TRANSPOSES IN THE q-DEFORMED MODULAR GROUP AND THEIR APPLICATIONS TO q-DEFORMED RATIONAL NUMBERS

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Dedicated to the memory of Professor Mitsuyasu Hashimoto

Abstract. The (right) q-deformed rational numbers was introduced by Morier-Genoud and Ovsienko, and its left variant, whose numerators and denominators are essentially the normalized Jones polynomials of rational links, by Bapat, Becker and Licata. These notions are based on continued fractions and the q-deformed modular group $\mathrm{PSL}_q(2,\mathbb{Z})$-actions. In this paper, we introduce the *q-transpose* for matrices in $\mathrm{PSL}_q(2,\mathbb{Z})$ to refine the basic perspective of the theory. For example, we present a new proof and a refinement of a theorem of Leclere and Morier-Genoud stating that the trace of $A \in \mathrm{PSL}_q(2,\mathbb{Z})$ is always palindromic and sign coherent. We also show arithmetic/combinatorial results on left q-deformed rationals (e.g., the criterion for their palindromicity). Finally, we discuss the connection to the conjecture of Kantarcı Oğuz on circular fence posets.

1. Introduction

For an irreducible fraction $\frac{r}{s}$ and a formal parameter q, the *right q-deformed rational number* $\left[\frac{r}{s}\right]_q^\sharp$ is defined as a rational function in $\mathbb{Z}(q)$. This notion was originally introduced by Morier-Genoud and Ovsienko [13, 15] via continued fractions and the q-deformed modular group $\mathrm{PSL}_q(2,\mathbb{Z})$-action. Here $\mathrm{PSL}_q(2,\mathbb{Z})$ is a quotient group of a subgroup of $\mathrm{GL}(2,\mathbb{Z}[q^{\pm 1}])$. They further extended this notion to arbitrary real numbers [14], and Gaussian integers [16]. Inspired by their works, Bapat, Becker and Licata [1] gave another q-deformation $\left[\frac{r}{s}\right]_q^\flat$ of $\frac{r}{s}$, which is called the *left q-deformed rational number.* The relationship between the left and right q-deformed rational numbers are further studied in [6, 19]. These notions are related to many directions including 2-Calabi-Yau categories [1], the Markov-Hurwitz approximation theory [8, 10, 17], the modular group [9, 15], Jones polynomials of rational knots [7, 11, 13, 1, 18] and combinatoris of posets [12, 3, 2].

For a matrix $A \in \mathrm{PSL}_q(2,\mathbb{Z})$, we define two operations the *q-transpose* A^{T_q} and the *orthogonal q-transpose* A^{O_q} as follows. Let

$$\begin{pmatrix} \mathcal{R}(q) & \mathcal{V}(q) \\ \mathcal{S}(q) & \mathcal{U}(q) \end{pmatrix}$$

2020 *Mathematics Subject Classification.* 05A30, 11A55, 11F06, and 57K14.

Key words and phrases. q-deformed rational numbers, continued fractions, modular group, Jones polynomials, rational knots, and circular fence posets.

be a matrix whose entries are elements in $\mathbb{Z}[q^{\pm 1}]$, we define

$$A^{T_q} := \begin{pmatrix} \mathcal{R}(q) & q^{-1}\mathcal{S}(q) \\ q\mathcal{V}(q) & \mathcal{U}(q) \end{pmatrix} \quad \text{and} \quad A^{O_q} := \begin{pmatrix} \mathcal{U}(q^{-1}) & q^{-1}\mathcal{V}(q^{-1}) \\ q\mathcal{S}(q^{-1}) & \mathcal{R}(q^{-1}) \end{pmatrix}.$$

It can be checked that $A \in \mathrm{PSL}_q(2,\mathbb{Z})$, if and only if $A^{T_q} \in \mathrm{PSL}_q(2,\mathbb{Z})$, if and only if $A^{O_q} \in \mathrm{PSL}_q(2,\mathbb{Z})$ (see Lemmas 3.2 and 3.7).

As for applications of these operations, we give some new proofs of existing results. For a polynomial $f(q) \in \mathbb{Z}[q]$, we write its *reciprocal polynomial* by $f(q)^\vee$, that is,

$$f(q)^\vee := q^{\deg(f)} f(q^{-1}).$$

Moreover, we say $f(q)$ is *palindrmoic* if $f(q) = f(q)^\vee$. A theorem of Leclere and Morier-Genoud ([9, Theorem 3]) states that, for $A \in \mathrm{PSL}_q(2,\mathbb{Z})$, the trace $\mathrm{tr}\, A$ is a palindromic polynomial with non-negative coefficients after $\pm q^N$ times for some integer N. We give a new proof of this fact in Theorem 4.9, and a refinement in Proposition 4.10.

Some arithmetic properties on the right q-deformed rational numbers were recently studied in [5]. In this paper, we also give their left versions as follows. For any irreducible fraction $\frac{r}{s}$, we write its left and right q-deformed rational numbers as

$$\left[\frac{r}{s}\right]^\flat_q = \frac{\mathcal{R}^\flat_{\frac{r}{s}}(q)}{\mathcal{S}^\flat_{\frac{r}{s}}(q)}, \qquad \left[\frac{r}{s}\right]^\sharp_q = \frac{\mathcal{R}^\sharp_{\frac{r}{s}}(q)}{\mathcal{S}^\sharp_{\frac{r}{s}}(q)},$$

respectively, where $\mathcal{R}^\flat_{\frac{r}{s}}(q),\ \mathcal{R}^\sharp_{\frac{r}{s}}(q) \in \mathbb{Z}[q^{\pm 1}]$, $\mathcal{S}^\flat_{\frac{r}{s}}(q),\ \mathcal{S}^\sharp_{\frac{r}{s}}(q) \in \mathbb{Z}[q]$, $\mathcal{R}^\flat_{\frac{r}{s}}(1) = \mathcal{R}^\sharp_{\frac{r}{s}}(1) = r$, and $\mathcal{S}^\flat_{\frac{r}{s}}(1) = \mathcal{S}^\sharp_{\frac{r}{s}}(1) = s$. While we only treat the denominator $\mathcal{S}^\flat_{\frac{r}{s}}(q)$ below, the similar hold for the numerator $\mathcal{R}^\flat_{\frac{r}{s}}(q)$.

Theorem 1.1 (see Theorem 4.8). *For irreducible fractions* $\frac{r}{s}, \frac{r'}{s}$ *with* $rr' \equiv 1 \pmod{s}$ (*resp.* $rr' \equiv -1 \pmod{s}$), *we have* $\mathcal{S}^\flat_{\frac{r}{s}}(q) = \mathcal{S}^\flat_{\frac{r'}{s}}(q)$ (*resp.* $\mathcal{S}^\flat_{\frac{r}{s}}(q) = \mathcal{S}^\flat_{\frac{r'}{s}}(q)^\vee$).

Theorem 1.2 (see Theorem 4.13). *For an irreducible fraction* $\frac{r}{s}$, $\mathcal{S}^\flat_{\frac{r}{s}}(q)$ *is palindromic if and only if* $r^2 \equiv -1 \pmod{s}$.

In [5], we showed that $\mathcal{S}^\sharp_{\frac{r}{s}}(q) = \mathcal{S}^\sharp_{\frac{r'}{s}}(q)$ (resp. $\mathcal{S}^\sharp_{\frac{r}{s}}(q) = \mathcal{S}^\sharp_{\frac{r'}{s}}(q)^\vee$) if $rr' \equiv -1 \pmod{s}$ (resp. $rr' \equiv 1 \pmod{s}$), and $\mathcal{S}^\sharp_{\frac{r}{s}}(q)$ is palindromic if and only if $r^2 \equiv 1 \pmod{s}$. So, the left and right q-deformed ration numbers behave oppositely in this sense.

An irreducible fraction $\alpha = \frac{r}{s} > 1$ determines a rational link $L(\alpha)$ via the continued fraction. We consider the *normalized Jones polynomial* $J_\alpha(q) \in \mathbb{Z}[q]$ of $L(\alpha)$, which can be computed by the left and right q-deformed rational numbers [11, 13, 1, 18]. The simplest expression is $J_\alpha(q) = \mathcal{R}^\flat_\alpha(q)^\vee$, and hence $J_\alpha(q)$ is palindromic if and only if $r^2 \equiv -1 \pmod{s}$. When $J_\alpha(q)$ is *not* palindromic, set

$$I_\alpha(q) := \pm q^N \frac{J_\alpha(q)^\vee - J_\alpha(q)}{1-q}$$

for some $N \in \mathbb{Z}$, where we take $\pm q^N$ so that $I_\alpha(q) \in \mathbb{Z}[q]$ with $I_\alpha(0) > 0$. Then we have that $I_\alpha(q)$ equals $\mathrm{tr}(A)$ for some $A \in \mathrm{PSL}_q(2,\mathbb{Z})$. In particular, $I_\alpha(0) = 1$ and $I_\alpha(q) \in \mathbb{Z}_{\geq 0}[q]$. Moreover, we show that Kantarcı Oğuz's conjecture [2, Conjecture 1.4] on circular fence posets implies a conjecture on $I_\alpha(q)$ (Conjecture 4.11). In particular, $I_\alpha(q)$ is conjectured to be at most bimodal (unimodal in most cases).

2. The left and right q-deformed rationals and $\mathrm{PSL}_q(2,\mathbb{Z})$

At the beginning, we briefly recall the definitions of the left and right q-deformed rational numbers. For details, see [13, 15, 1, 18]. We also give a few supplemental results.

2.1. **Continued fractions, $\mathrm{PSL}(2,\mathbb{Z})$-actions and its q-deformation.** In this paper, we regard $0 = \frac{0}{1}$ and $\frac{1}{0}$ as irreducible fractions. For an irreducible fraction $\frac{r}{s} \in \mathbb{Q}$, we always assume that s is positive. An irreducible fraction $\frac{r}{s} \in \mathbb{Q} \cup \left\{\frac{1}{0}\right\}$ has unique regular and negative continued fraction expansions as follows:

$$\frac{r}{s} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{2m}}}} = c_1 - \cfrac{1}{c_2 - \cfrac{1}{\ddots - \cfrac{1}{c_k}}}$$

with the following setting. (The latter is often called the *Hirzebruch-Jung continued fraction.*) If $\frac{r}{s}$ is positive, then we set $a_1 \in \mathbb{Z}_{\geq 0}$, $a_i \in \mathbb{Z}_{\geq 1}$ ($i \geq 2$), and $c_1 \in \mathbb{Z}_{\geq 1}$ and $c_j \in \mathbb{Z}_{\geq 2}$ ($j \geq 2$). If $\frac{r}{s}$ is negative, then we set $a_1 \in \mathbb{Z}_{\leq 0}$, $a_i \in \mathbb{Z}_{\leq -1}$ ($i \geq 2$), and $c_1 \in \mathbb{Z}_{\leq -1}$ and $c_j \in \mathbb{Z}_{\leq -2}$ ($j \geq 2$). We write $[a_1, \dots, a_{2m}]$ and $[[c_1, \dots, c_k]]$ for these expansions, respectively. As exceptional cases, we set $\frac{0}{1} = [-1, 1] = [[1, 1]]$ and $\frac{1}{0} = [\ ] = [[\ ]]$ (i.e., empty expansions). For a regular continued fraction of odd length, we use the symbol $[a_1, \dots, a_{2m+1}]$ also. Note that $[a_1, \dots, a_n + 1] = [a_1, \dots, a_n, 1]$, and this is the reason we can assume that the length is even.

Consider the group $\mathrm{SL}(2,\mathbb{Z}) = \langle R, S\rangle = \langle R, L\rangle$, where

$$R := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad L := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

This group (and its quotient $\mathrm{PSL}(2,\mathbb{Z})$) acts on $\mathbb{Q} \cup \left\{\frac{1}{0}\right\}$ by the fractional linear transformation:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \left(\frac{r}{s}\right) = \frac{ar + bs}{cr + ds},$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ and $\frac{r}{s} \in \mathbb{Q} \cup \left\{\frac{1}{0}\right\}$. Hence, a rational number $\frac{r}{s} = [a_1, \dots, a_{2m}] = [[c_1, \dots, c_k]]$ can be expressed by the following formulas:

$$\frac{r}{s} = M(a_1, \dots, a_{2m}) \left(\frac{1}{0}\right) := R^{a_1} L^{a_2} R^{a_3} L^{a_4} \cdots R^{a_{2m-1}} L^{a_{2m}} \left(\frac{1}{0}\right); \tag{2.1}$$

$$\frac{r}{s} = M^{-}(c_1, \dots, c_k) \left(\frac{1}{0}\right) := R^{c_1} S R^{c_2} S \cdots R^{c_k} S \left(\frac{1}{0}\right). \tag{2.2}$$

Lemma 2.1. *Let $\frac{r}{s} = [a_1, \dots, a_{2m}] > 0$ be an irreducible fraction with $s \geq 2$. For $\begin{pmatrix} r & v \\ s & u \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$, the following are equivalent.*

(1) $M(a_1,\dots,a_{2m})=\begin{pmatrix} r & v\\ s & u\end{pmatrix}$.

(2) $0<u<s$.

Proof. First, note that since $r,s>0$ and $ru-sv=1$, if the condition (2) holds then $r>v>0$. (1) $\Rightarrow$ (2): We can prove this by induction on m. (2) $\Rightarrow$ (1): Any element in $\mathrm{SL}(2,\mathbb{Z})$ of the form $\begin{pmatrix} r & *\\ s & *\end{pmatrix}$ satisfies $\begin{pmatrix} r & v+nr\\ s & u+ns\end{pmatrix}$ for some $n\in\mathbb{Z}$. (Here we employ Bézout's identity. This result will be freely used throughout the remainder of this paper.) There is only one n satisfying $0<u+ns<s$. □

We first consider the q-deformation of (2.1).

Proposition 2.2 ([9, Proposition 1.1]). *Let q be a formal parameter, and consider the matrices*

$$R_q:=\begin{pmatrix} q & 1\\ 0 & 1\end{pmatrix},\quad S_q=\begin{pmatrix} 0 & -q^{-1}\\ 1 & 0\end{pmatrix}.$$

The subgroup $\langle R_q,S_q\rangle$ of $\mathrm{GL}(2,\mathbb{Z}[q^{\pm1}])$ generated by R_q and S_q contains $\pm q^{\pm1}\mathrm{Id}$. We set

$$\mathrm{PSL}_q(2,\mathbb{Z}):=\langle R_q,S_q\rangle/\left\{\pm q^n\mathrm{Id}\mid n\in\mathbb{Z}\right\}.$$

Then, the assignment

$$R\longmapsto R_q,\qquad S\longmapsto S_q$$

induces an isomorphism from $\mathrm{PSL}(2,\mathbb{Z})$ to $\mathrm{PSL}_q(2,\mathbb{Z})$.

In the following, by abuse of notation, we sometimes identify $A\in\mathrm{PSL}_q(2,\mathbb{Z})$ with an explicit matrix in $\mathrm{GL}(2,\mathbb{Z}[q^{\pm1}])$ giving A.

We consider the matrix

$$L_q:=q^{-1}R_qS_qR_q=\begin{pmatrix} 1 & 0\\ 1 & q^{-1}\end{pmatrix},$$

then R_q and L_q also generate $\mathrm{PSL}_q(2,\mathbb{Z})$, and we set

$$M_q(a_1,\dots,a_{2m}):=R_q^{a_1}L_q^{a_2}R_q^{a_3}L_q^{a_4}\cdots R_q^{a_{2m-1}}L_q^{a_{2m}}\in\mathrm{GL}\left(2,\mathbb{Z}[q^{\pm1}]\right).$$

2.2. **The left and right q-deformed rational numbers.** We recall the definition of the left and right q-deformed rational numbers. We follow the notation in [1].

Definition 2.3 (c.f. [1]). We set

$$\left[\frac{0}{1}\right]^\flat_q=\frac{1-q^{-1}}{1},\quad \left[\frac{0}{1}\right]^\sharp_q=\frac{0}{1};\quad \left[\frac{1}{0}\right]^\flat_q=\frac{1}{1-q},\quad \left[\frac{1}{0}\right]^\sharp_q=\frac{1}{0}.$$

For an irreducible fraction $\frac{r}{s}=[a_1,\dots,a_{2m}]\in\mathbb{Q}$, then the *left q-deformed rational number* $\left[\frac{r}{s}\right]^\flat_q$ is defined by

$$\left[\frac{r}{s}\right]^\flat_q:=M_q(a_1,\dots,a_{2m})\left(\left[\frac{1}{0}\right]^\flat_q\right)=M_q(a_1,\dots,a_{2m})\left(\frac{1}{1-q}\right),$$

and the *right q-deformed rational number* $\left[\frac{r}{s}\right]_q^\sharp$ is defined by

$$\left[\frac{r}{s}\right]_q^\sharp := M_q(a_1,\dots,a_{2m})\left(\left[\frac{1}{0}\right]_q^\sharp\right) = M_q(a_1,\dots,a_{2m})\left(\frac{1}{0}\right).$$

Multiplying both the numerator and denominator by $\pm q^n$ for suitable $n \in \mathbb{Z}$, we have

$$\left[\frac{r}{s}\right]_q^\flat = \frac{\mathcal{R}^\flat_{\frac{r}{s}}(q)}{\mathcal{S}^\flat_{\frac{r}{s}}(q)}, \qquad \left[\frac{r}{s}\right]_q^\sharp = \frac{\mathcal{R}^\sharp_{\frac{r}{s}}(q)}{\mathcal{S}^\sharp_{\frac{r}{s}}(q)},$$

where

$$\mathcal{R}^\flat_{\frac{r}{s}}(q),\ \mathcal{R}^\sharp_{\frac{r}{s}}(q) \in \mathbb{Z}[q^{\pm 1}],\ \mathcal{S}^\flat_{\frac{r}{s}}(q), \mathcal{S}^\sharp_{\frac{r}{s}}(q) \in \mathbb{Z}[q]$$

with $\mathcal{S}^\flat_{\frac{r}{s}}(0) = \mathcal{S}^\sharp_{\frac{r}{s}}(0) = 1$.

We always have

(1) $\mathcal{R}^\flat_{\frac{r}{s}}(1) = \mathcal{R}^\sharp_{\frac{r}{s}}(1) = r$ and $\mathcal{S}^\flat_{\frac{r}{s}}(1) = \mathcal{S}^\sharp_{\frac{r}{s}}(1) = s$.
(2) $\mathcal{R}^\flat_{\frac{r}{s}}(q)$ and $\mathcal{S}^\flat_{\frac{r}{s}}(q)$ are coprime, and the same is true for $\mathcal{R}^\sharp_{\frac{r}{s}}(q)$ and $\mathcal{S}^\sharp_{\frac{r}{s}}(q)$.

It is known that if $\frac{r}{s} > 1$, then $\mathcal{R}^\flat_{\frac{r}{s}}(q),\ \mathcal{R}^\sharp_{\frac{r}{s}}(q) \in \mathbb{Z}[q]$ with $\mathcal{R}^\flat_{\frac{r}{s}}(0) = \mathcal{R}^\sharp_{\frac{r}{s}}(0) = 1$.

Easy calculation shows that $\mathcal{S}^\flat_{\frac{n}{1}}(q) = \mathcal{S}^\sharp_{\frac{n}{1}}(q) = 1$ for all $n \in \mathbb{Z}$, and

$$\mathcal{R}^\flat_{\frac{n}{1}}(q) = \begin{cases} q^n + q^{n-2} + q^{n-3} + \cdots + q + 1 & (n > 0) \\ 1 - q^{-1} & (n = 0) \\ -q^{-n-1} - q^{-n+1} - \cdots - q^{-1} & (n < 0), \end{cases}$$

while $\mathcal{R}^\sharp_{\frac{n}{1}}(q)$ coincides with the Euler q-integer $[n]_q = \dfrac{1-q^n}{1-q}$.

We can use negative continued fractions to compute the q-deformed rational numbers.

Proposition 2.4 (c.f. [13, 9])**.** *For the irreducible fraction $\frac{r}{s} = [[c_1,\dots,c_k]] > 0$, we consider the matrix*

$$M_q^-(c_1,\dots,c_k) := R_q^{c_1} S_q R_q^{c_2} S_q \cdots R_q^{c_k} S_q \in \mathrm{GL}\left(2, \mathbb{Z}[q^{\pm 1}]\right),$$

then we have

$$M_q^-(c_1,\dots,c_k) = \begin{pmatrix} \mathcal{R}^\sharp_{\frac{r}{s}}(q) & \mathcal{W}(q) \\ \mathcal{S}^\sharp_{\frac{r}{s}}(q) & \mathcal{X}(q) \end{pmatrix}.$$

Hence, the right q-deformed rational number of $\frac{r}{s}$ is

$$\left[\frac{r}{s}\right]_q^\sharp = M_q^-(c_1,\dots,c_k)\left(\frac{1}{0}\right) = \frac{\mathcal{R}^\sharp_{\frac{r}{s}}(q)}{\mathcal{S}^\sharp_{\frac{r}{s}}(q)}.$$

The first author of the present paper gave the left version of the above proposition (c.f. [18, Section 2.4]).

Proposition 2.4 can be extended as follows. The multiplicative group $G := \{\pm q^n \mid n \in \mathbb{Z}\}$ acts on the set

$$X := \left\{ \begin{pmatrix} \mathcal{V} \\ \mathcal{U} \end{pmatrix} \mid \mathcal{V}, \mathcal{U} \in \mathbb{Z}[q^{\pm 1}] \right\}$$

in the natural way. Let $\overline{X} := X/G$, and $\begin{bmatrix} \mathcal{V} \\ \hline \mathcal{U} \end{bmatrix} \in \overline{X}$ denote the orbit of $\begin{pmatrix} \mathcal{V} \\ \mathcal{U} \end{pmatrix}$. Clearly, $\mathrm{PSL}_q(2,\mathbb{Z})$ acts on $\overline{X}$. Consider the subset

$$\overline{Y} := \left\{ \left[\frac{\mathcal{R}^\sharp_{\frac{r}{s}}(q)}{\mathcal{S}^\sharp_{\frac{r}{s}}(q)} \right] \;\middle|\; \frac{r}{s} \in \mathbb{Q} \cup \left\{ \frac{1}{0} \right\} \right\}$$

of $\overline{X}$. The following seems to be well-known to experts.

Proposition 2.5. *With the above notation, the columns of any element of* $\mathrm{PSL}_q(2,\mathbb{Z})$ *belong to* $\overline{Y}$. *Hence, for*

$$\begin{pmatrix} \mathcal{R}(q) & \mathcal{V}(q) \\ \mathcal{S}(q) & \mathcal{U}(q) \end{pmatrix} \in \mathrm{PSL}_q(2,\mathbb{Z})$$

with $r := \mathcal{R}(1)$, $s := \mathcal{S}(1)$, $v := \mathcal{V}(1)$ *and* $u := \mathcal{U}(1)$, *we have*

$$\left[\frac{r}{s}\right]^\sharp_q = \left[\frac{\mathcal{R}(q)}{\mathcal{S}(q)}\right] \quad \textit{and} \quad \left[\frac{v}{u}\right]^\sharp_q = \left[\frac{\mathcal{V}(q)}{\mathcal{U}(q)}\right]$$

(strictly speaking, if $s < 0$, $\frac{r}{s}$ *should be* $\frac{-r}{-s}$, *and so on).*

Proof. Since R_q and S_q generate $\mathrm{PSL}_q(2,\mathbb{Z})$, it suffices to show that $\overline{Y}$ is closed under multiplication by these matrices. An easy computation shows that

$$R_q \left[\frac{\mathcal{R}^\sharp_{\frac{r}{s}}(q)}{\mathcal{S}^\sharp_{\frac{r}{s}}(q)} \right] = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix} \left[\frac{\mathcal{R}^\sharp_{\frac{r}{s}}(q)}{\mathcal{S}^\sharp_{\frac{r}{s}}(q)} \right] = \left[\frac{q\mathcal{R}^\sharp_{\frac{r}{s}}(q) + \mathcal{S}^\sharp_{\frac{r}{s}}(q)}{\mathcal{S}^\sharp_{\frac{r}{s}}(q)} \right] = \left[\frac{\mathcal{R}^\sharp_{\frac{r}{s}+1}(q)}{\mathcal{S}^\sharp_{\frac{r}{s}+1}(q)} \right] \in \overline{Y},$$

where the last equality follows from [9, Proposition 2.8 (2.6)]. Similarly, we have

$$S_q \left[\frac{\mathcal{R}^\sharp_{\frac{r}{s}}(q)}{\mathcal{S}^\sharp_{\frac{r}{s}}(q)} \right] = \begin{pmatrix} 0 & -q^{-1} \\ 1 & 0 \end{pmatrix} \left[\frac{\mathcal{R}^\sharp_{\frac{r}{s}}(q)}{\mathcal{S}^\sharp_{\frac{r}{s}}(q)} \right] = \left[\frac{-q^{-1}\mathcal{S}^\sharp_{\frac{r}{s}}(q)}{\mathcal{R}^\sharp_{\frac{r}{s}}(q)} \right] = \left[\frac{\mathcal{R}^\sharp_{-\frac{s}{r}}(q)}{\mathcal{S}^\sharp_{-\frac{s}{r}}(q)} \right] \in \overline{Y},$$

where the last equality follows from [9, Proposition 2.8 (2.8)]. □

Convention 2.6. For $f, g \in \mathbb{Z}[q^{\pm 1}]$, if $f = \pm q^n g$ for some $n \in \mathbb{Z}$, then we write $f \equiv g$. And for matrices A, B, if $A = \pm q^n B$ for some $n \in \mathbb{Z}$, then we write $A \equiv B$.

2.3. **Closures of a quiver for the left and right q-deformed rational numbers.** We recall a combinatorial interpretation of the left and right q-deformed rational numbers. For details, see [13, 1, 5, 12].

A *quiver* $Q = (Q_0, Q_1, s, t)$ consists of two sets Q_0, Q_1 and two maps $s, t : Q_1 \to Q_0$. Each element of Q_0 (resp. Q_1) is called a vertex (resp. an arrow). For an arrow $\alpha \in Q_1$, we call $s(\alpha)$ (resp. $t(\alpha)$) the source (resp. the target) of α. We will commonly write $\alpha : a \to b$ to indicate that an arrow α has the source a and the target b. In this paper, we always consider a *finite* quiver Q (i.e., both two sets Q_0 and Q_1 are finite sets). The *opposite quiver* of Q, say $Q^\vee$, is defined by $Q^\vee = (Q_0, Q_1, t, s)$.

Let Q be a finite quiver. A subset $C \subset Q_0$ is a *closure* if there is no arrow $\alpha \in Q_1$ such that $s(\alpha) \in C$ and $t(\alpha) \in Q_0 \setminus C$. A closure C is an *ℓ-closure* if the number of elements of C is ℓ. The number of ℓ-closures is denoted by $\rho_\ell(Q)$. Then the polynomial

$$\mathsf{cl}(Q;q) := \sum_{\ell=0}^{n} \rho_\ell(Q) q^\ell \in \mathbb{Z}[q],$$

where $n = |Q_0|$, is called the *closure polynomial* of Q.

Clearly, $C \subset Q_0$ is a closure of Q if and only if $Q_0 \setminus C$ is that of $Q^\vee$. Hence, one has

$$\mathsf{cl}\left(Q^\vee; q\right) = \mathsf{cl}\left(Q; q\right)^\vee. \tag{2.3}$$

For a tuple of integers $\mathbf{b} := (b_1, b_2, \ldots, b_d)$ with $b_1, b_d \geq 0, b_2, \ldots, b_{d-1} > 0$, we define the quiver $Q(\mathbf{b})$ by Figure 1 below (in the figure, we assume that d is even):

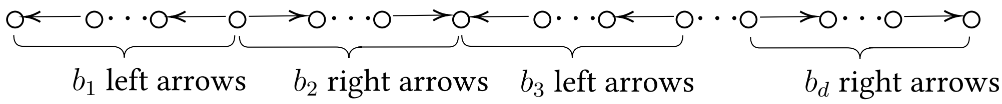

FIGURE 1. The quiver $Q(\mathbf{b})$.

For a rational number $\frac{r}{s} = [a_1, a_2, \ldots, a_{2m}] > 1$, we set

$$Q^{\sharp,\mathcal{R}}_{\frac{r}{s}} := Q(a_1 - 1, a_2, \ldots, a_{2m-1}, a_{2m} - 1),$$

$$Q^{\sharp,\mathcal{S}}_{\alpha} := \begin{cases} Q(0, a_2 - 1, a_3, \ldots, a_{2m-1}, a_{2m} - 1) & \text{if } m > 1, \\ Q(0, a_2 - 2) & \text{if } m = 1. \end{cases}$$

Here, if $a_2 = 1$ and $m > 1$ (resp. $a_2 = 2$ and $m = 1$, $a_2 = 1$ and $m = 1$), we understand that $Q^{\sharp,\mathcal{S}}_{\frac{r}{s}} = Q(a_3, \ldots, a_{2m-1}, a_{2m} - 1)$ (resp. $Q^{\sharp,\mathcal{S}}_{\frac{r}{s}} = Q^{\sharp,\mathcal{S}}_{\frac{r}{2}} = Q(0)$, $Q^{\sharp,\mathcal{S}}_{\frac{r}{s}} = Q^{\sharp,\mathcal{S}}_{\frac{r}{1}} = \varnothing$). The quiver $Q^{\sharp,\mathcal{S}}_{\frac{r}{s}}$ is obtained by deleting the first a_1 arrows from $Q^{\sharp,\mathcal{R}}_{\frac{r}{s}}$.

For the right q-deformed rational numbers, we have the following result.

Theorem 2.7 ([13, Theorem 4]). *Let $\frac{r}{s} > 1$ be an irreducible fraction. Then, the following equations hold:*

$$\mathcal{R}^\sharp_{\frac{r}{s}}(q) = \mathsf{cl}\left(Q^{\sharp,\mathcal{R}}_{\frac{r}{s}}; q\right), \tag{2.4}$$

$$\mathcal{S}^\sharp_{\frac{r}{s}}(q) = \mathsf{cl}\left(Q^{\sharp,\mathcal{S}}_{\frac{r}{s}}; q\right). \tag{2.5}$$

Let P be a finite poset. A subset I of P is said to be a *lower order ideal*, if $x \in I$, $y \in P$ and $y \leq x$ imply $y \in I$. Let $J(P)$ be the set of lower order ideals of P. The polynomial

$$\mathsf{rk}(P;q) := \sum_{I \in J(P)} q^{|I|}$$

is called the *rank polynomial* of P.

We can regard (the vertex set Q_0 of) the quiver $Q(\mathbf{b})$ as a poset as follows; for $a, b \in Q_0$, $a < b$ if and only if there is an arrow $b \to a$. Clearly, $I \subset Q(\mathbf{b})$ is a closure if and only if it is a lower order ideal, and hence

$$\mathsf{rk}(Q(\mathbf{b});q) = \mathsf{cl}(Q(\mathbf{b});q).$$

For the left q-deformed rational numbers, we also have a corresponding quiver.

For a tuple of integers $\mathbf{b} := (b_1, b_2, \dots, b_d)$ with $b_1, b_d \geq 0, b_2, \dots, b_{d-1} > 0$, we define the quiver $Q^\flat(\mathbf{b})$ adding a cycle to the right side of the quiver $Q(\mathbf{b})$ as follows (here we assume that d is even again). For a rational number $\frac{r}{s} = [a_1, a_2, \dots, a_{2m}] > 1$, set

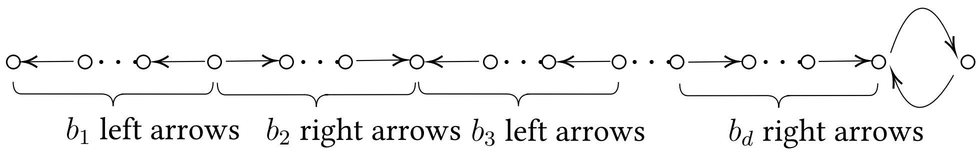

FIGURE 2. The quiver $Q^\flat(\mathbf{b})$.

$$Q^{\flat,\mathcal{R}}_{\frac{r}{s}} := Q^\flat(a_1 - 1, a_2, \dots, a_{2m-1}, a_{2m} - 1).$$

As in the right case, $Q^{\flat,\mathcal{S}}_{\frac{r}{s}}$ is obtained by deleting the first a_1 arrows from $Q^{\flat,\mathcal{R}}_{\frac{r}{s}}$.

Then for the left q-deformed rational numbers, we have the following result.

Theorem 2.8 ([1, Corollary A.2]). *Let $\frac{r}{s} > 1$ be an irreducible fraction. Then, the following equations hold:*

$$\mathcal{R}^\flat_{\frac{r}{s}}(q) = \mathsf{cl}\left(Q^{\flat,\mathcal{R}}_{\frac{r}{s}}; q\right), \tag{2.6}$$

$$\mathcal{S}^\flat_{\frac{r}{s}}(q) = \mathsf{cl}\left(Q^{\flat,\mathcal{S}}_{\frac{r}{s}}; q\right). \tag{2.7}$$

Example 2.9. For $\frac{11}{8} = [1, 2, 1, 2]$, we can draw the quiver $Q^{\flat,\mathcal{R}}_{\frac{11}{8}}$ as in Figure 3, and we

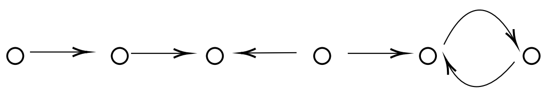

FIGURE 3. The quiver $Q^{\flat,\mathcal{R}}_{\frac{11}{8}}$.

have $\mathsf{cl}\left(Q^{\flat,\mathcal{R}}_{\frac{11}{8}}; q\right) = q^6 + 2q^5 + 2q^4 + 2q^3 + 2q^2 + q + 1 = \mathcal{R}^\flat_{\frac{11}{8}}(q)$.

Similarly, we have $\mathsf{cl}\left(Q^{\flat,\mathcal{S}}_{\frac{11}{8}}; q\right) = q^5 + 2q^4 + q^3 + 2q^2 + q + 1 = \mathcal{S}^\flat_{\frac{11}{8}}(q)$.

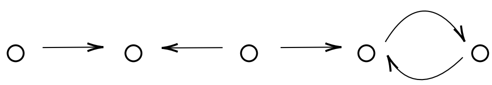

FIGURE 4. The quiver $Q^{\flat,\mathcal{S}}_{\frac{11}{8}}$.

Remark 2.10. If $\frac{r}{s} \notin \mathbb{Z}$ and $\frac{r}{s} > 1$, the above constructions of $Q^{\flat,\mathcal{R}}_{\frac{r}{s}}$, $Q^{\flat,\mathcal{S}}_{\frac{r}{s}}$ and $Q^{\sharp,\mathcal{S}}_{\frac{r}{s}}$, $Q^{\sharp,\mathcal{S}}_{\frac{r}{s}}$ do not care the parity of the length of the expression as a regular continued fraction. In fact, to adjust the parity of the length, we freely replace $[a_1, \dots, a_n + 1]$ by $[a_1, \dots, a_n, 1]$ (or vice versa), but the both expressions give the same quiver. For example, we have $Q(a_1 - 1, a_2, \dots, a_n + 1 - 1) = Q(a_1 - 1, a_2, \dots, a_n, 1 - 1)$.

2.4. **The basic properties of the left *q*-deformed rational numbers.** The left q-deformed rational numbers satisfy the following basic properties, which correspond to [9, Proposition 2.8] for the right variant, and the proof is also similar.

Proposition 2.11. *For* $\frac{r}{s} \in \mathbb{Q}, n \in \mathbb{Z}$, *one has*

(1) $\left[\frac{r}{s} + n\right]^\flat_q = q^n \left[\frac{r}{s}\right]^\flat_q + [n]_q$;

(2) $\left[-\frac{r}{s}\right]^\flat_q = -q^{-1} \left[\frac{r}{s}\right]^\flat_{q^{-1}}$;

(3) $\left[\frac{s}{r}\right]^\flat_q = \dfrac{1}{\left[\frac{r}{s}\right]^\flat_{q^{-1}}}$.

By Proposition 2.11 (1), for irreducible fractions $\frac{r}{s}, \frac{r'}{s}$ with $r \equiv r' \pmod{s}$, we have

$$\mathcal{S}^\flat_{\frac{r'}{s}}(q) = \mathcal{S}^\flat_{\frac{r}{s}}(q). \tag{2.8}$$

Proposition 2.12. *The following hold.*

(1) *For irreducible fractions* $\frac{r}{s}, \frac{r'}{s}$ *with* $r + r' \equiv 0 \pmod{s}$, *we have* $\mathcal{S}^\flat_{\frac{r}{s}}(q) = \mathcal{S}^\flat_{\frac{r'}{s}}(q)^\vee$.

(2) *For irreducible fractions* $\frac{r}{s}, \frac{r}{t} > 1$ *with* $s + t = r$, *we have* $\mathcal{R}^\flat_{\frac{r}{s}}(q) = \mathcal{R}^\flat_{\frac{r}{t}}(q)^\vee$.

Proof. (1) By (2.8), we may assume that $1 \leq n < \frac{r}{s} < \frac{r'}{s} < n+1$. If $\frac{r}{s} = [n, a_2, \dots, a_k]$, then $a_2 \geq 2$ and $\frac{r'}{s} = [n, 1, a_2 - 1, \dots, a_k]$ by [5, Lemma 3.1]. So we have $Q^{\flat,\mathcal{S}}_{\frac{r}{s}} = \left(Q^{\flat,\mathcal{S}}_{\frac{r'}{s}}\right)^\vee$, and the assertion follows from Theorem 2.8 and (2.3).

(2) By (1), we have $\mathcal{S}^\flat_{\frac{s}{r}}(q) = \mathcal{S}^\flat_{\frac{t}{r}}(q)^\vee$. By Proposition 2.11 (3), we have $\mathcal{R}^\flat_{\frac{r}{s}}(q) \equiv \mathcal{S}^\flat_{\frac{s}{r}}(q^{-1})$ and $\mathcal{R}^\flat_{\frac{r}{t}}(q) \equiv \mathcal{S}^\flat_{\frac{t}{r}}(q^{-1})$. Since $\mathcal{R}^\flat_{\frac{r}{s}}(q), \mathcal{R}^\flat_{\frac{r}{t}}(q) \in \mathbb{Z}[q]$ with $\mathcal{R}^\flat_{\frac{r}{s}}(0) = \mathcal{R}^\flat_{\frac{r}{t}}(0) = 1$ now, the assertion follows. $\square$

We remark that the right q-deformed rationals $\left[\frac{r}{s}\right]^\sharp_q$ also satisfy the equations corresponding to (2.8) and Proposition 2.12 (c.f. [13, 5]).

3. q-transposes and orthogonal q-transpose in $\mathrm{PSL}_q(2, \mathbb{Z})$

In this section, we define two operations the *q-transpose* A^{T_q} and the *orthogonal q-transpose* A^{O_q}, and give some basic properties and applications of them.

Definition 3.1. For a matrix

$$A = \begin{pmatrix} \mathcal{R}(q) & \mathcal{V}(q) \\ \mathcal{S}(q) & \mathcal{U}(q) \end{pmatrix}$$

whose entries are elements in $\mathbb{Z}[q^{\pm 1}]$, set

$$A^{T_q} := \begin{pmatrix} \mathcal{R}(q) & q^{-1}\mathcal{S}(q) \\ q\mathcal{V}(q) & \mathcal{U}(q) \end{pmatrix}.$$

We call it the *q-transpose* of A.

An easy calculation shows that $(A^{T_q})^{T_q} = A$, $\det(A^{T_q}) = \det A$, $\operatorname{tr}(A^{T_q}) = \operatorname{tr} A$, and $(AB)^{T_q} = B^{T_q}A^{T_q}$. The operation $(-)^{T_q}$ also makes sense up to the equivalence $\equiv$.

Lemma 3.2. *If $A \in \mathrm{PSL}_q(2,\mathbb{Z})$, then $A^{T_q} \in \mathrm{PSL}_q(2,\mathbb{Z})$.*

Proof. We have $R_q^{T_q} \equiv L_q$ and $S_q^{T_q} \equiv S_q$. Moreover, if A is regular, then A^{T_q} is also, and $(A^{-1})^{T_q} \equiv (A^{T_q})^{-1}$. So the assertion follows. $\square$

The next result immediately follows from Proposition 2.5 and Lemma 3.2.

Corollary 3.3. *For*

$$\begin{pmatrix} \mathcal{R}(q) & \mathcal{V}(q) \\ \mathcal{S}(q) & \mathcal{U}(q) \end{pmatrix} \in \mathrm{PSL}_q(2,\mathbb{Z})$$

with $r := \mathcal{R}(1)$, $s := \mathcal{S}(1)$, $v := \mathcal{V}(1)$ and $u := \mathcal{U}(1)$, we have

$$\left[\frac{r}{v}\right]_q^\sharp = \left[\frac{\mathcal{R}(q)}{q\mathcal{V}(q)}\right] \quad \textit{and} \quad \left[\frac{s}{u}\right]_q^\sharp = \left[\frac{\mathcal{S}(q)}{q\mathcal{U}(q)}\right].$$

Now we can get the third proof of [5, Theorem 3.5] ([5] already contains two proofs).

Corollary 3.4 ([5, Theorem 3.5]). *For a positive integer s and integers $v, w \in \mathbb{Z}$ with $vw \equiv -1 \pmod{s}$, we have $\mathcal{S}^\sharp_{\frac{v}{s}}(q) = \mathcal{S}^\sharp_{\frac{w}{s}}(q)$.*

Proof. Since $vw \equiv -1 \pmod{s}$, there is some $r \in \mathbb{Z}$ such that $\begin{pmatrix} r & v \\ w & s \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$. Now we can take $\mathcal{R}(q), \mathcal{V}(q), \mathcal{W}(q), \mathcal{S}(q) \in \mathbb{Z}[q^{\pm}]$ such that

$$\begin{pmatrix} \mathcal{R}(q) & \mathcal{V}(q) \\ \mathcal{W}(q) & \mathcal{S}(q) \end{pmatrix} \in \mathrm{PSL}_q(2,\mathbb{Z}) \quad \text{and} \quad \begin{pmatrix} \mathcal{R}(1) & \mathcal{V}(1) \\ \mathcal{W}(1) & \mathcal{S}(1) \end{pmatrix} \equiv \begin{pmatrix} r & v \\ w & s \end{pmatrix}.$$

By Proposition 2.5 and Corollary 3.3, we have

$$\left[\frac{v}{s}\right]_q^\sharp = \left[\frac{\mathcal{R}^\sharp_{\frac{v}{s}}(q)}{\mathcal{S}^\sharp_{\frac{v}{s}}(q)}\right] = \left[\frac{\mathcal{V}(q)}{\mathcal{S}(q)}\right] \quad \text{and} \quad \left[\frac{w}{s}\right]_q^\sharp = \left[\frac{\mathcal{R}^\sharp_{\frac{w}{s}}(q)}{\mathcal{S}^\sharp_{\frac{w}{s}}(q)}\right] = \left[\frac{\mathcal{W}(q)}{q\mathcal{S}(q)}\right].$$

Hence we have $\mathcal{S}^\sharp_{\frac{v}{s}}(q) \equiv \mathcal{S}(q) \equiv \mathcal{S}^\sharp_{\frac{w}{s}}(q)$. Since $\mathcal{S}^\sharp_{\frac{v}{s}}(q), \mathcal{S}^\sharp_{\frac{w}{s}}(q) \in \mathbb{Z}[q]$ with $\mathcal{S}^\sharp_{\frac{v}{s}}(0) = \mathcal{S}^\sharp_{\frac{w}{s}}(0) = 1$, we are done. $\square$

Corollary 3.5 ([5, Lemma 4.1]). *For $r \in \mathbb{Z}$ and positive integers v, w with $vw \equiv -1 \pmod{r}$, we have $\mathcal{R}^\sharp_{\frac{r}{v}}(q) \equiv \mathcal{R}^\sharp_{\frac{r}{w}}(q)$. If further $\frac{r}{v}, \frac{r}{w} > 1$, we have $\mathcal{R}^\sharp_{\frac{r}{v}}(q) = \mathcal{R}^\sharp_{\frac{r}{w}}(q)$.*

Proof. The first assertion can be shown in a similar way to Corollary 3.4. However, we focus on the $(1,1)$-entry of the matrices in the proof of the corollary. Since $\mathcal{R}^\sharp_\alpha(q) \in \mathbb{Z}[q]$ with $\mathcal{R}^\sharp_\alpha(0) = 1$ for $\alpha > 1$, the second assertion follows from the first. $\square$

Definition 3.6. For a matrix

$$A = \begin{pmatrix} \mathcal{R}(q) & \mathcal{V}(q) \\ \mathcal{S}(q) & \mathcal{U}(q) \end{pmatrix}$$

whose entries are elements in $\mathbb{Z}[q^{\pm 1}]$, set

$$A^{O_q} := \begin{pmatrix} \mathcal{U}(q^{-1}) & q^{-1}\mathcal{V}(q^{-1}) \\ q\mathcal{S}(q^{-1}) & \mathcal{R}(q^{-1}) \end{pmatrix}.$$

We call it the *orthogonal q-transpose* of A.

Easy calculation shows that $(A^{O_q})^{O_q} = A$, $(AB)^{O_q} = B^{O_q}A^{O_q}$. For $d(q) := \det A$ and $t(q) := \operatorname{tr} A$, we have $\det(A^{O_q}) = d(q^{-1})$ and $\operatorname{tr}(A^{O_q}) = t(q^{-1})$. Since the operation $(-)^{O_q}$ also makes sense up to the equivalence $\equiv$, and we have $(R_q)^{O_q} \equiv R_q$ and $(S_q)^{O_q} \equiv S_q$. By the same way as Lemma 3.2, we can show the following.

Lemma 3.7. *If $A \in \mathrm{PSL}_q(2,\mathbb{Z})$, then $A^{O_q} \in \mathrm{PSL}_q(2,\mathbb{Z})$.*

4. The trace of $A \in \mathrm{PSL}_q(2,\mathbb{Z})$ and the Jones polynomials of rational knots

In this section, we give further applications of the (orthogonal) q-transpose. These applications include arithmetic properties of the left q-deformed rationals, the trace of the matrices in $\mathrm{PSL}_q(2,\mathbb{Z})$, and the normalized Jones polynomials of rational links.

We recall some basic notions and results on rational links and their normalized Jones polynomials.

For an irreducible fraction $\frac{r}{s} = [a_1, \dots, a_{2m}] > 1$, the rational link associated with $\frac{r}{s}$ in the 3-sphere $\mathbb{S}^3$ is determined by the Figure 5, where each square is called a_i-half

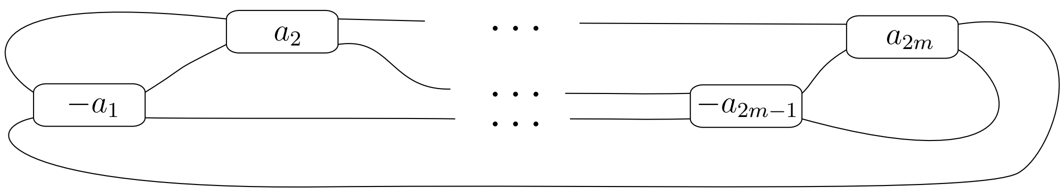

Figure 5. The rational link of $\frac{r}{s}$.

twists determined by Figure 6.

We denote by $L(\frac{r}{s})$ the rational link associated with $\frac{r}{s}$. See, for example, [4, 11]. The following theorem is due to H. Schubert in 1956.

Theorem 4.1 (c.f. [4, Theorem 2]). *For irreducible fractions $\frac{r}{s}$, $\frac{r'}{s'}$, the following are equivalent.*

(1) *$L(\frac{r}{s})$ and $L(\frac{r'}{s'})$ are isotopic.*
(2) *$r = r'$ and either $s \equiv s' \pmod r$ or $ss' \equiv 1 \pmod r$.*

As a useful isotopy invariant for an oriented link L in $\mathbb{S}^3$, the Jones polynomial $V_L(t) \in \mathbb{Z}[t^{\pm 1}] \cup t^{\frac{1}{2}}\mathbb{Z}[t^{\pm 1}]$ is well-studied. Lee and Schiffler [11, Proposition 1.2 (b)] introduced

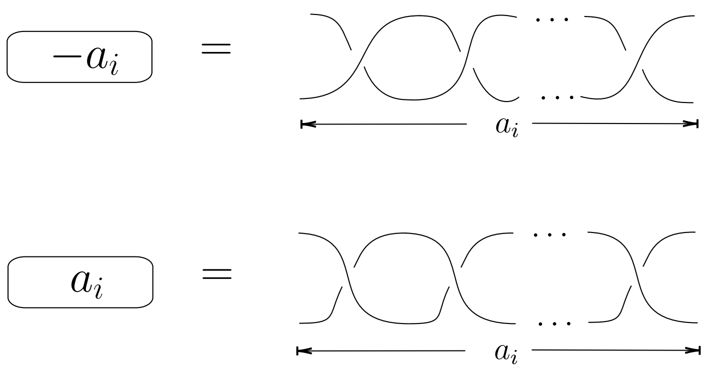

FIGURE 6. a_i-half twists

the following normalization $J_\alpha(q)$ of the Jones polynomial $V_{L(\alpha)}(t)$ of a rational link $L(\alpha)$:

$$J_\alpha(q) := \pm t^{-h} V_{L(\alpha)}(t)|_{t=-q^{-1}}, \tag{4.1}$$

where $\pm t^h$ is the leading term of $V_{L(\alpha)}(t)$. This indicates the normalization such that the constant term is 1 as a polynomial in q.

Theorem 4.2 ([13, Proposition A.1]). *For a rational number $\alpha > 1$, the normalized Jones polynomial $J_\alpha(q)$ can be computed by*

$$J_\alpha(q) = q\mathcal{R}^\sharp_\alpha(q) + (1-q)\mathcal{S}^\sharp_\alpha(q). \tag{4.2}$$

Now, we can give a new proof of the following result. This result was shown by Bapat, Becker and Licata [1, Theorem A3] by considering a homological interpretation of the left q-deformed rational numbers, and the first author of the present paper gave a combinatorial proof by using q-deformed Farey sums and induction on the size of negative continued fractions (c.f [18, Theorem 4.2]). We remark that all known proofs (including ours) use Theorem 4.2.

Theorem 4.3 (c.f. [1, Theorem A3], see also [18, Theorem 4.2]). *For a rational number $\alpha > 1$, we have*

$$J_\alpha(q) = \mathcal{R}^\flat_\alpha(q)^\vee.$$

Proof. We can take an irreducible fraction $\frac{r}{s} > 1$ with $\alpha = \frac{r}{r-s}$. If $\frac{r}{s} = [a_1, \ldots, a_{2m}]$, we have

$$M(a_1, \ldots, a_{2m}) = \begin{pmatrix} r & t \\ s & u \end{pmatrix}$$

for some $t, u \in \mathbb{N}$, and

$$M_q(a_1, \ldots, a_{2m}) = \begin{pmatrix} q\mathcal{R}^\sharp_{\frac{r}{s}}(q) & \mathcal{R}^\sharp_{\frac{t}{u}}(q) \\ q\mathcal{S}^\sharp_{\frac{r}{s}}(q) & \mathcal{S}^\sharp_{\frac{t}{u}}(q) \end{pmatrix}.$$

By Corollaries 3.5 and 3.3, we have $\mathcal{R}^\sharp_{\frac{r}{s}}(q) = \mathcal{R}^\sharp_{\frac{r}{t}}(q)$ and $\mathcal{R}^\sharp_{\frac{t}{u}}(q) = \mathcal{S}^\sharp_{\frac{r}{t}}(q)$. Hence we have

$$\mathcal{R}^\flat_{\frac{r}{s}}(q) = q\mathcal{R}^\sharp_{\frac{r}{s}}(q) + (1-q)\mathcal{R}^\sharp_{\frac{t}{u}}(q) = q\mathcal{R}^\sharp_{\frac{r}{t}}(q) + (1-q)\mathcal{S}^\sharp_{\frac{r}{t}}(q) = J_{\frac{r}{t}}(q), \tag{4.3}$$

where the last equality follows from Theorem 4.2. Since $t(r-s) \equiv -st \equiv 1 \pmod r$, we have $J_{\frac{r}{t}}(q) = J_{\frac{r}{r-s}}(q) = J_\alpha(q)$ by Theorem 4.1. By Proposition 2.12 (2), we have $\mathcal{R}^\flat_{\frac{r}{s}}(q) = \mathcal{R}^\flat_{\frac{r}{r-s}}(q)^\vee$. Summing up, we have

$$\mathcal{R}^\flat_\alpha(q)^\vee = \mathcal{R}^\flat_{\frac{r}{r-s}}(q)^\vee = \mathcal{R}^\flat_{\frac{r}{s}}(q) = J_{\frac{r}{t}}(q) = J_\alpha(q).$$

□

Remark 4.4. After the submission of the first version, the referee kindly suggested us results in [19] and [6] might give a more direct proof of Theorem 4.2. The following is a proof based on this suggestion.

A proof of Theorem 4.2 after [19, 6]**:** For a rational number $\alpha > 1$, we use the equation (c.f. [6, Equation (3)]) which was first observed in [19], and we replace q by q^{-1} as follows:

$$[\alpha]^\flat_{q^{-1}} = \frac{[\alpha]^\sharp_q + q^{-1} - 1}{(1-q^{-1})[\alpha]^\sharp_q + q^{-1}}.$$

It can be written as folllows:

$$\frac{\mathcal{R}^\flat_\alpha(q^{-1})}{\mathcal{S}^\flat_\alpha(q^{-1})} = \frac{\frac{\mathcal{R}^\sharp_\alpha(q)}{\mathcal{S}^\sharp_\alpha(q)} + q^{-1} - 1}{(1-q^{-1})\frac{\mathcal{R}^\sharp_\alpha(q)}{\mathcal{S}^\sharp_\alpha(q)} + q^{-1}} = \frac{q\mathcal{R}^\sharp_\alpha(q) + (1-q)\mathcal{S}^\sharp_\alpha(q)}{(q-1)\mathcal{R}^\sharp_\alpha(q) + \mathcal{S}^\sharp_\alpha(q)}. \tag{4.4}$$

We need to show that $J_\alpha(q) = q\mathcal{R}^\sharp_\alpha(q) + (1-q)\mathcal{S}^\sharp_\alpha(q)$ and $K_\alpha(q) := (q-1)\mathcal{R}^\sharp_\alpha(q) + \mathcal{S}^\sharp_\alpha(q)$ are coprime. Assume the contrary. Since

$$\begin{pmatrix} J_\alpha(q) \\ K_\alpha(q) \end{pmatrix} = \begin{pmatrix} q & 1-q \\ q-1 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{R}^\sharp_\alpha(q) \\ \mathcal{S}^\sharp_\alpha(q) \end{pmatrix},$$

and the determinant of the 2×2 matrix appearing in this equation is $\Delta := q^2 - q + 1$, we have

$$\begin{pmatrix} \Delta \cdot \mathcal{R}^\sharp_\alpha(q) \\ \Delta \cdot \mathcal{S}^\sharp_\alpha(q) \end{pmatrix} = \begin{pmatrix} 1 & q-1 \\ 1-q & q \end{pmatrix} \begin{pmatrix} J_\alpha(q) \\ K_\alpha(q) \end{pmatrix}.$$

Since $\mathcal{R}^\sharp_\alpha(q)$ and $\mathcal{S}^\sharp_\alpha(q)$ are coprime and $\Delta = q^2 - q + 1$ is irreducible, $q^2 - q + 1$ divides both $J_\alpha(q)$ and $K_\alpha(q)$. Hence, if ζ is a primitive 6th root of unity, then $J_\alpha(\zeta) = 0$. However, it is a classical result that $J_\alpha(\zeta) \neq 0$ always holds (see the last part of [5] for the proof using the theory of q-deformed rationals). This is a contradiction, and $J_\alpha(q)$ and $K_\alpha(q)$ are coprime. Hence (4.4) yields

$$\mathcal{R}^\flat_\alpha(q)^\vee \equiv \mathcal{R}^\flat_\alpha(q^{-1}) \equiv q\mathcal{R}^\sharp_\alpha(q) + (1-q)\mathcal{S}^\sharp_\alpha(q) = J_\alpha(q).$$

However, both $\mathcal{R}^\flat_\alpha(q)^\vee$ and $J_\alpha(q)$ are polynomilas whose constant terms are 1. Hence we are done. □

Lemma 4.5. *For irreducible fractions $\frac{r}{s}, \frac{r}{s'} > 1$ with $ss' \equiv 1 \pmod r$ (resp. $ss' \equiv -1 \pmod r$), we have $\mathcal{R}^\flat_{\frac{r}{s}}(q) = \mathcal{R}^\flat_{\frac{r}{s'}}(q)$ (resp. $\mathcal{R}^\flat_{\frac{r}{s}}(q) = \mathcal{R}^\flat_{\frac{r}{s'}}(q)^\vee$).*

Proof. If $ss' \equiv 1 \pmod r$, then the links $L(\frac{r}{s})$ and $L(\frac{r}{s'})$ are isotopic by Theorem 4.1, and have the same (normalized) Jones polynomial. By Theorem 4.2, we have

$$\mathcal{R}^\flat_{\frac{r}{s}}(q) = J_{\frac{r}{s}}(q)^\vee = J_{\frac{r}{s'}}(q)^\vee = \mathcal{R}^\flat_{\frac{r}{s'}}(q).$$

The case $ss' \equiv -1 \pmod r$ follows from the above equation and Proposition 2.12 (2). We also remark that this case also follows from Theorem 4.3 and the equation (4.3) in its proof. $\square$

Remark 4.6. For irreducible fractions $\frac{r}{s}, \frac{r}{s'} > 1$ with $ss' \equiv -1 \pmod r$, it is known that $\frac{r}{s} = [a_1, \dots, a_{2m}]$ implies $\frac{r}{s'} = [a_{2m}, \dots, a_1]$ (to see this, take the transpose of $M(a_1, \dots, a_{2m})$). Hence, the $ss' \equiv -1 \pmod r$ case of the above lemma states that, for a tuple of integers $(b_1, b_2, \dots, b_{2m})$ with $b_1, b_{2m} \geq 0$, $b_2, \dots, b_{2m-1} > 0$, the equation

$$\mathsf{cl}(Q^\flat(b_1, \dots, b_{2m}), q) = \mathsf{cl}(Q^\flat(b_{2m}, \dots, b_1), q)^\vee \tag{4.5}$$

holds. It is easy to see that (4.5) is equivalent to the following equation for $(b_1, \dots, b_{2m+1})$ with $b_1, b_{2m+1} \geq 0$, $b_2, \dots, b_{2m} > 0$:

$$\mathsf{cl}(Q^\flat(b_1, \dots, b_{2m+1}), q) = \mathsf{cl}(Q^\flat(b_{2m+1}, \dots, b_1), q). \tag{4.6}$$

It might be an interesting problem to find a bijective proof of the above equations.

Example 4.7. This is an example of the equation (4.6). Considering two tuples of integers $(1, 2, 0)$ and $(0, 2, 1)$, we can draw $Q^\flat(1, 2, 0)$ (for convenience, we number each vertex) and count its l-closures as follows.

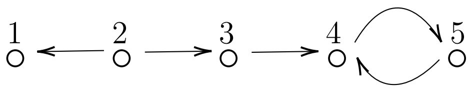

FIGURE 7. The quiver $Q^\flat(1, 2, 0)$.

l	l-closures	the number of l-closures
0	$\emptyset$	1
1	$\{1\}$	1
2	$\{4, 5\}$	1
3	$\{1, 4, 5\}, \{3, 4, 5\}$	2
4	$\{1, 3, 4, 5\}$	1
5	$\{1, 2, 3, 4, 5\}$	1

TABLE 1. The l-closure of $Q^\flat(1, 2, 0)$

Hence we have

$$\mathsf{cl}(Q^\flat(1, 2, 0)) = 1 + q + q^2 + 2q^3 + q^4 + q^5.$$

On the other hand, we can draw $Q^\flat(0, 2, 1)$, and count its l-closures as follows.

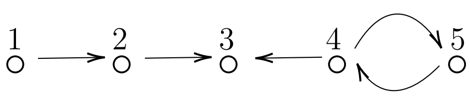

FIGURE 8. The quiver $Q^\flat(0, 2, 1)$.

l	l-closures	the number of l-closures
0	$\emptyset$	1
1	$\{3\}$	1
2	$\{2,3\}$	1
3	$\{1,2,3\},\{3,4,5\}$	2
4	$\{2,3,4,5\}$	1
5	$\{1,2,3,4,5\}$	1

TABLE 2. The l-closure of $Q^\flat(0,2,1)$

Hence we have

$$\mathsf{cl}(Q^\flat(0,2,1)) = 1 + q + q^2 + 2q^3 + q^4 + q^5 = \mathsf{cl}(Q^\flat(1,2,0)).$$

Theorem 4.8. *The following hold.*

(1) *For irreducible fractions $\frac{r}{s}, \frac{r'}{s}$ with $rr' \equiv 1 \pmod{s}$ (resp. $rr' \equiv -1 \pmod{s}$), we have $\mathcal{S}^\flat_{\frac{r}{s}}(q) = \mathcal{S}^\flat_{\frac{r'}{s}}(q)$ (resp. $\mathcal{S}^\flat_{\frac{r}{s}}(q) = \mathcal{S}^\flat_{\frac{r'}{s}}(q)^\vee$).*

(2) *For irreducible fractions $\frac{r}{s}, \frac{r}{s'}$ with $ss' \equiv 1 \pmod{r}$* (resp. $ss' \equiv -1 \pmod{r}$), we have $\mathcal{R}^\flat_{\frac{r}{s}}(q) \equiv \mathcal{R}^\flat_{\frac{r}{s'}}(q)$ (resp. $\mathcal{R}^\flat_{\frac{r}{s}}(q) \equiv \mathcal{R}^\flat_{\frac{r}{s'}}(q)^\vee$). (That is, if we replace $=$ by $\equiv$, the statement of Lemma 4.5 holds without the restriction that $\frac{r}{s}, \frac{r}{s'} > 1$.)

Proof. (1) By (2.8), we may assume that $0 < \frac{r}{s}, \frac{r'}{s} < 1$. Since we always have $\mathcal{S}^\flat_{\frac{r}{s}}(q) \in \mathbb{Z}[q]$ with $\mathcal{S}^\flat_{\frac{r}{s}}(0) = 1$, the assertion follows from Lemma 4.5 and Proposition 2.11 (3).

(2) The assertion follows from (1) and Proposition 2.11 (3). □

Using the above property of $\mathcal{R}^\flat_{\frac{r}{s}}(q)$ and (orthogonal) q-transposes, we can get a new proof of the following impressive result on $\operatorname{tr} A$ for $A \in \mathrm{PSL}_q(2,\mathbb{Z})$.

Theorem 4.9 ([9, Theorem 3])**.** *For $A \in \mathrm{PSL}_q(2,\mathbb{Z})$, $\operatorname{tr} A$ is equivalent to a palindromic polynomial whose coefficients are non-negative.*

In the following, we say $f(q) \in \mathbb{Z}[q]$ is *anti-palindromic* if $f(q) = -f(q)^\vee$.

Proof. In this proof, we treat

$$M := (-S_q)^{-1}A = \begin{pmatrix} \mathcal{R}(q) & \mathcal{T}(q) \\ \mathcal{S}(q) & \mathcal{U}(q) \end{pmatrix} \quad \text{with} \quad M|_{q=1} = \begin{pmatrix} r & t \\ s & u \end{pmatrix}$$

rather than A itself. Note that

$$M \equiv \begin{pmatrix} q\mathcal{R}^\sharp_{\frac{r}{s}}(q) & \mathcal{S}^\sharp_{\frac{r}{t}}(q) \\ q\mathcal{S}^\sharp_{\frac{r}{s}}(q) & \mathcal{U}(q) \end{pmatrix} \equiv \begin{pmatrix} \mathcal{R}(q) & \mathcal{R}^\sharp_{\frac{t}{u}}(q) \\ q\mathcal{R}^\sharp_{\frac{s}{u}}(q) & \mathcal{S}^\sharp_{\frac{t}{u}}(q) \end{pmatrix}$$

by Proposition 2.5 and Corollary 3.3 (strictly speaking, since the entries of $A \in \mathrm{PSL}_q(2,\mathbb{Z})$ are determined only up to multiplication by units in $\mathbb{Z}[q^{\pm 1}]$, $\mathcal{U}(q)$ and $\mathcal{R}(q)$ should be adjusted by multiplying by $\pm q^n$ for some $n \in \mathbb{Z}$, but we omit it for simplicity),

$$A = (-S_q)M \equiv \begin{pmatrix} \mathcal{S}^\sharp_{\frac{r}{s}}(q) & q^{-1}\mathcal{U}(q) \\ -q\mathcal{R}^\sharp_{\frac{r}{s}}(q) & -\mathcal{S}^\sharp_{\frac{r}{t}}(q) \end{pmatrix} \equiv \begin{pmatrix} \mathcal{R}^\sharp_{\frac{s}{u}}(q) & q^{-1}\mathcal{S}^\sharp_{\frac{t}{u}}(q) \\ -\mathcal{R}(q) & -\mathcal{R}^\sharp_{\frac{t}{u}}(q) \end{pmatrix}$$

and
$$\operatorname{tr} A \equiv \mathcal{S}^\sharp_{\frac{r}{s}}(q) - \mathcal{S}^\sharp_{\frac{r}{t}}(q) \equiv \mathcal{R}^\sharp_{\frac{s}{u}}(q) - \mathcal{R}^\sharp_{\frac{t}{u}}(q).$$
So we may assume that $s \neq t$.

First, consider the case $u = 0$. Then we have $(s, t) = (\pm 1, \mp 1)$, and $(\mathcal{S}(q), \mathcal{T}(q)) \equiv (q^n, -1)$ for some $n \in \mathbb{Z}$. Therefore $\operatorname{tr} A \equiv \mathcal{S}(q) - \mathcal{T}(q) = q^n + 1$. So we may assume that $u \neq 0$.

Replacing A by either $-A$, A^{O_q} or $-A^{O_q}$, if necessary, we may assume that $u > 0$ and $s > t$. By Proposition 2.11 (1) and Corollary 3.4 (note that $st \equiv -1 \pmod u$ now), we have
$$\begin{aligned}
\mathcal{R}^\sharp_{\frac{s}{u}+n}(q) - \mathcal{R}^\sharp_{\frac{t}{u}+n}(q) &= (q^n\mathcal{R}^\sharp_{\frac{s}{u}}(q) + [n]_q \cdot \mathcal{S}^\sharp_{\frac{s}{u}}(q)) - (q^n\mathcal{R}^\sharp_{\frac{t}{u}}(q) + [n]_q \cdot \mathcal{S}^\sharp_{\frac{t}{u}}(q)) \\
&= (q^n\mathcal{R}^\sharp_{\frac{s}{u}}(q) + [n]_q \cdot \mathcal{S}^\sharp_{\frac{s}{u}}(q)) - (q^n\mathcal{R}^\sharp_{\frac{t}{u}}(q) + [n]_q \cdot \mathcal{S}^\sharp_{\frac{s}{u}}(q)) \\
&= q^n(\mathcal{R}^\sharp_{\frac{s}{u}}(q) - \mathcal{R}^\sharp_{\frac{t}{u}}(q)) \equiv \operatorname{tr} A.
\end{aligned}$$
We also remark that there is some $\mathcal{R}'(q) \in \mathbb{Z}[q^{\pm 1}]$ such that
$$\begin{pmatrix} \mathcal{R}'(q) & \mathcal{R}^\sharp_{\frac{t}{u}+n}(q) \\ q\mathcal{R}^\sharp_{\frac{s}{u}+n}(q) & \mathcal{U}(q) \end{pmatrix} \in \mathrm{PSL}_q(2, \mathbb{Z}).$$

So keeping $\operatorname{tr} A$ up to $\equiv$, we may assume that $s, t > u > 0$. More precisely, we replace s (resp. t) by $s + nu$ (resp. $t + nu$) for $n \gg 0$, if necessary. Then $r > s, t > u$, and
$$M_q(a_1, \dots, a_{2m}) = \begin{pmatrix} q\mathcal{R}^\sharp_{\frac{r}{s}}(q) & \mathcal{S}^\sharp_{\frac{r}{t}}(q) \\ q\mathcal{S}^\sharp_{\frac{r}{s}}(q) & \mathcal{S}^\sharp_{\frac{t}{u}}(q) \end{pmatrix}$$
for $\frac{r}{s} = [a_1, \dots, a_{2m}] > 1$ by Lemma 2.1. Then $\frac{r}{t} = [a_{2m}, \dots, a_1] > 1$ (see Remark 4.6) and
$$M_q(a_{2m}, \dots, a_1) = \begin{pmatrix} q\mathcal{R}^\sharp_{\frac{r}{s}}(q) & \mathcal{S}^\sharp_{\frac{r}{s}}(q) \\ q\mathcal{S}^\sharp_{\frac{r}{t}}(q) & \mathcal{S}^\sharp_{\frac{t}{u}}(q) \end{pmatrix}.$$
Hence
$$\mathcal{R}^\flat_{\frac{r}{s}}(q) = q\mathcal{R}^\sharp_{\frac{r}{s}} + (1-q)\mathcal{S}^\sharp_{\frac{r}{t}} \quad \text{and} \quad \mathcal{R}^\flat_{\frac{r}{t}}(q) = q\mathcal{R}^\sharp_{\frac{r}{s}} + (1-q)\mathcal{S}^\sharp_{\frac{r}{s}}.$$
Since $\mathcal{R}^\flat_{\frac{r}{s}}(q) = \mathcal{R}^\flat_{\frac{r}{t}}(q)^\vee$ by Lemma 4.8 (note that $st \equiv -1 \pmod r$ now),
$$\mathcal{R}^\flat_{\frac{r}{s}}(q) - \mathcal{R}^\flat_{\frac{r}{t}}(q) = (q-1)(\mathcal{S}^\sharp_{\frac{r}{s}}(q) - \mathcal{S}^\sharp_{\frac{r}{t}}(q)) \tag{4.7}$$
is equivalent to an anti-palindromic polynomial. Since $(q-1)$ is also anti-palindromic, $\mathcal{S}^\sharp_{\frac{r}{s}}(q) - \mathcal{S}^\sharp_{\frac{r}{t}}(q)$ $(\equiv \operatorname{tr} A)$ is equivalent to a palindromic polynomial.

Next we will show that all non-zero coefficients of $\operatorname{tr} A$ have the same sign. Using the above equation $\mathcal{R}^\sharp_{\frac{s}{u}+n}(q) - \mathcal{R}^\sharp_{\frac{t}{u}+n}(q) \equiv \mathcal{R}^\sharp_{\frac{s}{u}}(q) - \mathcal{R}^\sharp_{\frac{t}{u}}(q)$ for all $n \in \mathbb{Z}$, we may assume that $0 \geq t > -u$. Then all non-zero coefficients of $\mathcal{R}^\sharp_{\frac{t}{u}}(q)$ are negative. So if $s \geq 0$, all non-zero coefficients of $\mathcal{R}^\sharp_{\frac{s}{u}}(q)$ are positive, and the same is true for $\mathcal{R}^\sharp_{\frac{s}{u}}(q) - \mathcal{R}^\sharp_{\frac{t}{u}}(q)$ $(\equiv \operatorname{tr} A)$.

It remains to show the case $0 > s > t > -u$. In this case, $|s|, |t| < u$. Since $ru - st = \pm 1$ now, we have $u > r > 0$. Since

$$M^{O_q} \equiv \begin{pmatrix} \mathcal{S}^\sharp_{\frac{t}{u}}(q^{-1}) & q^{-1}\mathcal{R}^\sharp_{\frac{t}{u}}(q^{-1}) \\ \mathcal{R}^\sharp_{\frac{s}{u}}(q^{-1}) & \mathcal{R}(q^{-1}) \end{pmatrix} \equiv \begin{pmatrix} \mathcal{U}'(q) & \mathcal{R}^\sharp_{\frac{t}{r}}(q) \\ q\mathcal{R}^\sharp_{\frac{s}{r}}(q) & \mathcal{S}^\sharp_{\frac{t}{r}}(q) \end{pmatrix} \text{ with } M^{O_q}|_{q=1} = \begin{pmatrix} u & t \\ s & r \end{pmatrix},$$

if we set $\tau(q) := \operatorname{tr} A$ then we have

$$\tau(q^{-1}) \equiv \mathcal{R}^\sharp_{\frac{s}{u}}(q^{-1}) - \mathcal{R}^\sharp_{\frac{t}{u}}(q^{-1}) \equiv \mathcal{R}^\sharp_{\frac{s}{r}}(q) - \mathcal{R}^\sharp_{\frac{t}{r}}(q).$$

In this way, it suffices to show that all non-zero coefficients of $\mathcal{R}^\sharp_{\frac{s}{r}}(q) - \mathcal{R}^\sharp_{\frac{t}{r}}(q)$ have the same sign. So we apply the above argument to M^{O_q}. That is, the reduction using

$$\mathcal{R}^\sharp_{\frac{s}{r}+n}(q) - \mathcal{R}^\sharp_{\frac{t}{r}+n}(q) \equiv \mathcal{R}^\sharp_{\frac{s}{r}}(q) - \mathcal{R}^\sharp_{\frac{t}{r}}(q).$$

If we cannot make $s > 0 > t$, we replace M by its orthogonal q-transpose (and repeat this process if necessary). The $(2,2)$ entry of $M|_{q=1}$, which is a positive integer, strictly decreases at each step, and cannot decrease indefinitely. So the process terminates after finitely many steps, and we are reduced to the case $s > 0 > t$. $\square$

Proposition 4.10. *For $A \in \mathrm{PSL}_q(2,\mathbb{Z})$, the trace $\operatorname{tr} A$ is equivalent to one of the following polynomials.*

1. $1 + q^n$ *for some $n \in \mathbb{Z}_{\geq 0}$.*
2. $[n]_q$ *for some $n \in \mathbb{Z}_{\geq 0}$ (note that $[0]_q = 0$).*
3. $\operatorname{tr} M_q(a_1, \dots, a_{2m})$ *for $a_1, \dots, a_{2m} > 0$.*

Proof. We reuse the proof of Theorem 4.9. In the notation there, the case $u = 0$ corresponds to (1). If $s = t$, then $\operatorname{tr} A = 0$. So, in that proof, assuming $u \neq 0$ and $s \neq t$, we reduced to the case

$$M|_{q=1} = \begin{pmatrix} r & t \\ s & u \end{pmatrix} \quad \text{and} \quad A^{T_q}|_{q=1} = \begin{pmatrix} s & -r \\ u & -t \end{pmatrix}$$

with $0 \geq t > -u$ and $s \geq 0$. If $s = 0$ or $t = 0$, then $\operatorname{tr} A \equiv [n]_q$ for some $n \in \mathbb{N}$. So it remains to consider the case $-u < t < 0 < s$. Now we have $A^{T_q} = M_q(a_1, \dots, a_{2m})$ for some $a_1 \geq 0$ and $a_2, \dots, a_{2m} > 0$ by Lemma 2.1. If $s > u$, then we have $a_1 > 1$ and $\operatorname{tr} A = \operatorname{tr} A^{T_q}$ is the type (3). So assume that $0 < s < u$. Since $|s|, |t| < u$ now, we have $0 < |r| < u$. Like the last step of the proof of Theorem 4.9, we replace M by M^{O_q} keeping $\operatorname{tr} A$ up to $\equiv$. And then we "adjust" M again to satisfy $0 \geq t > -u$ and $s \geq 0$. By repeating this procedure if necessary, we can reduce to the case $s > u$. $\square$

For a sequence of positive integers $a_1, \dots, a_{2m}$, Kantarcı Oğuz and Ravichandran [3] defined the *circular fence poset* $\overline{Q}(a_1, \dots, a_{2m})$, and Kantarcı Oğuz [2] showed that

$$\operatorname{tr}(M_q(a_1, \dots, a_{2m})) = \mathsf{rk}(\overline{Q}(a_1, \dots, a_{2m}), q)$$

([2, Proposition 5.5]), and hence

$$\operatorname{tr}(M_q(a_1, \dots, a_{2m}))|_{q=0} = \mathsf{rk}(\overline{Q}(a_1, \dots, a_{2m}), 0) = 1.$$

She also raised the following conjecture. For a polynomial $f(q) = \sum_{i=0}^n a_i q^i \in \mathbb{Z}[q]$, we say $f(q)$ is *unimodal* if $a_0 \leq \cdots \leq a_m \geq \cdots \geq a_n$ for some m with $0 \leq m \leq n$.

Conjecture 4.11 ([2, Conjecture 1.4])**.** For any sequence $a_1, \dots, a_{2m} > 0$, the trace $\operatorname{tr}(M(a_1, \dots, a_{2m}))$ is unimodal except for the cases $(a_1, a_2, \dots, a_{2m}) = (1, k, 1, k)$ or $(k, 1, k, 1)$ for some k. ($m = 2$ holds in these cases.)

We use the above observation to analyze the behavior of $\mathcal{R}^{\flat}_{\frac{r}{s}}(q)$.

Lemma 4.12. *For an irreducible fraction $\frac{r}{s} > 1$, $\mathcal{R}^{\flat}_{\frac{r}{s}}(q)$ is palindromic if and only if $s^2 \equiv -1 \pmod r$.*

In the above situation, if $s^2 \equiv -1 \pmod r$, then $\mathcal{R}^{\flat}_{\frac{r}{s}}(q) = \mathcal{R}^{\flat}_{\frac{r}{s}}(q)^{\vee}$ by Lemma 4.5, and hence $\mathcal{R}^{\flat}_{\frac{r}{s}}(q)$ is palindromic. So the sufficiency of Lemma 4.12 follows from Lemma 4.5. The similar is true for $\mathcal{R}^{\sharp}_{\frac{r}{s}}(q)$, which is palindromic if and only if $s^2 \equiv 1 \pmod r$.

Proof. Set $\frac{r}{s} = [a_1, \dots, a_{2m}]$ and $M(a_1, \dots, a_{2m}) = \begin{pmatrix} r & t \\ s & u \end{pmatrix}$. Then we have

$$\mathcal{R}^{\flat}_{\frac{r}{s}}(q) - \mathcal{R}^{\flat}_{\frac{r}{s}}(q)^{\vee} = \mathcal{R}^{\flat}_{\frac{r}{s}}(q) - \mathcal{R}^{\flat}_{\frac{r}{t}}(q) = (q-1)(\mathcal{S}^{\sharp}_{\frac{r}{s}}(q) - \mathcal{S}^{\sharp}_{\frac{r}{t}}(q))$$

by (4.7). Since $\mathcal{S}^{\sharp}_{\frac{r}{s}}(1) = s$ and $\mathcal{S}^{\sharp}_{\frac{r}{t}}(1) = t$, $\mathcal{S}^{\sharp}_{\frac{r}{s}}(q) - \mathcal{S}^{\sharp}_{\frac{r}{t}}(q) = 0$ if and only if $s = t$. The former condition is equivalent to that $\mathcal{R}^{\flat}_{\frac{r}{s}}(q)$ is palindromic, and the latter condition is equivalent to $s^2 \equiv -1 \pmod r$. (Note that $0 < s, t < r$ now.) $\square$

Theorem 4.13. *The following hold.*

(1) *For an irreducible fraction $\frac{r}{s}$, $\mathcal{S}^{\flat}_{\frac{r}{s}}(q)$ is palindromic if and only if $r^2 \equiv -1 \pmod s$;*
(2) *For an irreducible fraction $\frac{r}{s}$, $\mathcal{R}^{\flat}_{\frac{r}{s}}(q)$ is equivalent to a palindromic polynomial if and only if $s^2 \equiv -1 \pmod r$;*
(3) *For an irreducible fraction $\frac{r}{s} > 1$, the normalized Jones polynomial $J_{\frac{r}{s}}(q)$ is palindromic if and only if $s^2 \equiv -1 \pmod r$.*

Proof. (1) By Proposition 2.11 (1), we may assume that $0 < \frac{r}{s}, \frac{r'}{s} < 1$. Now the assertion follows from Lemma 4.12 and Proposition 2.11 (3).

(2) The assertion follows from (1) and Proposition 2.11 (3).

(3) The assertion follows from Theorem 4.3 and Lemma 4.12. $\square$

Let $\frac{r}{s} = [a_1, \dots, a_{2m}] > 1$ be an irreducible fraction with

$$M := M_q(a_1, \dots, a_{2m}) = \begin{pmatrix} r & t \\ s & u \end{pmatrix} \quad \text{and} \quad A := (-S_q)M. \tag{4.8}$$

By our proof of Theorem 4.9, we have

$$(q-1)\operatorname{tr} A = \mathcal{R}^{\flat}_{\frac{r}{s}}(q) - \mathcal{R}^{\flat}_{\frac{r}{s}}(q)^{\vee} = J_{\frac{r}{s}}(q)^{\vee} - J_{\frac{r}{s}}(q). \tag{4.9}$$

For an irreducible fraction $\alpha = \frac{r}{s} > 1$ with $s^2 \not\equiv -1 \pmod r$, $J_\alpha(q)$ is not palindromic by Theorem 4.13 (3), and $J_\alpha(q)^{\vee} - J_\alpha(q)$ is equivalent to a non-zero anti-palindromic polynomial which can be divided by $(1-q)$. So there is a unique palindromic polynomial $I_\alpha(q) \in \mathbb{Z}[q]$ with $I_\alpha(0) > 0$ and

$$I_\alpha(q) \equiv \frac{J_\alpha(q)^{\vee} - J_\alpha(q)}{1-q}.$$

If $J_\alpha(q)$ is palindromic, we set $I_\alpha(q) = 0$. By (4.9), we have $I_\alpha(q) \equiv \operatorname{tr} A$.

Corollary 4.14. *With the above notation, we have $I_\alpha(q) \in \mathbb{Z}_{\geq 0}[q]$ and $I_\alpha(0) = 1$.*

Proof. By (4.9), the assertion follows from Proposition 4.10, while it remains to show that $I_\alpha(q) \neq 2$. By the proof of Theorem 4.9, we see that if $\operatorname{tr} A \equiv 2$ then A is equivalent to the identity matrix. In this case, we have $M \equiv S_q$, but it is not possible since $M = M_q(a_1, \dots, a_{2m})$ for $a_1, \dots, a_{2m} > 0$. □

By Proposition 4.10, Conjecture 4.11 implies the following conjecture on the normalized Jones polynomials of rational links (equivalently, left q-deformed rationals).

Conjecture 4.15. $I_\alpha(q)$ is unimodal except for the following two types.

(1) $1 + q^n$ for some $n \geq 2$.
(2) $\sum_{i=0}^{2k} a_i q^i$ with $(a_0, \dots, a_{2k+1}) = (1, 2, \dots, k, k-1, k, k-1, k-2, \dots, 2, 1)$ for some $k \geq 2$.

Especially, $I_\alpha(q)$ is at most bimodal.

The case (1) (resp. (2)) corresponds to the case

$$A \equiv R^n \quad (\text{ resp. } A = M_q(1, k, 1, k), M_q(k, 1, k, 1) \text{ }).$$

Note that $\operatorname{tr} M_q(1, k, 1, k) (= \operatorname{tr} M_q(k, 1, k, 1))$ is computed in [3, Example 1.8.].

Example 4.16. Easy calculation shows that $I_{\frac{12}{5}}(q)$ and $I_{\frac{15}{4}}(q)$ are not unimodal. In fact, $I_{\frac{12}{5}}(q) = I_{\frac{12}{7}}(q) = q^3 + 1$ and $I_{\frac{15}{4}}(q) = I_{\frac{15}{11}}(q) = q^4 + 2q^3 + q^2 + 2q + 1$.

Finally, we give the following proposition.

Proposition 4.17. *For any irreducible fraction $\alpha > 1$, there is an infinite sequence of irreducible fractions $\alpha_1, \alpha_2, \dots$(every $\alpha_i > 1$ are distinct) such that $I_{\alpha_i}(q) = I_\alpha(q)$ for all $i \geq 1$.*

Proof. With the same notation as (4.8), we have

$$I_\alpha(q) \equiv \mathcal{S}^\sharp_{\frac{r}{s}}(q) - \mathcal{S}^\sharp_{\frac{r}{t}}(q) = \mathcal{R}^\sharp_{\frac{s}{u}}(q) - \mathcal{R}^\sharp_{\frac{t}{u}}(q)$$

by (4.7). For $\alpha_1 := \dfrac{r+s+t+u}{s+u} = [b_1, \dots, b_{2l}] > 1$, we have

$$M(b_1, \dots, b_{2l}) = \begin{pmatrix} r+s+t+u & t+u \\ s+u & u \end{pmatrix}$$

and

$$I_{\alpha_1}(q) \equiv \mathcal{R}^\sharp_{\frac{s+u}{u}}(q) - \mathcal{R}^\sharp_{\frac{t+u}{u}}(q) \equiv \mathcal{R}^\sharp_{\frac{s}{u}}(q) - \mathcal{R}^\sharp_{\frac{t}{u}}(q) \equiv I_\alpha(q).$$

Hence $I_{\alpha_1}(q) = I_\alpha(q)$. Repeating this procedure, we get the sequence $\{\alpha_i\}_{i \geq 1}$ with

$$\alpha_i = \frac{r + i(s+t) + i^2 u}{s + iu},$$

which satisfies the desired property. □

Acknowledgments

We would like to thank Professor Michihisa Wakui for stimulating discussion, and the anonymous referee for careful reading of the paper and helpful comments, in particular, for bringing the references [6] and [19] to the authors' attention. The first author is supported by JSPS KAKENHI Grant Number 21H04994, 19K03456. The second author is supported by JSPS KAKENHI Grant Number 22K03258.

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