MORSE THEORY AND THE HOMOLOGY OF LAGRANGIAN GRASSMANNIANS

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Abstract. We apply Morse homology theory to compute the integral homology groups of the Grassmannians $\Lambda(n)$ and $\widetilde{\Lambda}(n)$ of unoriented and oriented Lagrangian planes in the real $2n$ dimensional symplectic space. It is shown that all torsion coefficients of homology are 2, and thus $H_k(\Lambda(n);\mathbb{Z}) \cong \mathbb{Z}^{a(k)} \oplus {\mathbb{Z}_2}^{b(k)}$ and $H_k(\widetilde{\Lambda}(n);\mathbb{Z}) \cong \mathbb{Z}^{\tilde{a}(k)} \oplus {\mathbb{Z}_2}^{\tilde{b}(k)}$. The exponents $a(k)$, $b(k)$, $\tilde{a}(k)$ and $\tilde{b}(k)$ are expressed explicitly in terms of generating function.

1. Introduction

Morse theory provides useful methods for extracting the information about topology of manifolds from the context of the local behaviors of smooth functions. A fundamental idea is to construct a chain complex, called *Morse complex*. If the gradient flow satisfies certain transversality condition, the gradient lines between two critical points with Morse indices of difference one are isolated. The Morse complex is a chain complex whose chain group is generated by critical points and whose boundary map is defined by counting isolated gradient lines with sign (see Section 2). For a finite dimensional closed manifold, the homology group of the Morse complex is isomorphic to the singular homology group of the manifold (for the proof, see [2], [3] and the references given there). The idea of Morse complex, whose origins can be traced to the works of R. Thom, S. Smale and E. Witten (see, for instance, [4]), has been expanded by A. Fleor [5], K. Fukaya [6] and others.

A Morse function is called *perfect* if the boundary map of its Morse complex is a zero homomorphism. In that case, all the critical points represent independent homology cycles. For instance, if a Morse function has no critical points of odd indices, the homology group of the manifold can be fully read by counting the critical points, which is called *Morse lacuna principle*, and played a crucial role in the discovery of Bott's periodicity theorem. On the other hand, the Grassmannians of both unoriented and oriented planes in the real vector space are examples of manifolds which do not admit perfect Morse functions, because they have torsions in the homology groups. Nevertheless, they admit Morse functions whose Morse complexes have a fairly simple structure, and allow us to compute matrix representations of the boundary maps (see [8] and [9]).

The Grassmannians $\Lambda(n)$ and $\widetilde{\Lambda}(n)$ of unoriented and oriented Lagrangian planes in the real symplectic spaces play important roles in classical mechanics, geometric optics and quantum mechanics, etc. Their topological structures have been deeply studied by several authors (see, for example, V. Arnol'd [1] and A. Vassiliev [10]).

Although $\Lambda(n)$ and $\widetilde{\Lambda}(n)$ do not admit perfect Morse functions, there exist functions on them whose Morse complexes faithfully reflect the topological structure of the manifolds. The function introduced in Section 3 is one such example. We analyze the Morse complex of this function, and using combinatorial method, we

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compute the homology group, and finally obtain the generating functions of the free and torsion ranks of the integral homology groups of $\Lambda(n)$ and $\widetilde{\Lambda}(n)$ in explicit form (see Theorems 8.3 and 9.7).

2. Overview of the Morse homology theory

Consider a Morse function f of a closed Riemannian manifold. If the stable manifolds $W^s(x)$ and unstable manifolds $W^u(y)$ intersect transversally for all critical points x and y; $W^s(x) \pitchfork W^u(y)$, the function f is called a *Morse-Smale* function. In that case, a gradient line is isolated if the end points have the Morse indices differing by one. If the manifold is closed there are a finite number of critical points and a finite number of isolated gradient lines. In order to define an integral Morse complex of a Morse-Smale function, we fix an orientation for each unstable manifold, which also fix a co-orientation of the stable manifold. According to the orientations of unstable manifolds we define the sign of isolated gradient lines.

Definition 2.1. Let γ be a gradient line between critical points x and y with Morse index k and $k-1$ respectively, and take a local transversal slice $\mathcal{S}$ of γ at the midpoint $\gamma(0)$ in the unstable manifold $W^u(x)$. Let $\{v_1, \cdots, v_{k-1}\}$ be a basis of the tangent space $T_{\gamma(0)}\mathcal{S}$ such that $\{\frac{d\gamma}{dt}(0), v_1, \cdots, v_{k-1}\}$ is an oriented basis of $T_{\gamma(0)}W^u(x)$. We define the *sign* $\mathrm{sgn}(\gamma) = 1$ or -1 according to whether or not the order of vectors $\{v_1, \cdots, v_{k-1}\}$ agrees with the co-orientation of the stable manifold $W^s(y)$.

Denote by $\mathrm{Crt}_k(f)$ the set of all critical points of f with Morse index k, and by $C(f)_k$ the $\mathbb{Z}$-module formally generated over $\mathrm{Crt}_k(f)$. Let $\nu(x,y)$ denote the number of isolated gradient lines γ from x to y counted with sign;

$$\nu(x,y) = \sum \mathrm{sgn}(\gamma),$$

and define a homomorphism $\partial_k : C(f)_k \to C(f)_{k-1}$ by

$$\partial_k \langle x \rangle = \sum_{y \in \mathrm{Crt}_{k-1}(f)} \nu(x,y) \langle y \rangle .$$

The graded complex $\mathcal{MC}_f = (C_* = \bigoplus_k C(f)_k, \partial_* = \bigoplus_k \partial_k)$ is called the *Morse complex* of f. For the proof of the following theorem, see for instance [2, Ch. 3] and [3].

Theorem 2.2. *Let $f : M \to \mathbb{R}$ be a Morse-Smale function of a closed Riemannian manifold M. Then the map ∂_* defined above satisfies ${\partial_*}^2 = 0$, and the homology group $H_*(\mathcal{MC}_f)$ is isomorphic to the singular homology group $H_*(M;\mathbb{Z})$.*

Remark 2.3. The Smale transversality requires $W^s(p) \pitchfork W^u(q)$ for all pairs (p,q) of critical points of f. On the other hand, the *partial transversality* requires $W^s(p) \pitchfork W^u(q)$ only for pairs of critical points with Morse indices of difference one. We remark that, in order to define the Morse complex, it is sufficient that the gradient vector field satisfies the partial transversality. This is ensured by perturbation argument of gradient vector fields, since the existing transversality continue to hold in sufficiently small perturbations.

3. Definition of the Morse function f

3.1. **An embedding of the Lagrangian Grassmannian.** Let $\mathrm{Mat}(k,\ell)$ be the space of $k\times\ell$ matrices with real entries. The complex structure J of $\mathbb{R}^{2n}$, and the subspace $E(n)\subset\mathrm{Mat}(2n,2n)$ are defined by

$$J=\begin{pmatrix}0 & -I_n\\ I_n & 0\end{pmatrix},\qquad E(n)=\{A\,;\,A^T=A,\ AJ+JA=0\},\tag{1}$$

where $I_n\in\mathrm{Mat}(n,n)$ is the unit matrix. Using the linear symplectic form ω on $\mathbb{R}^{2n}$ defined by $\omega(u,v)=-u^TJv$, we regard $\mathbb{R}^{2n}$ a linear symplectic space.

Define a submanifold $\mathcal{L}(n)$ by

$$\mathcal{L}(n)=\{X\in E(n)\,;\,X^2=1\}.$$

The eigenvalues of $X\in\mathcal{L}(n)$ are ±1, and both eigenspaces are Lagrangian planes, and conversely for each Lagrangian plane λ, there exists a unique $X\in\mathcal{L}(n)$ which has λ as the eigenspace of eigenvalue 1. Hence the Lagrangian Grassmannian $\Lambda(n)$ is diffeomorphic to $\mathcal{L}(n)$. Using this diffeomorphism, we identify $\Lambda(n)$ and $\mathcal{L}(n)$, and endow $\Lambda(n)$ with a Riemannian structure by the inner product $(A,B)=\mathrm{tr}(AB)$ of $E(n)$.

For each element $Y\in\mathcal{L}(n)$, the line $\varepsilon\mapsto Y+\varepsilon A$ is tangent to $\mathcal{L}(n)$ at $\varepsilon=0$ if and only if $YA+AY=0$. The decomposition

$$E(n)=\{A\in E(n)\,;\,AY+YA=0\}\oplus\{A\in E(n)\,;\,AY-YA=0\}$$

is an orthogonal decomposition, and the first subspace $\{A\in E(n)\,;\,AY+YA=0\}$ coincides with the tangent space $T_Y\mathcal{L}(n)$. Hence the map

$$A\mapsto\frac{1}{2}(A-YAY)$$

is the orthogonal projection of a vector $A\in E(n)$ to the tangent space $T_Y\mathcal{L}(n)$.

3.2. **The Morse function and its gradient lines.** Let $D\in E(n)$ be a diagonal matrix whose diagonal entries are $-d_1,\cdots,-d_n,\,d_1,\cdots,d_n$, which satisfy $0<d_1<\cdots<d_n$;

$$D=\mathrm{diag}(-d_1,\cdots,-d_n,d_1,\cdots,d_n).\tag{2}$$

Throughout the paper, we will investigate the Morse complex of the following function $f:\Lambda(n)\cong\mathcal{L}(n)\to\mathbb{R}$:

$$f(X)=2\,\mathrm{tr}(XD).\tag{3}$$

The gradient vector field $X\mapsto\nabla_f(X)$ of f is the orthogonal projection of $-D$ to the tangent space $T_X\mathcal{L}(n)$, thus the gradient flow equation is

$$\frac{dX}{dt}=\nabla_f(X)=-D+XDX.\tag{4}$$

Lemma 3.1. *X is a critical point of f if and only if X is a diagonal matrix with diagonal entries $x_{1,1},\cdots,x_{2n,2n}$ satisfying $x_{n+i,n+i}=-x_{i,i}=\pm1$ for all $i=1,\cdots,n$.*

Proof. Decompose $X=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\mathcal{L}(n)$ into $n\times n$ matrices. The condition that X is in $E(n)$ implies that a and b are symmetric and $a+d=0=b-c$. Further, the equation $-D+XDX=0$ implies that a is diagonal and that $b=0$. Since $X^2=1$, the diagonals of a are ±1. □

Given a matrix $X_0 \in \mathcal{L}(n)$, define a curve $t \mapsto X(t) = -Y(t)Z(t)^{-1}$ by setting

$$\begin{cases} Y(t) = \big(\exp(tD)(I - X_0) - \exp(-tD)(I + X_0)\big)/2, \\ Z(t) = \big(\exp(tD)(I - X_0) + \exp(-tD)(I + X_0)\big)/2. \end{cases} \tag{5}$$

Lemma 3.2. *The curve $t \mapsto X(t) = -Y(t)Z(t)^{-1}$ is the gradient line of f with initial value $X(0) = X_0$.*

Proof. Since $Y(0) = -X_0$ and $Z(0) = I$, we have $X(0) = X_0$. The derivatives of Y and Z by t equal $Y' = DZ$ and $Z' = DY$ respectively, and $(Z^{-1})' = -Z^{-1}DYZ^{-1}$. Thus we have

$$X' = -DZZ^{-1} + YZ^{-1}DYZ^{-1} = -D + XDX = \nabla_f(X).$$

□

Proposition 3.3. *The map $\varphi : \mathbb{R} \to GL(2n, \mathbb{R})$ defined by $\varphi(t) = \exp(-tD)$ induces an action of $\mathbb{R}$ on $\Lambda(n)$ which generates the integral of the gradient flow ∇_f. In other words, if $V(t)$ is an integral curve of the gradient vector field ∇_f with initial value $V(0) = V$, then $V(t)$ equals the subspace of $\mathbb{R}^{2n}$ given by $\{\exp(-tD)v\,;\, v \in V\}$.*

Proof. Let Y and Z be the curves given in (5), and v an eigenvector of $X_0 \in \mathcal{L}(n)$ with eigenvalue 1. Then it holds that $Zv = \exp(-tD)v$. The equation

$$(YZ^{-1} + I)Z = Y + Z = \exp(tD)(I - X_0)$$

implies that $(I - X)Z = \exp(tD)(I - X_0)$. Therefore, the equation $(I - X_0)v = 0$ implies

$$0 = (I - X)Zv = (I - X)\exp(-tD)v.$$

Hence, if $X(t)$ is a gradient line in $\mathcal{L}(n)$, and if v is an eigenvector of $X(0) = X_0$ with eigenvalue 1, $\exp(-tD)v$ is contained in the eigenspace of $X(t)$ of eigenvalue 1 for all $t \in \mathbb{R}$. □

4. Ends of gradient lines

In this section, we set up notation and terminology to describe the critical points and isolated gradient lines of the Morse function $f : \Lambda(n) \cong \mathcal{L}(n) \to \mathbb{R}$ defined in (3).

4.1. **A partial order of the power set of $\{1, \cdots, n\}$.** Denote by $[n]$ the set $\{1, \cdots, n\}$, and for each $j \in [n]$, denote by $[j, n]$ the set $\{j, \cdots, n\}$. Let σ be a subset of $[n]$. Denote by $|\sigma|$ and $\|\sigma\|$ the numbers defined by

$$|\sigma| = \sum_{a \in \sigma} 1, \qquad \|\sigma\| = \sum_{a \in \sigma} a.$$

For each $k \in [n]$, using the symbol such as $\sigma_{<k}$, we represent the set $\sigma \cap [k-1]$.

Definition 4.1 (Partial order in $2^{[n]}$)**.** Define a partial order ">" in the power set $2^{[n]}$ by letting

$$\sigma > \tau \quad \text{if and only if} \quad |\sigma_{>j}| \geqslant |\tau_{>j}| \text{ for all } j.$$

In other words, if $\sigma = \{a_1, \cdots, a_{|\sigma|}\}$ and $\tau = \{b_1, \cdots, b_{|\tau|}\}$ such that $a_1 > \cdots > a_{|\sigma|}$ and $b_1 > \cdots > b_{|\tau|}$, the relation $\sigma > \tau$ means that $|\sigma| > |\tau|$ and $a_i \geqslant b_i$ for all $i = 1, \cdots, |\tau|$.

If a subset $\sigma \in 2^{[n]}$ contains 1, we call $(\sigma, \sigma\backslash\{1\})$ a *Type I pair*. If a subset σ contains k and does not contain $k-1$, then we call $(\sigma, (\sigma\backslash\{k\}) \cup \{k-1\})$ a *Type II pair* with difference at k.

Lemma 4.2. *Suppose $\|\sigma\| = \|\tau\| + 1$. The relation $\sigma > \tau$ holds if and only if (σ, τ) is a Type I or a Type II pair.*

Proof. Let $\sigma = \{a_1, \cdots, a_{|\sigma|}\}$ and $\tau = \{b_1, \cdots, b_{|\tau|}\}$, and suppose that $a_1 > \cdots > a_{|\sigma|}$ and $b_1 > \cdots > b_{|\tau|}$. We have $1 = \|\sigma\| - \|\tau\| \geqslant \sum_{i=1}^{|\tau|}(a_i - b_i) \geqslant 0$, which implies that either $|\sigma| = |\tau| + 1$, $a_i = b_i$ for all $i \in [|\sigma|]$ and $a_{|\sigma|} = 1$ (i.e. (σ, τ) is a Type I pair), or $|\sigma| = |\tau|$ and there exists a unique $k \in [|\sigma|]$ such that $a_k = b_k + 1$ and $a_i = b_i$ for all $i \in [|\sigma|]\backslash\{k\}$ (i.e. (σ, τ) is a Type II pair). □

Type I and II pairs are visualized by Maya diagram and by Type I and II moves in a Maya diagram.

Definition 4.3 (Maya diagram)**.** Arrange n boxes in a single row and number them from left to right as 1 through n. Given a subset $\sigma \in 2^{[n]}$, place one ball in the k^{th} box if $k \in \sigma$. Otherwise, leave the box empty. We call this sequence of boxes the *Maya diagram* representing σ.

Consider the following moves:

Type I move : Remove the ball from the leftmost box,

Type II move : Move a ball to the empty box on the left.

It is clear that moves of Type I and II correspond to pairs (σ, τ) of Type I and II, respectively. It is easy to see that if $\sigma > \tau$ then the Maya diagram of τ is obtained from that of σ by a sequence of Type I and Type II moves.

4.2. **Critical points of the function f.** Let $\{e_1, \cdots, e_{2n}\}$ be the canonical basis of $\mathbb{R}^{2n}$. For each $\sigma \in 2^{[n]}$, denote by λ_σ the coordinate plane spanned by vectors e_i with $i \in [n]\backslash\sigma$ and e_{j+n} with $j \in \sigma$;

$$\lambda_\sigma = \operatorname{span}\{e_i\,;\, i \in [n]\backslash\sigma \text{ or } i - n \in \sigma\}.$$

Lemma 4.4. *The set of critical points of $f : \Lambda(n) \to \mathbb{R}$ defined in (3) is given by*

$$\operatorname{Crt}(f) = \{\lambda_\sigma\,;\, \sigma \in 2^{[n]}\}.$$

Proof. Let X be a matrix obtained in Lemma 3.1. The 1-eigenspace of X equals λ_σ for some $\sigma \subset [n]$. □

We define the complete flag $\mathcal{F} = \{F_i\}$ of subspaces $F_i \subset \mathbb{R}^{2n}$ as follows: let $F_0 = \{0\}$ and

$$F_i = \begin{cases} F_{i-1} \oplus \mathbb{R}e_{n-i+1} & \text{for } i = 1, \cdots, n \\ F_{i-1} \oplus \mathbb{R}e_i & \text{for } i = n+1, \cdots, 2n. \end{cases} \tag{6}$$

Let $\mathcal{F}^\perp = \{(F_i)^\perp\}$ be the opposite flag, where $(F_i)^\perp$ is the orthogonal complement of F_i with respect to the canonical inner product of $\mathbb{R}^{2n}$.

Given any $\lambda \in \Lambda(n)$, let $\{\alpha_i\}$ and $\{\beta_i\}$ be the sequences of indices defined by

$$\alpha_i = \min\{j\,;\, \dim(\lambda \cap F_j) = i\}, \quad \beta_i = \min\{j\,;\, \dim(\lambda \cap (F_j)^\perp) = n - i\}, \tag{7}$$

and σ and τ the subsets of $[n]$ defined by

$$\sigma = \{\alpha_i - n\,;\, \alpha_i > n\}, \quad \tau = \{\beta_i - n\,;\, \beta_i > n\}. \tag{8}$$

We notice that the sequences $\{\alpha_i\}$ and $\{\beta_i\}$ are increasing with respect to i, and that

$$\{\alpha_i\,;\, \alpha_i \leqslant n\} = [n]\backslash\sigma \quad \text{and} \quad \{\beta_i\,;\, \beta_i \leqslant n\} = [n]\backslash\tau$$

hold provided λ is a Lagrangian plane.

Lemma 4.5. *Let γ be the gradient line with $\gamma(0) = \lambda$, and σ and τ defined by λ in (8). Then*

(1) $\gamma_\lambda(-\infty) = \lambda_\sigma$ *and* $\gamma_\lambda(\infty) = \lambda_\tau$,

(2) $\sigma > \tau$.

Proof. Let $\{\alpha_i\}$ and $\{\beta_i\}$ be the sequences obtained in (7) for λ. There exists a basis $\{v_i\}$ of λ consisting of vectors $v_i = (v_{i,1}, \cdots, v_{i,2n})$ such that $v_{i,\alpha_i} = 1$, and $v_{i,j} = 0$ for all j with $F_j \subset F_{\alpha_i}$. Then, by Proposition 3.3, we have

$$\lim_{t\to-\infty} \|\exp(-tD)v_i\|^{-1} \exp(-tD)v_i = \begin{cases} e_i & \text{if } i \in [n]\backslash\sigma, \\ e_{i+n} & \text{if } i \in \sigma. \end{cases} \tag{9}$$

and find the limits $\gamma(-\infty) = \lambda_\sigma$. The limit $\gamma(\infty) = \lambda_\tau$ is found similarly.

For any subspace $\mu \subset \mathbb{R}^{2n}$, it holds that

$$\dim(\mu \cap F_j) + \dim(\mu \cap (F_j)^\perp) \leqslant \dim \mu.$$

This implies $\alpha_i \geqslant \beta_i$ for all i with $\beta_i > n$, from which the inequality $\sigma > \tau$ follows. □

5. Local coordinate systems centered at critical points

Given $\sigma \subset [n]$, define the affine space $\mathrm{Mat}_\sigma \subset \mathrm{Mat}(n, 2n)$ consisting of matrices $(X\,Y)$ with $X = (x_{i,j})$ and $Y = (y_{i,j}) \in \mathrm{Mat}(n, n)$ that satisfy

(1) if $j \in [n]\backslash\sigma$, then $x_{i,j} = \delta_{i,j}$ for all $i \in [n]$, and $y_{i,j} = \begin{cases} y_{j,i} & \text{for all } i \in [n]\backslash\sigma, \\ -x_{j,i} & \text{for all } i \in \sigma, \end{cases}$

(2) if $j \in \sigma$, then $y_{i,j} = \delta_{i,j}$ for all $i \in [n]$, and $x_{i,j} = \begin{cases} x_{j,i} & \text{for all } i \in \sigma, \\ -y_{j,i} & \text{for all } i \in [n]\backslash\sigma, \end{cases}$

where $\delta_{i,j}$ is the Kronecker delta.

Define the index sets L_σ and $R_\sigma \subset [n] \times [n]$ by

$$L_\sigma = \{(i,j) \in [n] \times [n]\backslash\sigma \mid i \leqslant j\}, \quad R_\sigma = \{(i,j) \in [n] \times \sigma \mid i \leqslant j\}. \tag{10}$$

We fix an affine isomorphism from Mat_σ to $\mathbb{R}^{n(n+1)/2}$ defined by extracting entries $x_{i,j}$ with $(i,j) \in L_\sigma$ and $y_{i,j}$ with $(i,j) \in R_\sigma$ from $(X\,Y) = ((x_{i,j})\,(y_{i,j}))$.

Define the map $\Psi_\sigma : Mat_\sigma \to \Lambda(n)$ by

$$\Psi_\sigma \ni (X\,Y) \mapsto \mathrm{span}\{v_i\} \in \Lambda(n),$$

where $\mathrm{span}\{v_i\}$ is the subspace of $\mathbb{R}^{2n}$ spanned by the row vectors v_i of $(X\,Y)$, and denote by U_σ the image of Ψ_σ.

Lemma 5.1. *The map $\Psi_\sigma : \mathrm{Mat}_\sigma \longrightarrow U_\sigma$ is bijective, and the map $\Phi_\sigma : U_\sigma \xrightarrow{\Psi_\sigma^{-1}} \mathrm{Mat}_\sigma \xrightarrow{\cong} \mathbb{R}^{n(n+1)/2}$ is a local chart of $\Lambda(n)$ centered at λ_σ. It holds that $\bigcup_\sigma U_\sigma = \Lambda(n)$.*

The proof and a detailed explanation of the above lemma is found in [1, Section 3.3]. See also [7].

The transition map $\varphi_{\sigma\tau}$ on $U_\sigma \cap U_\tau$

$$\begin{array}{ccc} & U_\sigma \cap U_\tau & \\ \Psi_\sigma^{-1} \swarrow & & \searrow \Psi_\tau^{-1} \\ \mathrm{Mat}_\sigma & \xrightarrow{\varphi_{\sigma\tau}} & \mathrm{Mat}_\tau \end{array}$$

is obtained by elementary row operations to matrices in Mat_σ and Mat_τ.

5.1. **Stable and unstable manifolds.** Let v_i be the i^{th} row of $(X\,Y) \in \mathrm{Mat}_\sigma$. Suppose $y_{k,\ell} = 0$ for all $(k,\ell) \in R_\sigma$. Then $\lim_{t\to-\infty} \|\exp(-tD)v_i\|^{-1} \exp(-tD)v_i$ is found in (9), and the same is applied for L_σ, which proves the following :

Lemma 5.2. *In the neighborhood U_σ, the unstable manifold $W^u_\sigma = W^u(\lambda_\sigma)$ and stable manifold $W^s_\sigma = W^s(\lambda_\sigma)$ of the critical point λ_σ are determined by*

$$W^u_\sigma = \{y_{k,\ell} = 0 \quad \textit{for all } (k,\ell) \in R_\sigma\},$$
$$W^s_\sigma = \{x_{i,j} = 0 \quad \textit{for all } (i,j) \in L_\sigma\},$$

and the functions $x_{i,j}$'s with $(i,j) \in L_\sigma$ and $y_{k,\ell}$'s with $(k,\ell) \in R_\sigma$ form coordinate systems of W^u_σ and W^s_σ, respectively.

Corollary 5.3. *The Morse index of λ_σ equals $\|\sigma\|$.*

Proof. The Morse index is the dimension of the unstable manifold W^u_σ, which equals the cardinality of L_σ and equals $\|\sigma\|$. □

Example 5.4. As an example, consider the case that $n = 3$ and $\sigma = \{1,3\}$. We have

$$(X\,Y) = \begin{pmatrix} x_{1,1} & 0 & x_{1,3} & 1 & y_{1,2} & 0 \\ -y_{1,2} & 1 & x_{2,3} & 0 & y_{2,2} & 0 \\ x_{1,3} & 0 & x_{3,3} & 0 & -x_{2,3} & 1 \end{pmatrix} \in \mathrm{Mat}_{\{1,3\}}\,.$$

The elements $x_{1,1}$, $x_{1,3}$, $x_{2,3}$, $x_{3,3}$, $y_{1,2}$ and $y_{2,2}$ form a coordinate system of $U_{\{1,3\}}$ in $\mathcal{L}(3)$. Among these, $y_{1,2}$ and $y_{2,2}$ form the coordinate system of the stable manifold $W^s_{\{1,3\}}$, and the remaining elements form the coordinate system of the unstable manifold $W^u_{\{1,3\}}$.

5.2. **Isolated gradient lines.** Let γ be a gradient line such that $\gamma(\infty) = \lambda_\tau$ and $\gamma(-\infty) = \lambda_\sigma$. Lemmas 4.2, 4.4 and 4.5 imply that (σ,τ) is of Type I or II. Conversely, the following holds:

Lemma 5.5. *If (σ,τ) is of Type I or Type II, then there exist two gradient lines between λ_σ and λ_τ.*

Proof. Suppose that (σ,τ) is of Type I. Let γ be a gradient line between λ_σ and λ_τ. Then the sequences $\{\alpha_i\}$ and $\{\beta_i\}$ obtained in (7) coincide except at one position $\alpha_k = \beta_k + 1 = n+1$, which implies that the Lagrangian plane $\gamma(0)$ is spanned by $n-1$ vectors of the standard basis $\{e_i\}$ of $\mathbb{R}^{2n}$ and a vector $ae_1 + be_{n+1}$. Hence there exist two gradient lines; one when a and b have the same sign, and the other when they have different signs. Under the mapping Ψ^{-1}_σ, the image of a gradient line $\gamma(t)$ between λ_σ and λ_τ is a diagonal matrix;

$$\Psi^{-1}_\sigma(\gamma(t)) = \left(\begin{array}{ccc|ccc} s & & & 1 & & \\ & \ddots & & & \ddots & \\ & & & & & \end{array}\right), \tag{11}$$

where s is a function of t, whose derivative is non vanishing and $\lim_{t\to-\infty} s = 0$, so that $\Psi^{-1}_\sigma(\gamma(-\infty)) = \lambda_\sigma$.

Next, suppose (σ,τ) is of Type II. In this case, the plane $\gamma(0)$ is still spanned by $n-1$ vectors of $\{e_i\}$ and a vector $ae_{k-1} + be_k - be_{n+k-1} + ae_{n+k}$. There exist two gradient lines; one when a and b have the same sign, and the other when different signs. In this case, the images of gradient lines $\gamma(t)$ between λ_σ and λ_τ

under Ψ_σ^{-1} are block diagonal matrices;

$$\Psi_\sigma^{-1}(\gamma(t)) = \left(\begin{array}{ccc|ccc} \ddots & & & \ddots & & \\ & 1 \quad s & & & 0 \quad 0 & \\ & 0 \quad 0 & & & -s \quad 1 & \\ & & \ddots & & & \ddots \end{array}\right) \begin{matrix} \\ k-1 \\ k \\ \\ \end{matrix} \tag{12}$$

where s is similar to that in (11). □

6. The signs of isolated gradient lines

Consider a function $f : M \to \mathbb{R}$ on a closed manifold of dimension n that satisfies partial transversality condition (see Remark 2.3). Let γ be an isolated gradient line of f whose limits $\gamma(-\infty)$ and $\gamma(\infty)$ have the Morse indices k and $k-1$, respectively. Suppose that the unstable manifolds of $\gamma(-\infty)$ and $\gamma(\infty)$ are endowed with orientations. Let $\{x_1, \cdots, x_k\}$ and $\{y_1, \cdots, y_n\}$ be coordinate systems respectively of the unstable manifold $W^u(\gamma(-\infty))$ and of a neighborhood $U(\gamma(\infty))$ in M such that $y_1 = \cdots = y_{k-1} = 0$ is the defining equation of the stable manifold $W^s(\gamma(\infty))$ in a neighborhood of $\gamma(0)$, compatible with the co-orientation of $W^s(\gamma(\infty))$ in this order.

Assume that γ coincides with the coordinate line of x_m, in other words, $\frac{dx_m(\gamma(t))}{dt}$ is nonzero and $x_j(\gamma(t))$ is constant in t for all $j \neq m$. A local transversal slice $\eta : \mathbb{R}^{k-1} \to W^u(\gamma(-\infty))$ of γ at $\gamma(0)$ is given by

$$z = (z_1, \cdots, z_{k-1}) \mapsto \eta(z) = (z_1, \cdots, z_{m-1}, x_m(\gamma(0)), z_m, \cdots, z_{k-1}).$$

In order to check whether or not the basis $\{\dot\gamma(0), \eta_*\big(\frac{\partial}{\partial z_1}\big), \cdots, \eta_*\big(\frac{\partial}{\partial z_{k-1}}\big)\}$ of $T_{\gamma(0)}W^u(\gamma(-\infty))$ is positively oriented (see Definition 2.1), the following three factors s_1, s_2 and s_3 will be checked:

- $s_1 = 1$ or -1 depending on whether $\frac{dy_m(\gamma(t))}{dt}|_{t=0}$ is positive or negative,
- $s_2 = (-1)^{m-1}$, where m is the index of the coordinate x_m that describes the gradient line,
- $s_3 = 1$ or -1 depending on whether the sign of the Jacobian $\det \frac{\partial(y_1, \cdots, y_{k-1})}{\partial(z_1, \cdots, z_{k-1})}$ is positive or negative.

Then the sign $\mathrm{sgn}(\gamma)$ equals the product $\mathrm{sgn}(\gamma) = s_1 s_2 s_3$ (see Definition 2.1).

6.1. **Orientation of unstable manifolds.**

Let σ be a subset of $[n]$, and consider the neighborhood U_σ of λ_σ defined in Section 5. The matrix entries $x_{i,j}$ with $(i,j) \in L_\sigma$ form a local coordinate system of the unstable manifold W^u_σ (see Lemma 5.2), and thus a linear order of the index set L_σ gives an orientation of W^u_σ.

Definition 6.1. Define the orientation of unstable manifold W^u_σ by choosing the *dictionary ordering* of the index set L_σ; for (i_1, j_1) and $(i_2, j_2) \in L_\sigma$, we define $(i_1, j_1) < (i_2, j_2)$ if $i_1 < i_2$ or if $i_1 = i_2$ and $j_1 < j_2$.

6.2. **Computation of the signs.**

Let (σ, τ) be a pair of Type I. Consider the matrix representation Ψ_σ^{-1} of $\lambda \in U_\sigma$. We denote by $\gamma_\pm$ the gradient line with the midpoint $\gamma_\pm(0)$ represented by the matrix with $s = \pm 1$ in (11). For a Type II pair (σ, τ), we denote by $\gamma_\pm$ the gradient line with the midpoint $\gamma_\pm(0)$ represented by the matrix with $s = \pm 1$ in (12).

Proposition 6.2. *If (σ,τ) is of Type I, then*

$$\operatorname{sgn}(\gamma) = \begin{cases} 1 & \text{for } \gamma = \gamma_+, \\ (-1)^{|\sigma|} & \text{for } \gamma = \gamma_-. \end{cases} \tag{13}$$

If (σ,τ) is of Type II, then

$$\operatorname{sgn}(\gamma) = \begin{cases} (-1)^{\|\sigma_{<k-1}\|+(k-1)|\sigma_{>k}|} & \text{for } \gamma = \gamma_+, \\ (-1)^{\|\sigma_{<k-1}\|+(k-1)|\sigma_{>k}|+k-1} & \text{for } \gamma = \gamma_-. \end{cases} \tag{14}$$

6.2.1. *Proof of Formula (13).* We use the local coordinate system $(x_{i,j}, y_{k,\ell})$ in Section 5. Then the coordinate $x_{1,1}$ represents the gradient line $\gamma_\pm$ between λ_σ and λ_τ. A local slice $\mathcal{S}$ of the gradient line $\gamma_\pm$ at the midpoint $\gamma_\pm(0)$ in W^u_σ is given by equating $x_{1,1} = \pm 1$. The following matrices show the transition map $\varphi_{\sigma\tau} : \mathrm{Mat}_\sigma \to \mathrm{Mat}_\tau$ between local charts (U_σ, Φ_σ) and (U_τ, Φ_τ) restricted to W^u_σ:

$$\left(\begin{array}{cc|cc} \pm 1 & u & 1 & 0 \\ * & A & 0 & B \end{array}\right) \overset{\varphi_{\sigma\tau}}{\longmapsto} \left(\begin{array}{cc|cc} 1 & \pm u & \pm 1 & 0 \\ 0 & A' & * & B' \end{array}\right), \tag{15}$$

where the matrix on the right is obtained by row operation from that on the left. Note that all entries of A' are the same as A modulo second order terms.

Since we have $x_{1,1}(\gamma_\pm(0)) = \pm 1$ and thus $\pm \frac{x_{1,1}(\gamma_\pm(t))}{dt} > 0$, the term s_1 equals ± 1. Among the index set L_σ, $(1,1)$ is the first with respect to the dictionary order. Thus we have $s_2 = 1$. For γ_+, we easily see $s_3 = 1$. For γ_-, we have $-u$ in the first row of $\Psi_\tau^{-1}(\lambda)$ (the matrix on the right of (15)). The vector u contains $|\sigma| - 1$ coordinates of $\mathcal{S}$, thus we have $s_3 = (-1)^{|\sigma|-1}$, completing the proof of Formula (13). □

6.2.2. *Proof of Formula (14).* We continue to use the local coordinate system $(x_{i,j}, y_{k,\ell})$ in Section 5. Let (σ,τ) be of Type II, and $\tau = (\sigma\backslash\{k\}) \cup \{k-1\}$.

The coordinate of the gradient line $\gamma = \gamma_\pm$ is $x_{k-1,k}$. A local slice $\mathcal{S}$ of the gradient line $\gamma_\pm$ at the midpoint $\gamma_\pm(0)$ in W^u_σ is given by equating $x_{k-1,k} = \pm 1$. The following matrices show the transition map $\varphi_{\sigma\tau} : \mathrm{Mat}_\sigma \to \mathrm{Mat}_\tau$ between local charts (U_σ, Φ_σ) and (U_τ, Φ_τ) restricted to W^u_σ:

$$\left(\begin{array}{cccc|cccc} A & 0 & u & B & D & 0 & 0 & 0 \\ * & 1 & \pm 1 & v & * & 0 & 0 & 0 \\ * & 0 & x_{k,k} & w & * & \mp 1 & 1 & 0 \\ * & 0 & * & C & * & * & 0 & E \end{array}\right) \overset{\varphi_{\sigma\tau}}{\longmapsto} \left(\begin{array}{cccc|cccc} A' & \mp u & 0 & B' & D' & 0 & 0 & 0 \\ * & x_{k,k} & 0 & \mp w' & * & 1 & \mp 1 & 0 \\ * & \mp 1 & 1 & \pm v & * & 0 & 0 & 0 \\ * & * & 0 & C' & * & 0 & * & E' \end{array}\right), \tag{16}$$

where the matrix on the right is obtained from that on the left by row operation.

As in the case of Type I, the factor s_1 equals 1 if $\gamma = \gamma_+$, or -1 if $\gamma = \gamma_-$. The numbers of coordinate functions of U_σ contained in A, u, B, v and w of (16) are $\|\sigma_{<k-1}\|$, $k-2$, $(k-2)|\sigma_{>k}|$, $|\sigma_{>k}|$ and $|\sigma_{>k}|$, respectively. The coordinate functions preceding $x_{k-1,k}$ with respect to the dictionary order of L_σ are contained in A, u and B. Thus $s_2 = (-1)^{\|\sigma_{<k-1}\|+(k-2)+(k-2)|\sigma_{>k}|}$.

There is a permutation of coordinates $(v, x_{k,k}, w) \mapsto (x_{k,k}, w, v)$, whose sign is $+1$. There are changes of signs for u, v and w. The coordinates in u and w change sign for γ_+, and the coordinates in v change sign for γ_-, which implies that s_3 is equal to $(-1)^{k-2+|\sigma_{>k}|}$ if $\gamma = \gamma_+$, or equal to $(-1)^{|\sigma_{>k}|}$ if $\gamma = \gamma_-$. Thus we obtain $\operatorname{sgn}(\gamma) = s_1 s_2 s_3$ as shown in (14), and complete the proof. □

Example 6.3. As in Example 5.4, consider the case where $n = 3$ and $\sigma = \{1,3\}$, and let $\tau = \{1,2\}$. The pair $(\{1,3\},\{1,2\})$ is of Type II. We compute the sign of the gradient line γ_+ between $\lambda_{\{1,3\}}$ and$\lambda_{\{1,2\}}$.

The entries $(x_{1,1}, x_{1,3}, x_{2,3}, x_{3,3})$ and $(y_{1,2}, y_{2,2})$ are coordinate systems of the unstable manifold $W^u_{\{1,3\}}$ and the stable manifold $W^s_{\{1,3\}}$. The coordinate function $x_{2,3}$ corresponds to the gradient line between the critical points $\lambda_{\{1,3\}}$ and $\lambda_{\{1,2\}}$. Let $\gamma = \gamma_+$ be the gradient line with midpoint on $x_{2,3} = 1$. In this case, it holds that $s_1 = s_2 = 1$.

The transition map $\varphi_{\{1,3\},\{1,2\}} : \mathrm{Mat}_{\{1,3\}} \to \mathrm{Mat}_{\{1,2\}}$

$$\left(\begin{array}{ccc|ccc} x_{1,1} & 0 & x_{1,3} & 1 & y_{1,2} & 0 \\ -y_{1,2} & 1 & x_{2,3} & 0 & y_{2,2} & 0 \\ x_{1,3} & 0 & x_{3,3} & 0 & -x_{2,3} & 1 \end{array}\right) \mapsto \left(\begin{array}{ccc|ccc} z_{1,1} & z_{1,2} & 0 & 1 & 0 & w_{1,3} \\ z_{1,2} & z_{2,2} & 0 & 0 & 1 & w_{2,3} \\ -w_{1,3} & -w_{2,3} & 1 & 0 & 0 & w_{3,3} \end{array}\right)$$

is obtained by row operation. The unstable manifold $W^u_{\{1,3\}}$ and the stable manifold $W^s_{\{1,2\}}$ are defined by $y_{1,2} = y_{2,2} = 0$ and $z_{1,1} = z_{1,2} = z_{2,2} = 0$, respectively.

A local slice of γ_+ at the midpoint $\gamma_+(0)$ in $W^u_{\{1,3\}}$ is given by the map

$$\xi : (x_{1,1}, x_{1,3}, x_{3,3}) \mapsto \left(\begin{array}{ccc|ccc} x_{1,1} & 0 & x_{1,3} & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ x_{1,3} & 0 & x_{3,3} & 0 & -1 & 1 \end{array}\right).$$

Applying row operation to this matrix, we obtain the transition map $\varphi_{\{1,3\},\{1,2\}} \circ \xi$ whose Jacobian is equal to

$$\det \frac{\partial(z_{1,1}, z_{1,2}, z_{2,2})}{\partial(x_{1,1}, x_{1,3}, x_{3,3})} = \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -1,$$

which implies that $s_3 = -1$, and $\mathrm{sgn}(\gamma_+) = s_1 s_2 s_3 = -1$.

The following is a consequence of the proof of Proposition 6.2 (see Remark 2.3):

Corollary 6.4. *The gradient flow satisfies the partial transversality condition, that is, W^u_σ and W^s_τ intersect transversally for all pairs (σ, τ) with $\|\sigma\| = \|\tau\| + 1$.*

Proof. In the proof of Proposition 6.2, in order to find the value s_3, we have computed the Jacobian $\det \frac{\partial(y_1, \cdots, y_{k-1})}{\partial(z_1, \cdots, z_{k-1})}$, whose nondegeneracy implies the transversality of the stable and unstable manifolds. □

Let $\nu(\sigma, \tau)$ denote the coefficient of the boundary map $\partial : \mathbb{Z}\langle \mathrm{Crt}(f) \rangle \to \mathbb{Z}\langle \mathrm{Crt}(f) \rangle$;

$$\partial \langle \lambda_\sigma \rangle = \sum_{\|\tau\| = \|\sigma\| - 1} \nu(\sigma, \tau) \langle \lambda_\tau \rangle.$$

Proposition 6.5.

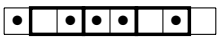
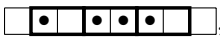
$$\nu(\sigma, \tau) = \begin{cases} 2, & \text{if } (\sigma, \tau) \text{ is of type I and } |\sigma| \text{ is even,} \\ (-1)^{\|\sigma < j-1\|} 2, & \text{if } (\sigma, \tau) \text{ is of type II with difference at odd } j, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. There exist isolated gradient lines between λ_σ and λ_τ, provided (σ, τ) is of Type I or Type II (see Lemma 4.2 and 4.5). Thus the coefficient $\nu(\sigma, \tau)$ equals 0 for other pairs. By Lemma 5.5, for pairs (σ, τ) of Type I and II, there exist two gradient lines $\gamma_\pm$ between λ_σ and λ_τ, and the coefficient $\nu(\sigma, \tau)$ is the sum $\mathrm{sgn}(\gamma_+) + \mathrm{sgn}(\gamma_-)$. The formula of the proposition is deduced from Proposition 6.2. □

7. Morse complex $\mathcal{MC}_f$

7.1. **Colored Maya diagram.** In a Maya diagram, treat each pair of even-numbered box and the box immediately to its right as a *domino*. A Maya diagram of length n has $\lfloor (n-1)/2 \rfloor$ dominos. Depending on whether the number of balls in a domino is 0, 1 or 2, assign it a color *white*, *gray* or *black*. When n is even, a Maya diagram has the rightmost single box, to which assign a color *white* or *black* depending on whether it contains no ball or one ball. This assigns to a Maya diagram a *color sequence*.

Example 7.1. The subsets $\sigma = \{1,3,4,5,7\}$ and $\tau = \{2,4,5,6\} \in 2^{[8]}$ are represented by the Maya diagrams

 and ,

and have the same color sequence $c =$ (gray, black, gray, white). They satisfy $\sigma > \tau$ and $\|\sigma\| = \|\tau\| + 3$. The Maya diagram of τ is obtained from that of σ by two Type II moves, each in a gray domino, and one Type I move.

7.2. **Decomposition of the Morse complex $\mathcal{MC}_f$.** By Proposition 6.5, the coefficient $\nu(\sigma,\tau)$ of boundary map is zero if τ is obtained by a Type II move of the ball in an even numbered box of σ. In other words, if the movement of balls go beyond the dominos, we have $\nu(\sigma,\tau) = 0$. Conversely, for a Type II move with $\nu(\sigma,\tau) \neq 0$, σ and τ share a common color sequence.

Let I_c be the subset consisting of $\sigma \in 2^{[n]}$ with color sequence c. Then $2^{[n]}$ is the disjoint union of those subsets; $2^{[n]} = \bigcup_c I_c$. Proposition 6.5 shows that the Morse complex $\mathcal{MC}_f$ decomposes as a direct sum into subcomplexes

$$\mathcal{MC}_f = \bigoplus_c \mathcal{C}_c, \tag{17}$$

and that the homology group of $\mathcal{MC}_f$ is the direct sum of the homology groups of these subcomplexes.

We say that the parity of a color sequence c is *even* or *odd* according to whether $|\sigma_{>1}|$ is even or odd for some σ with color sequence c. Let σ_c be the element $\sigma \in I_c$ with the maximum $\|\sigma\|$.

Proposition 7.2. *Let c be a color sequence. Suppose the parity of c is even, and let j be the number of gray dominos in c. If $j = 0$, then*

$$H_{\|\sigma_c\|-k}(\mathcal{C}_c) = \begin{cases} \mathbb{Z} & \text{for } k = 0,1, \\ 0 & \text{otherwise,} \end{cases} \tag{18}$$

and if $j > 0$, then

$$H_{\|\sigma_c\|-k-1}(\mathcal{C}_c) = \begin{cases} {\mathbb{Z}_2}^{\binom{j}{k}} & \text{for } k = 0,\cdots,j, \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

Suppose that the parity of c is odd, and that c has j gray dominos ($j \geqslant 0$). Then

$$H_{\|\sigma_c\|-k-1}(\mathcal{C}_c) = \begin{cases} {\mathbb{Z}_2}^{\binom{j+1}{k}} & \text{for } k = 0,\cdots,j+1, \\ 0 & \text{otherwise.} \end{cases} \tag{20}$$

Proof. Let p be a positive integer, and $\mathcal{B}_p$ the chain complex defined by

$$\mathcal{B}_p : 0 \longrightarrow \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \longrightarrow 0.$$

By Proposition 6.5, a Type II move of a ball in a domino corresponds to $\mathcal{B}_2$. A Type I move of a ball corresponds to $\mathcal{B}_2$ if $|\sigma_{>1}|$ is even, and to $\mathcal{B}_0$ if $|\sigma_{>1}|$ is odd. Then it is straightforward to verify that the complex $\mathcal{C}_c$ is isomorphic to $\mathcal{B}_p \otimes {\mathcal{B}_2}^{\otimes j}$ with $p = 0$ or 2, up to a shift of degree. Their homology groups are easily computed by the Künneth formula. □

Example 7.3. Figure 1 shows a diagram of subcomplex $\mathcal{C}_c$ which is isomorphic to $\mathcal{B}_0 \otimes {\mathcal{B}_2}^{\otimes 2}$. The solid line arrows connect Type II pairs, and the broken line arrows connect Type I pairs. The color sequence c of $\{1, 3, 5\}$ is (gray,gray), and the parity of c is even. We have $H_8(\mathcal{C}_c) = H_6(\mathcal{C}_c) = \mathbb{Z}_2$ and $H_7(\mathcal{C}_c) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

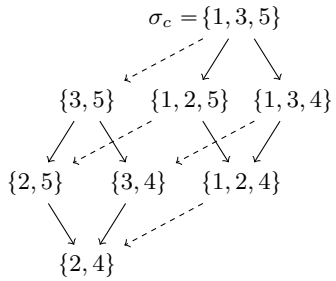


FIGURE 1

8. The homology group $H_k(\Lambda(n); \mathbb{Z})$

Proposition 7.2 shows that all the torsions of $H_*(\Lambda(n); \mathbb{Z}) = \bigoplus_c H_*(\mathcal{C}_c)$ are equal to 2. Let $a(k)$ and $b(k)$ be the free and 2-torsion ranks of the k dimensional integral homology group $H_k(\Lambda(n); \mathbb{Z})$; $H_k(\Lambda(n); \mathbb{Z}) \cong \mathbb{Z}^{a(k)} \oplus {\mathbb{Z}_2}^{b(k)}$. Let $A(x)$ and $B(x)$ be the generating functions

$$A(x) = \sum a(k)x^k, \quad B(x) = \sum b(k)x^k.$$

Lemma 8.1.

$$A(x) = (1 + x) \prod_{\ell=1}^{m} (1 + x^{4\ell+1}),$$

where $m = \lfloor (n - 1)/2 \rfloor$.

Proof. Let $\mathcal{A}$ be the subset of $2^{[n]}$ defined by

$$\mathcal{A} = \{\sigma \in 2^{[n]} ; \sigma \text{ has no gray dominos, and } |\sigma_{>1}| \text{ is even } \}.$$

Formula (18) implies $A(x) = \sum_{\sigma \in \mathcal{A}} x^{\|\sigma\|}$. Factorization of $\sum_{\sigma \in \mathcal{A}} x^{\|\sigma\|}$ is completed through combinatorial arguments concerning colored Maya diagrams: the factor $1 + x^{4\ell+1}$ is derived from the ℓ^{th} domino of $\sigma \in \mathcal{A}$, which is either white (giving the term 1) or black (giving the term $x^{4\ell+1}$), but not gray, since σ is in $\mathcal{A}$. Furthermore, $A(x)$ has the factor $1 + x$, since each $\sigma \in \mathcal{A}$ may or may not contain 1. This establishes the formula of the lemma. □

The formula for $B(x)$ can be obtained in the same way as $A(x)$ by combinatorial argument with formulas (19) and (20). It is also possible to obtain it through $\mathbb{Z}_2$-perfectness of the Morse function f combined with the universal coefficient theorem, since all torsion coefficients of $\mathcal{MC}_f$ are 2 (see Equations (19) and (20)).

Lemma 8.2. *The $\mathbb{Z}_2$-Poincaré polynomial of $\Lambda(n)$ equals $\prod_{i=1}^{n}(1+x^i)$.*

Proof. All coefficients of the boundary maps of the Morse complex $\mathcal{MC}_f$ are zero modulo 2 by Proposition 6.5, and thus the boundary map of $\mathcal{MC}_f \otimes \mathbb{Z}_2$ is trivial. In other words, all the critical points of f represent independent $\mathbb{Z}_2$-cycles. Hence, the combinatorial equation $\sum_{\sigma\in 2^{[n]}} x^{\|\sigma\|} = \prod_{i=1}^{n}(1+x^i)$ yields the formula. □

The universal coefficient theorem is applied to obtain

$$H_k(\Lambda(n);\mathbb{Z}_2) \cong \big(H_k(\Lambda(n);\mathbb{Z})\otimes\mathbb{Z}_2\big) \oplus \mathrm{Tor}(H_{k-1}(\Lambda(n);\mathbb{Z}),\mathbb{Z}_2),$$

which implies

$$\prod_{i=1}^{n}(1+x^i) = \big(A(x)+B(x)\big) + xB(x).$$

This together with Lemmas 8.1 and 8.2 proves the following:

Theorem 8.3. *The integral homology group of the Lagrangian Grassmannian $\Lambda(n)$ has only 2-torsions;*

$$H_k(\Lambda(n);\mathbb{Z}) \cong \mathbb{Z}^{a(k)} \oplus {\mathbb{Z}_2}^{b(k)}.$$

The generating functions $A(x) = \sum a(k)x^k$ and $B(x) = \sum b(k)x^k$ are given by

$$A(x) = (1+x)\prod_{\ell=1}^{m}(1+x^{4\ell+1}), \quad B(x) = \prod_{\ell=2}^{n}(1+x^\ell) - \prod_{\ell=1}^{m}(1+x^{4\ell+1}),$$

where $m = \lfloor (n-1)/2 \rfloor$.

9. Oriented Lagrangian Grassmannians

The Grassmannian $\widetilde{\Lambda}(n)$ of oriented Lagrangian planes in $\mathbb{R}^{2n}$ has the obvious two-sheeted covering map $\pi : \widetilde{\Lambda}(n) \to \Lambda(n)$. Suppose that $\widetilde{\Lambda}(n)$ is endowed with a Riemannian metric such that π is a local isometry. Let $\mathcal{MC}_{\widetilde{f}}$ be the Morse complex of the function $\widetilde{f} = f\circ\pi : \widetilde{\Lambda}(n) \to \Lambda(n) \to \mathbb{R}$.

For a subset $\sigma \in 2^{[n]}$, the n vectors

$$v_i = \begin{cases} e_{n+i} & \text{if } i \in \sigma, \\ e_i & \text{if } i \in [n]\backslash\sigma \end{cases}$$

span the Lagrangian plane λ_σ. Let $\widetilde{\lambda}_{\sigma,+}$ denote the oriented Lagrangian plane where $\{v_1,\cdots,v_n\}$ is a positively oriented basis, and $\widetilde{\lambda}_{\sigma,-}$ the Lagrangian plane with opposite orientation. The critical point set $\mathrm{Crt}_k(\widetilde{f})$ of index k is given by

$$\mathrm{Crt}_k(\widetilde{f}) = \{\widetilde{\lambda}_{\sigma,\epsilon}\,;\, \sigma\in 2^{[n]} \text{ with } \|\sigma\| = k, \epsilon\in\{+,-\}\}.$$

The orientation of the unstable manifold $\widetilde{W}^u_{\sigma,\epsilon} = W^u(\lambda_{\sigma,\epsilon})$ is so chosen that the map $\pi|_{\widetilde{W}^u_{\sigma,\epsilon}} : \widetilde{W}^u_{\sigma,\epsilon} \to W^u_\sigma$ is orientation preserving (see, for the definition of orientation of W^u_σ, Definition 6.1).

9.1. **Ends of the isolated gradient lines in $\widetilde{\Lambda}(n)$.** Let (σ,τ) be a Type I or Type II pair, and $\gamma_\pm$ the two gradient lines connecting λ_σ and λ_τ in $\Lambda(n)$. There exist four gradient lines in $\widetilde{\Lambda}(n)$ that cover $\gamma_\pm$. Suppose γ_1 and γ_2 are the two leaving $\widetilde{\lambda}_{\sigma,+}$ among those four.

Lemma 9.1. *If (σ,τ) is of Type I, then γ_1 and γ_2 have different ends $\widetilde{\lambda}_{\tau,+}$ and $\widetilde{\lambda}_{\tau,-}$ at $t\to+\infty$. In contrast, if (σ,τ) is of Type II, then they have the same end $\widetilde{\lambda}_{\tau,+}$. (See Figure 2.)*

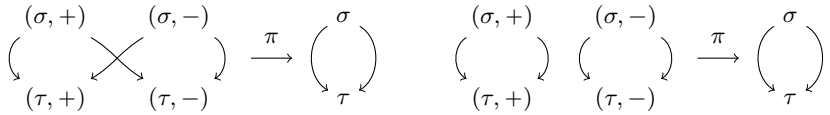


FIGURE 2. The left shows the case of Type I, and the right shows the case of Type II.

Proof. Let (σ, τ) be of Type I. Suppose that γ_1 and γ_2 are represented by the matrix in (11) with $s > 0$ and $s < 0$, respectively. The gradient flow is induced by the action of the matrix $\exp(-tD)$ on $\mathbb{R}^{2n}$ (see Proposition 3.3). By multiplying the first row by a scalar, we see that γ_1 and γ_2 converge to $\widetilde{\lambda}_{\tau,+}$ and $\widetilde{\lambda}_{\tau,-}$ as t tends to ∞, respectively.

For a Type II pair (σ, τ), γ_1 and γ_2 are represented by the matrix in (12) with $s > 0$ and $s < 0$, respectively. To find the limits $\gamma_i(\infty)$, we exchange the $(k-1)^{\text{th}}$ and k^{th} rows, and multiply by -1 the $(k-1)^{\text{th}}$ row if $s > 0$, or the k^{th} row if $s < 0$. In either case, we find the limits $\gamma_i(\infty)$ equal to $\widetilde{\lambda}_{\tau,+}$. □

9.2. **Coefficients of the boundary map.** Let $\nu((\sigma,\epsilon),(\tau,\delta))$ be the coefficient of the boundary map $\widetilde{\partial} : \mathbb{Z}\langle \mathrm{Crt}(\widetilde{f})\rangle \to \mathbb{Z}\langle \mathrm{Crt}(\widetilde{f})\rangle$;

$$\widetilde{\partial}\langle \widetilde{\lambda}_{\sigma,\epsilon}\rangle = \sum_{\|\tau\|=\|\sigma\|-1, \delta=\pm} \nu((\sigma,\epsilon),(\tau,\delta))\langle \widetilde{\lambda}_{\tau,\delta}\rangle.$$

Proposition 9.2.

$$\nu((\sigma,\epsilon),(\tau,\delta)) = \begin{cases} 1 & \text{if } (\sigma,\tau) \text{ is of Type I and } \epsilon\delta = -, \\ (-1)^{|\sigma|} & \text{if } (\sigma,\tau) \text{ is of Type I and } \epsilon\delta = +, \\ (-1)^{\|\sigma_{<j-1}\|}2 & \text{if } \epsilon\delta = + \text{ and } (\sigma,\tau) \text{ is of Type II with difference at odd } j, \\ 0 & \text{otherwise} \end{cases}$$

Proof. The proposition follows immediately from Proposition 6.2 and Lemma 9.1. □

9.3. **Decomposition of the Morse complex $\mathcal{MC}_{\widetilde{f}}$.** We decompose the Morse complex $\mathcal{MC}_{\widetilde{f}}$ in the same way as $\mathcal{MC}_f$ in (17). For a color sequence c, denote by $\widetilde{\mathcal{C}}_c$ the subcomplex consisting of all chains $\langle \widetilde{\lambda}_{\sigma,\epsilon}\rangle$ with $\sigma \in I_c$. Proposition 9.2 implies that if σ and τ have different color sequences, then $\nu((\sigma,\epsilon),(\tau,\delta)) = 0$. Hence the Morse complex $\mathcal{MC}_{\widetilde{f}}$ decomposes into subcomplexes $\widetilde{\mathcal{C}}_c$, and $H_*(\mathcal{MC}_{\widetilde{f}})$ is the direct sum $\bigoplus_c H_*(\widetilde{\mathcal{C}}_c)$. Let σ_c denote the element $\sigma \in I_c$ with the maximum $\|\sigma\|$.

Let $\widetilde{\mathcal{B}}_\pm$ be the chain complexes defined by

$$\widetilde{\mathcal{B}}_\pm : 0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\partial_\pm} \mathbb{Z}^2 \longrightarrow 0, \quad \text{where } \partial_\pm = \begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}.$$

Proposition 9.3. *Suppose a color sequence c has no gray dominos. Then*

$$H_{\|\sigma_c\|-k}(\widetilde{\mathcal{C}}_c) = \begin{cases} \mathbb{Z} & \text{for } k = 0, 1, \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

If c has j gray dominos with $j > 0$, then

$$H_{\|\sigma_c\|-k}(\widetilde{\mathcal{C}}_c) = \begin{cases} \mathbb{Z}_2{}^{\binom{j}{k}} & \text{for } k = 0, \cdots, j, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. When c has no gray dominos, the complex $\widetilde{\mathcal{C}}_c$ is isomorphic to $\widetilde{\mathcal{B}}_+$ or $\widetilde{\mathcal{B}}_-$ depending on whether $|\sigma_c|$ is even or odd (see Proposition 9.2). In either case, the homology group $H_k(\widetilde{\mathcal{B}}_\pm)$ is $\mathbb{Z}$ for $k = 0, 1$.

Suppose $j > 0$. It is also straightforward, by Proposition 9.2, to verify that $\widetilde{\mathcal{C}}_c$ is isomorphic to $\widetilde{\mathcal{B}}_\pm \otimes \mathcal{B}_2{}^{\otimes j}$ (see the proof of Proposition 7.2), whose homology group is computed by the Künneth formula. □

Example 9.4. Figure 3 shows a diagram of subcomplex $\widetilde{\mathcal{C}}_c$ which is isomorphic to $\widetilde{\mathcal{B}}_\pm \otimes \mathcal{B}_2$. The solid line arrows connect Type II pairs, and the broken line arrows connect Type I pairs.

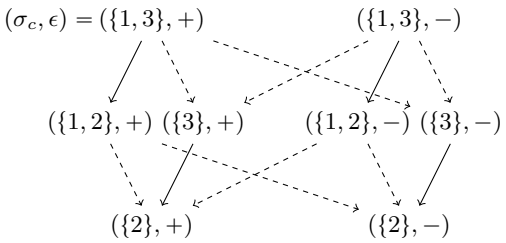


FIGURE 3

We compute the integral homology group of $\widetilde{\Lambda}(n)$ using a similar method to the case of $\Lambda(n)$.

Lemma 9.5. *The $\mathbb{Z}_2$-Poincaré polynomial of $\widetilde{\Lambda}(n)$ equals $\prod_{i=1}^n (1 + x^i)$.*

Proof. By Proposition 9.2, the coefficients $\nu((\sigma, \epsilon), (\tau, \delta))$ of the boundary map of $\mathcal{MC}_{\tilde{f}} \otimes \mathbb{Z}_2$ are 0 for Type II pairs (σ, τ), and 1 for Type I pairs. If (σ, τ) is a Type I pair, 4 critical points $(\sigma, \pm), (\tau, \pm)$ form a subcomplex isomorphic to $\widetilde{\mathcal{B}}_\pm \otimes \mathbb{Z}_2$, whose homology group has one generator each in degree 0 and 1. Thus all degree i cycles in $H_*(\mathcal{MC}_{\tilde{f}} \otimes \mathbb{Z}_2)$ can be paired bijectively with all elements $\sigma \in 2^{[n]}$ with $\|\sigma\| = i$. Hence the combinatorial equation $\sum_{\sigma \in 2^{[n]}} x^{\|\sigma\|} = \prod_{i=1}^n (1 + x^i)$ completes the proof. □

9.4. **The homology group $H_*(\widetilde{\Lambda}(n); \mathbb{Z})$.** We remark that $\widetilde{\Lambda}(2m+1)$ is diffeomorphic to $\Lambda(2m+1)$. Actually, there exists a fibration $\zeta : \Lambda(n) \to S^1$ whose fiber is $SU(n)/SO(n)$, which gives an isomorphism on the fundamental groups. The pull back of this fiber bundle by the double cover $S^1 \ni z \mapsto z^2 \in S^1$ equals $\widetilde{\Lambda}(n)$. If n is odd, the fibration ζ is trivial, and thus $\Lambda(n)$ and $\widetilde{\Lambda}(n)$ are diffeomorphic to $S^1 \times \big(SU(n)/SO(n)\big)$.

Corollary 9.6. *The integral homology group of $\widetilde{\Lambda}(2m+1)$ has only 2-torsion;*

$$H_k(\widetilde{\Lambda}(2m+1); \mathbb{Z}) \cong \mathbb{Z}^{\tilde{c}(k)} \oplus \mathbb{Z}_2{}^{\tilde{d}(k)}.$$

The generating functions $\widetilde{C}(x) = \sum \tilde{c}(k) x^k$ and $\widetilde{D}(x) = \sum \tilde{d}(k) x^k$ are given by

$$\widetilde{C}(x) = (1+x) \prod_{\ell=1}^{m} (1 + x^{4\ell+1}), \quad \widetilde{D}(x) = \prod_{\ell=2}^{2m+1} (1 + x^\ell) - \prod_{\ell=1}^{m} (1 + x^{4\ell+1}).$$

All that remains is to compute $H_*(\widetilde{\Lambda}(2m); \mathbb{Z})$.

Theorem 9.7. *The integral homology group of the oriented Lagrangian Grassmannian $\widetilde{\Lambda}(2m)$ has only 2-torsion;*

$$H_k(\widetilde{\Lambda}(2m); \mathbb{Z}) \cong \mathbb{Z}^{\tilde{a}(k)} \oplus \mathbb{Z}_2{}^{\tilde{b}(k)}.$$

The generating functions $\widetilde{A}(x) = \sum \tilde{a}(k) x^k$ and $\widetilde{B}(x) = \sum \tilde{b}(k) x^k$ are given by

$$\widetilde{A}(x) = (1+x)(1+x^{2m}) \prod_{\ell=1}^{m-1} (1 + x^{4\ell+1}), \quad \widetilde{B}(x) = \prod_{\ell=2}^{2m} (1 + x^\ell) - (1 + x^{2m}) \prod_{\ell=1}^{m-1} (1 + x^{4\ell+1}).$$

Proof. Maya diagrams of $\sigma \in 2^{[2m]}$ have single boxes in the leftmost and rightmost. Let $\widetilde{\mathcal{A}}$ be the subset of $2^{[2m]}$ defined by

$$\widetilde{\mathcal{A}} = \{\sigma \in 2^{[2m]} \,;\, \sigma \text{ has no gray dominos}\}.$$

Then Formula (21) yields the equation $\widetilde{A}(x) = \sum_{\sigma\in\widetilde{\mathcal{A}}} x^{\|\sigma\|}$. The factorization of this polynomial is obtained by combinatorial arguments, where the factors $1+x$ and $1+x^{2m}$ relate to the leftmost and rightmost boxes of σ, and $1+x^{4\ell+1}$ to the ℓ^{th} domino of σ's in $\widetilde{\mathcal{A}}$, which are white or black.

The formula of $\widetilde{B}(x)$ is derived from $\widetilde{A}(x)$ by Lemma 9.5 and the universal coefficient theorem. □

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