A CENTRAL LIMIT THEOREM FOR THE STOCHASTIC CABLE EQUATION

SOMA NISHINO

Abstract. We consider one-dimensional nonlinear stochastic cable equations driven by multiplicative space-time white noise. Using the Malliavin–Stein method, we establish a fixed-time central limit theorem for the spatial average of the solution, with convergence in total variation distance under mild conditions. We further show that the spatial average is asymptotically independent of the solution at any fixed time and position. A functional central limit theorem is also obtained under a technical assumption, which we verify for affine coefficients.

1. Introduction and main results

In this paper, we consider the nonlinear stochastic cable equation

$$\frac{\partial u}{\partial t} = \frac{\beta}{2}\frac{\partial^2 u}{\partial x^2} - \alpha u + \sigma(u)\,\dot{W}, \tag{1}$$

on $[0,T]\times[0,L]$ for constants $\alpha\in\mathbb{R}$ and $\beta>0$, where $T>0$ is fixed and $\dot{W}$ denotes space–time white noise on $[0,T]\times[0,L]$. The initial condition is $u_0(x)=1$, and we impose either Neumann, Dirichlet, or periodic boundary conditions. The coefficient σ is assumed to be globally Lipschitz.

By Walsh [25], (1) admits a unique mild solution, adapted to the filtration generated by W and satisfying $E[u(t,x)^2]<\infty$. The mild solution is given by

$$u(t,x) = \int_0^L u_0(y)G_t(x,y)\,\mathrm{d}y + \int_0^t\int_0^L G_{t-s}(x,y)\,\sigma(u(s,y))\,W(\mathrm{d}s,\mathrm{d}y), \tag{2}$$

where the stochastic integral is understood in the sense of Itô–Walsh, and G denotes the heat kernel associated with (5).

We investigate the large-L asymptotics of the spatial average

$$F_L(t) := \frac{1}{L}\int_0^L \big(u(t,x) - E[u(t,x)]\big)\,\mathrm{d}x. \tag{3}$$

For random variables X and Y, the total variation distance is defined by

$$d_{\mathrm{TV}}(X,Y) := \sup_{B\in\mathcal{B}(\mathbb{R})} |P(X\in B) - P(Y\in B)|,$$

where the supremum is taken over Borel sets $B\in\mathcal{B}(\mathbb{R})$. We write $d_{\mathrm{TV}}(F,\mathcal{N}(0,1))$ for the total variation distance between the law of F and the standard normal distribution.

For random variables X and Y such that $h(X),h(Y)\in L^1(P)$, the Wasserstein distance between their laws is defined by

$$d_{\mathrm{W}}(\mathcal{L}(X),\mathcal{L}(Y)) := \sup_{h\in\mathrm{Lip}(1)} \big|E[h(X)] - E[h(Y)]\big|,$$

2020 *Mathematics Subject Classification.* Primary 60H15; Secondary 60F05, 60H07.

where Lip(1) denotes the set of 1-Lipschitz functions, and $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ denote the laws of X and Y, respectively. We are now ready to state the first main result of this paper. First, under the assumption that $\sigma(1) \neq 0$, we present a quantitative fixed-time central limit theorem with respect to the total variation distance.

Theorem 1. *Suppose that $\sigma(1) \neq 0$. Then, for every $t > 0$ there exists a constant $c = c(t) > 0$ such that for all $L \geq 1$,*

$$d_{\mathrm{TV}}\left(\frac{F_L(t)}{\sqrt{\mathrm{Var}(F_L(t))}}, \mathcal{N}(0,1)\right) \leq \frac{c}{\sqrt{L}}. \tag{4}$$

REMARK 1.1. *Condition $\sigma(1) \neq 0$ guarantees that $\mathrm{Var}(F_L(t)) > 0$. This follows from* (8), (12), (14), (15), (37), (38) *and* (39).

This implies that the rescaled spatial average $F_L(t)/\sqrt{\mathrm{Var}(F_L(t))}$ converges in distribution to the standard normal distribution as $L \to \infty$. Moreover, we show that the spatial average is asymptotically independent of $u(t, x_0)$ for any fixed time t and position x_0. Specifically, under the assumption that $\sigma(1) \neq 0$, we establish the following result.

Theorem 2. *Suppose that $\sigma(1) \neq 0$. Then, for all $L \geq 1$, $t > 0$, and $x_0 \in [0, L]$, there exists a constant C such that*

$$d_{\mathrm{W}}\left(\mathcal{L}\left(\frac{F_L(t)}{\sqrt{\mathrm{Var}(F_L(t))}}, u(t, x_0)\right), \mathcal{L}(Z, u(t, x_0))\right) \leq \frac{C}{\sqrt{L}},$$

where $C = C(t) > 0$ depends only on t, and $Z \sim \mathcal{N}(0,1)$ is independent of $u(t, x_0)$.

We assume the following technical condition, which is required for the proof of the functional central limit theorem.

Assumption 1. *There exists a measurable function f_σ, positive almost everywhere and belongs to $L^1([0,T])$ for every $T > 0$, such that for all $t \in [0,T]$,*

$$\lim_{L\to\infty} \frac{1}{L} \int_0^L E[\sigma(u(t,x))^2] \, \mathrm{d}x = f_\sigma(t).$$

Finally, under Assumption 1, we state the following functional central limit theorem.

Theorem 3. *Fix $T > 0$. Suppose that Assumption 1 holds. Then, as $L \to \infty$,*

$$\left(\sqrt{L} F_L(t)\right)_{t\in[0,T]} \to \left(\int_0^t e^{-\alpha(t-s)} \sqrt{f_\sigma(s)} \, \mathrm{d}W_s\right)_{t\in[0,T]}$$

where $f_\sigma(t)$ is the limit in Assumption 1 and $W = \{W_s\}_{s\in[0,T]}$ denotes a standard one-dimensional Brownian motion, and the convergence is in law on the space of continuous functions $C([0,T])$.

REMARK 1.2. *Since Assumption 1 is established only when σ is affine, it follows that Theorem 3 holds only for that case, and the general case remains open.*

1.1. **Model background and conceptual significance.** The nonlinear stochastic cable equation arises in the mathematical modeling of neurons, where it describes the propagation of electrical signals along their cylindrical structure. Neurons, the fundamental components of the nervous system, operate through a sophisticated interplay of chemical, biological, and electrical phenomena. In a common mathematical simplification, a neuron is idealized as a long, thin cylinder, similar to an electrical cable. For such a cylinder on the interval $[0, L]$, we assume the electrical potential $u(t, x)$ depends only on position $x \in [0, L]$ and time t. While this potential $u(t, x)$ is accurately governed by the Hodgkin–Huxley equations, for specific ranges of u, these equations can be approximated by the cable equation:

$$\frac{\partial u}{\partial t} = \frac{\beta}{2}\frac{\partial^2 u}{\partial x^2} - \alpha u, \tag{5}$$

with $\beta/2 > 0$ representing the diffusion rate within the neuron and $\alpha \in \mathbb{R}$ being the rate of ion leakage through the membrane [11]. The neuron's surface receives current impulses through synapses. If this incoming current at (t, x) is represented by $F(t, x)$, the system obeys the inhomogeneous PDE:

$$\frac{\partial u}{\partial t} = \frac{\beta}{2}\frac{\partial^2 u}{\partial x^2} - \alpha u + F.$$

Even in a resting state, occasional random impulses may occur, which implies that F generally contains a stochastic component. For instance, Walsh studied the case where F is a compound Poisson process or a space-time white noise [24]. The nonlinear stochastic cable equation with $F = \sigma(u)\dot{W}$ was analyzed in [25].

In the electrochemical context of the cable equation, $F_L(t)$ represents the spatial average of the deviation of the potential from its expectation, while $u(t, x_0)$ denotes the potential at a fixed point $x_0 \in [0, L]$. Thus, $F_L(t)$ captures the macroscopic fluctuation, whereas $u(t, x_0)$ represents the local state. These correspondences provide a physical interpretation of our main results. In particular, Theorem 2 establishes that as the domain size $L \to \infty$, these two quantities become asymptotically independent at a rate of $O(1/\sqrt{L})$. This suggests that the macroscopic fluctuation decouples from the local potential in the large-system limit.

1.2. **Related work.** In recent years, considerable attention has been paid to the asymptotic behavior of spatial averages for solutions to stochastic partial differential equations. This area of inquiry was first explored by Huang, Nualart, and Viitasaari in [9]. Utilizing the Malliavin–Stein method (see [16]), they proved a central limit theorem, along with its functional counterpart, for the one-dimensional nonlinear stochastic heat equation with space-time white noise. Following their seminal work, analogous central limit theorems for various types of stochastic partial differential equations have been established. For example, Huang, Nualart, Viitasaari, and Zheng studied the d-dimensional stochastic heat equation with colored noise [10]. There has been much research on the stochastic heat equation under different settings; see, for example, [1, 2, 4, 13, 14, 18, 19, 21]. Similar results are also known for the stochastic wave equation; see, for instance, [5] for the one-dimensional case, [8] for the two-dimensional case, and [6, 7] for the three and higher dimensions. For other types of SPDEs, Liu and Shen studied a central limit theorem for the solution to an SPDE related to a pseudo-differential operator that generates a stable-like process [15].

The study most relevant to our setting is that of Pu [21], whose proof strategy we also substantially rely on. Pu considered the one-dimensional stochastic heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + u\,\dot{W},$$

on $[0,T]\times[0,L]$ with Neumann, Dirichlet, or periodic boundary conditions and analyzed its solution via its Wiener chaos decomposition. This equation corresponds to the SPDE obtained from (1) by setting $\alpha = 0$, $\beta = 1$, and $\sigma(u) = u$. For the spatial average of this solution, Pu established the counterparts to Theorems 1 and 3 of the present work [21, Theorems 1.1 and 1.2]. It follows from (8), (11), and (36) that, in contrast to Pu's setting, the term $-\alpha u$ decreases the covariance of the spatial average of the solution when $\alpha > 0$, whereas it increases the covariance when $\alpha < 0$. In addition, the parameter $\beta > 0$ affects the time scaling.

Despite these analytical differences, Assumption 1 remains essential for establishing the functional central limit theorem for SPDEs on bounded domains, both in the present work and in the work of Pu [21]. We verify Assumption 1 in Section 7 for the case $\sigma(u) = \sigma_1 u + \sigma_0$, where σ_1 and σ_0 are constants with $(\sigma_1, \sigma_0) \neq (0,0)$. Our argument relies on the Wiener chaos decomposition of the solution to the stochastic cable equation (1). Verifying Assumption 1 for general σ remains an open problem, as the Wiener chaos decomposition becomes significantly more complex due to the nonlinearity. The same method was employed in [21] to implicitly verify the condition corresponding to Assumption 1 for the stochastic heat equation with $\sigma(u) = u$. In contrast, in [9] and related works, since the solution to the SPDE is spatially stationary, the limit in Assumption 1 reduces to $E[\sigma(u(t,0))^2]$, and thus technical conditions such as those in Assumption 1 are not required.

1.3. **Structure of the paper.** We outline the structure of the paper. Section 2 collects several tools from Malliavin calculus that will be used in the sequel. In particular, we recall the Clark–Ocone formula, the associated Poincaré-type inequality, and the Malliavin–Stein method. We also present moment estimates for the solution to the stochastic cable equation, as well as Gaussian-type upper bounds for the moments of its Malliavin derivatives. Section 3 establishes several auxiliary lemmas and analyzes the asymptotic behavior of the covariance $\mathrm{Cov}[F_L(t_1), F_L(t_2)]$ of the spatial average $F_L(t)$. Section 4 analyzes the Malliavin derivative $DF_L(t)$; combined with the results of Section 3, this yields the proof of Theorem 1. Section 5 shows that supplementary arguments to those in Section 4 lead to asymptotic independence (Theorem 2). Section 6 establishes the functional central limit theorem (Theorem 3) under Assumption 1, by proving suitable moment inequalities and the convergence of finite-dimensional distributions. Section 7 verifies Assumption 1 in the case $\sigma(u) = \sigma_1 u + \sigma_0$. Appendix A summarizes the properties of the heat kernel G used in the paper.

2. Preliminaries

For any $Z \in L^k(\Omega)$, we define $\|Z\|_k := E[|Z|^k]^{1/k}$.

Throughout this paper, we assume that σ is globally Lipschitz continuous with constant K_σ. This implies the following linear growth condition:

$$|\sigma(y)| \le M_\sigma(1 + |y|)$$

where $M_\sigma = \max\{K_\sigma, |\sigma(0)|\}$.

2.1. **Clark–Ocone formula.** Let $\mathcal{H}$ be the Hilbert space $L^2([0,T]\times\mathbb{R})$. The family of stochastic integrals

$$X(h) = \int_{[0,T]\times\mathbb{R}} h(s,x)\ W(\mathrm{d}s,\mathrm{d}x)$$

constitutes an isonormal Gaussian process $\{X(h)\}_{h\in\mathcal{H}}$. In this context, we employ the tools of Malliavin calculus (see, e.g., [17]), and let D denote the associated Malliavin derivative operator. Let $\{\mathcal{F}_s\}_{s\in[0,T]}$ be the filtration generated by the space-time white noise $\dot{W}$. A key result is the Clark–Ocone formula, which provides the following representation for any $\mathcal{F}_T$-measurable random variable F in the Sobolev space $\mathbb{D}^{1,2}$:

$$F = E[F] + \int_{[0,T]\times\mathbb{R}} E[D_{s,z}F \mid \mathcal{F}_s]\ W(\mathrm{d}s,\mathrm{d}z) \quad \text{a.s.}$$

A well-known consequence of this formula, obtained via Jensen's inequality for conditional expectations, is the following Poincaré-type inequality:

$$|\mathrm{Cov}[F,G]| \le \int_0^T \int_{\mathbb{R}} \|D_{s,z}F\|_2 \|D_{s,z}G\|_2\ \mathrm{d}z\mathrm{d}s, \tag{6}$$

which holds for any pair of $\mathcal{F}_T$-measurable random variables $F, G \in \mathbb{D}^{1,2}$.

2.2. **The Malliavin–Stein method.** We now recall some key results from the Malliavin–Stein method. This method merges the Malliavin calculus with Stein's method to provide quantitative bounds on the distance between probability measures, proving particularly effective for central limit theorems [16].

Proposition 2.1. *Let $F = \delta(v)$ for some $v \in \mathrm{Dom}(\delta)$, where $\mathrm{Dom}(\delta)$ is the $L^2(\Omega)$-domain of the adjoint of the Malliavin derivative operator. Suppose that $F \in \mathbb{D}^{1,2}$ with $E[F^2] = 1$. Then, the following inequality holds:*

$$d_{\mathrm{TV}}(F, \mathcal{N}(0,1)) \le 2\sqrt{\mathrm{Var}(\langle DF, v\rangle_{\mathcal{H}})}.$$

The following proposition, which is a generalization of Proposition 2.1, was proved in [23].

Proposition 2.2. *Let $m \ge 1$ be an integer. Let $F = \delta(v)$ for some $v \in \mathrm{Dom}(\delta)$ and let $Y = (Y_1,\dots,Y_m)$ be a random vector. Suppose that $F \in \mathbb{D}^{1,2}$ with $E[F^2] = \rho^2$, and $Y_j \in \mathbb{D}^{1,2}$ for each $j = 1,\dots,m$. Then there exists a constant $C' > 0$, depending only on ρ^2, such that*

$$d_{\mathrm{W}}(\mathcal{L}(X,Y), \mathcal{L}(Z,Y)) \le C' \left\{ \sqrt{\mathrm{Var}(\langle DF, v\rangle_{\mathcal{H}})} + \sum_{j=1}^m \sqrt{E[\langle v, DY_j\rangle_{\mathcal{H}}^2]} \right\},$$

where $Z \sim \mathcal{N}(0,\rho^2)$ is independent of Y.

The proof of the functional CLT in Theorem 3 requires a multivariate version of Proposition 2.1, which we state below.

Proposition 2.3. *Let $m \ge 2$ be an integer. Let $F = (F_1,\dots,F_m)$ be a random vector where each component has the form $F_i = \delta(v_i)$ for some $v_i \in \mathrm{Dom}(\delta)$ and $F_i \in \mathbb{D}^{1,2}$. Let Z denote an m-dimensional centered Gaussian random vector with a given covariance matrix $(C_{i,j})_{1\le i,j\le m}$. Then,*

for any function $h \in C^2(\mathbb{R}^m)$ with bounded second partial derivatives, we have

$$|E[h(F)] - E[h(Z)]| \le \frac{1}{2}\|h''\|_\infty \sqrt{\sum_{i,j=1}^m E\left[|C_{i,j} - \langle DF_i, v_j\rangle_{\mathcal{H}}|^2\right]},$$

where

$$\|h''\|_\infty := \max_{1\le i,j\le m} \sup_{x\in\mathbb{R}^m} \left|\frac{\partial^2 h(x)}{\partial x_i \partial x_j}\right|.$$

2.3. **Moments and Malliavin derivative of the solution.** Recently, Pu proved that the moments of $u(t,x)$ are uniformly bounded for all $L \ge 1$ in the case where $\alpha = 0$ and $\beta = 1$ [21, Lemma 2.3.]. Similarly, a corresponding result has been obtained for the case $\alpha \in \mathbb{R}$ and $\beta > 0$.

Lemma 2.1. *Fix $T > 0$. Then for all $k \ge 2$, there exists $c_{T,k} > 0$ such that*

$$\sup_{L\ge 1} \sup_{(t,x)\in[0,T]\times[0,L]} \|u(t,x)\|_k \le c_{T,k} < \infty. \tag{7}$$

Pu also proved that the moments of the Malliavin derivative of $u(t,x)$ satisfy a Gaussian-type upper bound uniformly over $L \ge 1$ in the case where $\alpha = 0$ and $\beta = 1$ [21, Lemma 2.4.]. This result can similarly be extended to the case $\alpha \in \mathbb{R}$ and $\beta > 0$.

Lemma 2.2. *Fix $T > 0$. Then, for all $k \ge 2$, there exists $C_{T,k} > 0$ such that for all $L \ge 1$, $(t,x) \in [0,T]\times[0,L]$, and for almost every $(s,y) \in (0,t)\times\mathbb{R}$,*

$$\|D_{s,y}u(t,x)\|_k \le \begin{cases} C_{T,k} 1_{[0,L]}(y) p_{\beta(t-s)}(x-y) & \text{(Neumann/Dirichlet case)} \\ C_{T,k} 1_{[0,L]}(y) G_{t-s}(x,y) & \text{(periodic case)} \end{cases}$$

where $p_t(z)$ denotes the Gaussian heat kernel, defined in (35). In particular, we have $u(t,x) \in \bigcap_{k\ge 2} \mathbb{D}^{1,k}$ for all $(t,x) \in [0,T]\times[0,L]$.

3. Asymptotic behavior of the covariance

In this section, we analyze the asymptotic behavior of the covariance of the spatial integral of the solution to (1).

Since (36), by the same arguments as [21, Lemma 3.2. and Lemma A.4.], we obtain the following supporting lemma.

Lemma 3.1. *Fix $T > 0$. Denote for $(t,x) \in (0,T]\times[0,L]$,*

$$\mathcal{I}_0(t,x) = \int_0^L G_t(x,y)\, \mathrm{d}y, \ \text{and} \ \ \mathcal{I}_0(0,x) = 1. \tag{8}$$

Then

$$\sup_{L\ge 1} \sup_{(t,x)\in[0,T]\times[0,L]} \mathcal{I}_0(t,x) \le e^{|\alpha|T},$$

and for all $t > 0$,

$$\lim_{L\to\infty} \frac{1}{L}\int_0^L \mathcal{I}_0(t,x)\, \mathrm{d}x = e^{-\alpha t},$$

and for all $t_1, t_2 > 0$,

$$\lim_{L\to\infty} \frac{1}{L}\int_0^L \mathcal{I}_0(t_1,x)\mathcal{I}_0(t_2,x)\, \mathrm{d}x = e^{-\alpha(t_1+t_2)}.$$

The following lemmas are prepared for the proof of Proposition 3.1.

Lemma 3.2. *In the case of Neumann or periodic boundary conditions,*

$$\lim_{t\to 0}\sup_{L\geq 1}\sup_{x\in[0,L]}|E[\sigma(u(t,x))^2]-\sigma(1)^2|=0.$$

Proof. Fix $T>0$. For every $(t,x)\in[0,T]\times[0,L]$,

$$\begin{aligned}
&|E[\sigma(u(t,x))^2]-\sigma(1)^2|\\
&=|E[(\sigma(u(t,x))-\sigma(1))(\sigma(u(t,x))+\sigma(1))]|\\
&\leq\sqrt{E[|\sigma(u(t,x))-\sigma(1)|^2]}\sqrt{E[|\sigma(u(t,x))+\sigma(1)|^2]}\\
&\leq\sqrt{E[K_\sigma^2|u(t,x)-1|^2]}\sqrt{E[2\sigma(u(t,x))^2+2\sigma(1)^2]}\\
&\leq K_\sigma\sqrt{E[|u(t,x)-1|^2]}\sqrt{4M_\sigma^2(c_{T,2}^2+1)+2\sigma(1)^2},
\end{aligned}$$

where K_σ is the Lipschitz constant of σ, and M_σ is the constant in the linear growth condition. From (2) and Itô's isometry, we get

$$\begin{aligned}
&E[|u(t,x)-1|^2]\\
&\leq 2\left|\int_0^L G_t(x,y)\,\mathrm{d}y-1\right|^2+2\int_0^t\int_0^L G_{t-s}(x,y)^2E[\sigma^2(u(s,y))]\,\mathrm{d}s\mathrm{d}y\\
&\leq 2\left|\int_0^L G_t(x,y)\,\mathrm{d}y-1\right|^2+4M_\sigma^2(1+c_{T,2}^2)\int_0^t\int_0^L G_{t-s}(x,y)^2\,\mathrm{d}s\mathrm{d}y.
\end{aligned}\tag{9}$$

Since we assume Neumann or periodic boundary conditions, using (40) and semigroup property of G, we have

$$\begin{aligned}
&E[|u(t,x)-1|^2]\\
&\leq 2|e^{-\alpha t}-1|^2+4M_\sigma^2(1+c_{T,2}^2)\int_0^t G_{2(t-s)}(x,x)\,\mathrm{d}s
\end{aligned}$$

In the case of Neumann boundary conditions, using (41), we get

$$\begin{aligned}
\int_0^t G_{2(t-s)}(x,x)\,\mathrm{d}s&\leq K_{2T}e^{2|\alpha|T}\int_0^t\frac{1}{\sqrt{4\pi\beta(t-s)}}\,\mathrm{d}s\\
&=K_{2T}e^{2|\alpha|T}\sqrt{\frac{t}{\pi\beta}}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
&E[|u(t,x)-1|^2]\\
&\leq 2|e^{-\alpha t}-1|^2+4M_\sigma^2(1+c_{T,2}^2)K_{2T}e^{2|\alpha|T}\sqrt{\frac{t}{\pi\beta}}
\end{aligned}$$

In the case of periodic boundary conditions, using (39) and the following identity [21, A.17]

$$\sum_{j\in\mathbb{Z}}p_r(j)=\sum_{n\in\mathbb{Z}}e^{-2r\pi^2n^2}\quad\text{for all } r>0,$$

we have

$$\begin{aligned}
&\int_0^t G_{2(t-s)}(x,x)\ \mathrm{d}s \\
&= \int_0^t e^{-2\alpha(t-s)} \sum_{j\in\mathbb{Z}} p_{2\beta(t-s)}(jL)\ \mathrm{d}s \\
&\le \int_0^t e^{-2\alpha(t-s)} \sum_{j\in\mathbb{Z}} p_{2\beta(t-s)}(j)\ \mathrm{d}s \\
&= \int_0^t e^{-2\alpha(t-s)} \sum_{n\in\mathbb{Z}} e^{-4\beta(t-s)\pi^2n^2}\ \mathrm{d}s \\
&\le e^{2|\alpha|T} \sum_{n\in\mathbb{Z}} \int_0^t e^{-4\beta(t-s)\pi^2n^2}\ \mathrm{d}s \\
&= e^{2|\alpha|T} \sum_{n\in\mathbb{Z}} \frac{1-e^{-4\beta t\pi^2n^2}}{4\beta\pi^2n^2}.
\end{aligned}$$

Then,

$$\begin{aligned}
&E[|u(t,x)-1|^2] \\
&\le 2|e^{-\alpha t}-1|^2 + 4M_\sigma^2(1+c_{T,2}^2)e^{2|\alpha|T} \sum_{n\in\mathbb{Z}} \frac{1-e^{-4\beta t\pi^2n^2}}{4\beta\pi^2n^2}.
\end{aligned}$$

Applying the Lebesgue dominated convergence theorem, we get

$$\lim_{t\to0} \sum_{n\in\mathbb{Z}} \frac{1-e^{-4\beta t\pi^2n^2}}{4\beta\pi^2n^2} = \sum_{n\in\mathbb{Z}} \lim_{t\to0} \frac{1-e^{-4\beta t\pi^2n^2}}{4\beta\pi^2n^2} = 0.$$

Therefore, in the case of Neumann or periodic boundary conditions,

$$\lim_{t\to0} \sup_{L\ge1} \sup_{x\in[0,L]} |E[\sigma(u(t,x))^2] - E[\sigma(1)^2]| = 0.$$

□

Lemma 3.3. *In the case of Dirichlet boundary conditions,*

$$\lim_{t\to0} \sup_{L\ge1} \left| \frac{1}{L} \int_0^L E[\sigma(u(t,x))^2]\ \mathrm{d}x - \sigma(1)^2 \right| = 0.$$

Proof. Fix $T > 0$. For every $(t, x) \in [0, T] \times [0, L]$,

$$\begin{aligned}
&\left| \frac{1}{L} \int_0^L E[\sigma(u(t,x))^2] \, \mathrm{d}x - \sigma(1)^2 \right| \\
&= \left| \frac{1}{L} \int_0^L E[(\sigma(u(t,x)) - \sigma(1))(\sigma(u(t,x)) + \sigma(1))] \, \mathrm{d}x \right| \\
&\leq \sqrt{\frac{1}{L} \int_0^L E[|\sigma(u(t,x)) - \sigma(1)|^2] \, \mathrm{d}x} \sqrt{\frac{1}{L} \int_0^L E[|\sigma(u(t,x)) + \sigma(1)|^2] \, \mathrm{d}x} \\
&\leq \sqrt{\frac{1}{L} \int_0^L E[K_\sigma^2 |u(t,x) - 1|^2] \, \mathrm{d}x} \sqrt{\frac{1}{L} \int_0^L E[2\sigma(u(t,x))^2 + 2\sigma(1)^2] \, \mathrm{d}x} \\
&\leq K_\sigma \sqrt{\frac{1}{L} \int_0^L E[|u(t,x) - 1|^2] \, \mathrm{d}x} \sqrt{4M_\sigma^2 (c_{T,2}^2 + 1) + 2\sigma(1)^2}.
\end{aligned}$$

From (2) and Itô's isometry, we get (9). Since we assume Dirichlet boundary conditions, using semigroup property of G and (41), we have

$$\begin{aligned}
&E[|u(t,x) - 1|^2] \\
&\leq 2 \left| \int_0^L G_t(x,y) \, \mathrm{d}y - 1 \right|^2 + 4M_\sigma^2 (1 + c_{T,2}^2) \int_0^t G_{2(t-s)}(x,x) \, \mathrm{d}s \\
&\leq 2 \left| \int_0^L G_t(x,y) \, \mathrm{d}y - 1 \right|^2 + 4M_\sigma^2 (1 + c_{T,2}^2) K_{2T} e^{2|\alpha|T} \int_0^t \frac{1}{\sqrt{4\pi\beta(t-s)}} \, \mathrm{d}s \\
&\leq 2 \left| \int_0^L G_t(x,y) \, \mathrm{d}y - 1 \right|^2 + 4M_\sigma^2 (1 + c_{T,2}^2) K_{2T} e^{2|\alpha|T} \sqrt{\frac{t}{\pi\beta}}.
\end{aligned}$$

By (38), we get

$$\begin{aligned}
&\int_0^L G_t(x,y) \, \mathrm{d}y \\
&= 2e^{-\alpha t} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \frac{1 - (-1)^n}{n\pi} e^{-\frac{n^2\pi^2\beta t}{2L^2}}.
\end{aligned}$$

Then, applying the $L^2([0,L])$-orthogonality of the functions $\{x \mapsto \sin(n\pi x/L)\}_{n=1}^{\infty}$,

$$
\begin{aligned}
&\frac{1}{L}\int_0^L \left| \int_0^L G_t(x,y)\ \mathrm{d}y - 1 \right|^2 \ \mathrm{d}x \\
&= 4e^{-2\alpha t} \sum_{n=1}^{\infty} \frac{1}{L} \int_0^L \sin^2\left(\frac{n\pi x}{L}\right)\ \mathrm{d}x \left(\frac{1-(-1)^n}{n\pi}\right)^2 e^{-\frac{n^2\pi^2\beta t}{L^2}} \\
&\quad + 1 - 4e^{-\alpha t} \sum_{n=1}^{\infty} \frac{1}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right)\ \mathrm{d}x \frac{1-(-1)^n}{n\pi} e^{-\frac{n^2\pi^2\beta t}{2L^2}} \\
&= 2e^{-2\alpha t} \sum_{n=1}^{\infty} \left(\frac{1-(-1)^n}{n\pi}\right)^2 e^{-\frac{n^2\pi^2\beta t}{L^2}} + 1 - 4e^{-\alpha t} \sum_{n=1}^{\infty} \left(\frac{1-(-1)^n}{n\pi}\right)^2 e^{-\frac{n^2\pi^2\beta t}{2L^2}}.
\end{aligned} \tag{10}
$$

Now, for every $L \geq 1$,

$$
\begin{aligned}
&\left| \sum_{n=1}^{\infty} \left(\frac{1-(-1)^n}{n\pi}\right)^2 e^{-\frac{n^2\pi^2\beta t}{L^2}} - \sum_{n=1}^{\infty} \left(\frac{1-(-1)^n}{n\pi}\right)^2 \right| \\
&= \left| \sum_{n=1}^{\infty} \left(\frac{1-(-1)^n}{n\pi}\right)^2 (e^{-\frac{n^2\pi^2\beta t}{L^2}} - 1) \right| \\
&\leq \sum_{n=1}^{\infty} \left(\frac{1-(-1)^n}{n\pi}\right)^2 |e^{-\frac{n^2\pi^2\beta t}{L^2}} - 1| \\
&\leq \sum_{n=1}^{\infty} \left(\frac{1-(-1)^n}{n\pi}\right)^2 |e^{-n^2\pi^2\beta t} - 1|.
\end{aligned}
$$

Applying the dominated convergence theorem,

$$
\lim_{t\to 0} \sum_{n=1}^{\infty} \left(\frac{1-(-1)^n}{n\pi}\right)^2 |e^{-n^2\pi^2\beta t} - 1| = 0.
$$

Combining the above with the following identity

$$
\sum_{n=1}^{\infty} \left(\frac{1-(-1)^n}{n\pi}\right)^2 = \sum_{k=1}^{\infty} \frac{4}{(2k-1)^2\pi^2} = \frac{1}{2}
$$

yields

$$
\limsup_{t\to 0} \sup_{L\geq 1} \left| \sum_{n=1}^{\infty} \left(\frac{1-(-1)^n}{n\pi}\right)^2 e^{-\frac{n^2\pi^2\beta t}{L^2}} - \frac{1}{2} \right| = 0.
$$

Similarly, we get

$$
\limsup_{t\to 0} \sup_{L\geq 1} \left| \sum_{n=1}^{\infty} \left(\frac{1-(-1)^n}{n\pi}\right)^2 e^{-\frac{n^2\pi^2\beta t}{2L^2}} - \frac{1}{2} \right| = 0.
$$

These together with (10) imply

$$
\limsup_{t\to 0} \sup_{L\geq 1} \frac{1}{L}\int_0^L \left| \int_0^L G_t(x,y)\ \mathrm{d}y - 1 \right|^2 \ \mathrm{d}x = 0.
$$

Therefore, we have

$$\limsup_{t\to 0}\sup_{L\geq 1}\frac{1}{L}\int_0^L E[|u(t,x)-1|^2]\,\mathrm{d}x = 0.$$

This completes the proof. □

The following results provide the asymptotic behavior of the covariance function of the renormalized sequence of processes $F_L(t)$ as L tends to infinity. By using Lemma 3.2 in the case of Neumann or periodic boundary conditions, and Lemma 3.3 in the case of Dirichlet boundary conditions, we can prove the following proposition.

Proposition 3.1. *There exists $\delta > 0$, for every $t_1, t_2 > 0$,*

$$\liminf_{L\to\infty}\mathrm{Cov}\left[\sqrt{L}F_L(t_1),\sqrt{L}F_L(t_2)\right] \geq \frac{\sigma(1)^2}{2}\int_0^{t_1\wedge t_2\wedge\delta} e^{-\alpha(t_1+t_2-2s)}\,\mathrm{d}s. \tag{11}$$

Proof. Using the mild form in (2) and Itô's isometry, we have

$$\begin{aligned}
&\mathrm{Cov}\left[\sqrt{L}F_L(t_1),\sqrt{L}F_L(t_2)\right]\\
&=\frac{1}{L}\int_{[0,L]^2}\mathrm{Cov}\left[u(t_1,x),u(t_2,y)\right]\,\mathrm{d}x\mathrm{d}y\\
&=\frac{1}{L}\int_{[0,L]^2}\int_0^{t_1\wedge t_2}\int_0^L G_{t_1-s}(x,z)G_{t_2-s}(y,z)E[\sigma(u(s,z))^2]\,\mathrm{d}z\mathrm{d}s\mathrm{d}x\mathrm{d}y\\
&=\frac{1}{L}\int_0^{t_1\wedge t_2}\int_0^L \mathcal{I}_0(t_1-s,z)\mathcal{I}_0(t_2-s,z)E[\sigma(u(s,z))^2]\,\mathrm{d}z\mathrm{d}s,
\end{aligned} \tag{12}$$

where $\mathcal{I}_0$ is defined in Lemma 3.1. Moreover, we obtain

$$\begin{aligned}
&\mathrm{Cov}\left[\sqrt{L}F_L(t_1),\sqrt{L}F_L(t_2)\right]\\
&=\frac{1}{L}\int_0^{t_1\wedge t_2}\int_0^L\left[\mathcal{I}_0(t_1-s,z)\mathcal{I}_0(t_2-s,z)-e^{-\alpha(t_1+t_2-2s)}\right]E[\sigma(u(s,z))^2]\,\mathrm{d}z\mathrm{d}s\\
&\quad+\int_0^{t_1\wedge t_2}e^{-\alpha(t_1+t_2-2s)}\frac{1}{L}\int_0^L E[\sigma(u(s,z))^2]\,\mathrm{d}z\mathrm{d}s.
\end{aligned}$$

By (7),

$$\begin{aligned}
&\left|\frac{1}{L}\int_0^{t_1\wedge t_2}\int_0^L\left[\mathcal{I}_0(t_1-s,z)\mathcal{I}_0(t_2-s,z)-e^{-\alpha(t_1+t_2-2s)}\right]E[\sigma(u(s,z))^2]\,\mathrm{d}z\mathrm{d}s\right|\\
&\leq 2M_\sigma^2(1+c_{T,2}^2)\int_0^{t_1\wedge t_2}\left|\frac{1}{L}\int_0^L\mathcal{I}_0(t_1-s,z)\mathcal{I}_0(t_2-s,z)\,\mathrm{d}z-e^{-\alpha(t_1+t_2-2s)}\right|\,\mathrm{d}s,
\end{aligned}$$

where $M_\sigma > 0$ is the constant in the linear growth condition for σ. Hence, applying Lemma 3.1 and dominated convergence theorem, we obtain

$$\begin{aligned}
&\frac{1}{L}\int_0^{t_1\wedge t_2}\int_0^L\left[\mathcal{I}_0(t_1-s,z)\mathcal{I}_0(t_2-s,z)-e^{-\alpha(t_1+t_2-2s)}\right]E[\sigma(u(s,z))^2]\,\mathrm{d}z\mathrm{d}s\\
&\to 0 \text{ as } L\to\infty.
\end{aligned}$$

Then, we have

$$\begin{aligned}&\liminf_{L\to\infty} \mathrm{Cov}\left[\sqrt{L}F_L(t_1), \sqrt{L}F_L(t_2)\right]\\ &= \liminf_{L\to\infty} \int_0^{t_1\wedge t_2} e^{-\alpha(t_1+t_2-2s)} \frac{1}{L}\int_0^L E[\sigma(u(s,z))^2]\ \mathrm{d}z\mathrm{d}s\end{aligned} \tag{13}$$

.

In the case of Neumann or periodic boundary conditions, from Lemma 3.2, there exists $\delta > 0$, for every $L \geq 1$ and $(s, z) \in [0, \delta] \times [0, L]$,

$$E[\sigma(u(s,z))^2] \geq \frac{\sigma(1)^2}{2}. \tag{14}$$

In the case of Dirichlet boundary conditions, from Lemma 3.3, there exists $\delta > 0$, for every $L \geq 1$ and $s \in [0, \delta]$,

$$\frac{1}{L}\int_0^L E[\sigma(u(s,z))^2]\ \mathrm{d}z \geq \frac{\sigma(1)^2}{2}. \tag{15}$$

These together with (13) implies (11). □

Under Assumption 1, the following proposition can be proved by the same argument for all three cases of boundary conditions.

Proposition 3.2. *Suppose that Assumption 1 holds. For every $t_1, t_2 > 0$,*

$$\lim_{L\to\infty} \mathrm{Cov}\left[\sqrt{L}F_L(t_1), \sqrt{L}F_L(t_2)\right] = \int_0^{t_1\wedge t_2} e^{-\alpha(t_1+t_2-2s)} f_\sigma(s)\ \mathrm{d}s, \tag{16}$$

where the function f_σ is defined in Assumption 1.

Proof. By the same argument as in the proof of Proposition 3.1, for every $t_1, t_2 > 0$,

$$\begin{aligned}&\lim_{L\to\infty} \mathrm{Cov}\left[\sqrt{L}F_L(t_1), \sqrt{L}F_L(t_2)\right]\\ &= \lim_{L\to\infty} \int_0^{t_1\wedge t_2} e^{-\alpha(t_1+t_2-2s)} \frac{1}{L}\int_0^L E[\sigma(u(s,z))^2]\ \mathrm{d}z\mathrm{d}s\end{aligned}$$

This together with Assumption 1 implies (16). □

4. Proof of Theorem 1

Using stochastic Fubini's theorem, we have

$$F_L(t) = \int_0^t \int_0^L v_{L,t}(s,y)\ W(\mathrm{d}s, \mathrm{d}y) = \delta(v_{L,t})\ \text{a.s.}, \tag{17}$$

where

$$\begin{aligned}&v_{L,t}(s,y)\\ &:= \frac{1}{L} 1_{(0,t)}(s) 1_{[0,L]}(y) \sigma(u(s,y)) \int_0^L G_{t-s}(x,y)\ \mathrm{d}x\\ &= \frac{1}{L} 1_{(0,t)}(s) 1_{[0,L]}(y) \sigma(u(s,y)) \mathcal{I}_0(t-s,y),\end{aligned} \tag{18}$$

and δ is the adjoint of the Malliavin derivative operator.

We prepare the following proposition for the proof of our main results. Since the proof of Proposition 4.1 relies on Lemma 2.2, it should be noted that the argument is split into cases midway through: the case of Neumann or Dirichlet boundary conditions, and the case of periodic boundary conditions.

Proposition 4.1. *For every $T > 0$, there exists $A_T > 0$ such that*

$$\sup_{t,\tau\in[0,T]} \mathrm{Var}\left(\langle DF_L(t), v_{L,\tau}\rangle_{\mathcal{H}}\right) \le \frac{A_T}{L^3} \text{ for all } L \ge 1.$$

Proof. From Proposition 1.3.2 of [17] and (17), we have

$$D_{r,z}F_L(t) = 1_{(0,t)}(r)v_{L,t}(r,z) + 1_{(0,t)}(r)\int_r^t \int_0^L D_{r,z}v_{L,t}(s,y)\ W(\mathrm{d}s,\mathrm{d}y).$$

Hence, using the stochastic Fubini's theorem, we obtain

$$\begin{aligned}
&\langle DF_L(t), v_{L,\tau}\rangle_{\mathcal{H}} \\
&= \langle v_{L,t}, v_{L,\tau}\rangle_{\mathcal{H}} + \int_0^\tau \int_0^L v_{L,\tau}(r,z)\left(\int_r^t \int_0^L D_{r,z}v_{L,t}(s,y)\ W(\mathrm{d}s,\mathrm{d}y)\right)\mathrm{d}z\mathrm{d}r \\
&= \langle v_{L,t}, v_{L,\tau}\rangle_{\mathcal{H}} + \int_0^t \int_0^L \left(\int_0^{\tau\wedge s} \int_0^L v_{L,\tau}(r,z)D_{r,z}v_{L,t}(s,y)\ \mathrm{d}z\mathrm{d}r\right) W(\mathrm{d}s,\mathrm{d}y).
\end{aligned}$$

Therefore,

$$\mathrm{Var}\left(\langle DF_L(t), v_{L,\tau}\rangle_{\mathcal{H}}\right) \le 2(\Phi^{(1)}_{L,t,\tau} + \Phi^{(2)}_{L,t,\tau}),$$

where

$$\Phi^{(1)}_{L,t,\tau} = \mathrm{Var}\left(\langle v_{L,t}, v_{L,\tau}\rangle_{\mathcal{H}}\right),$$

$$\Phi^{(2)}_{L,t,\tau} = \mathrm{Var}\left(\int_0^t \int_0^L \left(\int_0^{\tau\wedge s} \int_0^L v_{L,\tau}(r,z)D_{r,z}v_{L,t}(s,y)\ \mathrm{d}z\mathrm{d}r\right) W(\mathrm{d}s,\mathrm{d}y)\right).$$

From (18),

$$\begin{aligned}
&\Phi^{(1)}_{L,t,\tau} \\
&= \frac{1}{L^4}\,\mathrm{Var}\left(\int_0^{t\wedge\tau} \int_0^L \sigma(u(s,y))^2\mathcal{I}_0(t-s,y)\mathcal{I}_0(\tau-s,y)\ \mathrm{d}y\mathrm{d}s\right) \\
&= \frac{1}{L^4}\int_{[0,t\wedge\tau]^2} \int_{[0,L]^2} \mathrm{Cov}\left[\sigma(u(s_1,y_1))^2, \sigma(u(s_2,y_2))^2\right] \\
&\quad \times \mathcal{I}_0(t-s_1,y_1)\mathcal{I}_0(\tau-s_1,y_1)\mathcal{I}_0(t-s_2,y_2)\mathcal{I}_0(\tau-s_2,y_2)\ \mathrm{d}y_1\mathrm{d}y_2\mathrm{d}s_1\mathrm{d}s_2.
\end{aligned}$$

By Lemma 3.1, Hölder's inequality and Minkowski inequality, we have

$$\begin{aligned}
&\Phi^{(1)}_{L,t,\tau} \\
&\leq \frac{4e^{4|\alpha|T}}{L^4} \int_{[0,t\wedge\tau]^2} \int_{[0,L]^2} \int_0^{s_1\wedge s_2} \int_{\mathbb{R}} \|\sigma(u(s_1,y_1))D_{r,z}\sigma(u(s_1,y_1))\|_2 \\
&\quad \times \|\sigma(u(s_2,y_2))D_{r,z}\sigma(u(s_2,y_2))\|_2 \,\mathrm{d}z\mathrm{d}r\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}s_1\mathrm{d}s_2 \\
&\leq \frac{4e^{4|\alpha|T}M_\sigma^2(1+c_{T,4})^2}{L^4} \int_{[0,t\wedge\tau]^2} \int_{[0,L]^2} \int_0^{s_1\wedge s_2} \int_{\mathbb{R}} \|D_{r,z}\sigma(u(s_1,y_1))\|_4 \|D_{r,z}\sigma(u(s_2,y_2))\|_4 \\
&\quad \times \mathrm{d}z\mathrm{d}r\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}s_1\mathrm{d}s_2,
\end{aligned}$$

where M_σ is the constant in the linear growth condition for σ. From the chain rule of Malliavin derivative for a Lipschitz function with constant K_σ [17],

$$D_{r,z}\sigma(u(s,y)) = G_{\sigma,u(s,y)}D_{r,z}u(s,y),$$

where $G_{\sigma,u(s,y)}$ is a random variable bounded by K_σ. Then, we obtain

$$\begin{aligned}
&\Phi^{(1)}_{L,t,\tau} \\
&\leq \frac{4e^{4|\alpha|T}M_\sigma^2(1+c_{T,4})^2K_\sigma^2}{L^4} \int_{[0,t\wedge\tau]^2} \int_{[0,L]^2} \int_0^{s_1\wedge s_2} \int_{\mathbb{R}} \|D_{r,z}u(s_1,y_1)\|_4 \|D_{r,z}u(s_2,y_2)\|_4 \\
&\quad \times \mathrm{d}z\mathrm{d}r\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}s_1\mathrm{d}s_2,
\end{aligned}$$

where K_σ is Lipschitz constant of σ. In the case of Neumann/Dirichlet boundary conditions, by Lemma 2.2, we have

$$\begin{aligned}
&\Phi^{(1)}_{L,t,\tau} \\
&\leq \frac{4e^{4|\alpha|T}M_\sigma^2(1+c_{T,4})^2K_\sigma^2C_{T,4}^2}{L^4} \int_{[0,t\wedge\tau]^2} \int_{[0,L]^2} \int_0^{s_1\wedge s_2} \int_{\mathbb{R}} p_{\beta(s_1-r)}(y_1-z)p_{\beta(s_2-r)}(y_2-z) \\
&\quad \times \mathrm{d}z\mathrm{d}r\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}s_1\mathrm{d}s_2 \\
&= \frac{4e^{4|\alpha|T}M_\sigma^2(1+c_{T,4})^2K_\sigma^2C_{T,4}^2}{L^4} \int_{[0,t\wedge\tau]^2} \int_{[0,L]^2} \int_0^{s_1\wedge s_2} p_{\beta(s_1+s_2-2r)}(y_1-y_2) \\
&\quad \times \mathrm{d}z\mathrm{d}r\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}s_1\mathrm{d}s_2,
\end{aligned}$$

where the equality holds by the semigroup property of Gaussian heat kernel $p_t(z)$. By using the identity

$$\int_{\mathbb{R}} p_{\beta(s_1+s_2-2r)}(y_1)\mathrm{d}y_1 = 1, \tag{19}$$

we obtain

$$\begin{aligned}
&\Phi^{(1)}_{L,t,\tau}\\
&\leq \frac{4e^{4|\alpha|T}M_\sigma^2(1+c_{T,4})^2K_\sigma^2C_{T,4}^2}{L^4}\int_{[0,t\wedge\tau]^2}\int_{[0,L]}\int_{\mathbb{R}}\int_0^{s_1\wedge s_2} p_{\beta(s_1+s_2-2r)}(y_1)\\
&\quad\times \mathrm{d}z\mathrm{d}r\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}s_1\mathrm{d}s_2\\
&= \frac{4e^{4|\alpha|T}M_\sigma^2(1+c_{T,4})^2K_\sigma^2C_{T,4}^2}{L^3}\int_{[0,t\wedge\tau]^2}\int_0^{s_1\wedge s_2}\mathrm{d}r\mathrm{d}s_1\mathrm{d}s_2\\
&\leq \frac{4T^3e^{4|\alpha|T}M_\sigma^2(1+c_{T,4})^2K_\sigma^2C_{T,4}^2}{L^3}.
\end{aligned}$$

In the case of periodic boundary conditions, from Lemma 2.2, we have

$$\begin{aligned}
&\Phi^{(1)}_{L,t,\tau}\\
&\leq \frac{4e^{4|\alpha|T}M_\sigma^2(1+c_{T,4})^2K_\sigma^2C_{T,4}^2}{L^4}\int_{[0,t\wedge\tau]^2}\int_{[0,L]^2}\int_0^{s_1\wedge s_2}\int_0^L G_{s_1-r}(y_1,z)G_{s_2-r}(y_2,r)\\
&\quad\times \mathrm{d}z\mathrm{d}r\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}s_1\mathrm{d}s_2\\
&= \frac{4e^{4|\alpha|T}M_\sigma^2(1+c_{T,4})^2K_\sigma^2C_{T,4}^2}{L^3}\int_{[0,t\wedge\tau]^2}\int_0^{s_1\wedge s_2} e^{-\alpha(s_1+s_2-r)}\mathrm{d}r\mathrm{d}s_1\mathrm{d}s_2\\
&\leq \frac{4T^3e^{6|\alpha|T}M_\sigma^2(1+c_{T,4})^2K_\sigma^2C_{T,4}^2}{L^3},
\end{aligned}$$

where the equality follows from (40).

We proceed to estimate $\Phi^{(2)}_{L,t,\tau}$. By Itô's isometry, we have

$$\begin{aligned}
&\Phi^{(2)}_{L,t,\tau}\\
&= \int_0^t\int_0^L\left\|\int_0^{\tau\wedge s}\int_0^L v_{L,\tau}(r,z)D_{r,z}v_{L,t}(s,y)\right\|_2^2\mathrm{d}y\mathrm{d}s\\
&= \frac{1}{L^4}\int_0^t\int_0^L\int_{[0,\tau\wedge s]^2}\int_{[0,L]^2}\mathcal{I}_0(\tau-r_1,z_1)\mathcal{I}_0(\tau-r_2,z_2)\mathcal{I}_0^2(t-s,y)\\
&\quad\times E[\sigma(u(r_1,z_1))(D_{r_1,z_1}\sigma(u(s,y)))\sigma(u(r_2,z_2))(D_{r_2,z_2}\sigma(u(s,y)))]\,\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}r_1\mathrm{d}r_2\mathrm{d}y\mathrm{d}s.
\end{aligned}$$

By Lemma 3.1, Hölder's inequality and Minkowski inequality,

$$\begin{aligned}
&\Phi^{(2)}_{L,t,\tau}\\
&\leq \frac{e^{4|\alpha|T}}{L^4}\int_0^t\int_0^L\int_{[0,\tau\wedge s]^2}\int_{[0,L]^2}\|\sigma(u(r_1,z_1))\|_4\|D_{r_1,z_1}\sigma(u(s,y))\|_4\\
&\quad\times\|\sigma(u(r_2,z_2))\|_4\|D_{r_2,z_2}\sigma(u(s,y))\|_4\,\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}r_1\mathrm{d}r_2\mathrm{d}y\mathrm{d}s\\
&\leq \frac{e^{4|\alpha|T}M_\sigma^2(1+c_{T,4})^2}{L^4}\int_0^t\int_0^L\int_{[0,\tau\wedge s]^2}\int_{[0,L]^2}\|D_{r_1,z_1}\sigma(u(s,y))\|_4\\
&\quad\times\|D_{r_2,z_2}\sigma(u(s,y))\|_4\,\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}r_1\mathrm{d}r_2\mathrm{d}y\mathrm{d}s,
\end{aligned}$$

where M_σ is the constant in the linear growth condition for σ. From the chain rule of Malliavin derivative for a Lipschitz function [17], we obtain

$$\begin{aligned}
&\Phi^{(2)}_{L,t,\tau} \\
&\leq \frac{e^{4|\alpha|T}M_\sigma^2(1+c_{T,4})^2}{L^4}\int_0^t\int_0^L\int_{[0,\tau\wedge s]^2}\int_{[0,L]^2}\|G_{\sigma,u(s,y)}D_{r_1,z_1}u(s,y)\|_4 \\
&\quad\times\|G_{\sigma,u(s,y)}D_{r_2,z_2}u(s,y)\|_4\,\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}r_1\mathrm{d}r_2\mathrm{d}y\mathrm{d}s \\
&\leq \frac{e^{4|\alpha|T}M_\sigma^2(1+c_{T,4})^2K_\sigma^2}{L^4}\int_0^t\int_0^L\int_{[0,\tau\wedge s]^2}\int_{[0,L]^2}\|D_{r_1,z_1}u(s,y)\|_4 \\
&\quad\times\|D_{r_2,z_2}u(s,y)\|_4\,\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}r_1\mathrm{d}r_2\mathrm{d}y\mathrm{d}s,
\end{aligned}$$

where K_σ is Lipschitz constant of σ and $G_{\sigma,u(s,y)}$ is a random variable bounded by K_σ. In the case of Neumann/Dirichlet boundary conditions, applying Lemma 2.2, we have

$$\begin{aligned}
&\Phi^{(2)}_{L,t,\tau} \\
&\leq \frac{e^{4|\alpha|T}M_\sigma^2(1+c_{T,4})^2K_\sigma^2C_{T,4}^2}{L^4}\int_0^t\int_0^L\int_{[0,\tau\wedge s]^2}\int_{[0,L]^2} \\
&\quad\times p_{\beta(s-r_1)}(y-z_1)p_{\beta(s-r_2)}(y-z_2)\,\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}r_1\mathrm{d}r_2\mathrm{d}y\mathrm{d}s \\
&\leq \frac{e^{4|\alpha|T}M_\sigma^2(1+c_{T,4})^2K_\sigma^2C_{T,4}^2}{L^4}\int_0^t\int_{[0,\tau\wedge s]^2}\int_{[0,L]^2}\int_{\mathbb{R}} \\
&\quad\times p_{\beta(s-r_1)}(y-z_1)p_{\beta(s-r_2)}(y-z_2)\,\mathrm{d}y\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}r_1\mathrm{d}r_2\mathrm{d}s \\
&= \frac{e^{4|\alpha|T}M_\sigma^2(1+c_{T,4})^2K_\sigma^2C_{T,4}^2}{L^4}\int_0^t\int_{[0,\tau\wedge s]^2}\int_{[0,L]^2} \\
&\quad\times p_{\beta(2s-r_1-r_2)}(z_1-z_2)\,\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}r_1\mathrm{d}r_2\mathrm{d}s,
\end{aligned}$$

where the equality holds by the semigroup property of Gaussian heat kernel $p_t(z)$. By using the identity (19), we obtain

$$\begin{aligned}
&\Phi^{(2)}_{L,t,\tau} \\
&\leq \frac{e^{4|\alpha|T}M_\sigma^2(1+c_{T,4})^2K_\sigma^2C_{T,4}^2}{L^4}\int_0^t\int_{[0,\tau\wedge s]^2}\int_0^L\int_{\mathbb{R}} \\
&\quad\times p_{\beta(2s-r_1-r_2)}(z_1-z_2)\,\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}r_1\mathrm{d}r_2\mathrm{d}s \\
&= \frac{e^{4|\alpha|T}M_\sigma^2(1+c_{T,4})^2K_\sigma^2C_{T,4}^2}{L^3}\int_0^t\int_{[0,\tau\wedge s]^2}\mathrm{d}r_1\mathrm{d}r_2\mathrm{d}s \\
&\leq \frac{T^3e^{4|\alpha|T}M_\sigma^2(1+c_{T,4})^2K_\sigma^2C_{T,4}^2}{L^3}
\end{aligned}$$

In the case of periodic boundary conditions, applying Lemma 2.2 and identity (40), we have

$$\begin{aligned}
&\Phi^{(2)}_{L,t,\tau} \\
&\leq \frac{e^{4|\alpha|T} M_\sigma^2 (1+c_{T,4})^2 K_\sigma^2 C_{T,4}^2}{L^4} \int_0^t \int_0^L \int_{[0,\tau\wedge s]^2} \int_{[0,L]^2} \\
&\quad \times G_{s-r_1}(y,z_1) G_{s-r_2}(y,z_2)\, \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}r_1 \mathrm{d}r_2 \mathrm{d}y \mathrm{d}s \\
&= \frac{e^{4|\alpha|T} M_\sigma^2 (1+c_{T,4})^2 K_\sigma^2 C_{T,4}^2}{L^3} \int_0^t \int_{[0,\tau\wedge s]^2} e^{-\alpha(2s-r_1-r_2)}\, \mathrm{d}r_1 \mathrm{d}r_2 \mathrm{d}s \\
&\leq \frac{T^3 e^{6|\alpha|T} M_\sigma^2 (1+c_{T,4})^2 K_\sigma^2 C_{T,4}^2}{L^3}.
\end{aligned}$$

This completes the proof. □

We are now ready to prove Theorem 1.

Proof of Theorem 1. Applying Proposition 4.1 with $t = \tau$, we have

$$\sup_{t\in[0,T]} \mathrm{Var}\left(\langle DF_L(t), v_{L,t}\rangle_{\mathcal{H}}\right) \leq \frac{A_T}{L^3}$$

for all $L \geq 1$. Then, by using (17) and Proposition 2.1,

$$\begin{aligned}
&d_{\mathrm{TV}}\left(\frac{F_L(t)}{\sqrt{\mathrm{Var}(F_L(t))}}, \mathcal{N}(0,1)\right) \\
&\leq 2\sqrt{\mathrm{Var}\left(\left\langle \frac{DF_L(t)}{\sqrt{\mathrm{Var}(F_L(t))}}, \frac{v_{L,t}}{\sqrt{\mathrm{Var}(F_L(t))}} \right\rangle_{\mathcal{H}}\right)} \\
&\leq \frac{2\sqrt{A_t}}{L^{3/2}\,\mathrm{Var}(F_L(t))}.
\end{aligned} \tag{20}$$

From Proposition 3.1, the asymptotic behavior of the variance is given by

$$\mathrm{Var}(F_L(t)) \gtrsim \frac{1}{L}\frac{\sigma(1)^2}{2} \int_0^{t\wedge\delta} e^{-2\alpha(t-s)}\, \mathrm{d}s, \tag{21}$$

as $L \to \infty$. Consequently, using this result along with $\sigma(1) \neq 0$ and (20), we can deduce (4). □

5. Proof of Theorem 2

Since the following proof employs Lemma 2.2, the argument is split into cases midway through: the case of Neumann or Dirichlet boundary conditions, and the case of periodic boundary conditions.

Proof. Using (17) and the Proposition 2.2 with $m = 1$, we have

$$
\begin{aligned}
&d_{\mathrm{W}}\left(\mathcal{L}\left(\frac{F_L(t)}{\sqrt{\mathrm{Var}(F_L(t))}}, u(t,x_0)\right), \mathcal{L}(Z, u(t,x_0))\right) \\
&\leq C'\left\{\sqrt{\mathrm{Var}\left(\left\langle \frac{DF_L(t)}{\sqrt{\mathrm{Var}(F_L(t))}}, \frac{v_{L,t}}{\sqrt{\mathrm{Var}(F_L(t))}}\right\rangle_{\mathcal{H}}\right)} + \sqrt{E\left[\left\langle \frac{v_{L,t}}{\sqrt{\mathrm{Var}(F_L(t))}}, Du(t,x_0)\right\rangle_{\mathcal{H}}^2\right]}\right\} \\
&= C'\left\{\frac{\sqrt{\mathrm{Var}(\langle DF_L(t), v_{L,t}\rangle_{\mathcal{H}})}}{\mathrm{Var}(F_L(t))} + \frac{\sqrt{E[\langle v_{L,t}, Du(t,x_0)\rangle_{\mathcal{H}}^2]}}{\sqrt{\mathrm{Var}(F_L(t))}}\right\}
\end{aligned}
$$

From (18), we can obtain

$$
\begin{aligned}
&\langle v_{L,t}, Du(t,x_0)\rangle_{\mathcal{H}} \\
&= \int_0^t \int_0^L \frac{1}{L}\sigma(u(s,y))\mathcal{I}_0(t-s,y)D_{s,y}u(t,x_0)\ \mathrm{d}y\mathrm{d}s.
\end{aligned}
$$

Then, using Hölder's inequality, we get

$$
\begin{aligned}
&E[\langle v_{L,t}, Du(t,x_0)\rangle_{\mathcal{H}}^2] \\
&= \int_0^t \int_0^L \int_0^t \int_0^L \frac{1}{L^2} E[\sigma(u(s_1,y_1))\sigma(u(s_2,y_2))D_{s_1,y_1}u(t,x_0)D_{s_2,y_2}u(t,x_0)] \\
&\quad \times \mathcal{I}_0(t-s_1,y_1)\mathcal{I}_0(t-s_2,y_2)\ \mathrm{d}y_1\mathrm{d}s_1\mathrm{d}y_2\mathrm{d}s_2 \\
&\leq \int_0^t \int_0^L \int_0^t \int_0^L \frac{1}{L^2} \|\sigma(u(s_1,y_1))\|_4 \|\sigma(u(s_2,y_2))\|_4 \\
&\quad \times \|D_{s_1,y_1}u(t,x_0)\|_4 \|D_{s_2,y_2}u(t,x_0)\|_4 \mathcal{I}_0(t-s_1,y_1)\mathcal{I}_0(t-s_2,y_2)\ \mathrm{d}y_1\mathrm{d}s_1\mathrm{d}y_2\mathrm{d}s_2.
\end{aligned}
$$

By Minkowski's inequality, Lemma 7 and Lemma 3.1,

$$
\begin{aligned}
&E[\langle v_{L,t}, Du(t,x_0)\rangle_{\mathcal{H}}^2] \\
&\leq \frac{4e^{2|\alpha|t}M_\sigma^2(1+c_{t,4})^2}{L^2} \\
&\quad \times \int_0^t \int_0^L \int_0^t \int_0^L \|D_{s_1,y_1}u(t,x_0)\|_4 \|D_{s_2,y_2}u(t,x_0)\|_4\ \mathrm{d}y_1\mathrm{d}s_1\mathrm{d}y_2\mathrm{d}s_2.
\end{aligned}
$$

In the case of Neumann/Dirichlet boundary conditions, using Lemma 2.2, we have

$$\begin{aligned}
&E[\langle v_{L,t}, Du(t,x_0)\rangle_{\mathcal{H}}^2] \\
&\le \frac{4e^{2|\alpha|t}M_\sigma^2(1+c_{t,4})^2C_{t,4}^2}{L^2} \\
&\quad \times \int_0^t\int_0^L\int_0^t\int_0^L p_{\beta(t-s_1)}(x_0-y_1)p_{\beta(t-s_2)}(x_0-y_2)\,\mathrm{d}y_1\mathrm{d}s_1\mathrm{d}y_2\mathrm{d}s_2 \\
&\le \frac{4e^{2|\alpha|t}M_\sigma^2(1+c_{t,4})^2C_{t,4}^2}{L^2}\int_0^t\int_0^t \mathrm{d}s_1\mathrm{d}s_2 \\
&= \frac{4e^{2|\alpha|t}t^2M_\sigma^2(1+c_{t,4})^2C_{t,4}^2}{L^2}.
\end{aligned}$$

In the case of periodic boundary conditions, using Lemma 2.2 and (40), we get

$$\begin{aligned}
&E[\langle v_{L,t}, Du(t,x_0)\rangle_{\mathcal{H}}^2] \\
&\le \frac{4e^{2|\alpha|t}M_\sigma^2(1+c_{t,4})^2C_{t,4}^2}{L^2} \\
&\quad \times \int_0^t\int_0^L\int_0^t\int_0^L G_{t-s_1}(x_0,y_1)G_{t-s_2}(x_0,y_2)\,\mathrm{d}y_1\mathrm{d}s_1\mathrm{d}y_2\mathrm{d}s_2 \\
&= \frac{4e^{2|\alpha|t}M_\sigma^2(1+c_{t,4})^2C_{t,4}^2}{L^2} \\
&\quad \times \int_0^t\int_0^t e^{-\alpha(t-s_1)}e^{-\alpha(t-s_2)}\,\mathrm{d}s_1\mathrm{d}s_2 \\
&\le \frac{4e^{4|\alpha|t}t^2M_\sigma^2(1+c_{t,4})^2C_{t,4}^2}{L^2}
\end{aligned}$$

Hence, by setting

$$D_t := 4e^{4|\alpha|t}t^2M_\sigma^2(1+c_{t,4})^2C_{t,4}^2,$$

we can obtain

$$E[\langle v_{L,t}, Du(t,x_0)\rangle_{\mathcal{H}}^2] \le \frac{D_t}{L^2}$$

in the case of Neumann/Dirichlet/periodic boundary conditions. Therefore, combining the above with Proposition 4.1 and (21), we have

$$\begin{aligned}
&d_{\mathrm{W}}\left(\mathcal{L}\left(\frac{F_L(t)}{\sqrt{\mathrm{Var}(F_L(t))}}, u(t,x_0)\right), \mathcal{L}(Z, u(t,x_0))\right) \\
&\le C'\left\{\frac{\sqrt{\mathrm{Var}(\langle DF_L(t), v_{L,t}\rangle_{\mathcal{H}})}}{\mathrm{Var}(F_L(t))} + \frac{\sqrt{E[\langle v_{L,t}, Du(t,x_0)\rangle_{\mathcal{H}}^2]}}{\sqrt{\mathrm{Var}(F_L(t))}}\right\} \\
&\le C'\left\{\frac{1}{\mathrm{Var}(F_L(t))}\frac{\sqrt{A_t}}{L^{3/2}} + \frac{1}{\sqrt{\mathrm{Var}(F_L(t))}}\frac{\sqrt{D_t}}{L}\right\} \\
&\le \frac{C(t)}{\sqrt{L}}.
\end{aligned}$$

This completes the proof. □

6. Proof of Theorem 3

We prepare the following proposition for the proof of Theorem 3. To proceed with the argument using the properties of the heat kernel for Neumann or periodic boundary conditions (40) and the expression for the Dirichlet heat kernel (38), the proof splits midway into two cases: the case of Neumann/periodic boundary conditions, and the case of Dirichlet boundary conditions.

Proposition 6.1. *For every $T > 0$ and $k \geq 2$, there exists $A_{T,k} > 0$ such that for all $t_1, t_2 \in [0, T]$,*

$$\|F_L(t_2) - F_L(t_1)\|_k \leq A_{T,k}|t_2 - t_1|^{1/2} L^{-1/2} \tag{22}$$

uniformly for all $L \geq 1$.

Proof. From (3),

$$F_L(t) = \frac{1}{L}\int_0^t \int_0^L \mathcal{I}_0(t-s, y)\sigma(u(s,y))\ W(\mathrm{d}s, \mathrm{d}y). \tag{23}$$

We assume that $t_1 \leq t_2$. Applying Burkholder's inequality, for all $k \geq 2$,

$$\begin{aligned}
&\|F_L(t_2) - F_L(t_1)\|_k^2 \\
&\leq \frac{2z_k^2}{L^2}\left(\int_{t_1}^{t_2}\int_0^L \mathcal{I}_0^2(t_2 - s, y)\|\sigma(u(s,y))\|_k^2\ \mathrm{d}y\mathrm{d}s\right. \\
&\quad \left. + \int_0^{t_1}\int_0^L (\mathcal{I}_0(t_2 - s, y) - \mathcal{I}_0(t_1 - s, y))^2\|\sigma(u(s,y))\|_k^2\ \mathrm{d}y\mathrm{d}s\right),
\end{aligned}$$

where z_k is the constant in Burkholder's inequality. By using (7), Minkowski's inequality and the change of variables,

$$\begin{aligned}
&\|F_L(t_2) - F_L(t_1)\|_k^2 \\
&\leq \frac{2z_k^2 M_\sigma^2(1 + c_{T,k})^2}{L^2}\left(\int_{t_1}^{t_2}\int_0^L \mathcal{I}_0^2(t_2 - s, y)\ \mathrm{d}y\mathrm{d}s\right. \\
&\quad \left. + \int_0^{t_1}\int_0^L (\mathcal{I}_0(t_2 - s, y) - \mathcal{I}_0(t_1 - s, y))^2\ \mathrm{d}y\mathrm{d}s\right) \\
&= \frac{2z_k^2 M_\sigma^2(1 + c_{T,k})^2}{L^2}\left(\int_{t_1}^{t_2}\int_0^L \mathcal{I}_0^2(t_2 - s, y)\ \mathrm{d}y\mathrm{d}s\right. \\
&\quad \left. + \int_0^{t_1}\int_0^L (\mathcal{I}_0(t_2 - t_1 + s, y) - \mathcal{I}_0(s, y))^2\ \mathrm{d}y\mathrm{d}s\right),
\end{aligned} \tag{24}$$

where M_σ is the constant in the linear growth condition for σ. In the case of Neumann/periodic boundary conditions, using $\mathcal{I}_0(t,y) = e^{-\alpha t}$(see Lemma 3.1 and (40) in Lemma A.1),

$$\begin{aligned}
&\|F_L(t_2) - F_L(t_1)\|_k^2 \\
&\leq \frac{2z_k^2 M_\sigma^2 (1+c_{T,k})^2}{L^2} \left(\int_{t_1}^{t_2} \int_0^L e^{-2\alpha(t_2-s)} \, \mathrm{d}y\mathrm{d}s \right. \\
&\quad \left. + \int_0^{t_1} \int_0^L (e^{-\alpha(t_2-t_1+s)} - e^{-\alpha s})^2 \, \mathrm{d}y\mathrm{d}s \right).
\end{aligned}$$

From the following inequality

$$|e^{-\alpha(t_2-t_1)} - 1| \leq |\alpha| e^{|\alpha|T} |t_2 - t_1| \quad \text{for } t_1, t_2 \in [0,T],$$

we have

$$\begin{aligned}
&\|F_L(t_2) - F_L(t_1)\|_k^2 \\
&\leq \frac{2z_k^2 M_\sigma^2 (1+c_{T,k})^2}{L^2} \left(\int_{t_1}^{t_2} \int_0^L e^{2|\alpha|T} \, \mathrm{d}y\mathrm{d}s \right. \\
&\quad \left. + \int_0^{t_1} \int_0^L e^{4|\alpha|T} |\alpha|^2 |t_2 - t_1|^2 \, \mathrm{d}y\mathrm{d}s \right) \\
&\leq \frac{2z_k^2 M_\sigma^2 (1+c_{T,k})^2}{L} \left(e^{2|\alpha|T} |t_2 - t_1| + T e^{4|\alpha|T} |\alpha|^2 |t_2 - t_1|^2 \right) \\
&\leq \frac{2z_k^2 M_\sigma^2 (1+c_{T,k})^2}{L} \left(e^{2|\alpha|T} + T^2 e^{4|\alpha|T} |\alpha|^2 \right) |t_2 - t_1|.
\end{aligned}$$

In the case of Dirichlet boundary conditions, using the representation of Dirichlet heat kernel (38),

$$\begin{aligned}
&\mathcal{I}_0(t_2 - t_1 + s, y) - \mathcal{I}_0(s, y) \\
&= \frac{2}{L} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi y}{L}\right) \int_0^L \sin\left(\frac{n\pi z}{L}\right) \, \mathrm{d}z \left[e^{-\alpha(t_2-t_1+s)} e^{-\frac{n^2\pi^2(t_2-t_1+s)}{2L^2}} - e^{-\alpha s} e^{-\frac{n^2\pi^2 s}{2L^2}} \right] \\
&= \frac{2e^{-\alpha s}}{L} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi y}{L}\right) \frac{1 - \cos(n\pi)}{n\pi} e^{-\frac{n^2\pi^2 s}{2L^2}} \left[e^{-\alpha(t_2-t_1)} e^{-\frac{n^2\pi^2(t_2-t_1)}{2L^2}} - 1 \right].
\end{aligned}$$

Then, applying the $L^2([0,L])$-orthogonality of the functions $\{y \mapsto \sin(n\pi y/L)\}_{n=1}^{\infty}$, we obtain

$$\begin{aligned}
&\int_0^{t_1} \int_0^L (\mathcal{I}_0(t_2 - t_1 + s, y) - \mathcal{I}_0(s, y))^2 \, \mathrm{d}y\mathrm{d}s \\
&= 4 \int_0^{t_1} e^{-2\alpha s} \sum_{n=1}^{\infty} \int_0^L \sin^2\left(\frac{n\pi y}{L}\right) \, \mathrm{d}y \left(\frac{1 - \cos(n\pi)}{n\pi} \right)^2 e^{-\frac{n^2\pi^2 s}{L^2}} \left[e^{-\alpha(t_2-t_1)} e^{-\frac{n^2\pi^2(t_2-t_1)}{2L^2}} - 1 \right]^2 \, \mathrm{d}s \\
&= 2L \int_0^{t_1} e^{-2\alpha s} \sum_{n=1}^{\infty} \left(\frac{1 - \cos(n\pi)}{n\pi} \right)^2 e^{-\frac{n^2\pi^2 s}{L^2}} \left[e^{-\alpha(t_2-t_1)} e^{-\frac{n^2\pi^2(t_2-t_1)}{2L^2}} - 1 \right]^2 \, \mathrm{d}s \\
&\leq 8L \int_0^{t_1} e^{-2\alpha s} \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} e^{-\frac{n^2\pi^2 s}{L^2}} \left[e^{-\alpha(t_2-t_1)} e^{-\frac{n^2\pi^2(t_2-t_1)}{2L^2}} - 1 \right]^2 \, \mathrm{d}s.
\end{aligned}$$

By using the following inequality

$$\begin{aligned}
&\left|e^{-\alpha(t_2-t_1)}e^{-\frac{n^2\pi^2(t_2-t_1)}{2L^2}}-1\right| \\
&\le \left|e^{-\alpha(t_2-t_1)}e^{-\frac{n^2\pi^2(t_2-t_1)}{2L^2}}-e^{-\frac{n^2\pi^2(t_2-t_1)}{2L^2}}\right|+\left|e^{-\frac{n^2\pi^2(t_2-t_1)}{2L^2}}-1\right| \\
&= e^{-\frac{n^2\pi^2(t_2-t_1)}{2L^2}}\left|e^{-\alpha(t_2-t_1)}-1\right|+\left|e^{-\frac{n^2\pi^2(t_2-t_1)}{2L^2}}-1\right| \\
&\le |\alpha|e^{|\alpha|T}|t_2-t_1|+\left|e^{-\frac{n^2\pi^2(t_2-t_1)}{2L^2}}-1\right| \quad \text{for } t_1,t_2\in[0,T],
\end{aligned}$$

we have

$$\begin{aligned}
&\int_0^{t_1}\int_0^L \left(\mathcal{I}_0(t_2-t_1+s,y)-\mathcal{I}_0(s,y)\right)^2\,\mathrm{d}y\mathrm{d}s \\
&\le 16Le^{2|\alpha|T}\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}\int_0^{t_1}e^{-\frac{n^2\pi^2 s}{L^2}}\,\mathrm{d}s\left(|\alpha|^2e^{2|\alpha|T}|t_2-t_1|^2+\left|e^{-\frac{n^2\pi^2(t_2-t_1)}{2L^2}}-1\right|^2\right) \\
&= \frac{16}{6}L|\alpha|^2Te^{4|\alpha|T}|t_2-t_1|^2+16Le^{2|\alpha|T}\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}\int_0^{t_1}e^{-\frac{n^2\pi^2 s}{L^2}}\,\mathrm{d}s\left|e^{-\frac{n^2\pi^2(t_2-t_1)}{2L^2}}-1\right|^2.
\end{aligned}$$

Using the fact that $1-e^{-x}\le 1\wedge x$ for all $x\ge 0$,

$$\begin{aligned}
&16Le^{2|\alpha|T}\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}\int_0^{t_1}e^{-\frac{n^2\pi^2 s}{L^2}}\,\mathrm{d}s\left|e^{-\frac{n^2\pi^2(t_2-t_1)}{2L^2}}-1\right|^2 \\
&= 16Le^{2|\alpha|T}\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}\frac{L^2}{n^2\pi^2}\left(1-e^{-\frac{n^2\pi^2 t_1}{L^2}}\right)\left|e^{-\frac{n^2\pi^2(t_2-t_1)}{2L^2}}-1\right|^2 \\
&\le 16Le^{2|\alpha|T}\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}\left|1\wedge\frac{n^2\pi^2(t_2-t_1)}{2L^2}\right|^2\frac{L^2}{n^2\pi^2} \\
&\le \frac{16e^{2|\alpha|T}}{L}\sum_{n=1}^{\infty}\left(\frac{L^4}{n^4\pi^4}\wedge|t_2-t_1|^2\right).
\end{aligned}$$

Moreover, we obtain

$$\begin{aligned}
&\frac{16e^{2|\alpha|T}}{L}\sum_{n=1}^{\infty}\left(\frac{L^4}{n^4\pi^4}\wedge|t_2-t_1|^2\right) \\
&\le \frac{16e^{2|\alpha|T}}{L}\left(\sum_{n\le|t_2-t_1|^{-1/2}L/\pi}|t_2-t_1|^2+\frac{L^4}{\pi^4}\sum_{n>|t_2-t_1|^{-1/2}L/\pi}\frac{1}{n^4}\right) \\
&\le \frac{16e^{2|\alpha|T}}{L}\left(\frac{L}{\pi}|t_2-t_1|^{3/2}+\frac{L^4}{\pi^4}\int_{|t_2-t_1|^{-1/2}L/(2\pi)}^{\infty}\frac{1}{y^4}\,\mathrm{d}y\right) \\
&= \frac{176}{3\pi}e^{2|\alpha|T}|t_2-t_1|^{3/2}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\int_0^{t_1}\int_0^L (\mathcal{I}_0(t_2-t_1+s,y)-\mathcal{I}_0(s,y))^2 \,\mathrm{d}y\mathrm{d}s\\
&\le \frac{16}{6}L|\alpha|^2Te^{4|\alpha|T}|t_2-t_1|^2+\frac{176}{3\pi}e^{2|\alpha|T}|t_2-t_1|^{3/2}\\
&\le \left(\frac{16}{6}L|\alpha|^2T^2e^{4|\alpha|T}+\frac{176}{3\pi}T^{1/2}e^{2|\alpha|T}\right)|t_2-t_1|.
\end{aligned}$$

Hence, from (24) and Lemma 3.1, we obtain

$$\begin{aligned}
&\|F_L(t_2)-F_L(t_1)\|_k^2\\
&\le \frac{2z_k^2M_\sigma^2(1+c_{T,k})^2}{L^2}\left(\int_{t_1}^{t_2}\int_0^L \mathcal{I}_0^2(t_2-s,y)\,\mathrm{d}y\mathrm{d}s\right.\\
&\quad\left.+\int_0^{t_1}\int_0^L (\mathcal{I}_0(t_2-t_1+s,y)-\mathcal{I}_0(s,y))^2\,\mathrm{d}y\mathrm{d}s\right)\\
&\le \frac{2z_k^2M_\sigma^2(1+c_{T,k})^2}{L^2}\left(Le^{2|\alpha|T}|t_2-t_1|+\left(\frac{16}{6}L|\alpha|^2T^2e^{4|\alpha|T}+\frac{176}{3\pi}T^{1/2}e^{2|\alpha|T}\right)|t_2-t_1|\right)\\
&\le \frac{2z_k^2M_\sigma^2(1+c_{T,k})^2}{L}\left(e^{2|\alpha|T}+\frac{16}{6}|\alpha|^2T^2e^{4|\alpha|T}+\frac{176}{3\pi}T^{1/2}e^{2|\alpha|T}\right)|t_2-t_1|.
\end{aligned}$$

Therefore, in the case of Neumann, Dirichlet, or periodic boundary conditions, we have (22). This completes the proof. □

Proof of Theorem 3. The tightness of the family of processes $\{\sqrt{L}F_L(\cdot)\}_{L\ge1}$ in $C([0,T])$ is guaranteed by Proposition 6.1 (see, e.g., [12]). What remains is to show that the finite-dimensional distributions of $(\sqrt{L}F_L(t))_{t\in[0,T]}$ converge to those of $(\int_0^t e^{-\alpha(t-s)}\sqrt{f_\sigma(s)}\,\mathrm{d}W_s)_{t\in[0,T]}$ as $L\to\infty$

Let us fix $T>0$ and $m\ge1$ points $t_1,\dots,t_m\in(0,T]$. According to Proposition 3.2, as $L\to\infty$, the covariance satisfies

$$\mathrm{Cov}\left[F_L(t_i),F_L(t_j)\right]\sim\frac{1}{L}\int_0^{t_i\wedge t_j}e^{-\alpha(t_i+t_j-2s)}f_\sigma(s)\,\mathrm{d}s \tag{25}$$

for all $i,j=1,\dots,m$. To proceed, let us define the vector $F:=(F_1,\dots,F_m)$ with components

$$F_i:=\frac{F_L(t_i)}{\sqrt{\mathrm{Var}(F_L(t_i))}}.$$

We then define $G=(G_1,\dots,G_m)$ to be a centered Gaussian random vector whose covariance matrix $C=(C_{i,j})$ is given by

$$C_{i,j}:=\mathrm{Cov}\left[F_i,F_j\right].$$

Let us introduce the rescaled random fields $V_1,\dots,V_m$ by setting

$$V_i:=\frac{v_{L,t_i}}{\sqrt{\mathrm{Var}(F_L(t_i))}},\quad i=1,\dots,m,$$

where the fields v_{L,t_i} are given in (18). From (17), we have the relation $F_i=\delta(V_i)$. By duality, $E[\langle DF_i,V_j\rangle_{\mathcal{H}}]=E[F_i\delta(V_j)]=C_{i,j}$ for all $i,j=1,\dots,m$. Applying Proposition 2.3 then yields the

desired bound for any $h \in C^2(\mathbb{R}^m)$:

$$|E[h(F)] - E[h(G)]| \le \frac{1}{2}\|h''\|_\infty \sqrt{\sum_{i,j=1}^m \operatorname{Var}\left(\langle DF_i, V_j\rangle_{\mathcal{H}}\right)}.$$

From Proposition 4.1,

$$\begin{aligned}\operatorname{Var}\left(\langle DF_i, V_j\rangle_{\mathcal{H}}\right) &= \frac{\operatorname{Var}\left(\langle DF_L(t_i), v_{L,t_j}\rangle_{\mathcal{H}}\right)}{\operatorname{Var}(F_L(t_i))\operatorname{Var}(F_L(t_j))} \\ &\le \frac{A_T}{L^3 \operatorname{Var}(F_L(t_i))\operatorname{Var}(F_L(t_j))},\end{aligned}$$

which together with (25) implies that

$$\lim_{L\to\infty} |E[h(F)] - E[h(G)]| = 0$$

for all $h \in C^2(\mathbb{R}^m)$.

We also examine the limit of the covariances $C_{i,j}$. Using the result from (25), we find that as $L \to \infty$,

$$C_{i,j} \to \frac{\int_0^{t_i \wedge t_j} e^{-\alpha(t_i+t_j-2s)} f_\sigma(s)\, \mathrm{d}s}{\sqrt{\int_0^{t_i} e^{-2\alpha(t_i-s)} f_\sigma(s)\, \mathrm{d}s}\sqrt{\int_0^{t_j} e^{-2\alpha(t_j-s)} f_\sigma(s)\, \mathrm{d}s}}.$$

It follows that the law of the Gaussian vector G, which is determined by its covariance structure, converges weakly to that of

$$\left(\frac{\int_0^{t_1} e^{-\alpha(t_1-s)}\sqrt{f_\sigma(s)}\, \mathrm{d}W_s}{\sqrt{\int_0^{t_1} e^{-2\alpha(t_1-s)} f_\sigma(s)\, \mathrm{d}s}}, \dots, \frac{\int_0^{t_m} e^{-\alpha(t_m-s)}\sqrt{f_\sigma(s)}\, \mathrm{d}W_s}{\sqrt{\int_0^{t_m} e^{-2\alpha(t_m-s)} f_\sigma(s)\, \mathrm{d}s}}\right). \tag{26}$$

Therefore, F converges weakly to the random vector in (26) as $L \to \infty$. Recalling the definition of F_i and using the asymptotic variance from (25), the random vector

$$\sqrt{L}\left(\frac{F_L(t_1)}{\sqrt{\int_0^{t_1} e^{-2\alpha(t_1-s)} f_\sigma(s)\, \mathrm{d}s}}, \dots, \frac{F_L(t_m)}{\sqrt{\int_0^{t_m} e^{-2\alpha(t_m-s)} f_\sigma(s)\, \mathrm{d}s}}\right)$$

converges to the random vector in (26) as $L \to \infty$. This completes the proof. □

7. On the limit in Assumption 1

If $\sigma(u) = \sigma_1 u + \sigma_0$ for some constants σ_1 and σ_0 with $(\sigma_1, \sigma_0) \neq (0, 0)$, we can calculate the limit $f_\sigma(t)$ in Assumption 1. Our calculation is based on the Wiener chaos decomposition of $u(t, x)$. Let

$$u_0(t, x) = \int_0^L G_t(x, y)\, \mathrm{d}y$$

for every $(0, T] \times [0, L]$ and $u_0(0, x) = u_0(x) = 1$ for all $x \in [0, L]$. Let

$$\mathcal{I}_0(t, x) := u_0(t, x)$$

and

$$\begin{aligned}\mathcal{I}_k(t,x) :=& \sigma_1^{k-1} \int_0^t \int_0^L \cdots \int_0^{r_{k-1}} \int_0^L G_{t-r_1}(x,z_1) \dots G_{r_{k-1}-r_k}(z_{k-1},z_k) \\ & \times (\sigma_1 \mathcal{I}_0(r_k,z_k) + \sigma_0)\ W(\mathrm{d}r_k, \mathrm{d}z_k) \dots W(\mathrm{d}r_1, \mathrm{d}z_1)\end{aligned}$$

for all $k \geq 1$. We define the Picard iteration $\{u_n(t,x)\}_{n=0}^{\infty}$ for $u(t,x)$. Define iteratively, for every $n \geq 0$,

$$u_{n+1}(t,x) := u_0(t,x) + \int_0^t \int_0^L G_{t-r}(x,z)\sigma(u_n(r,z))\ W(\mathrm{d}r, \mathrm{d}z). \tag{27}$$

Since we only consider the case $\sigma(u) = \sigma_1 u + \sigma_0$ through this section, we have

$$\begin{aligned}& u_{n+1}(t,x) \\ &= u_0(t,x) + \int_0^t \int_0^L G_{t-r_1}(x,z_1)(\sigma_1 u_n(r_1,z_1) + \sigma_0)\ W(\mathrm{d}r_1, \mathrm{d}z_1).\end{aligned}$$

Then, by using mathematical induction, we obtain the Wiener chaos decomposition

$$u_{n+1}(t,x) = \sum_{k=0}^{n} \mathcal{I}_k(t,x). \tag{28}$$

Therefore, $u_n \to u$ in L^p $(p \geq 2)$ (see [25]) and applying (28), we have

$$u(t,x) = \sum_{k=0}^{\infty} \mathcal{I}_k(t,x). \tag{29}$$

Moreover, by multiple Itô's isometry,

$$\|u(t,x)\|_2^2 = \sum_{k=0}^{\infty} \|\mathcal{I}_k(t,x)\|_2^2 \tag{30}$$

where

$$\begin{aligned}\|\mathcal{I}_k(t,x)\|_2^2 =& \sigma_1^{2(k-1)} \int_0^t \int_0^L \cdots \int_0^{r_{k-1}} \int_0^L G_{t-r_1}^2(x,z_1) \dots G_{r_{k-1}-r_k}^2(z_{k-1},z_k) \\ & \times (\sigma_1 \mathcal{I}_0(r_k,z_k) + \sigma_0)^2\ \mathrm{d}r_k \mathrm{d}z_k \dots \mathrm{d}r_1 \mathrm{d}z_1\end{aligned}$$

for all $k \geq 1$.

Using Lemma 3.1, we get the following Propositions.

Proposition 7.1. *Fix $T > 0$. In the case of Neumann/Dirichlet boundary conditions, for every $k \geq 1$*

$$\sup_{L \geq 1} \sup_{(t,x) \in [0,T] \times [0,L]} \|\mathcal{I}_k(t,x)\| \leq \sigma_1^{2(k-1)} (|\sigma_1| e^{|\alpha|T} + |\sigma_0|)^2 \frac{K_T^{2k} (4\beta)^{-k/2} T^{k/2}}{\Gamma((k+2)/2)}, \tag{31}$$

where the constant K_T is defined in (41)*. Moreover, for every $t > 0$ and $k \geq 1$, there exists $f_k(t)$ such that*

$$\lim_{L \to \infty} \frac{1}{L} \int_0^L \|\mathcal{I}_k(t,x)\|_2^2\ \mathrm{d}x = f_k(t).$$

Proof.

$$
\begin{aligned}
&\|\mathcal{I}_k(t,x)\|_2^2 \\
&\le \sigma_1^{2(k-1)} K_T^{2k} (|\sigma_1| e^{|\alpha| T} + |\sigma_0|)^2 \\
&\quad \times \int_0^t \int_{\mathbb{R}} \cdots \int_0^{r_{k-1}} \int_{\mathbb{R}} p^2_{\beta(t-r_1)}(x - z_1) \dots p^2_{\beta(r_{k-1}-r_k)}(z_{k-1} - z_k) \, \mathrm{d}r_k \mathrm{d}z_k \dots \mathrm{d}r_1 \mathrm{d}z_1 \\
&= \sigma_1^{2(k-1)} K_T^{2k} (|\sigma_1| e^{|\alpha| T} + |\sigma_0|)^2 \left(\frac{t}{4\pi\beta} \right)^{k/2} \\
&\quad \times \int_{0 < r_k < \cdots < r_1 < 1} \sqrt{\frac{1}{(1-r_1) \times \cdots \times (r_{k-1} - r_k)}} \, \mathrm{d}r_k \dots \mathrm{d}r_1 \\
&= \sigma_1^{2(k-1)} K_T^{2k} (|\sigma_1| e^{|\alpha| T} + |\sigma_0|)^2 \left(\frac{t}{4\pi\beta} \right)^{k/2} \frac{\Gamma(1/2)^k}{\Gamma((k+2)/2)} \\
&\le \sigma_1^{2(k-1)} (|\sigma_1| e^{|\alpha| T} + |\sigma_0|)^2 \frac{K_T^{2k} (4\beta)^{-k/2} T^{k/2}}{\Gamma((k+2)/2)}
\end{aligned}
$$

where the first equality follows from semigroup property of Gaussian heat kernel $p_t(z)$ and change of variables, and the second one holds by the following identity

$$
\int_{0 < r_k < \cdots < r_1 < 1} \sqrt{\frac{1}{(1-r_1) \times \cdots \times (r_{k-1} - r_k)}} \, \mathrm{d}r_k \mathrm{d}z_k \dots \mathrm{d}r_1 \mathrm{d}z_1 = \frac{\Gamma(1/2)^k}{\Gamma((k+2)/2)}, \tag{32}
$$

see [20, 5.14.1]. Then, we have (31).

Using the same arguments as [21, Proposition 3.4.], we obtain

$$
\begin{aligned}
&\frac{1}{L} \int_0^L \|\mathcal{I}_k(t,x)\|_2^2 \, \mathrm{d}x \\
&= \sigma_1^{2(k-1)} \int_{0 < r_k < \cdots < r_1 < t} e^{-2\alpha(t-r_1)} p_{2\beta(t-r_1)}(0) \dots e^{-2\alpha(r_{k-1}-r_k)} p_{2\beta(r_{k-1}-r_k)}(0) \\
&\quad \times \frac{1}{L} \int_0^L (\sigma_1 \mathcal{I}_0(r_k, z_k) + \sigma_0)^2 \, \mathrm{d}z_k \mathrm{d}r_k \dots \mathrm{d}r_1 \\
&\quad + o(L).
\end{aligned}
$$

Therefore, using dominated convergence theorem, (31) and Lemma 3.1, we have

$$
\begin{aligned}
&\lim_{L\to\infty}\frac{1}{L}\int_0^L \|\mathcal{I}_k(t,x)\|_2^2\,\mathrm{d}x\\
&=\sigma_1^{2(k-1)}\int_{0<r_k<\cdots<r_1<t} e^{-2\alpha(t-r_1)}p_{2\beta(t-r_1)}(0)\ldots e^{-2\alpha(r_{k-1}-r_k)}p_{2\beta(r_{k-1}-r_k)}(0)\\
&\quad\times\lim_{L\to\infty}\frac{1}{L}\int_0^L(\sigma_1\mathcal{I}_0(r_k,z_k)+\sigma_0)^2\,\mathrm{d}z_k\mathrm{d}r_k\ldots\mathrm{d}r_1\\
&=\sigma_1^{2(k-1)}\int_{0<r_k<\cdots<r_1<t} e^{-2\alpha(t-r_1)}p_{2\beta(t-r_1)}(0)\ldots e^{-2\alpha(r_{k-1}-r_k)}p_{2\beta(r_{k-1}-r_k)}(0)\\
&\quad\times(\sigma_1^2e^{-2\alpha r_k}+2\sigma_1\sigma_0e^{-\alpha r_k}+\sigma_0^2)\,\mathrm{d}r_k\ldots\mathrm{d}r_1\\
&=\sigma_1^{2(k-1)}e^{-2\alpha t}\int_{0<r_k<\cdots<r_1<t} p_{2\beta(t-r_1)}(0)\ldots p_{2\beta(r_{k-1}-r_k)}(0)\\
&\quad\times(\sigma_1+e^{\alpha r_k}\sigma_0)^2\,\mathrm{d}r_k\ldots\mathrm{d}r_1\\
&=:f_k(t).
\end{aligned}
\tag{33}
$$

□

Proposition 7.2. *In the case of periodic boundary conditions, for all $k\geq 1$ and $t>0$, $\|\mathcal{I}_k(t,x)\|_2^2$ does not depend on $x\in[0,L]$ and*

$$\lim_{L\to\infty}\|\mathcal{I}_k(t,x)\|_2^2=f_k(t)$$

where $f_k(t)$ is the limit in (33).

Proof. Since $\mathcal{I}_0(t,x)=e^{-\alpha t}$(see Lemma 3.1 and (40) in Lemma A.1), we have

$$
\begin{aligned}
&\|\mathcal{I}_k(t,x)\|_2^2\\
&=\sigma_1^{2(k-1)}\int_0^t\int_0^L\cdots\int_0^{r_{k-1}}\int_0^L G_{t-r_1}^2(x,z_1)\ldots G_{r_{k-1}-r_k}^2(z_{k-1},z_k)\\
&\quad\times(\sigma_1e^{-\alpha r_k}+\sigma_0)^2\,\mathrm{d}r_k\mathrm{d}z_k\ldots\mathrm{d}r_1\mathrm{d}z_1\\
&=\sigma_1^{2(k-1)}\int_{0<r_k<\cdots<r_1<t} G_{2(t-r_1)}(0,0)\ldots G_{2(r_{k-1}-r_k)}(0,0)(\sigma_1e^{-\alpha r_k}+\sigma_0)^2\,\mathrm{d}r_1\ldots\mathrm{d}r_k
\end{aligned}
$$

where in the second equality follows from semigroup property of G. Then, it follows that $\|\mathcal{I}_k(t,x)\|_2^2$ does not depend on $x \in [0,L]$. Moreover, by dominated convergence theorem,

$$\begin{aligned}
&\lim_{L\to\infty} \|\mathcal{I}_k(t,x)\|_2^2 \\
&= \sigma_1^{2(k-1)} \int_{0<r_k<\cdots<r_1<t} e^{-2\alpha(t-r_1)} p_{2\beta(t-r_1)}(0) \dots e^{-2\alpha(r_{k-1}-r_k)} p_{2\beta(r_{k-1}-r_k)}(0) \\
&\quad \times (\sigma_1 e^{-\alpha r_k} + \sigma_0)^2 \, \mathrm{d}r_1 \dots \mathrm{d}r_k \\
&= \sigma_1^{2(k-1)} e^{-2\alpha t} \int_{0<r_k<\cdots<r_1<t} p_{2\beta(t-r_1)}(0) \dots p_{2\beta(r_{k-1}-r_k)}(0) \\
&\quad \times (\sigma_1 + \sigma_0 e^{\alpha r_k})^2 \, \mathrm{d}r_1 \dots \mathrm{d}r_k \\
&= f_k(t),
\end{aligned}$$

where f_k is the same as in Proposition 7.1. □

By Propositions 7.1 and 7.2, Assumption 1 holds for all three boundary conditions when σ is affine. Specifically, we obtain the following statement.

Proposition 7.3. *For every $t > 0$,*

$$\begin{aligned}
&\lim_{L\to\infty} \frac{1}{L} \int_0^L E[\sigma(u(t,x))^2] \, \mathrm{d}x \\
&= \sigma_1^2 \sum_{k=0}^{\infty} f_k(t) + 2\sigma_1\sigma_0 e^{-\alpha t} + \sigma_0^2 \\
&=: f_\sigma(t),
\end{aligned}$$

where $\sigma(u) = \sigma_1 u + \sigma_0$, $f_0(t) := e^{-2\alpha t}$ and f_k is defined in (33)*. Moreover,*

$$\begin{aligned}
f_\sigma(t) \le e^{-2\alpha t}(|\sigma_1| + e^{|\alpha| t}|\sigma_0|)^2 (f(\sigma_1^4 t/\beta) - 1) \\
+ (\sigma_1 e^{-\alpha t} + \sigma_0)^2,
\end{aligned}$$

and

$$\begin{aligned}
f_\sigma(t) \ge e^{-2\alpha t} \inf_{r\in[0,t]} (\sigma_1 + e^{\alpha r}\sigma_0)^2 (f(\sigma_1^4 t/\beta) - 1) \\
+ (\sigma_1 e^{-\alpha t} + \sigma_0)^2,
\end{aligned}$$

where

$$f(t) := 2e^{t/4} \int_{-\infty}^{\sqrt{t/2}} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, \mathrm{d}y. \tag{34}$$

In particular, $f_\sigma \in L^1([0,T])$ for all $T > 0$, and f_σ is positive almost everywhere.

Proof. We first consider the Neumann/Dirichlet case. By Proposition (30) and Tonelli's theorem,

$$\frac{1}{L} \int_0^L E[u(t,x)^2] \, \mathrm{d}x = \sum_{k=0}^{\infty} \frac{1}{L} \int_0^L \|\mathcal{I}_k(t,x)\|_2^2 \, \mathrm{d}x.$$

Since the series $\sum_{k=1}^{\infty} \sigma_1^{2(k-1)}(|\sigma_1|e^{|\alpha|T} + |\sigma_0|)^2 \frac{K_T^{2k}(4\beta)^{-k/2}T^{k/2}}{\Gamma((k+2)/2)}$ converges, by Proposition 7.1, dominated convergence theorem and Lemma 3.1, we have

$$\begin{aligned}
&\lim_{L\to\infty} \frac{1}{L}\int_0^L E[u(t,x)^2]\,\mathrm{d}x \\
&= \sum_{k=0}^{\infty} \lim_{L\to\infty} \frac{1}{L}\int_0^L \|\mathcal{I}_k(t,x)\|_2^2\,\mathrm{d}x \\
&= e^{-2\alpha t} + \sum_{k=1}^{\infty} f_k(t) \\
&= \sum_{k=0}^{\infty} f_k(t),
\end{aligned}$$

where $f_0(t) := e^{-2\alpha t}$.

Similarly, in the case of periodic boundary conditions, Proposition 7.2 and (30) imply that $E[u(t,x)^2]$ does not depend on $x \in [0,L]$. Hence, we have

$$\begin{aligned}
&\lim_{L\to\infty} \frac{1}{L}\int_0^L E[u(t,x)^2]\,\mathrm{d}x \\
&= \lim_{L\to\infty} E[u(t,0)^2] \\
&= \sum_{k=0}^{\infty} \lim_{L\to\infty} \|\mathcal{I}_k(t,0)\|_2^2 \\
&= e^{-2\alpha t} + \sum_{k=1}^{\infty} f_k(t) \\
&= \sum_{k=0}^{\infty} f_k(t),
\end{aligned}$$

where in the second equality, we apply dominated convergence theorem in order to exchange the limit and the sum.

Therefore, in the case of Neumann, Dirichlet, or periodic boundary conditions, we obtain

$$\lim_{L\to\infty} \frac{1}{L}\int_0^L E[u(t,x)^2]\,\mathrm{d}x = \sum_{k=0}^{\infty} f_k(t).$$

Hence, we have

$$\begin{aligned}
&\lim_{L\to\infty}\frac{1}{L}\int_0^L E[\sigma(u(t,x))^2]\,\mathrm{d}x\\
&=\sigma_1^2\lim_{L\to\infty}\frac{1}{L}\int_0^L E[u(t,x)^2]\,\mathrm{d}x\\
&\quad+2\sigma_1\sigma_0\lim_{L\to\infty}\frac{1}{L}\int_0^L E[u(t,x)]\,\mathrm{d}x+\sigma_0^2\\
&=\sigma_1^2\sum_{k=0}^{\infty}f_k(t)+2\sigma_1\sigma_0e^{-\alpha t}+\sigma_0^2,
\end{aligned}$$

where the second equality holds by $E[u(t,x)]=\mathcal{I}_0(t,x)$ and Lemma 3.1.

From (32), (33) and change of variables, for all $k\geq 1$, we have

$$\begin{aligned}
\sigma_1^2f_k(t)&\leq\sigma_1^{2k}e^{-2\alpha t}(|\sigma_1|+e^{|\alpha|t}|\sigma_0|)^2\\
&\quad\times\int_{0<r_k<\cdots<r_1<t}p_{2\beta(t-r_1)}(0)\dots p_{2\beta(r_{k-1}-r_k)}(0)\,\mathrm{d}r_k\dots\mathrm{d}r_1\\
&=e^{-2\alpha t}(|\sigma_1|+e^{|\alpha|t}|\sigma_0|)^2\frac{(\sigma_1^4t/(4\beta))^{k/2}}{\Gamma((k+2)/2)}.
\end{aligned}$$

By using the following identity(see [3, Lemma 2.3.4])

$$\sum_{n=1}^{\infty}\frac{\lambda^{n-1}}{\Gamma((n+1)/2)}=2e^{\lambda^2}\int_{-\infty}^{\sqrt{2}\lambda}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}\,\mathrm{d}y,\ \text{for all }\lambda\geq 0$$

with $\lambda=\sqrt{\sigma_1^4t/(4\beta)}$, we obtain

$$\begin{aligned}
&\sum_{k=0}^{\infty}\frac{(\sigma_1^4t/(4\beta))^{k/2}}{\Gamma((k+2)/2)}\\
&=\sum_{n=1}^{\infty}\frac{(\sigma_1^4t/(4\beta))^{(n-1)/2}}{\Gamma((n+1)/2)}\\
&=f(\sigma_1^4t/\beta),
\end{aligned}$$

where f is defined in (34). Hence, we have

$$\begin{aligned}
&\sigma_1^2 \sum_{k=0}^{\infty} f_k(t) \\
&= \sigma_1^2 e^{-2\alpha t} + \sum_{k=1}^{\infty} \sigma_1^2 f_k(t) \\
&\leq \sigma_1^2 e^{-2\alpha t} + e^{-2\alpha t}(|\sigma_1| + e^{|\alpha| t}|\sigma_0|)^2 \sum_{k=1}^{\infty} \frac{(\sigma_1^4 t/(4\beta))^{k/2}}{\Gamma((k+2)/2)} \\
&= \sigma_1^2 e^{-2\alpha t} + e^{-2\alpha t}(|\sigma_1| + e^{|\alpha| t}|\sigma_0|)^2 \sum_{k=0}^{\infty} \frac{(\sigma_1^4 t/(4\beta))^{k/2}}{\Gamma((k+2)/2)} \\
&\quad - e^{-2\alpha t}(|\sigma_1| + e^{|\alpha| t}|\sigma_0|)^2 \\
&= \sigma_1^2 e^{-2\alpha t} + e^{-2\alpha t}(|\sigma_1| + e^{|\alpha| t}|\sigma_0|)^2 (f(\sigma_1^4 t/\beta) - 1).
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
f_\sigma(t) &= \sigma_1^2 \sum_{k=0}^{\infty} f_k(t) + 2\sigma_1\sigma_0 e^{-\alpha t} + \sigma_0^2 \\
&\leq e^{-2\alpha t}(|\sigma_1| + e^{|\alpha| t}|\sigma_0|)^2 (f(\sigma_1^4 t/\beta) - 1) \\
&\quad + (\sigma_1 e^{-\alpha t} + \sigma_0)^2.
\end{aligned}$$

In the same way, since

$$\begin{aligned}
\sigma_1^2 f_k(t) &\geq \sigma_1^{2k} e^{-2\alpha t} \inf_{r\in[0,t]} (\sigma_1 + e^{\alpha r}\sigma_0)^2 \\
&\quad \times \int_{0<r_k<\cdots<r_1<t} p_{2\beta(t-r_1)}(0) \ldots p_{2\beta(r_{k-1}-r_k)}(0)\, \mathrm{d}r_k \ldots \mathrm{d}r_1 \\
&= e^{-2\alpha t} \inf_{r\in[0,t]} (\sigma_1 + e^{\alpha r}\sigma_0)^2 \frac{(\sigma_1^4 t/(4\beta))^{k/2}}{\Gamma((k+2)/2)},
\end{aligned}$$

it follows that

$$\begin{aligned}
f_\sigma(t) &\geq e^{-2\alpha t} \inf_{r\in[0,t]} (\sigma_1 + e^{\alpha r}\sigma_0)^2 (f(\sigma_1^4 t/\beta) - 1) \\
&\quad + (\sigma_1 e^{-\alpha t} + \sigma_0)^2.
\end{aligned}$$

Combining these estimates with the fact that f is an increasing function and $f(0) = 1$, we conclude that f_σ is positive almost everywhere and belongs to $L^1([0,T])$ for all $T > 0$. □

Appendix A. Properties of heat kernels

Denote the Gaussian heat kernel on $\mathbb{R}$ as

$$p_t(z) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{z^2}{2t}}, \quad t > 0, z \in \mathbb{R}. \tag{35}$$

Let G (resp. g) be the heat kernel for the cable equation(resp. heat equation) with Neumann/Dirichlet/periodic boundary conditions. Note that

$$G_t(x,y) = e^{-\alpha t} g_{\beta t}(x,y). \tag{36}$$

Hence, the properties of G follow directly from those of g. For all $t > 0$ and $x, y \in [0, L]$, in the case of Neumann boundary conditions,

$$G_t(x,y) = e^{-\alpha t} \sum_{n\in\mathbb{Z}} \left(p_{\beta t}(x - y + 2nL) + p_{\beta t}(x + y + 2nL)\right), \tag{37}$$

or equivalently,

$$G_t(x,y) = \frac{e^{-\alpha t}}{\sqrt{L}} + \frac{2e^{-\alpha t}}{L} \sum_{n=1}^{\infty} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi y}{L}\right) e^{-\frac{n^2\pi^2\beta t}{2L^2}};$$

and in the case of Dirichlet boundary conditions,

$$G_t(x,y) = e^{-\alpha t} \sum_{n\in\mathbb{Z}} \left(p_{\beta t}(x - y + 2nL) - p_{\beta t}(x + y + 2nL)\right),$$

or equivalently,

$$G_t(x,y) = \frac{2e^{-\alpha t}}{L} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi y}{L}\right) e^{-\frac{n^2\pi^2\beta t}{2L^2}}; \tag{38}$$

and in the case of periodic boundary conditions,

$$G_t(x,y) = e^{-\alpha t} \sum_{n\in\mathbb{Z}} p_{\beta t}(x - y + nL). \tag{39}$$

Moreover, by using the properties of g [21, Lemma A.1.], we get some useful properties of G.

Lemma A.1.

(1) *Symmetry.* $G_t(x,y) = G_t(y,x)$ *for all* $t > 0$, $x, y \in [0, L]$.

(2) *In the case of Neumann and periodic boundary conditions, for all* $t > 0$ *and* $x \in [0, L]$,

$$\int_0^L G_t(x,y)\ \mathrm{d}y = e^{-\alpha t}. \tag{40}$$

(3) *Semigroup property. For all* $t, s > 0$ *and* $x, y \in [0, L]$,

$$\int_0^L G_t(x,z) G_s(z,y)\ \mathrm{d}z = G_{t+s}(x,y).$$

(4) *In the case of Neumann/Dirichlet boundary conditions, for every* $t > 0$ *and* $x, y \in [0, L]$,

$$G_t(x,y) \le e^{-\alpha t} p_{\beta t}(x - y) \left(4 + \frac{4}{1 - e^{-\frac{L^2}{\beta t}}}\right),$$

and as a consequence, for all $t \in (0, T]$, $L \ge 1$ *and* $x, y \in [0, L]$,

$$G_t(x,y) \le K_T p_{\beta t}(x - y),$$

where

$$K_T = e^{|\alpha| T} \left(4 + \frac{4}{1 - e^{-\frac{1}{\beta T}}}\right). \tag{41}$$

ACKNOWLEDGEMENTS. *The author would like to thank Yushi Hamaguchi and Masahisa Ebina for helpful discussions and valuable suggestions. The author is also grateful to the anonymous referee for carefully reviewing the manuscript and providing constructive comments. The author was supported by the MIYAKO-MIRAI Project of Tokyo Metropolitan University.*

References

[1] O. Assaad, D. Nualart, C. A. Tudor, L. Viitasaari: *Quantitative normal approximations for the stochastic fractional heat equation*, Stoch. Partial Differ. Equ. Anal. Comput. **10** (2022), 223–254.
[2] R. M. Balan and W. Yuan: *Central limit theorems for heat equation with timeindependent noise: the regular and rough cases*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **26** (2023), 2250029.
[3] L. Chen: *Moments, intermittency, and growth indices for nonlinear stochastic PDE's with rough initial conditions*, PhD Thesis, École Polytechnique Fédérale de Lausanne, No. 5712 (2013).
[4] L. Chen, D. Khoshnevisan, D. Nualart, F. Pu: *Central limit theorems for spatial averages of the stochastic heat equation via Malliavin-Stein's method*, Stoch. Partial Differ. Equ. Anal. Comput. **11** (2023), 122–176.
[5] F. Delgado-Vences, D. Nualart, G. Zheng: *A central limit theorem for the stochastic wave equation with fractional noise*, Ann. Inst. Henri Poincaré Probab. Stat. **56** (2020), 3020–3042.
[6] M. Ebina: *Central limit theorems for nonlinear stochastic wave equations in dimension three*, Stoch. Partial Differ. Equ. Anal. Comput. **12** (2024), 1141–1200.
[7] M. Ebina: *Central limit theorems for stochastic wave equations in high dimensions*, Electron. J. Probab. **30** (2025), 1–56.
[8] R. B. Guerrero, D. Nualart, G. Zheng: *Averaging 2d stochastic wave equation*, Electron. J. Probab. **26** (2021), 1–32.
[9] J. Huang, D. Nualart, L. Viitasaari: *A central limit theorem for the stochastic heat equation*, Stochastic Process. Appl. **130** (2020), 7170–7184.
[10] J. Huang, D. Nualart, L. Viitasaari, G. Zheng: *Gaussian fluctuations for the stochastic heat equation with colored noise*, Stoch. Partial Differ. Equ. Anal. Comput. **8** (2020), 402–421.
[11] G. Kallianpur, J. Xiong: *Stochastic differential equations in infinite-dimensional spaces*, IMS Lecture Notes Monogr. Ser., 1995.
[12] I. Karatzas and S. E. Shreve: *Brownian motion and Stochastic calculus*, Springer, 2nd ed., 1998.
[13] D. Khoshnevisan, D. Nualart, F. Pu: *Spatial stationarity, ergodicity, and CLT for parabolic Anderson model with delta initial condition in dimension* $d \geq 1$, SIAM J. Math. Anal. **53** (2021), 2084–2133.
[14] K. Kim and J. Yi: *Limit theorems for time-dependent averages of nonlinear stochastic heat equations*, Bernoulli, **28** (2022), 214–238.
[15] J. Liu and G. Shen: *Gaussian fluctuation for spatial average of the stochastic pseudo-partial differential equation with fractional noise*, ALEA, Lat. Am. J. Probab. Math. Stat., **20** (2023), 1483–1509.
[16] I. Nourdin and G. Peccati: *Normal approximations with Malliavin calculus. From Stein's method to universality*, Cambridge Univ. Press, 2012.
[17] D. Nualart: *The Malliavin Calculus and Related Topics*, Springer-Verlag, Berlin, 2006.
[18] D. Nualart, P. Xia, G. Zheng: *Quantitative central limit theorems for the parabolic Anderson model driven by colored noises*, Electron. J. Probab. **27** (2022), 1–43.
[19] D. Nualart and G. Zheng: *Averaging Gaussian functionals*, Electron. J. Probab., **25** (2020), 1–54.
[20] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (eds.): *NIST handbook of mathematical functions*, U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge, 2010.
[21] F. Pu: *Gaussian fluctuation for spatial average of parabolic Anderson model with Neumann/Dirichlet/periodic boundary conditions*, Trans. Amer. Math. Soc. **375** (2022), 2481–2509.
[22] R. Shakarchi and E. Stein: *Fourier analysis: an introduction*, Princeton Univ. Press, 2003.
[23] C. A. Tudor and J. Zurcher: *The spatial average of solutions to SPDEs is asymptotically independent of the solution*, Bull. Sci. Math. **205** (2025), 103719.
[24] J. B. Walsh: *A stochastic model of neural response*, Adv. Appl. Probab. **13** (1981), 231–281.

[25] J. B. Walsh: *An introduction to stochastic partial differential equations*, École d'été de probabilités de Saint-Flour, XIV–1984 pp. 265–439, Lecture Notes in Mathematics, vol. 1180, Springer, Berlin, 1986.

Soma Nishino
Department of Mathematical Sciences
Tokyo Metropolitan University
Hachioji, Tokyo 192-0397
Japan
e-mail: nishino-soma@ed.tmu.ac.jp