HARMONIC NON-HOLOMORPHIC MAPS FROM S^2 TO HIRZEBRUCH SURFACES

MONICA ALICE APRODU

Abstract. In this Note, we use the method of Arezzo–La Nave, [1], to prove the existence of non-holomorphic harmonic maps from S^2 to Hirzebruch surfaces in various homotopy classes.

1. Introduction

Let (M, g) and (N, h) be compact Riemannian manifolds, $\mathcal{C}^0(M, N)$ be the set of all continuous mappings from M to N, and $[M, N]$ denote the homotopy classes of $\mathcal{C}^0(M, N)$. A classical existence problem for harmonic maps asks whether a given continuous mapping $\psi : M \to N$ can be deformed continuously to a harmonic mapping $\varphi : M \to N$, see [7]. Affirmative answers for this question are known in several special cases, see [7]:

- If $\dim(M) = 1$, every homotopy class of maps from S^1 into N admits a harmonic mapping, see (5.3) in [7].
- If $\dim(N) = 1$, every homotopy class of maps of M into S^1 can be represented by a harmonic 1-form with integral periods, see (7.1) in [7].
- If M and N are compact Riemannian manifolds and N has non-positive sectional curvature (Theorem of Eells–Sampson 1964); this result was extended to noncompact complete Riemannian manifolds by Li–Tam in 1991.
- If M is compact and two–dimensional and N is compact with $\pi_2(N) = \{0\}$ (Theorem of Lemaire and Sacks–Uhlenbeck).
- If M and N are both Euclidean n-spheres with $n \leq 7$ (Theorem of Smith).

On the negative side, Eells and Wood proved that no harmonic representative exists when M is a torus, N is a 2–sphere, and the degree of ψ is ± 1.

All these important results motivate the following problem stated by L. Lemaire:

Problem (L. Lemaire). Are there examples of a Riemannian manifold (N, h) and a homotopy class in $\pi_2(N)$ that lack harmonic representatives?

It is worth noticing that such classes must contain no holomorphic representative.

2020 *Mathematics Subject Classification.* 58E20, 14J26.

Key words and phrases. harmonic maps, harmonic morphisms, Hirzebruch surfaces.

In [1], Arezzo and La Nave identified interesting examples of non-holomorphic, stably minimal 2-spheres in $S^2 \times S^2$. In this note, we use the same methods to provide a small extension of their results to Hirzebruch surfaces H_m. These surfaces are deformations of $H_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$ or H_1, the latter being the blow-up of the complex projective plane at a single point. We prove the existence of non-holomorphic harmonic maps from S^2 to H_m in various homotopy classes.

The outline of this Note is as follows. In Subsection 2.1, we review some fundamental concepts regarding harmonic maps, specifically those originating from S^2. Subsection 2.2 presents the classical theory of Hirzebruch surfaces and the holomorphic embeddings of S^2, following [5], [11], [9]. In Subsection 2.3, we discuss the deformations of Hirzebruch surfaces within $\mathbb{CP}^1 \times \mathbb{CP}^2$. Finally, Section 3 is devoted to the statement and proof of our main result.

2. Preliminaries

2.1. **Harmonic maps.** We begin by recalling fundamental concepts of weak conformality and harmonicity for maps between Riemannian manifolds, using [2], [14]. We fix (M, g) and (N, h) Riemannian manifolds of dimension m and n, respectively, and $\varphi : M \to N$ a smooth map.

Definition 2.1 (see [2]). The map φ is said *(weakly) conformal* if there exists a smooth function

$$\lambda : M \to [0, \infty),$$

such that

$$h(d\varphi(V), d\varphi(W)) = \lambda^2 g(V, W), \text{ for all } V, W \text{ smooth vector fields on } M.$$

The function λ is called the *conformality factor of* φ.

Note that the definition can be also given point–wise, as in [2, Definition 2.3.1], in which case we speak about weak conformality at a given point $x \in M$.

Remark 2.2. A fundamental example is the identity map $\mathrm{id} : (M, g) \to (M, h)$ which is conformal if and only if the metrics g and h are conformally equivalent (see [2, Example 2.3.7]). In particular, by the Riemann Uniformization Theorem (see [13, Corollary 5.4.1]), any metric on S^2 is conformally equivalent to the standard metric, and hence the identity map of S^2 is conformal.

Definition 2.3 (see [2], [14]). The *energy density* of φ is the smooth function

$$e(\varphi) : M \to [0, \infty),$$

defined by $e(\varphi)_x = \frac{1}{2}\|d\varphi_x\|^2$, for all $x \in M$. Here, $\|-\|$ denotes the Hilbert-Schmidt norm, given in an orthonormal basis $\{e_i\}$ of T_xM by $\|d\varphi_x\|^2 = \sum_{i=1}^m h(d\varphi_x(e_i), d\varphi_x(e_i))$. For a compact domain $D \subseteq M$, the *energy integral* of φ over D, with respect to the metric g, is:

$$E(\varphi, g) = \int_D e(\varphi) v_g.$$

Remark 2.4 (see [2], [14]). In general, the energy integral depends on the metrics g and h. However, when $\dim(M) = 2$, the energy is conformally invariant.

Definition 2.5 (see [2], [14]). The map φ is *harmonic* if it is a critical point of the energy functional $E(-, g) : \mathcal{C}^\infty(M, N) \to \mathbb{R}$, where $\mathcal{C}^\infty(M, N)$ denotes the space of smooth maps from M to N.

In the 2-dimensional case, the composition between conformal and harmonic maps yields the following:

Proposition 2.6 (see [2]). *Let $\varphi : M^2 \to N^2$ be a weakly conformal map between Riemannian manifolds of dimension two, and $\psi : N^2 \to P$ be a harmonic map with values in a Riemannian manifold. Then $\psi \circ \varphi : M^2 \to P$ is a harmonic map.*

Remark 2.7 (see [2]). If $M = S^2$ and N is an arbitrary Riemannian manifold, then any harmonic map $\varphi : S^2 \to N$ is necessarily weakly conformal.

Definition 2.8 (see [2], [14]). Assume M is a compact manifold. Then the *volume integral* of φ over M is the integral:

$$V(\varphi, g) = \int_M \sqrt{\det(\varphi^* h)} v_g.$$

By $\varphi^* h$ we denote the pull-back of the metric h defined by

$$\varphi^* h(V, W) = h(d\varphi_x(V), d\varphi_x(W), \text{ for all } V, W \in T_x M.$$

Viewing $\varphi^* h$ as an endomorphism of $T_x M$ with λ_i^2 its eigenvalues, and choosing an orthonormal basis of eigenvectors $\{e_i\}$ with $\varphi^* h(e_i, e_j) = \delta_{ij} \lambda_i^2$, the volume integral becomes:

$$V(\varphi) = \int_M \lambda_1 \lambda_2 \ldots \lambda_m v_g.$$

In the case $m = 2$, the volume is specifically referred to as the *area*.

For two dimensional domains, the relation between energy and area is given by the following result:

Proposition 2.9 (see [2]). *Let $\varphi : M^2 \to N^n$ be a smooth map from a compact two-dimensional Riemannian manifold to an arbitrary Riemannian manifold. Then $E(\varphi) \geq V(\varphi)$, with equality if and only if φ is weakly conformal.*

2.2. **Holomorphic 2-spheres in Hirzebruch surfaces.** For any integer $m \in \mathbb{Z}$, the natural complex line bundle $L_m \to \mathbb{CP}^n$ is defined by the total space

$$L_m = (\mathbb{C}^{n+1} \setminus \{0\} \times \mathbb{C}) / \sim, \text{ where } (z, v) \sim (\lambda z, \lambda^m v), \text{ for all } \lambda \in \mathbb{C} \setminus \{0\}.$$

The projection map $\pi : L_m \to \mathbb{CP}^n$ is given by $[(z, v)] \mapsto [z]$, see, for example, [3], [5], [10].

Over the open sets $U_i = \{[z_0 : z_1 : \ldots : z_n] | z_i \neq 0\}$, for $i = \overline{0, n}$, which together form an open covering of $\mathbb{CP}^n$, the trivialization maps of L_m are given by:

$$\begin{array}{rcl} h_i \ : \pi^{-1}(U_i) & \to & U_i \times \mathbb{C} \\ [(z, v)] & \to & \big([z], z_i^{-m} v\big) \end{array}$$

and the transition functions are (see, for example, [12] or [4]):

$$\begin{array}{rcl} h_i \circ h_j^{-1} \ : (U_i \cap U_j) \times \mathbb{C} & \to & (U_i \cap U_j) \times \mathbb{C} \\ ([z], v) & \to & \big([z], z_j^m z_i^{-m} v\big) \end{array}$$

By the form of the transition functions, the bundle L_{-m}, for positive m, can be described by the equations:

$$L_{-m} = \{([z_0 : \ldots : z_n], (y_0, \ldots, y_n)) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} | z_i^m y_j = z_j^m y_i, \text{ for all } i, j\}.$$

Let $\underline{1} = \mathbb{CP}^n \times \mathbb{C}$ denote the trivial line bundle. For the direct sum $L_m \oplus \underline{1}$ we consider the projective bundle $\mathbb{P}(L_m \oplus \underline{1}) \to \mathbb{CP}^n$. This space is obtained by adding a copy of $\mathbb{CP}^n$ to the total space of L_m, since $\mathbb{P}(L_m \oplus \underline{1}) = L_m \cup \mathbb{P}(L_m)$.

In the particular case where $n = 1$ and $m \geq 0$, the projective bundle $H_m = \mathbb{P}(L_{-m} \oplus \underline{1})$ over $\mathbb{CP}^1$ is the *m–th Hirzebruch surface*, and the projection $H_m \to \mathbb{CP}^1$ is called the *ruling*. Geometrically, H_m is a compactification of L_{-m} realized by adding a section τ_∞, called the *section at infinity*.

The geometry of a Hirzebruch surface is determined by the numerical invariant m; Hirzebruch (see [11], [5]) proved that $H_m \ncong H_n$, unless $m = n$. Topologically, $-m$ is the minimum self-intersection of a holomorphically embedded 2–sphere τ in H_m, which is negative for all cases except H_0. For $m \neq 0$, this 2–sphere is unique and is known as the *negative section*. Geometrically, it can be described starting from the observation that $H_m \cong \mathbb{P}(L_m \oplus \underline{1}) = L_m \cup \mathbb{CP}^1$. Then the minimal section is the copy of $\mathbb{CP}^1$ that compactifies the total space L_m. However, this section τ is clearly different from the section τ_∞ that appears above.

For H_0, the section τ is not unique anymore, and moves in a family as a fibre of the second ruling. From Hirzebruch's Theorem (see [11], [5]), we infer that H_m is diffeomorphic to H_0 if and only if m is even, and the Hirzebruch surfaces with odd m–invariant are all diffeomorphic to H_1. This partition by the parity of the m–invariant justifies the following terminology. The surfaces H_{2k} are called *even* Hirzebruch surfaces, and H_{2k+1} are *odd* Hirzebruch surfaces.

From the projective geometry viewpoint, the Hirzebruch surfaces admit the following description [3, p. 266]

$$H_m = \{([x_0 : x_1], [y_0 : y_1 : y_2]) \in \mathbb{CP}^1 \times \mathbb{CP}^2 | \ x_0^m y_1 - x_1^m y_0 = 0\}$$

such that the ruling is given by the projection to $\mathbb{CP}^1$ and the section τ_∞ is the inverse image of the point $[0 : 0 : 1]$ via the second projection onto $\mathbb{CP}^2$. This is directly connected to the realization of the bundle L_m as the set

$$\{([x_0 : x_1], (y_0, y_1)) \in \mathbb{CP}^1 \times \mathbb{C}^2 | \ x_0^m y_1 - x_1^m y_0 = 0\}.$$

The fiber of the ruling is denoted by f. It is well known (see [11], [5]) that

$$H_2(H_m,\mathbb{Z}) \cong \pi_2(H_m) \cong \mathbb{Z}\cdot[\tau]\oplus\mathbb{Z}\cdot[f].$$

Using the adjunction formula and the description of irreducible complex curves, it has been proved that for $m>0$, the homotopy class in $\pi_2(H_m)$ of any *holomorphically embedded* 2–sphere must be of type $[f]$, $[\tau]$, or $[\tau]+\mu[f]$ with $\mu \geq m$.

2.3. **Deformations of Hirzebruch surfaces.** There are several classical methods for deforming even Hirzebruch surfaces into one another. A particularly convenient approach uses the embeddings of Hirzebruch surfaces in $\mathbb{CP}^1\times\mathbb{CP}^2$, as detailed in [3, p. 266–267]. More precisely, for any $n\geq 2$ and $0\leq k\leq n/2$, we consider the following family X_t, indexed by $t\in\mathbb{C}$:

$$(1)\quad X_t := \left\{([x_0:x_1],[y_0:y_1:y_2])\in\mathbb{CP}^1\times\mathbb{CP}^2 \mid x_0^n y_1 - x_1^n y_0 + t x_0^{n-k}x_1^k y_2 = 0\right\}.$$

Each member of this family is isomorphic to a Hirzebruch surface via the first projection $pr_1 : X_t \to \mathbb{CP}^1$.

To identify these surfaces, we examine their transition functions. Consider the principal open sets:

$$U_0 = \{[x_0:x_1] | x_0\neq 0\}\subset\mathbb{CP}^1 \text{ and } U_1 = \{[x_0:x_1] | x_1\neq 0\}\subset\mathbb{CP}^1.$$

Over U_0 we can write

$$y_1 = \left(x_1x_0^{-1}\right)^n y_0 - t\left(x_1x_0^{-1}\right)^k y_2,$$

and over U_1 we write

$$y_0 = \left(x_0x_1^{-1}\right)^n y_1 + t\left(x_0x_1^{-1}\right)^{n-k} y_2.$$

The resulting transition functions of X_t take the form:

$$\begin{array}{rlcl} h_0\circ h_1^{-1} & : (U_0\cap U_1)\times\mathbb{CP}^1 & \longrightarrow & (U_0\cap U_1)\times\mathbb{CP}^1 \\ & ([x_0:x_1],[y_1:y_2]) & \longrightarrow & ([x_0:x_1],[z_1:z_2]) \end{array}$$

where $z_1 = \left(x_0x_1^{-1}\right)^n y_1 + t\left(x_0x_1^{-1}\right)^{n-k} y_2$, $z_2 = y_2$.

Comparing these transition functions with those of Hirzebruch surface H_{n-2k}:

$$\begin{array}{ccc} (U_0\cap U_1)\times\mathbb{CP}^1 & \longrightarrow & (U_0\cap U_1)\times\mathbb{CP}^1 \\ ([x_0:x_1],[y_1:y_2]) & \longrightarrow & \left([x_0:x_1],\left[\left(x_0x_1^{-1}\right)^{n-2k} y_1 : y_2\right]\right) \end{array}$$

we note that $X_0\cong H_n$ and $X_t\cong H_{n-2k}$ for all $t\neq 0$ [3]. In particular, from Ehresmann's theorem [8], H_n and H_{n-2k} are diffeomorphic, which is one part of Hirzebruch's theorem [11, section 2].

In the specific case $n=2k$, we obtain a family of complex structures $(S^2\times S^2,\mathcal{J}_t)_{t\in\mathbb{C}}$ where $(S^2\times S^2,\mathcal{J}_t)\cong H_0$ for $t\neq 0$ and $(S^2\times S^2,\mathcal{J}_0)\cong H_{2k}$. A similar interpretation is valid for the blowup of $\mathbb{CP}^2$ in one point.

3. Minimal non-holomorphic 2-spheres

With the preparations from the previous sections, we provide the proof of the existence of harmonic representatives in some classes that admit no holomorphic maps.

Theorem 3.1. *For any integers $k \geq 1$, $m \geq 0$, and $p \geq 1$, the homotopy class $p([\tau] - k[f]) \in \pi_2(H_m)$ can be represented by a non-holomorphic harmonic map. The map is harmonic with respect to a specific Kähler metric on H_m induced by an embedding into a projective space that arises from a deformation of H_{m+2k}.*

Proof. Our strategy is to view H_m as a member of a family of complex surfaces and construct a specific Kähler metric for which the desired harmonic map exists.

Let $n = m + 2k$ and consider the deformation family X_t described in (1). For $t \neq 0$, X_t is biholomorphic to $H_{n-2k} = H_m$, while for $t = 0$, X_0 is biholomorphic to H_n.

To avoid confusions, the negative section of the Hirzebruch surface X_t over the point t in a small neighbouhood of the origin, in the family above, will be denoted by τ_t and the fibre is denoted by f_t.

As explained before, the surfaces H_n and H_m are diffeomorphic, since n and $m = n-2k$ have the same parity [11, section 2]. This diffeomorphism is confirmed by Ehresmann's Theorem (see [8]), which implies that the family X_t is a smooth fibration over a connected base. Therefore, all fibres are diffeomorphic, and we can identify their homotopy groups, $\pi_2(H_n) \cong \pi_2(H_m)$. Under this identification, standard topological computations show that $[f_t] = [f_0]$ and $[\tau_t] = [\tau_0] + k[f_0]$.

The Hirzebuch surfaces are all embedded in $\mathbb{CP}^5$ via the Segre embedding:

$$\begin{array}{ccc} \mathbb{CP}^1 \times \mathbb{CP}^2 & \to & \mathbb{CP}^5 \\ ([x_0 : x_1], [y_0, y_1, y_2]) & \mapsto & ([x_0y_0 : x_1y_0 : x_0y_1 : x_1y_1 : x_0y_2 : x_1y_2]) \end{array}$$

Since X_t and X_0 are all diffeomorphic, let us denote the common $\mathcal{C}^\infty$ support by X, and hence the Fubini-Study metric on $\mathbb{CP}^5$ induces a family of metrics g_t on the same $\mathcal{C}^\infty$ manifold X.

The class $[\tau_0] \in \pi_2(X_0)$ has a holomorphic representative $\phi_0 : S^2 \to X_0$. However, for $t \neq 0$, the class $[\tau_t] - k[f_t]$ contains no holomorphic representatives. This is because its homology class corresponds to $[\tau_0] - k[f_0]$, which is not of the type permitted for holomorphic spheres, see [9, Ch. V, Corollary 2.18, p. 380].

Consider the family of energy functionals

$$\mathcal{E} : \mathcal{C}^\infty(S^2, X) \times \{\mathcal{C}^\infty - \text{ Riemannian metrics on } X\} \to \mathbb{R}$$

defined by

$$\mathcal{E}(\phi, g) := \frac{1}{2} \int_{S^2} ||d\phi||^2 v_g.$$

From [14, Theorem 3.1], the holomorphic mapping $\phi_0 : S^2 \to X_0$ minimizes the energy functional $\mathcal{E}(-, g_0)$ in its homotopy class.

Let g_{S^2} be the standard metric on S^2 and $\phi_0^* g_0$ the pullback metric on S^2 induced by the map ϕ_0. By Riemann's Uniformization Theorem (as noted in Remark 2.2), any two metrics on S^2 are conformally equivalent. Thus, the identity map id: $(S^2, \phi_0^* g_0) \to (S^2, g_{S^2})$ is a conformal map. The composition

$$\phi_0 \circ \mathrm{id} : (S^2, \phi_0^* g_0) \to (X_0, g_0),$$

between the harmonic map $\phi_0 : (S^2, g_{S^2}) \to (X_0, g_0)$ and the (conformal) identity map id : $(S^2, \phi_0^* g_0) \to (S^2, g_{S^2})$, is a harmonic map (see Proposition 2.6). Moreover, by Remark 2.7, $\phi_0 \circ id$ is weakly conformal and so its energy integral is equal to its area (see Proposition 2.9).

From [14, p. 142], the harmonic isometric immersion ϕ_0 is minimal, and hence is an absolute minimum for the functional $\mathcal{V}(-, g_0)$, where

$$\mathcal{V} : \mathcal{C}^\infty(S^2, X) \times \{\mathcal{C}^\infty - \text{ Riemannian metrics on } X\} \to \mathbb{R}$$

is defined by

$$\mathcal{V}(\phi, g) = \int_{S^2} \phi^*(dv_g).$$

We apply the argument of [1, proof of Theorem 5] that uses the main result of [15]. From White's Perturbation Theorem [15, p. 162], ϕ_0 can be perturbed in a small neighborhood of the origin to obtain an immersion $\phi_t : S^2 \to (X, g_t)$ which is a critical point of the area functional $V(-, g_t)$, i.e., a minimal immersion. Since the domain is two-dimensional, a map is a minimal immersion if and only if it is harmonic (cf. Proposition 2.9). Thus, ϕ_t is the desired harmonic map.

To obtain harmonic maps in the homotopy class $p[\tau_t] - kp[f_t]$ we compose ϕ_t with the degree p holomorphic map $\psi : S^2 \to S^2$, defined by $S^2 = \mathbb{CP}^1 \to \mathbb{CP}^1 = S^2$, $[z_0 : z_1] \mapsto [z_0^p : z_1^p]$, which is clearly weakly conformal, and hence it is a harmonic morphism [2, Chapter 4]. Therefore, $\phi_t \circ \psi : S^2 \to X_t$ is a harmonic, non–holomorphic map. $\square$

Remark 3.2. Returning to L. Lemaire's problem stated in the Introduction, we note the following. If there exists a Riemannian manifold N with a homotopy class in $\pi_2(N)$ without harmonic reprezentative, then this example must be outside the set of classes mentioned in the main result.

Acknowledgement. This work was supported by a grant of the Ministry of Research and Innovation CNCS-UEFISCDI, project no PN-III-P4-ID-PCE-2016-0019, within PNCDI III. The author thanks the anonymous referee for their constructive comments that helped improve the presentation.

References

[1] C. Arezzo, G. La Nave: *Minimal two spheres in Kähler-Einstein Fano manifolds*, Advances in Mathematics **191** (2005), 209–223.

[2] P. Baird, J.C. Wood: Harmonic Morphisms Between Riemannian Manifolds, Clarendon Press-Oxford, 2003.

[3] W.P. Barth, K. Hulek, C.A.M. Peters, A. Van de Ven: Compact Complex Surfaces, Springer-Verlag, 2004.

[4] S.K. Donaldson, P.B. Kronheimer: The Geometry of Four-Manifolds, Oxford Mathematical Monographs, Clarendon Press, 1990.

[5] D. Crowley, M. Kreck: *Hirzebruch surfaces*, Bulletin of the Manifold Atlas (2011), http://www.boma.mpim-bonn.mpg.de/articles/25.

[6] S.-S. Chern, W.-H. Chen, K.S. Lam: Lectures on Differential Geometry, Series on University Mathematics, Vol. 1, World Scientific, 1999.

[7] J. Eells, L. Lemaire: *A report on harmonic maps*, Bulletin of the London Mathematical Society **10**(1) (1978), 1–68.

[8] C. Ehresmann: *Sur les espaces fibrés différentiables*, C.R. Acad. Sci. Paris **224** (1947), 1611–1612.

[9] R. Hartshorne: Algebraic Geometry, Springer-Verlag, 1977.

[10] F. Hirzebruch: Topological Methods in Algebraic Geometry, Springer-Verlag, 1978.

[11] F. Hirzebruch: *Über eine Klasse von einfachzusammenhängenden komplexen Mannigfaltigkeiten*, Mathematische Annalen **124** (1951), 77–86.

[12] S. Kobayashi, K. Nomizu: Foundations of Differential Geometry, Vol. 1, Interscience Publishers, 1963.

[13] J. Jost: Compact Riemann Surfaces, Springer-Verlag, 2006.

[14] H. Urakawa: Calculus of Variations and Harmonic Maps, Translations of Mathematical Monographs 132 AMS, 1993.

[15] B. White: *The Space of Minimal Submanifolds for Varying Riemannian Metrics*, Indiana University Mathematics Journal, Vol. **40**, No. 1 (1991), 161–200.

Department of Mathematics and Computer Science, Faculty of Sciences and Environmental, University "Dunărea de Jos" of Galaţi, Ştiinţei Str. 111, RO-800189, Galaţi, Romania. e-mail: Monica.Aprodu@ugal.ro &

"Simion Stoilow" Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, Bucharest, Romania