, the following holds.

(i) $\dim V^P \geq 5$,

(ii) V satisfies the weak-gap condition: $\dim V^P \geq 2\dim V^L$ for all $L \in \mathscr{U}_G(P)$.

(iii) If $\dim V^P = 2\dim V^H$, where $H \in \mathscr{U}_G(P)$, then $|H : P| = 2$ and $\dim V^H > \dim V^L + 1$ for all $L \in \mathscr{U}_G(H)$.

(iv) If $2\dim V^H = 2\dim V^K = \dim V^P$, where H, $K \in \mathscr{U}_G(P)$ (possibly $H = K$), then $\langle H, K\rangle$ (the smallest subgroup of G containing H and K) belongs to $\mathscr{M}(G)$.

(A3) For any $H \in \mathscr{G}^1(G)$, it holds that $\dim V^H \geq 1$ and $\dim V^H > \dim V^{>H}$.

(A4) V is $\mathscr{L}(G)$-free.

We remark that if V is a preferable G-representation and W is an $\mathscr{L}(G)$-free weak-gap G-representation (possibly $W = \{0\}$) then $V \oplus W$ is preferable. Remark 2.1 immediately implies that the G-representations $V(G)^{\oplus n}$ with $n \geq 3$ are preferable. In this paper, we employ the following two lemmas which are fundamental in the study of $\mathfrak{R}(G)_1$.

Lemma 3.1. *For any Oliver group G, $V(G)$ is a preferable G-representation.*

The lemma is a corollary to [19, Theorems 1.1 and 2.2]. However we can obtain the lemma from Remark 2.1. For the sake of reader's convenience, we give a proof in Appendix I.

Lemma 3.2. *Let G be an Oliver group. If V is a preferable real G-representation, then V lies in $\mathfrak{R}(G)_1$.*

A preferable real G-representation is admissible in the sense of [15, Definition 1.1] and therefore the above lemma is a corollary to [15, Theorem 1.2]. However the lemma follows also from arguments in [9, Section 2], [11, Sections 4, 5], [13, p. 502] and [12, Theorem 1.1]. For the sake of readers' convenience, we give a proof of the lemma in Appendix II.

Now we are ready to prove Lemma 2.1.

Proof of Lemma 2.1. Let $W \in \mathfrak{R}(G)^{\mathrm{w.gap}}_{\mathscr{L}(G)}$. It is clear that $V(G) \oplus W \in \mathfrak{R}(G)^{\mathrm{w.gap}}_{\mathscr{L}(G)}$. Lemma 3.1 shows that $V(G)\oplus W$ is a preferable G-representation. Therefore Lemma 3.2 implies that $V(G) \oplus W \in \mathfrak{R}(G)_1$. □

4. Proof of Lemma 2.2

For integers $a_1, \ldots, a_n \in \overline{\mathbb{N}}$, let $\mathbb{N}(a_1, \ldots, a_n)$ denote the set of all integers $m = a_1x_1 + \cdots + a_nx_n$ satisfying $m > 0$, where $x_1, \ldots, x_n \in \overline{\mathbb{N}}$. For a finite set $F = \{a_1, \ldots, a_n\}$, we mean by $\mathbb{N}(F)$ the set $\mathbb{N}(a_1, \ldots, a_n)$. Let A be a subset of $\mathbb{N}$. We say that A is *additively closed* if $a + b$ belongs to A whenever $a, b \in A$.

Lemma 4.1. *Let m and n be relatively prime natural numbers and set $N = \max\{(m-1)(n-1), 1\}$. Then $[N..\infty)$ is contained in $\mathbb{N}(m, n)$.*

Proof. Let m and n be relatively prime natural numbers. If $m = 1$ or $n = 1$ then the conclusion of the lemma is obviously true. Thus we assume that $m \geq 2$ and $n \geq 2$. Then N is equal to $(m-1)(n-1)$. If a natural number x lies in $\mathbb{N}(m, n)$ then we have the inclusions

$$x + \min(m, n)\mathbb{N} \subseteq x + m\overline{\mathbb{N}} + n\overline{\mathbb{N}} \subseteq \mathbb{N}(m, n).$$

Therefore, for the proof of Lemma 4.1, it suffices to show that $mn - m - n + k$ belongs to $\mathbb{N}(m, n)$ for all $k \in \mathbb{N}$ with $1 \leq k \leq \min(m, n) - 1$.

Let k be a natural number such that $1 \leq k \leq \min(m, n) - 1$. We have clearly $mn - m - n + k = (m-1)(n-1) + (k-1) \geq 1$. There is a pair (a, b) of integers a and b such that $am - bn = k$. In the rest of the proof, we can suppose $a > 0$ and $b > 0$ without any loss of generality (otherwise, we replace m and n with n and m, respectively). In addition, we choose the natural number a as small as possible. Then it holds that $1 \leq a \leq n - 1$. Since $bn + k = am \leq (n-1)m = mn - m$, we get $1 \leq b \leq m - 1$. The number

$$\begin{aligned} mn - m - n + k &= mn - m - n + am - bn \\ &= (a-1)m + (m - 1 - b)n \quad (\geq 1) \end{aligned}$$

belongs to $\mathbb{N}(m, n)$. □

For $m, n \in \mathbb{N}$, $a, b \in \overline{\mathbb{N}}$, the integer $am + bn$ is equal to $d\,(a(m/d) + b(n/d))$, where $d = \gcd(m, n)$. Therefore we get the next corollary.

Corollary 4.2. *Let $m, n \in \mathbb{N}$, $d = \gcd(m, n)$ and $N = \max\{(m/d - 1)(n/d - 1), 1\}$. Then the set $d\,[N..\infty)$ is contained in $\mathbb{N}(m, n)$, i.e. $\mathbb{N}(m, n) \cap [dN..\infty) = d\,[N..\infty)$.*

This has the next corollary.

Corollary 4.3. *Let $a_1, \dots, a_n \in \mathbb{N}$ and set $d = \gcd(a_1, \dots, a_n)$. Then there exists an integer N such that $d\,[N..\infty)$ is contained in $\mathbb{N}(a_1, \dots, a_n)$, i.e. $\mathbb{N}(a_1, \dots, a_n) \cap [dN..\infty) = d\,[N..\infty)$ $(= (d\,\mathbb{N}) \cap [dN..\infty))$.*

Proposition 4.4. *Let A be an additively closed subset of $\mathbb{N}$. Then there exists a finite subset F of A such that $\mathbb{N}(F) = A$.*

Proof. Set $d = \gcd(A)$. There is a finite subset $F_0 = \{a_1, \dots, a_n\}$ of A such that $\gcd(F_0) = d$. We have $\mathbb{N}(F_0) \subseteq A \subseteq d\,\mathbb{N}$. Corollary 4.3 says that $(d\,\mathbb{N}) \smallsetminus \mathbb{N}(F_0)$ is a finite set, and therefore $F_1 := A \smallsetminus \mathbb{N}(F_0)$ is a finite set. Then the union $F = F_0 \cup F_1$ satisfies $\mathbb{N}(F) = A$. □

Proposition 4.5. *Let A be an additively closed subset of $\mathbb{N}$ and set $d = \gcd(A)$. Then A is cofinal with $d\,\mathbb{N}$, i.e. there is an integer N such that $d\,[N..\infty)$ is contained in A.*

Proof. In the proof of Proposition 4.4, we showed that $(d\,\mathbb{N}) \smallsetminus A$ is finite. Therefore there is an integer N such that $d\,[N..\infty)$ is contained in A. □

Proposition 4.6. *Let A and B be subsets of $\mathbb{N}$ such that $A \subseteq B$. Suppose that A is additively closed. Then there is an integer k such that*

$$(A \cap d\,\mathbb{N}) \cap [k..\infty) = (B \cap d\,\mathbb{N}) \cap [k..\infty) = (d\,\mathbb{N}) \cap [k..\infty) \tag{4.1}$$

for any multiple d of $\gcd(A)$.

Proof. For a multiple d of $\gcd(A)$, we have the equality $\gcd(A)\,\mathbb{N} \cap d\,\mathbb{N} = d\,\mathbb{N}$. Therefore it is sufficient to prove (4.1) in the case $d = \gcd(A)$. Suppose $d = \gcd(A)$. Then $A = A \cap d\,\mathbb{N}$. By Proposition 4.5, there is an integer k such that $(A \cap d\,\mathbb{N}) \cap [k..\infty) = (d\,\mathbb{N}) \cap [k..\infty)$. In addition, it is clear that

$$A \cap d\,\mathbb{N} \subseteq B \cap d\,\mathbb{N} \subseteq d\,\mathbb{N}.$$

Therefore we obtain the formula (4.1). □

Proposition 4.7. *Let A be a subset of $\mathbb{N}$ and $x \in \mathbb{N}$ such that kx and $(k+1)x$ both belong to A for some $k \in \mathbb{N}$. Then $\gcd(A + x)$, $\gcd(A + kx)$ and $\gcd(A + (k+1)x)$ all coincide with $\gcd(A)$. Set $d = \gcd(A)$. If A is additively closed, then $A + kx$ and $A + (k+1)x$ are additively closed and A, $A + x$, $A + kx$ and $A + (k+1)x$ all are cofinal with $d\,\mathbb{N}$.*

Proof. It holds that

$$\begin{aligned}\gcd(A+x) &= \gcd(A+x\cup\{kx+x,(k+1)x+x\})\\ &= \gcd(A+x\cup\{(k+1)x,(k+2)x\}) = \gcd(A+x\cup\{x\})\\ &= \gcd(A\cup\{x\}) = \gcd(A\cup\{kx,(k+1)x\})\\ &= \gcd(A).\end{aligned}$$

Similarly, we can prove $\gcd(A+kx)=\gcd(A)$ and $\gcd(A+(k+1)x)=\gcd(A)$.

Suppose A is additively closed. Then it is clear that $A+kx$ and $A+(k+1)x$ are additively closed. Proposition 4.5 implies that A, $A+kx$ and $A+(k+1)x$ are cofinal with $d\,\mathbb{N}$. Note that $A+kx\subseteq A\subseteq d\,\mathbb{N}$. It implies that $(A+kx)+x\subseteq A+x\subseteq d\,\mathbb{N}$. Since $A+(k+1)x\approx d\,\mathbb{N}$, we conclude $A+x\approx d\,\mathbb{N}$. □

Proposition 4.8. *Let $A\subseteq B$ and C be subsets of $\mathbb{N}$ such that A is additively closed and let x be an element in $A\cap C$ such that $A+x\subseteq C$. (Therefore $x\,\mathbb{N}\subseteq A\cap C$.) Suppose that for each element $y\in B$, there is $m(y)\in\mathbb{N}$ such that $y+m(y)x$ belongs to A. Then the following holds.*

(1) $\gcd(A)=\gcd(B)$.
(2) $A\approx B\approx C\cap\gcd(A)\,\mathbb{N}$.
(3) *There exists $N\in\mathbb{N}$ such that $B+Nx\subseteq C$.*

Proof. It holds that

$$\gcd(B)=\gcd(B\cup\{x\})=\gcd(A\cup\{x\})=\gcd(A).$$

Recall that $\gcd(A+x)=\gcd(A)$, $A+x\subseteq A\subseteq B\subseteq\gcd(B)\mathbb{N}$ and $A+x\approx A\approx\gcd(A)\,\mathbb{N}$. Therefore we get $B\approx\gcd(A)\,\mathbb{N}$. Since $A+x\subseteq C\cap\gcd(A)\,\mathbb{N}\subseteq\gcd(A)\,\mathbb{N}$, we get $A+x\approx C\cap\gcd(A)\,\mathbb{N}\approx\gcd(A)\,\mathbb{N}$. We have showed that $A\approx B\approx C\cap\gcd(A)\,\mathbb{N}$.

Note that $Y:=B\setminus A$ is a finite set. For $N=1+\sum_{y\in Y}m(y)$ (≥ 1), it is clear that

$$\begin{aligned}B+N\,x &= (A\cup(B\setminus A))+N\,x\\ &= (A+N\,x)\cup((B\setminus A)+N\,x)\\ &\subseteq C\cup C=C.\end{aligned}$$

□

Proof of Lemma 2.2. Let G be an Oliver group. Set $A=\mathrm{Dim}(G,\mathrm{w})\setminus\{0\}$, $B=\mathrm{Dim}(G,\mathrm{s})\setminus\{0\}$ and $C=\mathrm{Dim}(G,1)$. It is clear that A and B are additively closed and $A\subseteq B$. Set $x=\dim V(G)$. Then Lemma 2.1 shows that $x\in A\cap C$ and $A+x\subseteq C$.

By the definition of weak-gap stabilizer, for each $y \in B$, there exists $m(y) \in \mathbb{N}$ such that $y + m(y)\,x$ lies in A. It is clear that $d(G, \mathrm{w})$ divides x. Therefore Lemma 2.2 follows from Proposition 4.8. □

In the above proof we used the result that $V(G)$ is preferable. However, in order to prove Lemma 2.1, we need merely the fact that $V(G)^{\oplus k}$ is preferable for some $k \in \mathbb{N}$ (because setting $x = k \dim V(G)$, the above proof works well).

5. Preparation from Algebra

We begin this section with recalling several known facts in algebra which will be used in the subsequent sections. Let $\mathfrak{Irr}_{\mathbb{R}}(G)$ (resp. $\mathfrak{Irr}_{\mathbb{C}}(G)$) denote the family of irreducible real (resp. complex) G-representations and let $\mathfrak{Irr}_{\mathbb{R}}(G)^*$ (resp. $\mathfrak{Irr}_{\mathbb{C}}(G)^*$) denote a complete set of representatives of isomorphism classes of irreducible real (resp. complex) G-representations.

Lemma 5.1 ([23, 2.4 Corollary 2])**.** *The equality*

$$\sum_{W \in \mathfrak{Irr}_{\mathbb{C}}(G)^*} (\dim W)^2 = |G|$$

holds.

Lemma 5.2 ([23, 6.5 Corollary 2])**.** *The dimension of any irreducible complex G-representation divides $|G|$.*

Corollary 5.3. *Let W be an irreducible real G-representation of odd dimension. Then the complexification of W is an irreducible complex G-representation, and therefore $\dim W$ divides $|G|$.*

Lemma 5.4 ([23, 8.5 Theorem 16])**.** *Let G be a supersolvable group. Then each irreducible complex G-representation is induced from a one-dimensional representation of a subgroup of G.*

Lemma 5.5 (P. G. L. Dirichlet, cf. [22, Chapter VI, p. 61])**.** *Let a, $b \in \mathbb{N}$. If a and b are relatively prime, then there exist infinitely many primes p such that $p \equiv a \bmod b$.*

Corollary 5.6 (P. G. L. Dirichlet)**.** *Let p be a prime and $m \in \mathbb{N}$. Then there exist infinitely many primes q such that $q \equiv 1 \bmod p^m$.*

In the rest of this section, we prepare basic facts on the representation theory which will be needed in the subsequent sections.

Proposition 5.7. *Let H, K and L be subgroups of G, let W be a real K-representation, and set $V = \mathrm{ind}_K^G W$. Then the following holds.*

1. *If $W^{K\cap a^{-1}La} = \{0\}$ for all $a \in G$, then $V^L = \{0\}$.*
2. *If W satisfies the weak-gap condition at $K \cap a^{-1}Ha$ for all $a \in G$, then V satisfies the weak-gap condition at H.*

Proof. By the Mackey decomposition, we have

$$\mathrm{res}_L^G V = \bigoplus_{LaK \in L\backslash G/K} \mathrm{ind}_{L\cap aKa^{-1}}^L c_{a*}\mathrm{res}_{K\cap a^{-1}La}^K W.$$

Therefore $\dim V^L = \sum_{LaK} \dim W^{K\cap a^{-1}La}$, which shows the claim (1).

Next we are going to show $\dim V^H \geq 2\dim V^L$ for $H \in \mathscr{S}(G)$ and $L \in \min(\mathscr{U}_G(H))$.

$$\dim V^H - 2\dim V^L = \sum_{HaK\in H\backslash G/K} \dim W^{K\cap a^{-1}Ha} - \sum_{LbK\in L\backslash G/K} 2\dim W^{K\cap b^{-1}Lb}.$$

For arbitrarily fixed $b \in G$, either $LbK = HbK$ or $LbK = HbK \amalg \coprod_{i=1}^{m_b} Ha_{b,i}bK$ ($m_b \in \mathbb{N}$, $a_{b,i} \in L \smallsetminus H$) holds. We remark that $LbK = HbK$ holds if and only if $|L : H| = |K \cap b^{-1}Lb : K \cap b^{-1}Hb|$. Let $\mathscr{A}$ (resp. $\mathscr{B}$) be a complete set of representatives b of $L\backslash G/K$ such that $|L : H| = |K \cap b^{-1}Lb : K \cap b^{-1}Hb|$ (resp. $|L : H| \neq |K \cap b^{-1}Lb : K \cap b^{-1}Hb|$). Then we have

$$\begin{aligned}
\dim V^H - 2\dim V^L &= \sum_{b\in\mathscr{A}} (\dim W^{K\cap b^{-1}Hb} - 2\dim W^{K\cap b^{-1}Lb}) \\
&\quad + \sum_{b\in\mathscr{B}} \left(\dim W^{K\cap b^{-1}Hb} + \sum_{i=1}^{m_b} \dim W^{K\cap b^{-1}a_{b,i}^{-1}Ha_{b,i}b} - 2\dim W^{K\cap b^{-1}Lb} \right) \\
&\geq \sum_{b\in\mathscr{B}} \left(\sum_{i=1}^{m_b} \dim W^{K\cap b^{-1}a_{b,i}^{-1}Ha_{b,i}b} - \dim W^{K\cap b^{-1}Lb} \right).
\end{aligned}$$

Since $b^{-1}a_{b,i}^{-1}Ha_{b,i}b < b^{-1}Lb$, we see $\dim W^{K\cap b^{-1}a_{b,i}^{-1}Ha_{b,i}b} \geq \dim W^{K\cap b^{-1}Lb}$, which shows $\dim V^H - 2\dim V^L \geq 0$. □

For a finite group A and $m \in \mathbb{N}$, we mean by $A^{\times m}$ the m-fold direct product $A \times \cdots \times A$ of A. For groups P and Q and a homomorphism $\psi : P \to \mathrm{Aut}(Q)$, we say that ψ is *free* if the non-equality $\psi(g)(x) \neq x$ holds for any $g \in P \smallsetminus \{e\}$ and $x \in Q \smallsetminus \{e'\}$, where e and e' are the identity elements of P and Q, respectively. It is

clear that if $\psi : P \to \mathrm{Aut}(Q)$ is free then $\varphi : P \to \mathrm{Aut}(Q^{\times m})$ is free for $m \in \mathbb{N}$, where $\varphi = \psi^{\times m}$ $(= \psi \times \cdots \times \psi$ m-fold).

Proposition 5.8. *Let p be an odd prime, $n \in \mathbb{N}$ and P a cyclic group of order p^n. Then there exist infinitely many elementary abelian 2-groups T with homomorphism $\varphi : P \to \mathrm{Aut}(T)$ such that φ is free.*

Proof. Consider the regular representation $V = \mathbb{F}_2[P]$ of P over $\mathbb{F}_2$. Since $|P|$ is odd, there is a P-invariant inner-product on the $\mathbb{F}_2$-vector space V. Let N be the subgroup of P of order p. We have the orthogonal decomposition $V = V^N \oplus V_N$ as P-representations over $\mathbb{F}_2$. Let $\psi : P \to \mathrm{Aut}_{\mathbb{F}_2}(V_N)$ be the associated homomorphism with the $\mathbb{F}_2$-linear P-action on V_N. Set $Q = V_N$ as an abelian group. It is clear that $\psi : P \to \mathrm{Aut}(Q)$ is free and therefore $\varphi : P \to \mathrm{Aut}(T)$, where $\varphi = \psi^{\times m}$ and $T = Q^{\times m}$, is free for every $m \in \mathbb{N}$. □

We mean by a *subquotient group H/K of G* the quotient group H/K of subgroups $K \trianglelefteq H \in \mathscr{S}(G)$.

Proposition 5.9. *Let p be a prime, let G be a finite group not of prime-power order with $\mathfrak{P}(G) = \{p\}$ and set $N = G^{\{p\}}$. Invoke the following hypotheses:*

1. *G/N and N are abelian groups.*
2. *$|N|$ is not divisible by p.*
3. *Any element of $G \smallsetminus N$ has p-power order.*
4. *The order $|H/K|$ is a power of p for any cyclic subquotient group H/K of G such that $|H/K|$ is divisible by p.*

Let m be the integer given by $|G/N| = p^m$. Then for an arbitrary $V \in \mathfrak{Irr}_{\mathbb{C}}(G)_{\mathscr{L}(G)}$, there is $W \in \mathfrak{Irr}_{\mathbb{C}}(N)$ such that $V \cong \mathrm{ind}_N^G W$ as complex G-representations. Therefore $\mathrm{Dim}(\mathfrak{Irr}_{\mathbb{C}}(G)) = \{1, p^m\}$, $\mathrm{Dim}(\mathfrak{Irr}_{\mathbb{C}}(G)_{\mathscr{L}(G)}) = \{p^m\}$, and $\mathrm{Dim}(\mathfrak{R}(G)_{\mathscr{L}(G)}) \subseteq p^m\,\mathbb{N}$.

Proof. We have $\mathscr{L}(G) = \overline{\mathscr{U}}_G(N)$. Let $V \in \mathfrak{Irr}_{\mathbb{C}}(G)_{\mathscr{L}(G)}$. Then $V^N = \{0\}$ and $V = V_N$. Remark that for any non-abelian subquotient group H/K of G, where $K \triangleleft H \in \mathscr{S}(G)$, $(H \cap N)/(K \cap N)$ is an abelian subgroup of H/K which is not contained in the center of H/K. By the argument same as the proof of [23, Chapter 8, Theorem 16], we can see that there are a subgroup H and a one-dimensional complex H-representation W such that $V \cong \mathrm{ind}_H^G W$ as complex G-representations. Since N is

abelian and V is irreducible, for the proof of $H = N$, it is sufficient to show $H \subseteq N$. To prove the property $H \subseteq N$, we will find a contradiction under the assumption

$$H \smallsetminus N \neq \emptyset.$$

Let K be the kernel of the representation W, namely K is the maximal subgroup of H satisfying $W^K = W$. By definition, K is a normal subgroup of H. Since $\dim W = 1$, the group H/K is cyclic. Since $\dim V^N = |G/(HN)| \dim W^{H\cap N} = 0$, we get $W^{H\cap N} = \{0\}$. Therefore $H \cap N \smallsetminus K \neq \emptyset$. The hypothesis (2) implies that $|H/K|$ is not a power of p. Furthermore the hypothesis (4) implies that $|H/K|$ is not divisible by p. Let P be a Sylow p-subgroup of H. Then $H = (H \cap N) \rtimes P$, $K = (K \cap N) \rtimes P$ and the canonical homomorphism $(H \cap N)/(K \cap N) \to H/K$ is an isomorphism. Since $K \cap N = K^{\{p\}}$, we see $K \cap N \lhd H$ and obtain $H' = K' \rtimes P$, where $H' = H/(K \cap N)$ and $K' = (H \cap N)/(K \cap N)$. Observe the homomorphism

$$\psi : H' \to H/K \xrightarrow{\cong} (H \cap N)/(K \cap N) = K'.$$

The kernel of ψ is P, which implies $P \lhd H'$. It means $H' = K' \times P$. Set $H'' = K' \times C_p$ for a cyclic subgroup C_p with order p of P. Since the group H'' is a cyclic subquotient group of G, the hypothesis (4) shows that K' is the trivial group, which says $H \cap N = K \cap N$ as well as $H = K$ (contradiction). It completes the proof of $H = N$, which shows $\dim V = p^m$.

The above fact shows the equality $\mathrm{Dim}(\mathfrak{Irr}_{\mathbb{C}}(G)_{\mathscr{L}(G)}) = \{p^m\}$. Since G/N is abelian, we have $\mathrm{Dim}(\mathfrak{Irr}_{\mathbb{C}}(G)) = \{1, p^m\}$.

Next let V be an $\mathscr{L}(G)$-free real G-representation of positive dimension. Let $V_{\mathbb{C}}$ be the complexification of V. Then we have $\dim_{\mathbb{R}} V = \dim_{\mathbb{C}} V_{\mathbb{C}}$. Since $\dim_{\mathbb{C}} V_{\mathbb{C}}$ is divisible by p^m, so is $\dim_{\mathbb{R}} V$. □

Lemma 5.10 (due to S. Tamura)**.** *Let p be a prime, G a nontrivial finite group, and N a normal subgroup of G satisfying the hypotheses (2) and (3) of Proposition 5.9. Then the hypotheses (4) of Proposition 5.9 is fulfilled.*

Proof. Let H/K be an arbitrary cyclic subquotient group of G such that $|H/K|$ is divisible by p. There is an element $h \in H$ such that the coset hK is a generator of H/K. Since $|H/K|$ is divisible by p, the order of h in H is divisible by p. By the hypothesis (2), h lies in $G \smallsetminus N$. By the hypothesis (3) the order of h in H a power of p, and therefore $|H/K|$ is a power of p. □

Proposition 5.11. *Let p and q be distinct primes. Let P be a nontrivial abelian p-group, F an elementary abelian q-group, and $\varphi : P \to \mathrm{Aut}(F)$ a free homomorphism. Then the group $G = F \rtimes_\varphi P$ satisfies the hypotheses (1)–(4) of Proposition 5.9.*

Proof. It is clear that G satisfies the hypotheses (1)–(3). By Lemma 5.10, G satisfies the hypothesis (4). □

The next fact follows immediately from the above proposition.

Corollary 5.12. *Let p and q be distinct primes. Let P be a nontrivial cyclic p-group and $\varphi_1 : P \to \mathrm{Aut}(C_q)$ a monomorphism. Let G be the group $(C_q \times C_q) \rtimes_\varphi P$, where $\varphi = \varphi_1 \times \varphi_1 : P \to \mathrm{Aut}(C_q \times C_q)$. Then the group G satisfies the hypotheses (1)–(4) of Proposition 5.9.*

For a real G-representation V, we denote by $\mathrm{Sub}(V)$ the set of all G-subrepresentations of V. Remark that $V(G)$ is an $\mathcal{L}(G)$-free weak-gap representation and that $V(G)$ is isomorphic to $\bigoplus_W W^{\oplus a_W}$ for some $a_W \in \mathbb{N}$, where W runs over $\mathfrak{Irr}_{\mathbb{R}}(G)^*_{\mathscr{L}(G)}$. It is clear that

$$\gcd(\mathrm{Dim}(\mathfrak{R}(G)_{\mathscr{L}(G)})) = \gcd(\mathrm{Dim}(\mathrm{Sub}(V(G)))) = \gcd(\mathrm{Dim}(\mathfrak{Irr}_{\mathbb{R}}(G)_{\mathscr{L}(G)})).$$

Without difficulties, we can see that

$$d(G, \mathrm{s}) = \gcd(\mathrm{Dim}(\mathrm{Sub}(V(G)^{\oplus n})^{\mathrm{s.w.gap}}))$$

for sufficiently large $n \in \mathbb{N}$. If G is a gap group, then we have

$$\begin{aligned} d(G, \mathrm{s}) &= \gcd(\mathrm{Dim}(\mathfrak{R}(G)_{\mathscr{L}(G)})) \\ &= \gcd(\mathrm{Dim}(\mathrm{Sub}(V(G)))) \\ &= \gcd(\mathrm{Dim}(\mathfrak{Irr}_{\mathbb{R}}(G)_{\mathscr{L}(G)})). \end{aligned}$$

The following three propositions are concerned with $\mathscr{L}(G)$-free G-representations of odd dimension.

Proposition 5.13. *Let L be a nontrivial group of even order. Then there is $W \in \mathfrak{Irr}_{\mathbb{R}}(L)_{\{L\}}$ of odd dimension. In addition, $\gcd(\dim W, \dim \mathbb{R}[L]_L)$ is equal to 1 for the representation W.*

Proof. Set $V = \mathbb{R}[L]_L$. Then we have $\dim V = |L| - 1$ (an odd integer). The equality

$$1 + \sum_{W \in \mathfrak{Irr}_{\mathbb{R}}(L)^*_{\{L\}}} a_W \dim W = |L| \quad (a_W \in \mathbb{N})$$

implies that there is $W \in \mathfrak{Irr}_{\mathbb{R}}(L)_{\{L\}}$ of odd dimension. By Corollary 5.3, $\dim W$ divides $|L|$. Therefore $\gcd(\dim W, \dim V)$ divides $\gcd(|L|, |L|-1) = 1$, which shows $\gcd(\dim W, \dim V) = 1$. □

Corollary 5.14. *If G is a nontrivial perfect group, then there is an $\mathscr{L}(G)$-free irreducible real G-representation W of odd dimension and* $\gcd(\dim W, \dim V(G)) = 1$, *and therefore $d(G, \mathrm{s}) = 1$.*

Proposition 5.15. *Let G be an Oliver group such that $G^{\{2\}} = G$ (i.e. $2 \notin \mathfrak{P}(G)$) and* $\mathrm{Dim}(G, 1)$ *is cofinal with $\mathbb{N}$. Then $V(G)$ is an $\mathscr{L}(G)$-free real G-representation of odd dimension.*

Proof. Since $\mathrm{Dim}(G, 1)$ is cofinal with $\mathbb{N}$, $|G|$ is even (cf. Remark 1.1). Observe the equality

$$\dim V(G) = (|G| - 1) - \sum_{p \in \mathfrak{P}(G)} (|G/G^{\{p\}}| - 1).$$

Since $2 \notin \mathfrak{P}(G)$, the integers $|G/G^{\{p\}}| - 1$ above are even. Since $|G| - 1$ is odd, we conclude that $\dim V(G)$ is odd. □

Proposition 5.16. *Let G be a finite group not of prime-power order and let V be a $\{G^{\cap 2}\}$-free real G-representation of odd dimension. Then V contains an irreducible component of odd dimension. Moreover, for an arbitrary irreducible component W of V such that* $\dim W$ *is odd, W is $\{G^{\mathrm{nil}}\}$-free (therefore $\mathscr{L}(G)$-free).*

Proof. Since $\dim V$ is odd, there is an irreducible component W of V such that $\dim W$ is odd. By hypothesis, W is $\{G^{\cap 2}\}$-free. Set $N = G^{\mathrm{nil}}$. We have the orthogonal decomposition $W = W^N \oplus W_N$ with respect to a G-invariant inner-product on W. Since W is irreducible, we get $W = W^N$ or $W = W_N$. Assume $W = W^N$. Since W is $\{G^{\cap 2}\}$-free and G/N is nilpotent, $\dim W$ is even (contradiction). Therefore, W coincides with W_N, namely W is $\{N\}$-free. □

Proposition 5.17. *Let G be an Oliver group and V a real G-representation. If $2 \dim V^H \leq \dim V^P$ for all $(P, H) \in \mathscr{PH}_2(G)$, then V is a stably weak-gap representation.*

Proof. By [6, Theorem 2.3], $V \oplus V(G)^{\oplus k}$, where $k = \dim V$, is a weak-gap representation. □

Proposition 5.18. *Let G be an Oliver group, N a normal subgroup of G and φ the canonical projection $G \to \overline{G} = G/N$. Suppose that G is a gap group or $N \subseteq G^{\{2\}}$. Let $\overline{\mathscr{R}}$ be a non-empty subset of $\mathfrak{R}(\overline{G})_{\mathscr{L}(\overline{G})}$ such that $\mathrm{Dim}(\overline{\mathscr{R}}) \neq \{0\}$. Set $\overline{d} = \gcd(\mathrm{Dim}(\overline{\mathscr{R}}))$. Then $d(G, \mathrm{s})$ divides $\overline{d}$, and therefore the sets $\mathrm{Dim}(G, \mathrm{s}) \cap \overline{d}\,\mathbb{N}$ and $\mathrm{Dim}(G, 1) \cap \overline{d}\,\mathbb{N}$ are cofinal with $\overline{d}\,\mathbb{N}$.*

Proof. Let W be an element of $\overline{\mathscr{R}}$ and set $V = \varphi^* W$ (the induced real G-representation). Since $\varphi(G^{\{p\}}) = \overline{G}^{\{p\}}$ for any prime p, V is an $\mathscr{L}(G)$-free stably weak-gap G-representation. Therefore $d(G, \mathrm{s})$ divides $\overline{d}$. □

Proposition 5.19. *Let G be a finite group such that $|\mathfrak{P}(G)| \geq 2$. Then there is an $\mathscr{L}(G)$-free stably weak-gap irreducible real G-representation of dimension 2. Therefore $d(G, \mathrm{s})$ is equal to 1 or 2.*

Proof. Since $|\mathfrak{P}(G)| \geq 2$, there is a normal subgroup N of G such that G/N is a cyclic group of order pq for distinct primes p and q. Let φ be the canonical projection $G \to G/N$ and let W be a free real G/N-representation of dimension 2. For all $P \in \mathscr{P}(G/N)$ and $H \in \mathscr{U}_{G/N}(P)$, the inequality $2 \dim W^H \leq \dim W^P$ holds clearly, i.e. W is a weak-gap G/N-representation. Therefore $\varphi^* W$ is an $\mathscr{L}(G)$-free stably weak-gap irreducible real G-representation of dimension 2. □

Proposition 5.20 (cf. Proposition 2.4)**.** *Let G be a finite group not of prime power order. Suppose $\mathfrak{P}(G) = \{p\}$ for a prime p. Then $d(G, \mathrm{s})$ divides $2|G/G^{\{p\}}|$. Furthermore if $|G^{\{p\}}|$ is even, then $d(G, \mathrm{s})$ divides $|G/G^{\{p\}}|$.*

Proof. Set $N = G^{\{p\}}$.

Firstly suppose that $|N|$ is even. Then $V = \mathbb{R}[N]_N$ is an $\{N\}$-free gap real N-representation and there is an $\{N\}$-free gap real N-representation W such that $\dim W \equiv 1 \mod \dim V$. It says that $d(N, \mathrm{s})$ divides 1. Since $\mathrm{ind}_N^G V$ and $\mathrm{ind}_N^G W$ are contained in $\mathfrak{R}(G)^{\mathrm{s.w.gap}}_{\mathscr{L}(G)}$, $d(G, \mathrm{s})$ divides $|G/N|$.

Next suppose that $|N|$ is odd. Then since N is solvable, there is a normal subgroup M of N such that $|N/M| = q$, where q is a prime ($\neq p$). We have a free real N/M-representation U of dimension 2. Let φ be the canonical projection $N \to N/M$. Then $\varphi^* U$ lies in $\mathfrak{R}(N)^{\mathrm{s.w.gap}}_{\{N\}}$. Therefore $\mathrm{ind}_N^G \varphi^* U$ is an $\mathscr{L}(G)$-free stably weak-gap G-representation of dimension $2|G/N|$, which shows $d(G, \mathrm{s})$ divides $2|G/N|$. □

6. Preparation from topological transformation groups

For Oliver groups G having δ_G, we need information more than Lemma 2.2 to determine δ_G. In this section, we prepare several results which are helpful to estimate δ_G.

For a prime p, let $\mathbb{Z}_p$ stand for the ring (or sometimes the module) $\mathbb{Z}/(p)$. A compact connected manifold X is said to be $\mathbb{Z}_p$-*orientable* if $H_n(X, \partial X; \mathbb{Z}_p) \cong \mathbb{Z}_p$ for $n = \dim X$. A compact manifold X is said to be $\mathbb{Z}_p$-*orientable* if each connected component of X is $\mathbb{Z}_p$-orientable. Remark that any compact manifold X is $\mathbb{Z}_2$-orientable. Let us recall a classical result in the theory of transformation groups.

Proposition 6.1. *Let X be a homology sphere with G-action and H a subgroup of G. If a Sylow 2-subgroup Q of H is normal in H, then X^H is orientable (therefore $\mathbb{Z}_p$-orientable for any prime p).*

Proof. Since X^Q is a $\mathbb{Z}_2$-homology sphere, X^Q is orientable. Since H/Q has odd order, $X^H = (X^Q)^{H/Q}$ is orientable. □

The next lemma is a new generalization of [10, Lemma 2.1].

Lemma 6.2. *Let p be a prime, G a nontrivial cyclic group of p-power order, and X a connected closed G-manifold. If X is $\mathbb{Z}_p$-orientable, $|X^G| = 1$, and $|X^H| < \infty$ for all $H \in \mathscr{S}(G) \smallsetminus \{E\}$, then X is a singleton, i.e. $|X| = 1$.*

Proof. Contrarily assuming that $\dim X > 0$, we will find a contradiction. Let x_0 be the G-fixed point of X and set $V = T_{x_0}(X)$. Let U be a closed disk neighborhood with linear G-action of x_0 in X. Therefore U is G-diffeomorphic to $D(V)$. Let Σ be the pinched G-space $X/(X \smallsetminus \overset{\circ}{U})$ and let $f : X \to \Sigma$ be the canonical projection. Then Σ (resp. x_0) can be identified with $S(\mathbb{R} \oplus V)$ (resp. $(1, 0)$, where $1 \in \mathbb{R}$ and $0 \in V$). Let y_0 be the G-fixed point in Σ identified with $(-1, 0) \in S(\mathbb{R} \oplus V)$. Let H be a subgroup of G such that $H \neq G$, E, and let x be an arbitrary point in X with isotropy subgroup H, i.e. $G_x = H$. Then $T_x(X)^K = \{0\}$ for all $K \in \mathscr{S}(H)$ with $K \neq E$. It implies that there is an H-map $g_x : S(T_x(X)) \to S(V)$. We can identify a small closed H-disk (resp. G-disk) neighborhood $\Delta(x)$ (resp. $\Delta(y_0)$) of x (resp. y_0) in X (resp. $\Sigma = S(\mathbb{R} \oplus V)$) with $D(T_x(X))$ (resp. $D(V)$). We can choose $\Delta(ax)$ so that $\Delta(ax) = a\Delta(x)$ for $a \in G \smallsetminus H$. Set $Z = X \smallsetminus \bigcup_x \overset{\circ}{\Delta}(x)$, where x runs over

$\bigcup_{H\neq G,\,E} X^H$ and $\overset{\circ}{\Delta}(x)$ is the interior of $\Delta(x)$. Then f is G-homotopic to a G-map $f' : X \to \Sigma = S(\mathbb{R}\oplus V)$ such that

(1) the restriction of f' to a small neighborhood $\Delta(x_0)$ of x_0 is the identity map,
(2) the restrictions $f'|_x : \Delta(x) \to \Delta(y_0)$ of f' are the radial extension of g_x for all $x \in \bigcup_{H\neq G,\,E} X^H$,
(3) $f'(X \smallsetminus \overset{\circ}{\Delta}(x_0)) \subseteq \Sigma \smallsetminus \overset{\circ}{\Delta}(x_0)$, and
(4) the restriction $f'|_Z$ of f' to Z is transverse regular to y_0 in $\Sigma = S(\mathbb{R}\oplus V)$.

For $z \in {f'|_Z}^{-1}(y_0)$, let $\deg_z(f')$ denote the local mapping-degree of f' at z, namely $\deg_z(f') = 1$ (resp. -1) if f' is an orientation-preserving (resp. orientation-reversing) map from a small neighborhood of z to a small neighborhood of y_0. If p is odd then the G-action on X is orientation preserving. Therefore $\sum_{u\in Gz} \deg_u(f') = 0$ $(\in \mathbb{Z}_p)$, for all $z \in {f'|_Z}^{-1}(y_0)$. In addition, $\sum_{u\in Gx} \deg(g_u) = 0$ $(\in \mathbb{Z}_p)$ for all $x \in X^H$ $(H \neq G,\, E)$. Therefore the mod-p mapping-degree $\deg(f')$ is equal to 0. However this contradicts the fact $\deg(f') = \deg(f) = 1$. □

Remark that $G/G^{\cap p}$ is an elementary abelian group. Lemma 6.2 gives the following results.

Proposition 6.3 (cf. [10, Lemma 2.1])**.** *Let G be a finite group and X a connected closed G-manifold such that $X^G = \{x_0\}$. Set $V = T_{x_0}(X)$. Then the equality $V^{G^{\cap 2}} = \{0\}$ holds.*

Proof. Assume $V^{G^{\cap 2}} \neq \{0\}$. Then there is a normal subgroup $L \triangleleft G$ such that $|G/L| = 2$ and $V^L \neq \{0\}$. Clearly, X^L is $\mathbb{Z}_2$-orientable. Let Y be the connected component of X^L containing x_0. Then $\dim Y > 0$. set $P = G/L$. We have $Y^P = X^G = \{x_0\}$. Lemma 6.2 says $Y = Y^P$ $(= \{x_0\})$, which contradicts $\dim Y > 0$. Therefore, we conclude $V^{G^{\cap 2}} = \{0\}$. □

Corollary 6.4. *Let G be an Oliver group. For an arbitrary real G-representation V in $\mathfrak{R}(G)_1$, it holds that $V^{G^{\cap 2}} = \{0\}$, i.e. $\mathfrak{R}(G)_1 \subseteq \mathfrak{R}(G)_{\{G^{\cap 2}\}}$.*

Proposition 6.5. *Let G be an Oliver group. If $G^{\{2\}}$ has odd order, then* $\mathrm{Dim}(\mathfrak{R}(G)_1) \subseteq 2\,\mathbb{N}$.

Proof. Set $L = G^{\{2\}}$. Let V be a real G-representation in $\mathfrak{R}(G)_1$. We have the orthogonal decomposition $V = V^L \oplus V_L$ as G-representations. Since $|L|$ is odd, $\dim V_L$

is even. By Corollary 6.4, we have $(V^L)^{G^{\cap 2}} = \{0\}$, which implies that $\dim V^L$ is even. Putting these together, we see that $\dim V$ is even. It shows that $\mathrm{Dim}(G,1) \subseteq 2\,\mathbb{N}$. □

Proposition 6.6. *Let G be a finite group, Q a Sylow 2-subgroup of G, p an odd prime and X a connected closed orientable G-manifold. Suppose that $Q \trianglelefteq G$, X^Q is orientable and $X^G = \{x_0\}$. Set $V = T_{x_0}(X)$. Then the equality $V^{G^{\cap p}} = \{0\}$ holds.*

Proof. Assume $V^{G^{\cap p}} \neq \{0\}$. Then there is a normal subgroup $L \triangleleft G$ such that $|G/L| = p$ and $V^L \neq \{0\}$. Since $Q \trianglelefteq L$, X^L $(= (X^Q)^{L/Q})$ is orientable (therefore $\mathbb{Z}_p$-orientable). Let Y be the connected component of X^L containing x_0. Then $\dim Y > 0$. Set $P = G/L$. Then, by Lemma 6.2 we see $Y = Y^P = \{x_0\}$, which contradicts $\dim Y > 0$. □

Recall that if X is a $\mathbb{Z}_q$-homology sphere with G-action and Q is a q-subgroup of G for a prime q then X^Q is a $\mathbb{Z}_q$-homology sphere and therefore X^Q is orientable. We obtain the following corollaries from the above proposition.

Corollary 6.7. *Let G be an Oliver group such that a Sylow 2-subgroup of G is normal in G. For an arbitrary real G-representation V in $\mathfrak{R}(G)_1$ and any prime p, the equality $V^{G^{\cap p}} = \{0\}$ holds, and therefore $\mathfrak{R}(G)_1 \subseteq \mathfrak{R}(G)_{\{G^{\cap p}\}}$.*

Corollary 6.8. *Let G be an Oliver group such that a Sylow 2-subgroup of G is normal in G. If $G^{\cap p}$ coincides with $G^{\{p\}}$ for all $p \in \mathfrak{P}(G)$, then $\mathfrak{R}(G)_1 \subseteq \mathfrak{R}(G)_{\mathscr{L}(G)}$.*

Let V_1 and V_2 be real G-representations and $\mathscr{F}$ a set of subgroups of G. We say that (V_1, V_2) is an *$\mathscr{F}$-matched pair* if $\mathrm{res}^G_H V_1$ is isomorphic to $\mathrm{res}^G_H V_2$ for every $H \in \mathscr{F}$. The next lemma is a classical result in the theory of transformation groups.

Lemma 6.9 (due to P. A. Smith and T. Petrie)**.** *Let G be a finite group, p a prime dividing $|G|$, P a p-subgroup of G, X a $\mathbb{Z}_p$-homology sphere with G-action, and x_1, $x_2 \in X^G$. If $\dim X^P > 0$ then $(T_{x_1}(X), T_{x_2}(X))$ is a $\{P\}$-matched pair.*

Proof. By the Smith theorem, X^P is a $\mathbb{Z}_p$-homology sphere. Since $\dim X^P > 0$, X^P is connected. Therefore, $\mathrm{res}^G_P T_{x_1}(X)$ is isomorphic to $\mathrm{res}^G_P T_{x_2}(X)$. □

Corollary 6.10. *Let G be a finite group, X a homology sphere with G-action, and x_1, $x_2 \in X^G$. If the inequalities $\dim X^P > 0$ hold for all $P \in \mathscr{P}(G)$, then $(T_{x_1}(X), T_{x_2}(X))$ is a $\mathscr{P}(G)$-matched pair.*

Using Lemmas 6.2 and 6.9, we obtain the next interesting result.

Lemma 6.11. *Let G be an Oliver group and N a normal subgroup of G such that $G/N = C_{p^m}$ (a cyclic group of order p^m) for a prime p and $m \in \mathbb{N}$. Let X be a homology sphere with G-action such that $X^G = \{x_0\}$. Suppose the following conditions are fulfilled.*

1. *Each element of $G \setminus N$ has p-power order.*
2. *The connected component containing x_0 of X^L is $\mathbb{Z}_p$-orientable for any $L \in \overline{\mathscr{U}}_G(N)$.*

Then it holds that $T_{x_0}(X)^N = \{0\}$.

Proof. Set $V = T_{x_0}(X)$ as a real G-representation. Assume $V^N \neq \{0\}$ (we will find a contradiction). There is a (unique) maximal element $M \in \overline{\mathscr{U}}_G(N)$ satisfying $V^M \neq \{0\}$. It is clear that $M \lhd G$, G/M is a cyclic group of order p^k for some $k \in [1..m]$ and $V^L = \{0\}$ for all $L \in \mathscr{U}_G(M)$. Let Y be the connected component of X^M with $x_0 \in Y$. By hypothesis, Y is $\mathbb{Z}_p$-orientable. If $\dim Y^L = 0$ for all $L \in \mathscr{U}_G(M)$, then Lemma 6.2 says $Y = \{x_0\}$, which shows $V^M = \{0\}$ (contradiction). Therefore, under the assumption that there is $L \in \mathscr{U}_G(M)$ satisfying $\dim Y^L > 0$, we will find a contradiction. Let K be a (unique) maximal element of $\mathscr{U}_G(M)$ such that $\dim Y^K > 0$ (obviously $M < K < G$). Let x_1 be a point in Y^K such that $\dim W^K = \dim Y^K$ for $W := T_{x_1}(X)$ (as a real K-representation). Since Y is connected, we have $\mathrm{res}^G_M V \cong \mathrm{res}^K_M W$. Let g be an element of $K \setminus M$. Since $g \in G \setminus N$, the order of g is a power of p. Let P be the cyclic group generated by g. We have $\dim X^P \geq \dim Y^P \geq \dim Y^K > 0$. By the Smith theorem, X^P is a connected $\mathbb{Z}_p$-homology sphere. Therefore we get $\mathrm{res}^G_P V \cong \mathrm{res}^G_P W$. It follows that $\mathrm{res}^G_K V \cong \mathrm{res}^G_K W$ and $\dim V^K = \dim W^K > 0$ (contradiction). □

Corollary 6.12. *Let G be an Oliver group such that $\mathfrak{P}(G) = \{p\}$ for some prime p. Set $N = G^{\{p\}}$ $(= G^{\mathrm{nil}})$. Suppose that*

1. *G/N is a cyclic group of p-power order,*
2. *each element of $G \setminus N$ has p-power order, and*
3. *a Sylow 2-subgroup Q of N is normal in N if p is odd.*

Let X be a homology sphere with G-action such that $X^G = \{x_0\}$. Then $T_{x_0}(X)^N = \{0\}$. Therefore $\mathfrak{R}(G)_1$ is contained in $\mathfrak{R}(G)_{\mathscr{L}(G)}$.

Proof. Let $L \in \overline{\mathscr{U}}_G(N)$. If p is odd then, since Q is normal in L, $X^L = (X^Q)^{L/Q}$ is orientable. If $p = 2$ then X^L is $\mathbb{Z}_2$-orientable. Therefore it follows from Lemma 6.11 that $T_{x_0}(X)^N = \{0\}$. □

7. Examples of Oliver groups with determinable δ_G

In this section, applying Lemma 2.2, we determine the integers δ_G of several interesting Oliver groups G.

Example 7.1 (Perfect group). For the special linear group $G = \mathrm{SL}(2, \mathbb{F}_q)$ with $q > 4$, $\mathrm{Dim}(G, 1)$ is cofinal with $\mathbb{N}$, i.e. $\delta_G = 1$.

Proof. Since G is a perfect group, it follows from Corollary 5.14 and Lemma 2.2 that $\mathrm{Dim}(G, 1)$ is cofinal with $\mathbb{N}$. □

Example 7.2 (Case 2 (i)). For the small group $G = SG(1176, 220)$ given by GAP, $\mathrm{Dim}(G, 1)$ is cofinal with $\mathbb{N}$, i.e. $\delta_G = 1$.

Proof. We remark that $|G| = 1176$, $\mathfrak{P}(G) = \{3\}$, $|G/G^{\{3\}}| = 3$, G has a form $G = ((C_2{}^{\times 3} \rtimes C_7) \times C_7) \rtimes C_3$ and G has a quotient group $F = (C_2{}^{\times 3} \rtimes C_7) \rtimes C_3$ and a subquotient group $K = C_2{}^{\times 3} \rtimes C_3$. Moreover K has a quotient group L isomorphic to $C_2{}^{\times 2}$. Remark that $V(G)$ is an $\mathcal{L}(G)$-free gap representation and G is a supersolvable group. Since G fits to Proposition 5.20, $d(G, \mathrm{s})$ divides 3. There is an $\mathcal{L}(G)$-free complex G-representation W of dimension 7. The realification of W is an $\mathcal{L}(G)$-free stably weak-gap representation of dimension 14. Therefore we get $d(G, \mathrm{s}) = 1$. Lemma 2.2 shows $\mathrm{Dim}(G, 1) \approx \mathbb{N}$. □

Example 7.3 (Case 2 (ii)). Let p be an odd prime, let $m \in \mathbb{N}$, let q and r be distinct odd primes such that $q - 1$ and $r - 1$ are divisible by p^m, set $P = C_{p^m}$, let $\varphi_q : P \to \mathrm{Aut}(C_q)$ and $\varphi_r : P \to \mathrm{Aut}(C_r)$ be monomorphisms, set $C_{qr} = C_q \times C_r$, set $\varphi_{qr} = \varphi_q \times \varphi_r : P \to \mathrm{Aut}(C_{qr})$, let G be the semi-direct product $(C_{qr} \times C_{qr}) \rtimes_\varphi P$, where $\varphi = \varphi_{qr} \times \varphi_{qr} : P \to \mathrm{Aut}(C_{qr} \times C_{qr})$. Then G is an Oliver group of odd order, $\mathfrak{P}(G) = \{p\}$, $G/G^{\mathrm{nil}} \cong C_{p^m}$, $G^{\mathrm{nil}} = C_{qr} \times C_{qr}$, and $\mathrm{Dim}(G, 1)$ is cofinal with $2p^m\,\mathbb{N}$, i.e. $\delta_G = 2p^m$.

Proof. It is clear that $|G|$ is odd, $\mathfrak{P}(G) = \{p\}$, $G^{\mathrm{nil}} = G^{\{p\}}$, $\mathscr{L}(G) = \overline{\mathscr{U}}_G(G^{\{p\}})$, $G/G^{\mathrm{nil}} \cong C_{p^m}$ and $G^{\mathrm{nil}} = C_{qr} \times C_{qr}$. It follows that G is an Oliver group and $V(G)$ is an $\mathscr{L}(G)$-free gap G-representation. Using Corollary 5.12, we can see that

G satisfies the hypothesis (4) of Proposition 5.9. Therefore we obtain the equality $\mathrm{Dim}(\mathfrak{Irr}_{\mathbb{C}}(G)_{\mathscr{L}(G)}) = \{p^m\}$. Since $|G|$ is odd, we obtain $\mathrm{Dim}(\mathfrak{Irr}_{\mathbb{R}}(G)_{\mathscr{L}(G)}) = \{2p^m\}$, which shows $d(G, \mathrm{s}) = 2p^m$. By Corollary 6.12, $\mathfrak{R}(G)_1 \subseteq \mathfrak{R}(G)_{\mathscr{L}(G)}$ $(= \mathfrak{R}(G)^{\mathrm{s.w.gap}}_{\mathscr{L}(G)})$. We get $\mathrm{Dim}(G, 1) \subseteq 2p^m\,\mathbb{N}$. On the other hand, Lemma 2.2 says that $\mathrm{Dim}(G, 1) \cap 2p^m\,\mathbb{N}$ is cofinal with $2p^m\,\mathbb{N}$. Therefore we conclude that $\mathrm{Dim}(G, 1)$ is cofinal with $2p^m\,\mathbb{N}$. □

Example 7.4 (Case 2 (i)). Let p, m, q, r, P, φ_q and φ_r, C_{qr}, φ_{qr} be as in Example 7.3 and let F be a nontrivial elementary abelian 2-group, i.e. $F \cong C_2 \times \cdots \times C_2$. Suppose there is a free homomorphism $\varphi_2 : P \to \mathrm{Aut}(F)$ (cf. Proposition 5.8). Let G be the semi-direct product $(C_{qr} \times C_{qr} \times F) \rtimes_{\varphi} P$, where $\varphi = \varphi_{qr} \times \varphi_{qr} \times \varphi_2 : P \to \mathrm{Aut}(C_{qr} \times C_{qr} \times F)$. Then G is a gap Oliver group of even order, $\mathfrak{P}(G) = \{p\}$, $G/G^{\mathrm{nil}} \cong C_{p^m}$, $G^{\mathrm{nil}} = C_{qr} \times C_{qr} \times F$, and $\mathrm{Dim}(G, 1)$ is cofinal with $p^m\,\mathbb{N}$, i.e. $\delta_G = p^m$.

Proof. Set $N = G^{\{p\}}$. It is clear that $\mathfrak{P}(G) = \{p\}$, $G^{\mathrm{nil}} = N$, $\mathscr{L}(G) = \overline{\mathscr{U}}_G(N)$, $G/N \cong C_{p^m}$, $N = C_{qr} \times C_{qr} \times F$ and $|N|$ is even. Furthermore F is a Sylow 2-subgroup of N (as well as of G) and normal in G. It follows that G is an Oliver group and $V(G)$ is an $\mathscr{L}(G)$-free gap G-representation. Using Proposition 5.11 (and Corollary 5.12), we can see that G satisfies the hypothesis (4) of Proposition 5.9. Therefore we obtain the equality $\mathrm{Dim}(\mathfrak{Irr}_{\mathbb{C}}(G)_{\mathscr{L}(G)}) = \{p^m\}$. We can easily see that $\mathrm{Dim}(\mathfrak{Irr}_{\mathbb{R}}(G)_{\mathscr{L}(G)}) = \{p^m, 2p^m\}$ and $\gcd(\mathrm{Dim}(\mathfrak{R}(G)_{\mathscr{L}(G)})) = p^m$. By Corollary 6.12, $\mathfrak{R}(G)_1 \subseteq \mathfrak{R}(G)_{\mathscr{L}(G)}$ $(= \mathfrak{R}(G)^{\mathrm{s.w.gap}}_{\mathscr{L}(G)})$. We get $\mathrm{Dim}(G, 1) \subseteq p^m\,\mathbb{N}$. On the other hand, Lemma 2.2 says that $\mathrm{Dim}(G, 1) \cap p^m\,\mathbb{N}$ is cofinal with $p^m\,\mathbb{N}$. Therefore we conclude that $\mathrm{Dim}(G, 1)$ is cofinal with $p^m\,\mathbb{N}$. □

Example 7.5 (Case 3 (i)). For the group $G = S_n$ with $n \geq 5$, $\mathrm{Dim}(G, 1)$ is cofinal with $\mathbb{N}$, i.e. $\delta_G = 1$.

Proof. Since G is nonsolvable, G is an Oliver group. It is known that $\mathrm{Dim}(S_5, 1)$ is cofinal with $\mathbb{N}$, see [13, 14]. Here we consider the case where $n \geq 6$. It is clear that $\mathfrak{P}(G) = \{2\}$, $G^{\{2\}} = A_n$ and $\mathscr{L}(G) = \{G, G^{\{2\}}\}$. Note that $|G| \equiv 0 \mod 4$, $|G/G^{\{2\}}| = 2$ and $|G^{\{2\}}|$ is even. Proposition 5.20 shows that $d(G, \mathrm{s}) = 1$ or 2. By the equality

$$2 + \sum_{W \in \mathfrak{Irr}_{\mathbb{C}}(G)^*_{\mathscr{L}(G)}} (\dim W)^2 = |G|,$$

there is an $\mathscr{L}(G)$-free irreducible complex G-representation U of odd dimension. It is known [4, Part I, Chapter 4] that every $\mathscr{L}(G)$-free irreducible complex G-representation of odd dimension has the real type. It says that $U = \mathbb{C}\otimes_{\mathbb{R}} W$ with some real G-representation W. Since $G = S_n$ is a gap group [1, 18], W is an $\mathscr{L}(G)$-free, stably weak-gap G-representation of odd dimension. Therefore we get $d(G, \mathrm{s}) = 1$. Lemma 2.2 says that $\mathrm{Dim}(G, 1)$ is cofinal with $\mathbb{N}$. □

Example 7.6 (Case 3 (ii)). Let m be a positive odd integer divisible by at least 2 distinct primes, and $G = D_{2m} \times D_{2m}$, where D_{2m} is a dihedral group of order $2m$. Then G is an Oliver group, $\mathfrak{P}(G) = \{2\}$, $G/G^{\mathrm{nil}} \cong C_2 \times C_2$, $G^{\mathrm{nil}} = C_m \times C_m$, where C_m is the cyclic subgroup with order m of D_{2m}, and $\mathrm{Dim}(G, 1)$ is cofinal with $2\,\mathbb{N}$, i.e. $\delta_G = 2$.

Proof. It is clear that $\mathfrak{P}(G) = \{2\}$, $G^{\mathrm{nil}} = G^{\{2\}} = G^{\cap 2} = C_m \times C_m$, $G/G^{\mathrm{nil}} \cong C_2 \times C_2$ and $\mathscr{L}(G) = \overline{\mathscr{U}}_G(G^{\cap 2})$. We can see without difficulties that G is an Oliver group. Since $G = D_{2m} \times D_{2m}$, it is easy to see that $\mathrm{Dim}(\mathfrak{Irr}_{\mathbb{C}}(G)_{\mathscr{L}(G)}) = \{2, 4\}$, $\mathrm{Dim}(\mathfrak{Irr}_{\mathbb{R}}(G)_{\mathscr{L}(G)}) = \{2, 4\}$, and $d(G, \mathrm{s}) = 2$. Lemma 2.2 says that $\mathrm{Dim}(G, 1) \cap 2\,\mathbb{N}$ is cofinal with $2\,\mathbb{N}$. On the other hand, by Corollary 6.4 we get $\mathrm{Dim}(G, 1) \subseteq 2\,\mathbb{N}$. Therefore $\mathrm{Dim}(G, 1)$ is cofinal with $2\,\mathbb{N}$. □

Example 7.7 (Case 3 (ii)). Let $m \in \mathbb{N}$, let q and r be distinct odd primes such that $q-1$ and $r-1$ both are divisible by 2^m, set $F = C_{2^m}$, let $\varphi_q : F \to \mathrm{Aut}(C_q)$ and $\varphi_r : F \to \mathrm{Aut}(C_r)$ be monomorphisms, set $C_{qr} = C_q \times C_r$ and $\varphi_{qr} = \varphi_q \times \varphi_r : F \to \mathrm{Aut}(C_{qr})$, and let $G = (C_{pq} \times C_{pq}) \rtimes_{\varphi} F$, where $\varphi = \varphi_{qr} \times \varphi_{qr} : F \to \mathrm{Aut}(C_{pq} \times C_{pq})$. Then G is an Oliver group, $\mathfrak{P}(G) = \{2\}$, $G/G^{\mathrm{nil}} \cong C_{2^m}$, $G^{\mathrm{nil}} = C_{qr} \times C_{qr}$, and $\mathrm{Dim}(G, 1)$ is cofinal with $2^m\,\mathbb{N}$, i.e. $\delta_G = 2^m$.

Proof. It is clear that $\mathfrak{P}(G) = 2$, $G^{\mathrm{nil}} = G^{\{2\}}$, $\mathscr{L}(G) = \overline{\mathscr{U}}_G(G^{\mathrm{nil}})$, $G/G^{\mathrm{nil}} \cong C_{2^m}$, $G^{\mathrm{nil}} = C_{qr} \times C_{qr}$, and G is an Oliver group. Set $N = G^{\mathrm{nil}}$ and let $K = N \rtimes_{\varphi|_{C_2}} C_2$ be the normal subgroup of G with index 2^{m-1}. Proposition 5.9 says that an arbitrary $\mathscr{L}(G)$-free irreducible complex G-representation V is isomorphic to $\mathrm{ind}_N^G W$ for some nontrivial one-dimensional complex N-representation W and $\mathrm{Dim}(\mathfrak{Irr}_{\mathbb{C}}(G)_{\mathscr{L}(G)}) = \{2^m\}$. Remark that $\mathrm{ind}_N^G W$ coincides with $\mathrm{ind}_K^G(\mathrm{ind}_N^K W)$ and there is a real K-representation of which the complexification is isomorphic to $\mathrm{ind}_N^K W$. It follows that $\mathrm{Dim}(\mathfrak{Irr}_{\mathbb{R}}(G)_{\mathscr{L}(G)}) = \{2^m\}$. Since $V \oplus (\mathrm{ind}_N^G \mathbb{R}[N]_N)^n$, where $n = \dim V$, is a weak-gap G-representation for any $V \in \mathfrak{Irr}_{\mathbb{R}}(G)_{\mathscr{L}(G)}$, each element of $\mathfrak{R}(G)_{\mathscr{L}(G)}$ is a stably

weak-gap G-representation. Therefore we get $d(G,\mathrm{s}) = 2^m$. Lemma 2.2 says that $\mathrm{Dim}(G,1) \cap 2^m\,\mathbb{N}$ is cofinal with $2^m\,\mathbb{N}$.

On the other hand, Corollary 6.12 shows the inclusion $\mathfrak{R}(G)_1 \subseteq \mathfrak{R}(G)_{\mathscr{L}(G)}$. It shows $\mathrm{Dim}(G,1) \subseteq \mathrm{Dim}(G,\mathrm{s}) \subseteq 2^m\,\mathbb{N}$. Therefore $\mathrm{Dim}(G,1)$ is cofinal with $2^m\,\mathbb{N}$. □

Example 7.8 (Cases 1–3). Let q be a prime power > 4. For $G = \mathrm{GL}(2,\mathbb{F}_q)$, $\mathrm{Dim}(G,1)$ is cofinal with $\mathbb{N}$.

Proof. Remark that $G^{\mathrm{nil}} = \mathrm{SL}(2,\mathbb{F}_q)$ is a perfect group. In addition, we have $|G| = (q-1)^2q(q+1)$ and $|G/G^{\mathrm{nil}}| = q-1$. By [25, Theorem B, Proposition 3.4] and [26, Theorem 1.4], G is a gap group. Set $d = \mathrm{Dim}(G,\mathrm{s})$. By the character table of $\mathrm{GL}(2,\mathbb{F}_q)$ (which is given in [7]), we see that there are $\mathscr{L}(G)$-free irreducible complex G-representations of dimensions $q-1$, q, and $q+1$, respectively. It implies that $d = 1$ or 2. If q is a power of 2, then $V(G)$ is odd-dimensional, which implies $d = 1$. Suppose q is a power of an odd prime. Then there is an $\mathscr{L}(G)$-free irreducible real G-representation W of dimension q. Since d divides q as well as 2, we obtain $d = 1$. Therefore Lemma 2.2 says that $\mathrm{Dim}(G,1)$ is cofinal with $\mathbb{N}$. □

8. Proofs of theorems in Section 1

We devote this section to proving Theorems 1.1, 1.2 and 1.3.

Proof of Theorem 1.1. Let G be a finite group with a normal subgroup N such that $\overline{G} = G/N$ is a nontrivial perfect group. It follows from [24, Theorem 1.2] that G is a gap group. Let φ be the canonical projection $G \to \overline{G}$. By Corollary 5.14, we have $d(\overline{G},\mathrm{s}) = 1$. Since $\varphi^*\mathfrak{R}(\overline{G})^{\mathrm{s.w.gap}}_{\mathscr{L}(\overline{G})} \subseteq \mathfrak{R}(G)^{\mathrm{s.w.gap}}_{\mathscr{L}(G)}$, we obtain $d(G,\mathrm{s}) = 1$. Lemma 2.2 says that $\mathrm{Dim}(G,1) \approx \mathbb{N}$. □

Proof of Theorem 1.2. Suppose that $|G/[G,G]|$ is divisible by two odd primes p and q. Then $\mathfrak{P}(G) \supseteq \{p,q\}$ and $V(G)$ is an $\mathscr{L}(G)$-free gap representation. Since there is a normal subgroup N of G such that G/N is a cyclic group of order pq, there is an $\mathscr{L}(G)$-free real G-representation of dimension 2. It shows that $d(G,\mathrm{s}) = 1$ or 2. Lemma 2.2 says that $\mathrm{Dim}(G,1) \cap 2\,\mathbb{N} \approx 2\,\mathbb{N}$.

Suppose that $|G|$ is odd. Then it holds that

$$\mathrm{Dim}(G,1) \subseteq \mathrm{Dim}(\mathfrak{R}(G)_{\{G\}}) \subseteq 2\,\mathbb{N}.$$

Therefore we conclude that $\mathrm{Dim}(G,1) \approx 2\,\mathbb{N}$.

Suppose that $|G|$ is even. Set $L = G^{\{p\}} \cap G^{\{q\}}$. Since $|L|$ is even, the L-representation $U := \mathbb{R}[L]_L$ is odd dimensional. It follows that $\mathrm{ind}_L^G U$ is an $\mathscr{L}(G)$-free weak-gap G-representation of odd dimension. Therefore we get $d(G, \mathrm{s}) = 1$. Lemma 2.2 says that $\mathrm{Dim}(G, 1) \approx \mathbb{N}$. □

Proof of Theorem 1.3. Let G be an Oliver group such that $\mathrm{Dim}(G, 1) \approx \delta_G \mathbb{N}$, where $\delta_G \in \mathbb{N}$. By Lemma 2.2, we see that δ_G divides $\mathrm{Dim}(G, \mathrm{s})$.

If $\mathfrak{P}(G) = \emptyset$, then G is a nontrivial perfect group and therefore $\delta_G = d(G, \mathrm{s}) = 1$.

If $|\mathfrak{P}(G)| \geq 2$, then Proposition 5.19 says that $d(G, \mathrm{s}) = 1$ or 2, and therefore $\delta_G = 1$ or 2.

Suppose $\mathfrak{P}(G) = \{r\}$ for some prime r. Then $|G/G^{\{r\}}| = r^m$ for some $m \in \mathbb{N}$. Proposition 5.20 says that $d(G, \mathrm{s})$ divides $2r^m$. Since δ_G divides $d(G, \mathrm{s})$, we get $\delta_G = 2^a$, r^b or $2r^c$ for some a, b, $c \in \overline{\mathbb{N}}$. □

9. Proofs of propositions in Section 2

In this section, we prove Propositions 2.3–2.6.

Proof of Proposition 2.3. Suppose $\mathfrak{P}(G) = \{2, p\}$, where p is an odd prime. There is a normal subgroup N of G such that $N \supseteq G^{\{2\}} \cap G^{\{p\}}$ and $|G/N| = 2p$ and there is an $\mathscr{L}(G)$-free irreducible real G-representation W of dimension 2 such that $W^N = W$, $W^{G^{\{2\}}} = \{0\}$, and $W^{G^{\{p\}}} = \{0\}$. Let $(Q, H) \in \mathscr{PH}_2(G)$. If $\dim W^H > 0$, then $\dim W^Q > 0$ and therefore $Q \subseteq N$, which contradicts the condition $QG^{\{p\}} = G$ required in the definition of $\mathscr{PH}_2(G)$. It means that the inequality $2 \dim W^H \leq \dim W^Q$ holds. Proposition 5.17 says that W is a stably weak-gap representation. Moreover we can easily see that W is $\mathscr{L}(G)$-free. Therefore $d(G, \mathrm{s}) = 1$ or 2. Lemma 2.2 says that $\mathrm{Dim}(G, 1) \cap 2\mathbb{N}$ is cofinal with $2\mathbb{N}$.

Suppose that there is an $\mathscr{L}(G)$-free, stably weak-gap real G-representation of odd dimension. Then it follows that $d(G, \mathrm{s}) = 1$. Lemma 2.2 says that $\mathrm{Dim}(G, 1)$ is cofinal with $\mathbb{N}$. □

Proof of Proposition 2.4. The proposition follows from Lemma 2.2 and Proposition 5.20. □

Proof of Proposition 2.5. Suppose $\mathfrak{P}(G) = \{2\}$. Set $L = G^{\{2\}}$ and $d = |G/L|$. Then Proposition 5.20 says that $d(G, \mathrm{s})$ divides $2\,d$ (resp. d) if $|L|$ is odd (resp. even).

(i) Subcase: $|L|$ is even. Since $d(G,\mathrm{s})$ divides d, Lemma 2.2 says that $\mathrm{Dim}(G,1) \cap d\,\mathbb{N} \approx d\,\mathbb{N}$.

Let us consider the case where there is an $\mathscr{L}(G)$-free weak-gap G-representation V of odd dimension k. Then $d(G,\mathrm{s})$ divides k and $2d$. Therefore we get $d(G,\mathrm{s}) = 1$. By Lemma 2.2, $\mathrm{Dim}(G,1)$ is cofinal with $\mathbb{N}$.

(ii) Subcase: $|L|$ is odd. Proposition 6.5 says that $\mathrm{Dim}(G,1) \subseteq 2\,\mathbb{N}$. Since $d(G,\mathrm{s})$ divides $2d$, Lemma 2.2 says that $\mathrm{Dim}(G,1) \cap 2d\,\mathbb{N} \approx 2d\,\mathbb{N}$.

Suppose that there is an $\mathscr{L}(G)$-free weak-gap complex G-representation U of odd dimension k. Let V be the realification of U. Clearly, $\dim V = 2k$. Then $d(G,\mathrm{s})$ divides $2d$ and $2k$, which implies that $d(G,\mathrm{s}) = 1$ or 2. By Lemma 2.2, $\mathrm{Dim}(G,1) \cap 2\,\mathbb{N}$ is cofinal with $2\,\mathbb{N}$. Since $\mathrm{Dim}(G,1) \subseteq 2\,\mathbb{N}$, $\mathrm{Dim}(G,1)$ is cofinal with $2\,\mathbb{N}$. □

Proof of Proposition 2.6. Let G be an Oliver group.

(i) Suppose $\mathrm{Dim}(G,1)$ is cofinal with $\mathbb{N}$. By definition, we have $\mathrm{Dim}(G,1) = \mathrm{Dim}(\mathfrak{R}(G)_1)$. Corollary 6.4 says that $\mathfrak{R}(G)_1 \subseteq \mathfrak{R}(G)_{\{G^{\cap 2}\}}$. Therefore there is $V \in \mathfrak{R}(G)_{\{G^{\cap 2}\}}$ of odd dimension. By Proposition 5.16, there is an $\mathscr{L}(G)$-free G-representation of odd dimension.

(ii) Suppose that there is a $\{G^{\cap 2}\}$-free real G-representation V of odd dimension. By Proposition 5.16, there is an $\mathscr{L}(G)$-free irreducible real G-representation W of odd dimension. Additionally suppose that G is a gap group. It implies that $\mathfrak{R}(G)_{\mathscr{L}(G)} = \mathfrak{R}(G)^{\mathrm{s.w.gap}}_{\mathscr{L}(G)}$. If $|\mathfrak{P}(G)| \geq 2$, then $d(G,\mathrm{s})$ divides 2 as well as $\dim W$, which shows $d(G,\mathrm{s}) = 1$. If $\mathfrak{P}(G) = \{2\}$, then $d(G,\mathrm{s})$ divides $2|G/G^{\{2\}}|$ as well as $\dim W$, which shows $d(G,\mathrm{s}) = 1$. If $\mathfrak{P}(G) = \emptyset$, then G is perfect and therefore $d(G,\mathrm{s}) = 1$. Proposition 2.3 says that $\mathrm{Dim}(G,1)$ is cofinal with $\mathbb{N}$.

(iii) Suppose that $\mathfrak{P}(G) = \{p\}$ for an odd prime p. Then G is a gap group. Set $N = G^{\{p\}}$. We consider the case that there is a $\{G\}$-free irreducible real G-representation V of odd dimension relatively prime to p. Then $\dim V$ divides $|G|$. Let us prove that V is $\mathscr{L}(G)$-free. Assume $V^N \neq \{0\}$. Since V is irreducible, the equality $V = V^N$ holds. Since $\dim V$ divides $2\,|G/N|$ and is odd, $\dim V = p^a$ is for some $a \in \overline{\mathbb{N}}$. Since $(\dim V, p) = 1$, we get $\dim V = 1$. Since G/N is of odd order, we get $V^G = V$ (contradiction), Therefore V is $\mathscr{L}(G)$-free. Note that $d(G,\mathrm{s})$ divides

$$\gcd(\dim V, \dim V(G)) = \gcd(\dim V, |G| - |G/N|) = \gcd(\dim V, |G/N|) = 1,$$

i.e. $d(G,\mathrm{s}) = 1$. Proposition 2.3 says that $\mathrm{Dim}(G,1)$ is cofinal with $\mathbb{N}$. □

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Appendix I: Proof of Lemma 3.1

Let G be an Oliver group. We will prove that if $P \in \mathscr{P}(G)$ satisfies $\dim V(G)^P \leq 4$ then P belongs to $\mathscr{F}(G,1)$.

We begin with the assumption that there is $P \in \mathscr{P}(G) \smallsetminus \mathscr{F}(G,1)$ satisfying $\dim V(G)^P \leq 3$. Remark 2.1 (2) says that $\dim V(G)^P = 2$ or 3. Therefore, by Remark 2.1 (2) and (3), we conclude that $P \in \max(\mathscr{M}(G))$ and $N_G(P) = P$, i.e. $P \in \mathscr{F}(G,1)$ (contradiction).

Therefore, assuming that there is $P \in \mathscr{P}(G) \smallsetminus \mathscr{F}(G,1)$ satisfying $\dim V(G)^P = 4$, we will find contradictions. Since $P \notin \mathscr{F}(G,1)$, there is $H \in \mathscr{U}_G(P) \cap \mathscr{M}(G)$. By Remark 2.1 (3), we see that $\dim V(G)^H = 2$ and $|H : P| = 2$ (therefore $H \in \max(\mathscr{M}(G))$). Since $\dim V(G)^P = 2 \dim V(G)^H$, Remark 2.1 (6) shows that $(P,H) \in \mathscr{P}\mathscr{H}_2(G)$. It implies that $\mathfrak{P}(G) = \{2\}$ or $\{2,p\}$ for some odd prime p.

In the following, we mean by G_i, $i \in [0..\infty)$, the characteristic subgroups $G_0 = G$, $G_1 = G^{\mathrm{nil}}$, $G_2 = {G_1}^{\mathrm{nil}}$, $G_3 = {G_2}^{\mathrm{nil}}$, ... of G.

Case A. Suppose $\mathfrak{P}(G) = \{2,p\}$. Since $PG^{\{p\}} = G$, p divides $|P|$. Say $|G| = 2^a p^b m$, where a, b, $m \in \mathbb{N}$ such that m is not divisible by 2 nor p. Then $|G/G^{\{2\}}| = 2^{a'}$ and $|G/G^{\{p\}}| = p^{b'}$ for some a' and $b' \in \mathbb{N}$ such that $a' \leq a$ and $b' \leq b$. Clearly, P is a Sylow p-subgroup of G and therefore $|P| = p^b$ and $|H| = 2p^b$. Note that $\dim V(G)^P = 2^a m - 2^{a'} = 4$. It implies that $m = 1$ or 3.

Subcase A.1. Suppose $m = 1$. Then $a' = 2$ and $a = 3$, and therefore $|G| = 8p^b$, $|G/G^{\{2\}}| = 4$ and $|G/G_1| = 4p^{b'}$. Burnside's theorem says that G is solvable. Since G is an Oliver group and $|G|$ is not divisible by distinct three primes, G is not nilpotent (therefore $G_2 < G_1$). Let P' be a Sylow p-subgroup of G_1. Since $|G_1 : P'| = 2$, P' is a normal subgroup of G_1. Therefore we see $P' = (G_1)^{\{2\}} < G_1$ and P' is a characteristic

subgroup of G. In addition, $P' = P \cap G_1$ and P/P' is a Sylow p-subgroup of G/P'. Note that every element of P/P' commutes in G/G_1 with all elements of the 2-group $G_{\{2\}}/G_1$, where $G_{\{2\}} = G^{\{p\}}$, and that every element of P/P' commutes with the two elements of the group G_1/P' of order 2. By elementary calculation, we can see that every element of P/P' commutes with any element of a Sylow 2-subgroup of G/P'. It means that G/P' is nilpotent, which contradicts the definition that $G_1 = G^{\mathrm{nil}}$ is the smallest normal subgroup N of G such that G/N is nilpotent.

Subcase A.2. Suppose $m = 3$. Then we see $a' = a = 1$, $|G| = 6p^b$, $p \geq 5$, $|G/G_1| = 2p^{b'}$ and $|G_1| = 3p^{b-b'}$. It is clear that G is solvable and non-nilpotent and P is a Sylow p-subgroup of G. Note that $\mathfrak{P}(G_1) = \{3\}$, $\{p\}$ or $\{3, p\}$. For a while we assume that $\mathfrak{P}(G_1) = \{3\}$ or $\{3, p\}$. Then $P' := (G_1)^{\{3\}}$ is a characteristic subgroup of G. Therefore $P' \subseteq P$ and $P' = P \cap G_1$. Since $p \geq 5$, every element of P commutes with the three elements of $(G_1)_{\{3\}}/P'$, where $(G_1)_{\{3\}} = (G_1)^{\{p\}}$. By elementary calculation, we can see that P/P' is a normal subgroup of G/P'. Therefore we get the sequence $P \triangleleft G^{\{2\}} \triangleleft G$, where $G^{\{2\}}/P \cong C_3$, which shows that G is not an Oliver group (contradiction). Now we assume that $\mathfrak{P}(G_1) = \{p\}$. Then we have $G_2 = (G_1)^{\{p\}}$ and $|G_2|$ is divisible by 3. Let k be the largest integer such that $|G_k|$ is divisible by 3. Since $|G_1/G_k|$ is a power of p, we see $k = 2$. We observe $G_3 \triangleleft G_2 \triangleleft G_1 \triangleleft G$. Note that $G_1/(G_2)^{\{3\}} \cong C_3 \rtimes P''$, where P'' is a p-group. Since the conjugation action of P'' on C_3 is trivial, $G_1/G_2^{\{3\}}$ is nilpotent, which contradicts the definition of G_2 $(= (G_1)^{\mathrm{nil}})$.

Case B. Suppose $\mathfrak{P}(G) = \{2\}$.

Subcase B.1. Suppose $P = E$. Say $G = 2^a m$, where m is an odd integer ≥ 3. Then we have $\dim V(G)^P = 2^a m - 2^{a'} = 2^{a'}(2^{a-a'}m - 1) = 4$. It implies that $a' = 1$, $a = a'$ and $m = 3$. Therefore we get $|G| = 6$, which shows that G is not an Oliver group (contradiction).

Subcase B.2. Suppose that there is an odd prime p dividing $|P|$. Say $G = 2^a p^b m$, where m is an odd positive integer not divisible by p. Remark that P is a Sylow p-subgroup of G. Then we have $\dim V(G)^P = 2^a m - 2^{a'} = 2^{a'}(2^{a-a'}m - 1) = 4$. It implies that $a' = 1$, $a = a'$ and $m = 3$, and therefore $|G| = 6p^b$. Clearly G is solvable, and furthermore non-nilpotent. Assume that $\mathfrak{P}(G_1) = \{p\}$. Then the sequence $G_3 \triangleleft G_2 \triangleleft G_1 \triangleleft G$ satisfies that $|G/G_1| = 2$, $|G_1/G_2| = p^{b'}$, $3 \in \mathfrak{P}(G_2)$ and $|G_3| = p^{a''}$, where $a'' \geq 0$. Since m $(= 3)$ is not divisible by p, we see that $p \geq 5$. Clearly, $P' := (G_2)^{\{3\}}$ is a characteristic p-subgroup of G and therefore $P \supseteq P'$. Note

$G_1/P' \cong C_3 \rtimes P''$, where $P'' = P/P'$. Since the conjugation action by elements of P'' on C_3 is trivial, G_1/P' is nilpotent, which contradicts the definition of G_2.
Subcase B.3. Suppose that P is a nontrivial 2-group. Then H is a Sylow 2-subgroup of G. Say $|G| = 2^a m$, where $a \in [2..\infty)$ and m is an odd integer in $[3..\infty)$. Then it holds that $\dim V(G)^P = 2m - 2 = 4$, which shows $m = 3$ and $|G| = 2^a 3$. Therefore G is solvable and non-nilpotent. It is easy to see that $\mathfrak{P}(G_1) \ni 3$ and G is not an Oliver group (contradiction).

We have completed the proof of Lemma 3.1.

Appendix II: Proof of Lemma 3.2

Let G be an Oliver group and V a preferable real G-representation. In this appendix, we prove that there is a sphere Σ with effective one-fixed-point G-action such that $T_{x_0}(\Sigma) \cong V$, where $\Sigma^G = \{x_0\}$. For a G-space X, let $\mathrm{Iso}(G, X)$ denote the set of all isotropy subgroups G_x at x in X, where x runs over X. Since G is an Oliver group, there is $\beta \in \Omega(G)$ satisfying (3.1). Let Y be the (closed) unit disk $D(V)$ with respect to a G-invariant inner-product on V. Let y_0 denote the origin of V and I the closed unit interval $[0, 1]$ in $\mathbb{R}$. As in [9, Section 2], [11, Section 4], there is a G-framed map $\boldsymbol{f} = (f, b)$, where

$$f : (X, \partial X) \to (Y, \partial Y)$$

is a degree-one G-map and

$$b : \varepsilon_X(\mathbb{R}) \oplus T(X) \oplus \varepsilon_X(\mathbb{R}^{2m}) \to \varepsilon_X(\mathbb{R}) \oplus f^*T(Y) \oplus \varepsilon_X(\mathbb{R}^{2m})$$

is a G-vector bundle isomorphism for some $m \in \mathbb{N}$, such that $\partial f = f|_{\partial X}$ coincides with $id_{\partial Y}$ and X^G is the empty set. In addition, for each $M \in \max(\mathscr{M}(G))^*$, there is an M-framed cobordism $\boldsymbol{F}_M = (F_M, B_M)$ rel. ∂ from $\mathrm{res}^G_M \boldsymbol{f}$ to $\mathrm{res}^G_M \boldsymbol{id}_Y$, where

$$F_M : (W_M, \partial_0 W_M, \partial_1 W_M, \partial_{01} W_M) \to \mathrm{res}^G_M(Z, \partial_0 Z, \partial_1 Z, \partial_{01} Z),$$

$Z = I \times Y$, $\partial_0 Z = \{0\} \times Y$, $\partial_1 Z = \{1\} \times Y$, $\partial_{01} Z = I \times \partial Y$, and

$$B_M : T(W_M) \oplus \varepsilon_{W_M}(\mathbb{R}^{2m}) \to F_M^* \mathrm{res}^G_M T(Z) \oplus \varepsilon_{W_M}(\mathbb{R}^{2m})$$

satisfying the conditions (1)–(4) in [13, p. 502]. For each $H \in \mathscr{M}(G)^*$ such that $\dim V^H > 0$, we fix an orientation $\mathfrak{o}(Y^H)$ of Y^H and give the induced orientation $\mathfrak{o}(X^H)$ from $\mathfrak{o}(Y^H)$ by b^H. Then the mapping-degree of $f^H : (X^H, \partial X^H) \to (Y^H, \partial Y^H)$ is one for $H \in \mathscr{M}(G)^*$ such that $\dim V^H > 0$. The condition (A3) implies

that

$$\mathscr{P}(G) \subseteq \mathscr{G}^1(G) \subseteq \mathrm{Iso}(G, V) \smallsetminus \{G\} \subseteq \mathrm{Iso}(G, X).$$

Moreover, we can suppose without loss of generality that

(Q1) for each $H \in \mathscr{S}(G)^* \smallsetminus \{G\}$, $f^{-1}(y_0)^H$ contains points $a_{H,+}$ and $a_{H,-}$ with isotropy subgroup H satisfying the following conditions.

(Q1.1) For each $a_{H,\epsilon}$, where $\epsilon \in \{+,-\}$, f is a diffeomorphism on a neighborhood of $a_{H,\epsilon}$ to a neighborhood of y_0.

(Q1.2) For each $a_{H,\epsilon}$ with $\dim V^H = 0$, where $\epsilon \in \{+,-\}$, $df_{a_{H,\epsilon}} : T_{a_{H,\epsilon}}(X) \to T_{y_0}(Y)$ is orientation-preserving (resp. orientation-reversing) if $\epsilon = +$ (resp. $-$).

(Q1.3) For each $a_{H,\epsilon}$ with $\dim V^H > 0$, where $\epsilon \in \{+,-\}$, $df_{a_{H,\epsilon}} : T_{a_{H,\epsilon}}(X) \to T_{y_0}(Y)$ and $df^H_{a_{H,\epsilon}} : T_{a_{H,\epsilon}}(X^H) \to T_{y_0}(Y^H)$ both are orientation-preserving (resp. orientation-reversing) if $\epsilon = +$ (resp. $-$).

The hypothesis implies

(Q2) $\mathrm{Iso}(G, X) = \mathscr{S}(G) \smallsetminus \{G\} \supseteq \mathrm{Iso}(G, \beta)$.

Let $(-\beta)^{\%}$ be an element of $\Omega(G)$ having the form

$$(-\beta)^{\%} = \sum\nolimits_{H \in \mathrm{Iso}(G,\beta)^*} c_H[G/H] \quad (c_H \in \mathbb{N})$$

with

$$c_H \equiv -b_H \mod 2|G||\widetilde{K}_0(\mathbb{Z}[G])|$$

for all $H \in \mathrm{Iso}(G, \beta)^*$. For $\omega \in \Omega(G)$ having the form

$$\omega = \sum\nolimits_{H \in \mathscr{S}(G)^* \smallsetminus \{G\}} d_H[G/H] \quad (d_H \in \mathbb{Z}),$$

we use the notation $([G/G] + \omega)X$ and $([G/G] + \omega)f$ for the G-connected sums $(1 \dashv \omega)X$ and $(1 \dashv \omega)f$ in [11, p. 116], respectively.

Step 1. We will modify $\boldsymbol{f}$ so that

(Q3) $f^M : (X^M, \partial X^M) \to (Y^M, \partial Y^M)$ is homotopic rel. ∂ to a diffeomorphism

for all $M \in \mathscr{F}(G, 1)$ such that $\dim V^M > 0$.

Claim 1. *Let $M \in \mathscr{F}(G,1)^*$ such that $\dim V^M > 0$. Then $\boldsymbol{f}$ can be modified, by G-surgeries on $\boldsymbol{f}$ rel. ∂ of isotropy type $(M)_G$ so that $\boldsymbol{f}$ satisfies* (Q3) *(therefore X^M is diffeomorphic to Y^M).*

Proof. Recall that $N_G(M)/M$ is the trivial group, $X^{>M}$ is a finite set, and W_M^M is a cobordism from X^M to Y^M rel. ∂. By G-surgeries on $\boldsymbol{f}$ rel. ∂ of isotropy type $(M)_G$, we can suppose without loss of generality that X^M is connected. If $\dim V^M = 1$ then X^M is diffeomorphic to the disk D^1 of dimension 1 and $\boldsymbol{f}$ satisfies (Q3). If $\dim V^M = 2$ then X^M is diffeomorphic to an orientable connected surface with boundary S^1. Therefore, by G-surgeries on $\boldsymbol{f}$ rel. ∂ of isotropy type $(M)_G$, we can modify $\boldsymbol{f}$ so that $\boldsymbol{f}$ satisfies (Q3).

Now suppose $\dim V^M \geq 3$. We can modify $\boldsymbol{F}_M$ so that W_M^M is connected. Note that $\dim(I \times X^{>M}) = 1$ and $\dim W_M^M \geq 4$. There is an embedding $\psi : I \times X^{>M} \to W_M^M \smallsetminus \partial_{01} W_M$ such that $\psi|_{\{0\}\times X}$ is the canonical inclusion to $\partial_0 W_M$ and $F_M \circ \psi(t,x) = (t, g(t,x))$, where $g(t,x) \in Y^M$, for all $t \in I$ and $x \in X^{>M}$. By the reflection method (see Proof of [13, Lemma 6.1]) (therefore by G-surgeries) we can modify $\boldsymbol{f}$ so that $\boldsymbol{f}$ satisfies (Q3). □

Step 2. We will modify $\boldsymbol{f}$ so that

(Q4) $\chi(X^H) = 1$ for all $H \in \mathscr{G}^1(G) \smallsetminus (\mathscr{F}(G,1) \cup \mathscr{P}(G))$.

If $\dim V^H$ is odd, then we have $\chi(X^H \cup_\partial Y^H) = 0$ and $\chi(X^H \cup_\partial Y^H) = \chi(X^H) + \chi(Y^H) - 2$, and therefore $\chi(X^H) = 1$.

Let $\{H_1, \cdots, H_k\}$ be the set of subgroups $H \in \mathscr{G}^1(G)^* \smallsetminus (\mathscr{F}(G,1) \cup \mathscr{P}(G))$ such that $\dim V^H$ is even (≥ 2). We can assume without loss of generality that if $H_i \supseteq gH_jg^{-1}$ holds for some i, $j \in [1..k]$ with $i \neq j$ and $g \in G$, then $i < j$. Suppose that i is the smallest integer in $[1..k]$ such that $\chi(X^{H_i}) \neq 1$.

Claim 2. *Set $\boldsymbol{f}' = ([G/G] + (-\beta)^{\%})\boldsymbol{f}$. Then $\boldsymbol{f}'$ can be modified by G-surgeries on $\boldsymbol{f}'$ rel. ∂ of isotropy type $(H_i)_G$ so that $\chi(X'^H) = 1$ for all $H = H_j$, $j \in [1..i]$, where $X' = ([G/G] + (-\beta)^{\%})X$.*

Proof. As in [11, pp. 117, 118], we get $\chi(X'^{H_i}) \equiv 1 \mod 2|G|$. Say $\dim V^{H_i} = 2u$ with $u \in \mathbb{N}$. Then $\chi(S^{2u}) = 2$ and $\chi(S^{1\times 2u} \# S^{1\times 2u}) = -2$, where $S^{1\times 2u} = S^1 \times \cdots \times S^1$ ($2u$-fold). Therefore, we can modify $\boldsymbol{f}'$ by G-surgeries on $\boldsymbol{f}'$ rel. ∂ of isotropy type $(H_i)_G$ so that $\chi(X'^{H_i}) = 1$. □

Hereafter we suppose that $\boldsymbol{f}$ satisfies (Q4).

Step 3. Next we will modify $\boldsymbol{f}$ so that

(Q5) X^P is 1-connected and $f : X^P \to Y^P$ is a $\mathbb{Z}_{(p)}$-homology equivalence

for any $P \in \mathscr{P}(G) \smallsetminus (\mathscr{F}(G,1) \cup \{E\})$, where p is the prime dividing $|P|$ and

$$\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \in \mathbb{Q} \;\middle|\; a \in \mathbb{Z},\, b \in \mathbb{N} \text{ such that } b \text{ is not divisible by } p \right\}.$$

Let $\{P_1, \cdots, P_{k'}\}$ be the set of subgroups $P \in \mathscr{P}(G)^* \smallsetminus (\mathscr{F}(G,1) \cup \{E\})$. We can assume without loss of generality that if $P_i \supseteq gP_jg^{-1}$ holds for some $i, j \in [1..k']$ with $i \neq j$ and $g \in G$, then $i < j$. Suppose that i is the smallest integer in $[1..k']$ such that (Q5) is not satisfied for $P = P_i$. By G-surgeries on $\boldsymbol{f}$ rel. ∂ of isotropy type $(P_i)_G$, we can suppose without loss of generality that $f^{P_i} : X^{P_i} \to Y^{P_i}$ is $[n/2]$-connected, where $n = \dim V^{P_i}$ and $[n/2]$ is the greatest integer not exceeding $n/2$. Let t be the integer such that $|G| = 2^t u$ for some odd integer u.

Claim 3. *Set $\boldsymbol{f}' = ([G/G] - \beta)^{2t+3}\boldsymbol{f}$. Then $\boldsymbol{f}'$ can be modified, by G-surgeries on $\boldsymbol{f}'$ rel. ∂ of isotropy type $(P_i)_G$, so that $\boldsymbol{f}$ replaced with $\boldsymbol{f}'$ satisfies* (Q5) *for all $P = P_j$, $j \in [1..i]$.*

The proof is same as [11, Section 5, Case 2]. Therefore we can suppose without loss of generality that $\boldsymbol{f}$ satisfies (Q5) for all $P \in \mathscr{P}(G) \smallsetminus (\mathscr{F}(G,1) \cup \{E\ \})$.
Step 4. We will modify $\boldsymbol{f}$ so that X is contractible. We may suppose that $f : X \to Y$ is $[n/2]$-connected, where $n = \dim V$. Let t be the integer such that $|G| = 2^t u$ for some odd integer u.

Claim 4. *Set $\boldsymbol{f}' = ([G/G] + (-\beta)^{\%})\boldsymbol{f}$ and $\boldsymbol{f}'' = ([G/G] - \beta)^{2t+3}\boldsymbol{f}'$. Then $\boldsymbol{f}''$ can be modified, by G-surgeries on $\boldsymbol{f}''$ rel. ∂ of isotropy type $(E)_G$, so that X'' is contractible, where $\boldsymbol{f}'' = (f'', b'')$ and $f'' : (X'', \partial X'') \to (Y, \partial Y)$.*

The proof is same as [11, Section 5, Case 3]. Therefore we can suppose without loss of generality that $f : (X, \partial X) \to (Y, \partial Y)$ is a homotopy equivalence satisfying $\partial f = id_{\partial Y}$.
Step 5. Let X be a disk with fixed-point-free G-action such that $\partial X = \partial Y$ $(= \partial D(V))$. It is clear that the G-action on X is effective. Then $\Sigma_0 = X \cup_\partial Y$ is a homotopy sphere with G-action and $\Sigma_0^G = Y^G = \{y_0\}$. Note that $\max(\mathscr{P}(G))$ is the set of all Sylow subgroups of G. Clearly the greatest common divisor d of the set $\{|G/P| \mid P \in \max(\mathscr{P}(G))^*\}$ is one. Theorem 1.1 of [12] says that by taking

appropriately G-connected sum

$$\begin{aligned}\Sigma_0 \#_{G,(P_1)_G} (G/P_1 \times \Sigma_1) \#_{G,(P_1)_G} \cdots \#_{G,(P_1)_G} (G/P_1 \times \Sigma_{r_1}) \\ \#_{G,(P_2)_G} (G/P_2 \times \Sigma_1) \#_{G,(P_2)_G} \cdots \#_{G,(P_2)_G} (G/P_2 \times \Sigma_{r_2}) \\ \cdots\cdots\cdots\cdots\cdots \\ \#_{G,(P_n)_G} (G/P_n \times \Sigma_1) \#_{G,(P_n)_G} \cdots \#_{G,(P_n)_G} (G/P_n \times \Sigma_{r_n}),\end{aligned}$$

where Σ_i's are copies of Σ_0 and $\{P_1, \ldots, P_n\} = \max(\mathscr{P}(G))^*$, we obtain a sphere Σ with one-fixed-point G-action such that $T_{x_0}(\Sigma) \cong V$, where $\Sigma^G = \{x_0\}$.

We have finished the proof of Lemma 3.2.

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