BLOW UP AND LIFESPAN ESTIMATES FOR SYSTEMS OF SEMI-LINEAR WAVE EQUATIONS WITH DAMPING AND POTENTIAL

MENGLIANG LIU

ABSTRACT. In this paper, we investigate the finite time blow up phenomenon of semi-linear wave systems with power nonlinearities and a large class of space-dependent dampings and potentials. We obtain the blow up results as well as the lifespan estimates for three kinds of wave systems.

1. Introduction

In this paper, we study the finite time blow up phenomenon of three kinds of semi-linear wave systems with space dependent damping and potential. More precisely, we consider the following Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + D(x)u_t + V(x)u = N_1(v, v_t), & (t,x) \in (0,T) \times \mathbb{R}^n, \\ v_{tt} - \Delta v + D(x)v_t + V(x)v = N_2(u, u_t), & (t,x) \in (0,T) \times \mathbb{R}^n, \\ u(0,x) = \varepsilon u_0(x), \quad u_t(0,x) = \varepsilon u_1(x), & x \in \mathbb{R}^n \ , \\ v(0,x) = \varepsilon v_0(x), \quad v_t(0,x) = \varepsilon v_1(x), & x \in \mathbb{R}^n \ . \end{cases} \tag{1.1}$$

Here, the small parameter $\varepsilon > 0$ measures the size of the initial data. As usual, to show blow up, we assume $(u_0, u_1), (v_0, v_1) \in C_c^\infty(\mathbb{R}^n)$ and they are nontrivial, nonnegative and supported in $B_R := \{x \in \mathbb{R}^n : r \le R\}$ for some $R > 0$, where $|x| = r$. For the coefficient of dampings $D(x)$ and potentials $V(x)$, we assume $V(x)$, $D(x) \in C(\mathbb{R}^n) \cap C^\delta(B_\delta)$ for some $\delta > 0$. In addition, we assume

$$0 \le D(x) = D(|x|) \le \alpha_1(1 + |x|)^{-\beta}, \ 0 \le \alpha_1 \in \mathbb{R}, 1 < \beta \in \mathbb{R} \ , \tag{1.2}$$

$$0 \le V(x) = V(|x|) \le \alpha_2(1 + |x|)^{-\omega}, \ 0 \le \alpha_2 \in \mathbb{R}, 2 < \omega \in \mathbb{R} \ . \tag{1.3}$$

Concerning the nonlinear terms, we will consider the following three cases

$$\begin{cases} N_1(v, v_t) = |v|^p \\ N_2(u, u_t) = |u|^q \end{cases}, \quad \begin{cases} N_1(v, v_t) = |v_t|^p \\ N_2(u, u_t) = |u_t|^q \end{cases}, \quad \begin{cases} N_1(v, v_t) = |v|^q \\ N_2(u, u_t) = |u_t|^p \end{cases}, \tag{1.4}$$

where $1 < p, q \in \mathbb{R}$.

Before proceeding, we give a definition of the weak solution.

Definition of weak solutions: Let's take the case $N_1(v, v_t) = |v|^p, N_2(u, u_t) = |u|^q$ as an example. We say that (u, v) is a solution of (1.1) on $[0, T]$, if

$$u \in C([0,T], H^1(\mathbb{R}^n)) \cap C^1([0,T], L^2(\mathbb{R}^n)) \cap L^q_{loc}([0,T] \times \mathbb{R}^n) \ ,$$

$$v \in C([0,T], H^1(\mathbb{R}^n)) \cap C^1([0,T], L^2(\mathbb{R}^n)) \cap L^p_{loc}([0,T] \times \mathbb{R}^n) \ ,$$

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satisfying

$$\text{supp }(u,v)\subset\{(x,t)\in\mathbb{R}^n\times\ [0,T]:|x|\le t+R\}\ ,$$

and

$$\begin{aligned}\varepsilon\int_{\mathbb{R}^n}u_1(x)\Psi(0,x)dx+\varepsilon\int_{\mathbb{R}^n}u_0(x)D(x)\Psi(0,x)dx+\int_0^T\int_{\mathbb{R}^n}N_1(v,v_t)\Psi(t,x)dxdt\\ =-\int_0^T\int_{\mathbb{R}^n}u_t(t,x)\Psi_t(t,x)dxdt+\int_0^T\int_{\mathbb{R}^n}\nabla u(t,x)\nabla\Psi(t,x)dxdt\\ -\int_0^T\int_{\mathbb{R}^n}u(t,x)D(x)\Psi_t(t,x)dxdt+\int_0^T\int_{\mathbb{R}^n}V(x)u(t,x)\Psi(t,x)dxdt\end{aligned}\tag{1.5}$$

$$\begin{aligned}\varepsilon\int_{\mathbb{R}^n}v_1(x)\Psi(0,x)dx+\varepsilon\int_{\mathbb{R}^n}v_0(x)D(x)\Psi(0,x)dx+\int_0^T\int_{\mathbb{R}^n}N_2(u,u_t)\Psi(t,x)dxdt\\ =-\int_0^T\int_{\mathbb{R}^n}v_t(t,x)\Psi_t(t,x)dxdt+\int_0^T\int_{\mathbb{R}^n}\nabla v(t,x)\nabla\Psi(t,x)dxdt\\ -\int_0^T\int_{\mathbb{R}^n}v(t,x)D(x)\Psi_t(t,x)dxdt+\int_0^T\int_{\mathbb{R}^n}V(x)v(t,x)\Psi(t,x)dxdt\ ,\end{aligned}\tag{1.6}$$

for any $\Psi(t,x)\in C([0,T],H^1_{loc}(\mathbb{R}^n))\cap C^1([0,T],L^2_{loc}(\mathbb{R}^n))$.

The supremum of all such time of existence, T, is called to be the lifespan to the problem (1.1), denoted by T_ε. In the sequel, we use $A\lesssim B$ to denote that $A\le CB$ for some positive constant C, which is independent of ε and whose value may change from line to line. Moreover, we write $A\simeq B$ if there exist positive constants C_1 and C_2 such that $C_1B\le A\le C_2B$.

Before stating our results, let us briefly review the history of this problem. For the wave equation

$$\partial_t^2u-\Delta u=|u|^p\ ,(t,x)\in(0,T)\times\mathbb{R}^n\ ,u(0,x)=\varepsilon u_0(x)\ ,u_t(0,x)=\varepsilon u_1(x)\ ,$$

the problem has been extensively investigated and is known as the Strauss conjecture [40], which was initiated in the work of John [11]. It is known that, the problem admits small data global existence only if $p>p_S(n)$, where the critical power $p_S(n)$ is the positive root of equation

$$(n-1)p^2-(n+1)p-2=0\ .\tag{1.7}$$

We refer [26] for discussion of the history and related results.

For the Cauchy problem

$$\partial_t^2u-\Delta u=|u_t|^p\ ,(t,x)\in(0,T)\times\mathbb{R}^n\ ,u(0,x)=\varepsilon u_0(x)\ ,u_t(0,x)=\varepsilon u_1(x).$$

Glassey [5] made a conjecture that the critical power p for the problem to admit global solutions with small, smooth initial data is

$$p_G(n):=1+\frac{2}{n-1}\ .$$

We refer [27] for the history and related results.

Recently, there are many works concerning the damped wave equations, with typical damping term $D(x) = \mu(1+|x|)^{-\beta}$

$$\begin{cases} u_{tt} - \Delta u + \frac{\mu}{(1+|x|)^\beta} u_t = N(u, u_t), (t, x) \in (0, T) \times \mathbb{R}^n, \\ u(0, x) = \varepsilon u_0(x), \ u_t(0, x) = \varepsilon u_1(x), \end{cases} \tag{1.8}$$

where $\mu, \beta \in \mathbb{R}$. When $N(u, u_t) = |u|^p$ and $\beta \le 1$, (1.8) has been well investigated, see, e.g., [9, 32, 7, 33, 23]. We refer [18] for the history and related results. For the scattering case $\beta > 1$, Nishihara-Sobajima-Wakasugi [33] conjecture that the critical power to (1.8) is exactly the same as that of the Strauss conjecture, i.e., $p_c = p_S(n)$, see also the introduction in page 2 in Ikehata-Todorova-Yordanov [9]. This conjecture has been verified for the blow up part when $\beta > 2$ and $n \ge 3$ by Lai-Tu [21]. For the global solutions, when $n = 3, 4$, Metcalfe-Wang [31] obtained the global existence when $p > p_S(n)$ and $\beta > 1$ with sufficiently small $|\mu|$. Later, Liu-Wang [28] remove the restriction $|\mu| \ll 1$. Recently, Lai-Liu-Tu-Wang [18] proved the blow up results when $\beta > 1$ and $n \ge 2$. More generally, they studied the following problem

$$\begin{cases} u_{tt} - \Delta u + D(x) u_t + V(x) u = |u|^p, \qquad (t, x) \in (0, T) \times \mathbb{R}^n, \\ u(0, x) = \varepsilon f(x), \ u_t(0, x) = \varepsilon g(x), \end{cases} \tag{1.9}$$

and showed that for the coefficient of dampings and potentials satisfying (1.2), (1.3), the solution of (1.9) will blow up whenever $1 < p < p_S(n)$.

When the nonlinear term $N(u, u_t) = |u_t|^p$, (1.8) is closely related with Glassey conjecture. Lai-Tu [21] obtained the blow up results when $1 < p \le p_G(n)$ and $\beta > 2$. Besides, there are many works concerning time dependent damping, see, e.g., [19] [2] and references therein.

Recently, there are many works studied wave systems (1.1). When there are no damping and potential terms, that is $D(x) = V(x) = 0$, (1.1) has been well understood. When $N_1(v, v_t) = |v|^p$, $N_2(u, u_t) = |u|^q$, Santo-Georgiev-Mitidieri [38] proved there exist a critical curve in (p, q)-plane,

$$\Gamma_{SS}(n, p, q) = \max\{\frac{p + 2 + q^{-1}}{pq - 1}, \frac{q + 2 + p^{-1}}{pq - 1}\} - \frac{n-1}{2} \ , n \ge 2 \ ,$$

such that the system (1.1) has a global in time weak solution for sufficiently small ε when $\Gamma_{SS}(n, p, q) < 0$ and the weak solution will blow up in finite time if $\Gamma_{SS}(n, p, q) > 0$. Kurokawa-Takamura-Wakasa [16] obtained the upper bound of lifespan estimates:

$$T_\varepsilon \lesssim \begin{cases} \varepsilon^{-\Gamma_{SS}^{-1}(n,p,q)}, & \Gamma_{SS}(n, p, q) > 0, \\ \exp\left(\varepsilon^{-\min\{p(pq-1), q(pq-1)\}}\right), & \Gamma_{SS}(n, p, q) = 0, p \ne q, \\ \exp\left(\varepsilon^{-p(p-1)}\right), & \Gamma_{SS}(n, p, q) = 0, p = q = p_S(n). \end{cases} \tag{1.10}$$

See also [37, 39, 12, 8, 1, 4, 14, 15] for related works. Palmieri-Takamura [34] considered a system with the same nonlinearities but with time-dependent, scattering-producing damping terms:

$$\begin{cases} u_{tt} - \Delta u + b_1(t) u_t = N_1(v, v_t), & x \in \mathbb{R}^n, t > 0, \\ v_{tt} - \Delta v + b_2(t) v_t = N_2(u, u_t), & x \in \mathbb{R}^n, t > 0, \\ (u, u_t, v, v_t)(0, x) = (\varepsilon u_0, \varepsilon u_1, \varepsilon v_0, \varepsilon v_1)(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.11}$$

where $b_1, b_2 \in \mathscr{C}([0,\infty)) \cap L^1([0,\infty))$ are nonnegative functions, and $p, q > 1$, $n \geq 2$. They then obtained the same upper bound for the lifespan estimate as in (1.10).

When $D(x) = V(x) = 0$ and $N_1(v, v_t) = |v_t|^p$, $N_2(u, u_t) = |u_t|^q$, (1.1) has been investigated by [3, 8, 43, 13]. Deng [3] obtained the blow up results when

$$1 < pq < \infty \begin{cases} 1 \leq p < \infty, & n = 1, \\ 1 < p \leq 2, & n = 2, \\ p = 1, & n = 3, \end{cases}$$

and

$$1 < pq \leq \frac{(n+1)p}{(n-1)p - 2} \begin{cases} 2 < p \leq 3, & n = 2, \\ 1 < p \leq 2, & n = 3, \\ 1 < p \leq \frac{n+1}{n-1}, & n \geq 4. \end{cases}$$

Ikeda-Sobajima-Wakasa [8] proved the finite time blow up results when

$$\Gamma_{GG}(p, q, n) := \max\{\frac{p+1}{pq-1}, \frac{q+1}{pq-1}\} - \frac{n-1}{2} \geq 0,\ p, q > 1,\ n \geq 2\ ,$$

and obtained the upper bound of lifespan estimates:

$$T_\varepsilon \lesssim \begin{cases} \varepsilon^{-\Gamma_{GG}^{-1}(n,p,q)}, & \Gamma_{GG}(n,p,q) > 0, \\ \exp(\varepsilon^{-(pq-1)}), & \Gamma_{GG}(n,p,q) = 0, \quad p \neq q, \\ \exp(\varepsilon^{-(p-1)}), & \Gamma_{GG}(n,p,q) = 0, \quad p = q. \end{cases} \tag{1.12}$$

This matches the result obtained by Palmieri and Takamura [35], who considered system (1.11) with the same nonlinearities.

When $D(x) = V(x) = 0$ and $N_1(v, v_t) = |v|^q$, $N_2(u, u_t) = |u_t|^p$, Hidano-Yokoyama [6] obtained the blow up results when

$$\Big(\frac{(n-1)p}{2} - 1\Big)(pq-1) < p + 2\ , 1 < p < \frac{2n}{n-1}\ , q > 1\ , n \geq 2\ ,$$

as well as the upper bound of lifespan estimates:

$$T_\varepsilon \lesssim \varepsilon^{-\frac{p(pq-1)}{p+2-\left(\frac{(n-1)p}{2}-1\right)(pq-1)}}\ . \tag{1.13}$$

Let

$$\begin{cases} F_{SG,1}(n,p,q) = (\frac{1}{p} + 1 + q)(pq-1)^{-1} - \frac{n-1}{2}, \\ F_{SG,2}(n,p,q) = (\frac{1}{q} + 2)(pq-1)^{-1} - \frac{n-1}{2}\ . \end{cases}$$

See, e.g., [6, 8] for the introduction of the exponents. Ikeda-Sobojima-Wakasa [8] showed the blow up results when

$$\Gamma_{SG}(n,p,q) = \max\{F_{SG,1}(n,p,q), F_{SG,2}(n,p,q)\} \geq 0\ ,$$

and obtained the upper bound of lifespan estimates:

$$T_\varepsilon \lesssim \begin{cases} \varepsilon^{-\Gamma_{SG}^{-1}(n,p,q)}, & \Gamma_{SG}(n,p,q) > 0, \\ \exp(\varepsilon^{-p(pq-1)}), & F_{SG,1}(n,p,q) = 0 > F_{SG,2}(n,p,q), \\ \exp(\varepsilon^{-q(pq-1)}), & F_{SG,2}(n,p,q) = 0 > F_{SG,1}(n,p,q), \\ \exp(\varepsilon^{-(pq-1)}), & F_{SG,2}(n,p,q) = 0 = F_{SG,1}(n,p,q). \end{cases}$$

Palmieri and Takamura [36] considered the system (1.11) with the same nonlinearities and derived the following upper bound for the lifespan estimate:

$$T_\varepsilon \lesssim \varepsilon^{-\Gamma_{\mathrm{SG}}(n,p,q)^{-1}}, \quad \text{for } \Gamma_{\mathrm{SG}}(n,p,q) > 0.$$

Besides, there are many works studied the time dependent damping terms of system (1.1), see, e.g., [36, 34, 35, 22, 20, 30, 19] and references therein.

In this work, we shall consider the finite time blow up phenomenon of wave system (1.1) with a large class of damping and potential functions (1.2) (1.3), as well as the upper bound of lifespan estimates for three kinds of nonlinear terms (1.4). Unlike the classical systems without damping and potential [38, 16, 8], the presence of spatially decaying coefficients $D(x)$ and $V(x)$ in (1.1) introduces new regimes in the lifespan estimates. For instance, in Theorem 1.1, when p or q is below the Strauss exponent $p_S(n)$, we obtain estimates of the form $T_\varepsilon \lesssim \varepsilon^{-\frac{2p(q-1)}{2+2q+(n-1)p-(n-1)pq}}$ etc., which do not appear in the undamped theory. This reflects the additional decay induced by $D(x)$ and $V(x)$. Regarding the details, one can refer to [18] for the influence of damping and potential terms in a single equation on the blow-up region of solutions to semilinear wave equations.

The main results in this paper are illustrated by the following Theorems.

Theorem 1.1. *Let $n \geq 3$, $p, q > 1$, $\Gamma_{SS}(n,p,q) \geqslant 0$. Consider the semi-linear wave systems (1.1) with nonlinear terms $N_1(v, v_t) = |v|^p$, $N_2(u, u_t) = |u|^q$, damping (1.2), potential (1.3). Then any weak solution with nontrivial, nonnegative, compactly supported data will blow up in finite time. In addition, there exists a constant $\varepsilon_0 = \varepsilon_0(u_0, u_1, v_0, v_1, n, p, q, R)$ such that for any $\varepsilon \in (0, \varepsilon_0)$, the lifespan T_ε satisfies*

$$T_\varepsilon \lesssim \begin{cases} \varepsilon^{-\Gamma_{SS}^{-1}(n,p,q)}, & \Gamma_{SS}(n,p,q) > 0,\ p,q > \frac{n}{n-1}\ , \\ \exp\left(\varepsilon^{-\min\{p(pq-1),q(pq-1)\}}\right), & \Gamma_{SS}(n,p,q) = 0, p \neq q,\ p,q > \frac{n}{n-1}, \\ \exp\left(\varepsilon^{-p(p-1)}\right), & \Gamma_{SS}(n,p,q) = 0, p = q = p_S(n)\ , \\ \varepsilon^{-\frac{2p(q-1)}{2+2q+(n-1)p-(n-1)pq}}, & 1 < p \leq q, \frac{n}{n-1} \leq q < p_S(n), \\ \varepsilon^{-\frac{2p(q-1)}{(n-1)p+2nq-2(n-1)-(n-1)pq}}, & 1 < p \leq q \leq \frac{n}{n-1}, \\ \varepsilon^{-\frac{2q(p-1)}{2+2p+(n-1)q-(n-1)pq}}, & 1 < q \leq p, \frac{n}{n-1} \leq p < p_S(n), \\ \varepsilon^{-\frac{2q(p-1)}{(n-1)q+2np-2(n-1)-(n-1)pq}}, & 1 < q \leq p \leqslant \frac{n}{n-1}. \end{cases}$$

Theorem 1.2. *Let $n \geq 3$, $p, q > 1$, $\Gamma_{GG}(n,p,q) \geqslant 0$. Consider the semi-linear wave systems (1.1) with nonlinear terms $N_1(v, v_t) = |v_t|^p$, $N_2(u, u_t) = |u_t|^q$, damping (1.2), potential (1.3). Then any weak solution with nontrivial, nonnegative, compactly supported data will blow up in finite time. In addition, there exists a constant $\varepsilon_0 = \varepsilon_0(u_0, u_1, v_0, v_1, n, p, q, R)$ such that for any $\varepsilon \in (0, \varepsilon_0)$, the lifespan T_ε satisfies (1.12).*

Theorem 1.3. *Let $n \geq 3$, $p, q > 1$, $\Gamma_{SG}(n,p,q) \geqslant 0$. Consider the semi-linear wave systems (1.1) with nonlinear terms $N_1(v, v_t) = |v|^q$, $N_2(u, u_t) = |u_t|^p$, damping (1.2) and potential (1.3). Then any weak solution with nontrivial, nonnegative, compactly supported data will blow up in finite time. In addition, there exists a constant $\varepsilon_0 = \varepsilon_0(u_0, u_1, v_0, v_1, n, p, q, R)$ such that for any $\varepsilon \in (0, \varepsilon_0)$, the lifespan*

T_ε satisfies

$$T_\varepsilon \lesssim \begin{cases} \min\{\varepsilon^{-\frac{2p}{2n+p-np-2}}, \varepsilon^{-\frac{2q}{2n+q-nq-2}}\}\ , & 1<p,q\le n/(n-1)\ ,\\ \varepsilon^{-\frac{2q(q-1)}{(n+1)q+2-(n-1)q^2}}\ , & 1<p\le \frac{n}{n-1}, \frac{n}{n-2}<q<p_S(n)\ ,\\ \varepsilon^{-\frac{2p}{2n+p-np-2}}\ , & 1<p\le\frac{n}{n-1}, \frac{n}{n-1}\le q\le\frac{n}{n-2},\\ \varepsilon^{-\frac{2q(p-1)}{(n-1)q+2-(n-1)pq}}\ , & 1<q\le\frac{n}{n-1}\le p\le 1+\frac{2}{(n-1)q}\ ,\\ \varepsilon^{-\Gamma_{SG}^{-1}(n,p,q)}, & \Gamma_{SG}(n,p,q)>0\ , p,q>\frac{n}{n-1}\ ,\\ \exp(\varepsilon^{-p(pq-1)}), & F_{SG,1}(n,p,q)=0>F_{SG,2}(n,p,q),\ p,q>\frac{n}{n-1}. \end{cases}$$

In addition, when $D(x)=0$, we have

$$T_\varepsilon \lesssim \begin{cases} \exp(\varepsilon^{-q(pq-1)}), & F_{SG,2}(n,p,q)=0>F_{SG,1}(n,p,q),\\ \exp(\varepsilon^{-(pq-1)}), & F_{SG,2}(n,p,q)=F_{SG,1}(n,p,q)=0\ . \end{cases} \tag{1.14}$$

Remark 1.1. The local existence of weak solutions can be obtained by energy estimates, we do not discuss it here. For the single wave equation, the low-dimensional cases $n=1,2$ are known to be much more delicate than the higher-dimensional ones, see, e.g., [25, 24, 29, 10, 41, 42, 45, 17]. To avoid these technical complications and to focus on the new phenomena generated by the space-dependent damping $D(x)$ and potential $V(x)$, we restrict ourselves to $n\ge 3$. Here Lemma 2.1 tells us that ϕ_0 is essentially constant ($\phi_0\simeq 1$). This uniform estimate greatly simplifies the analysis and helps us focus on the real impact of $D(x)$ and $V(x)$ on the lifespan.

2. Preliminary

In this section, we collect some Lemmas we will use later. For the strategy of the proof, we use some special test functions, see, e.g., [46], [44], [8].

Lemma 2.1 (Lemma 2.2 in [18]). *Support $0\le V\in C(\mathbb{R}^n)$, $V=\mathcal{O}(\langle r\rangle^{-\omega})$ for some $\omega>2$, and $V\in C^\delta(B_\delta)$ for some $\delta>0$. In addition, assume V is nontrivial when $n=2$. Then there exist a C^2 solution of*

$$\Delta\phi_0 = V\phi_0,\quad x\in\mathbb{R}^n, n\ge 2, \tag{2.1}$$

satisfying

$$\phi_0\simeq\begin{cases}\ln(r+2), & n=2,\\ 1, & n\ge 3.\end{cases} \tag{2.2}$$

Moreover, when $n=2$ and $V=0$, it is clear that $\phi_0=1$ is a solution (2.1).

Lemma 2.2 (Lemma 2.4 in [18]). *Let $V=V(r)$, $D=D(r)\in C(\mathbb{R}^n)\cap C^\delta(B_\delta)$ for some $\delta>0$. Suppose $V\ge 0$, $D\ge 0$, and also that for some $d_\infty\in\mathbb{R}$, $R>1$ and $r\ge R$, we have*

$$V(r)\in L^1([R,\infty)),\quad D(r)=\frac{d_\infty}{r}+D_\infty(r),\quad D_\infty\in L^1([R,\infty)). \tag{2.3}$$

Then there exists a C^2 solution of

$$\Delta\phi_\lambda=(\lambda^2+\lambda D+V)\phi_\lambda,\quad \lambda>0,\quad x\in\mathbb{R}^n, \tag{2.4}$$

satisfying

$$\phi_\lambda\simeq\langle r\rangle^{-\frac{n-1-d_\infty}{2}}e^{\lambda r},\ n\ge 3\ . \tag{2.5}$$

Lemma 2.3. *Let $a > 0$ and ϕ_λ in Lemma 2.2. Define*

$$b_a(t,x) = \int_0^1 e^{-\lambda t}\phi_\lambda(x)\lambda^{a-1}d\lambda,\ x \in \mathbb{R}^n\ .$$

Then it is easy to verify that $b_a(t,x)$ satisfies

$$\partial_t^2 b_a - \Delta b_a - D(x)\partial_t b_a + V(x)b_a = 0,\ \partial_t b_a = -b_{a+1}\ . \tag{2.6}$$

In addition, when D and V satisfying (1.2) (1.3), we have

$$b_a(t,x) \gtrsim (t+R)^{-a}\ ,\ a > 0\ , \tag{2.7}$$

$$b_a(t,x) \lesssim \begin{cases} (t+R)^{-a}, & 0 < a < \frac{n-1}{2}, \\ (t+R)^{-\frac{n-1}{2}}(t+R+1-|x|)^{\frac{n-1}{2}-a}, & a > \frac{n-1}{2}\ , \end{cases} \tag{2.8}$$

for $n \geq 2$ and $|x| \leq t+R$.

Proof. By Lemma 2.2, we have that

$$\phi_\lambda \simeq \langle r\rangle^{-\frac{n-1}{2}}e^{\lambda r},\ \lambda > 0\ .$$

since $d_\infty = 0$ when D and V satisfy (1.2) (1.3). Then by the proof of (6.7) (6.8) in [18], we have (2.7) (2.8). ∎

Lemma 2.4 (**Lemma 3.1 in** [21])**.** *If $\beta > 0$, then for any $\alpha \in \mathbb{R}$ and a fixed constant R, we have*

$$\int_0^{t+R}(1+r)^\alpha e^{-\beta(t-r)}dr \lesssim (t+R)^\alpha\ .$$

Finally, we need the following ODE Lemma to show the finite time blow up in some critical cases.

Lemma 2.5 (**Lemma 3.10 in** [8])**.** *Let $2 < t_0 < T$, $0 \leq \phi \in C^1([t_0,T))$. Assume that*

$$\begin{cases} \delta \leq K_1 t\phi'(t),\ t \in (t_0,T), \\ \phi(t)^{p_1} \leq K_2 t(\ln t)^{p_2-1}\phi'(t),\ t \in (t_0,T), \end{cases}$$

with δ, K_1, $K_2 > 0$ and p_1, $p_2 > 1$. If $p_2 < p_1 + 1$, then there exists positive constant δ_0 and K_3 (independent of δ) such that

$$T \leq \exp(K_3\delta^{-\frac{p_1-1}{p_1-p_2+1}})$$

for all $\delta \in (0,\delta_0)$.

3. Proof of Theorems 1.1-1.3

In this section, we give the proof of Theorems 1.1-1.3.

3.1. **Proof of Theorem 1.1.**

3.1.1. $\Gamma_{SS}(n,p,q)>0$.

We first introduce a smooth cut-off function $\eta(t)\in C^\infty([0,\infty))$ such that

$$\eta(t)=1\ ,\ t\le 1/2\ ,\ \eta(t)=0\ ,\ t\ge 1\ ,$$

and $\eta(t)$ is decreasing in $(1/2,1)$. Then for $M\in(1,T_\varepsilon)$, we set $\eta_M(t)=\eta(\frac{t}{M})$. In addition, let

$$\theta(t)=\begin{cases}0, & 0\le t<\frac{1}{2},\\ \eta(t), & t\ge\frac{1}{2},\end{cases} \tag{3.1}$$

and $\theta_M(t)=\theta(\frac{t}{M})$. It is easy to see that $\eta'(t)$ is bounded and $0\le\theta_M(t)=\eta_M(t)\le 1$ when $M/2\le t\le M$.

(I) Test function 1: $\Psi(t,x)=\eta_M^{2p'}(t)\phi_0(x)$

Firstly, we choose the test function $\Psi(t,x)=\eta_M^{2p'}(t)\phi_0(x)$, $p\le q$, where ϕ_0 satisfying

$$\Delta\phi_0=V(x)\phi_0\ .$$

Then by (1.5) we get that

$$\begin{aligned}
&\varepsilon\int_{\mathbb{R}^n}u_1(x)\phi_0dx+\varepsilon\int_{\mathbb{R}^n}u_0(x)D(x)\phi_0dx+\int_0^M\int_{\mathbb{R}^n}|v|^p\eta_M^{2p'}\phi_0dxdt\\
&=\int_0^M\int_{\mathbb{R}^n}u\phi_0\big(\partial_t^2\eta_M^{2p'}-D(x)\partial_t\eta_M^{2p'}\big)dxdt\\
&\ -\int_0^M\int_{\mathbb{R}^n}u\eta_M^{2p'}(\Delta\phi_0-V(x)\phi_0)dxdt\\
&=\int_0^M\int_{\mathbb{R}^n}u\phi_0\big(\partial_t^2\eta_M^{2p'}\big)dxdt-\int_0^M\int_{\mathbb{R}^n}u\phi_0D(x)\big(\partial_t\eta_M^{2p'}\big)dxdt\\
&=I_1+I_2\ .
\end{aligned}\tag{3.2}$$

Recall that $p\le q$, then we have $(2p'-2)q\ge 2p'$. Thus by Hölder's inequality and (2.2) we have

$$\begin{aligned}
|I_1|=&|\int_0^M\int_{\mathbb{R}^n}u\phi_0\big(\partial_t^2\eta_M^{2p'}\big)dxdt|\\
\lesssim&M^{-2}\Big(\int_0^M\int_{\mathbb{R}^n}|u|^q\theta_M^{(2p'-2)q}dxdt\Big)^{\frac{1}{q}}\Big(\int_{\frac{M}{2}}^M\int_{B_{t+R}}dxdt\Big)^{\frac{1}{q'}}\\
\lesssim&M^{-2}\Big(\int_0^M\int_{\mathbb{R}^n}|u|^q\theta_M^{2p'}dxdt\Big)^{\frac{1}{q}}\Big(\int_{\frac{M}{2}}^M\int_{B_{t+R}}dxdt\Big)^{\frac{1}{q'}}\qquad (p\le q)\\
\lesssim&M^{-2}\Big(\int_0^M\int_{\mathbb{R}^n}|u|^q\theta_M^{2p'}dxdt\Big)^{\frac{1}{q}}\Big(\int_{\frac{M}{2}}^M\int_0^{t+R}r^{n-1}drdt\Big)^{\frac{1}{q'}}\\
\lesssim&M^{\frac{nq-n-q-1}{q}}\Big(\int_0^M\int_{\mathbb{R}^n}|u|^q\theta_M^{2p'}dxdt\Big)^{\frac{1}{q}},
\end{aligned}\tag{3.3}$$

$$
\begin{aligned}
|I_2| =& |\int_{\mathbb{R}^n} u\phi_0 D(x)\big(\partial_t \eta_M^{2p'}\big)dxdt| \\
\lesssim& M^{-1}\Big(\int_0^M\int_{\mathbb{R}^n}|u|^q\theta_M^{(2p'-1)q}dxdt\Big)^{\frac{1}{q}}\Big(\int_{\frac{M}{2}}^M\int_{B_{t+R}}(D(x))^{q'}dxdt\Big)^{\frac{1}{q'}} \\
\lesssim& M^{-1}\Big(\int_0^M\int_{\mathbb{R}^n}|u|^q\theta_M^{2p'}dxdt\Big)^{\frac{1}{q}}\Big(\int_{\frac{M}{2}}^M\int_0^{t+R}(1+r)^{-\beta q'}r^{n-1}drdt\Big)^{\frac{1}{q'}} \qquad (3.4)\\
\lesssim& M^{-1}\Big(\int_0^M\int_{\mathbb{R}^n}|u|^q\theta_M^{2p'}dxdt\Big)^{\frac{1}{q}}\begin{cases}(M\ln(M))^{\frac{1}{q'}}, & n-q'=0,\\ M^{(n+1-q')\frac{1}{q'}}, & n-q'>0,\\ M^{\frac{1}{q'}}, & n-q'<0.\end{cases} \\
\lesssim& \Big(\int_0^M\int_{\mathbb{R}^n}|u|^q\theta_M^{2p'}dxdt\Big)^{\frac{1}{q}}\begin{cases}M^{-1}(M\ln(M))^{\frac{1}{q'}}, & q\le\frac{n}{n-1},\\ M^{\frac{nq-n-q-1}{q}}, & q>\frac{n}{n-1}\ .\end{cases}
\end{aligned}
$$

When $q > \frac{n}{n-1}$, by combining (3.3) and (3.4), we get that

$$
\varepsilon C_1(u_0,u_1)+\int_0^M\int_{\mathbb{R}^n}|v|^p\eta_M^{2p'}dxdt\lesssim M^{\frac{nq-n-q-1}{q}}\Big(\int_0^M\int_{\mathbb{R}^n}|u|^q\theta_M^{2p'}dxdt\Big)^{\frac{1}{q}}, \tag{3.5}
$$

where

$$
C_1(u_0,u_1)=\int_{\mathbb{R}^n}u_1(x)\phi_0dx+\int_{\mathbb{R}^n}u_0(x)D(x)\phi_0dx.
$$

Then we get that

$$
\int_0^M\int_{\mathbb{R}^n}|v|^p\eta_M^{2p'}dxdt\lesssim M^{\frac{nq-n-q-1}{q}}\Big(\int_0^M\int_{\mathbb{R}^n}|u|^q\theta_M^{2p'}dxdt\Big)^{\frac{1}{q}}. \tag{3.6}
$$

Similarly, by applying test function $\Psi(t,x)=\eta_M^{2p'}(t)\phi_0(x)$ to (1.6), we can get

$$
\int_0^M\int_{\mathbb{R}^n}|u|^q\eta_M^{2p'}dxdt\lesssim M^{\frac{np-n-p-1}{p}}\Big(\int_0^M\int_{\mathbb{R}^n}|v|^p\theta_M^{2p'}dxdt\Big)^{\frac{1}{p}}. \tag{3.7}
$$

Thus (3.6) and (3.7) yields

$$
\int_0^M\int_{\mathbb{R}^n}|v|^p\eta_M^{2p'}dxdt\lesssim M^{\frac{nqp-2p-n-pq-1}{qp-1}}, \tag{3.8}
$$

$$
\int_0^M\int_{\mathbb{R}^n}|u|^q\eta_M^{2p'}dxdt\lesssim M^{\frac{nqp-2q-n-pq-1}{qp-1}}. \tag{3.9}
$$

(II) Test function 2: $\Psi(t,x)=\eta_M^{2p'}(t)e^{-t}\phi(x)=\eta_M^{2p'}(t)\Phi(t,x)$

Secondly, we choose another test function $\Psi=\eta_M^{2p'}\Phi(t,x)$, where $\Phi(t,x)=e^{-t}\phi(x)$ and ϕ satisfying

$$
\Delta\phi=(1+D+V)\phi\ .
$$

Then it is easy to see that $\Phi(t,x)=e^{-t}\phi(x)$ satisfying

$$
\partial_t^2\Phi-\Delta\Phi-D(x)\partial_t\Phi+V(x)\Phi=0\ .
$$

By taking $\Psi(t,x) = e^{-t}\phi(x)$ in (1.5), we get that

$$
\begin{aligned}
&\varepsilon C_2(u_0,u_1) + \int_0^M \int_{\mathbb{R}^n} |v|^p \eta_M^{2p'} \Phi dxdt \\
=& -\int_0^M \int_{\mathbb{R}^n} u_t \partial_t(\eta_M^{2p'}\Phi) dxdt + \int_0^M \int_{\mathbb{R}^n} \nabla u \nabla(\eta_M^{2p'}\Phi) dxdt \\
& -\int_0^M \int_{\mathbb{R}^n} D(x) u \partial_t(\eta_M^{2p'}\Phi) dxdt + \int_0^M \int_{\mathbb{R}^n} u V(x) \eta_M^{2p'} \Phi dxdt \\
=& \int_0^M \int_{\mathbb{R}^n} u\Big((\partial_t^2 \eta_M^{2p'})\Phi + 2(\partial_t \eta_M^{2p'})(\partial_t \Phi) - D(x)(\partial_t \eta_M^{2p'})\Phi\Big) dxdt \\
& + \int_0^M \int_{\mathbb{R}^n} u \eta_M^{2p'} \Big(\partial_t^2 \Phi - \Delta \Phi - D(x)\partial_t \Phi + V(x)\Phi\Big) dxdt \\
=& \int_0^M \int_{\mathbb{R}^n} u\Big((\partial_t^2 \eta_M^{2p'})\Phi + 2(\partial_t \eta_M^{2p'})(\partial_t \Phi) - D(x)(\partial_t \eta_M^{2p'})\Phi\Big) dxdt \\
=& I_3 + I_4 + I_5 \ .
\end{aligned}
\tag{3.10}
$$

By applying Hölder's inequality and Lemma 2.2 with $\lambda = 1$, we have that

$$
\begin{aligned}
|I_3| =& \Big| \int_0^M \int_{\mathbb{R}^n} u(\partial_t^2 \eta_M^{2p'})\Phi dxdt \Big| \\
\lesssim& M^{-2}\Big(\int_0^M \int_{\mathbb{R}^n} |u|^q \theta_M^{(2p'-2)q} dxdt\Big)^{\frac{1}{q}} \Big(\int_{\frac{M}{2}}^M \int_{B_{t+R}} \Phi^{q'} dxdt\Big)^{\frac{1}{q'}} \\
\lesssim& M^{-2}\Big(\int_0^M \int_{\mathbb{R}^n} |u|^q \theta_M^{2p'} dxdt\Big)^{\frac{1}{q}} \Big(\int_{\frac{M}{2}}^M \int_{B_{t+R}} \Phi^{q'} dxdt\Big)^{\frac{1}{q'}} \ (p \le q) \\
\lesssim& M^{-2}\Big(\int_0^M \int_{\mathbb{R}^n} |u|^q \theta_M^{2p'} dxdt\Big)^{\frac{1}{q}} \\
& \times \Big(\int_{\frac{M}{2}}^M \int_0^{t+R} (r+1)^{n-1-\frac{n-1}{2}q'} e^{-(t-r)q'} drdt\Big)^{\frac{1}{q'}} \ (Lemma\ 2.2) \\
\lesssim& M^{-2}\Big(\int_0^M \int_{\mathbb{R}^n} |u|^q \theta_M^{2p'} dxdt\Big)^{\frac{1}{q}} \Big(\int_{\frac{M}{2}}^M (2+t)^{n-1-\frac{n-1}{2}q'} dt\Big)^{\frac{1}{q'}} \ (Lemma\ 2.4) \\
\lesssim& M^{-2+(n-\frac{(n-1)q'}{2})\frac{1}{q'}} \Big(\int_0^M \int_{\mathbb{R}^n} |u|^q \theta_M^{2p'} dxdt\Big)^{\frac{1}{q}},
\end{aligned}
$$

$$
\begin{aligned}
|I_4| =& \Big| \int_0^M \int_{\mathbb{R}^n} 2u(\partial_t \eta_M^{2p'})(\partial_t \Phi) dxdt \Big| \\
\lesssim& M^{-1+(n-\frac{(n-1)q'}{2})\frac{1}{q'}} \Big(\int_0^M \int_{\mathbb{R}^n} |u|^q \theta_M^{2p'} dxdt\Big)^{\frac{1}{q}},
\end{aligned}
$$

$$
\begin{aligned}
|I_5| =&|\int_0^M\int_{\mathbb{R}^n} uD(x)(\partial_t\eta_M^{2p'})\Phi dxdt|\\
\lesssim& M^{-1}\int_0^M\int_{\mathbb{R}^n}|u\theta_M^{2p'-1}\Phi D(x)|dxdt\\
\lesssim& M^{-1}\Big(\int_0^M\int_{\mathbb{R}^n}|u|^q\theta_M^{2p'}dxdt\Big)^{\frac{1}{q}}\Big(\int_{\frac{M}{2}}^M\int_{B_{t+R}}D^{q'}(x)\Phi^{q'}dxdt\Big)^{\frac{1}{q'}}\\
\lesssim& M^{-1}\Big(\int_0^M\int_{\mathbb{R}^n}|u|^q\theta_M^{2p'}dxdt\Big)^{\frac{1}{q}}\Big(\int_{\frac{M}{2}}^M(1+t)^{n-1-(\frac{n-1}{2}+\beta)q'}dt\Big)^{\frac{1}{q'}}\ (Lemma\ 2.4)\\
\lesssim& M^{-2+(n-\frac{(n-1)q'}{2})\frac{1}{q'}}\Big(\int_0^M\int_{\mathbb{R}^n}|u|^q\theta_M^{2p'}dxdt\Big)^{\frac{1}{q}}.
\end{aligned}
$$

This in turn implies that

$$
(3.11)\qquad \big(C_2(u_0,u_1)\varepsilon\big)^q\lesssim M^{\frac{(n-1)q}{2}-n}\Big(\int_0^M\int_{\mathbb{R}^n}|u|^q\theta_M^{2p'}dxdt\Big)\ ,
$$

where

$$
C_2(u_0,u_1)=\int_{\mathbb{R}^n}u_1(x)\phi(x)dx+\int_{\mathbb{R}^n}\Big(1+D(x)\Big)u_0(x)\phi(x)dx\ ,
$$

$$
C_2'(v_0,v_1)=\int_{\mathbb{R}^n}v_1(x)\phi(x)dx+\int_{\mathbb{R}^n}\Big(1+D(x)\Big)v_0(x)\phi(x)dx\ .
$$

In an analogy way, by applying test function $\Psi=\eta_M^{2p'}e^{-t}\phi(x)$ to (1.6), we could get that

$$
(3.12)\qquad \big(C_2'(v_0,v_1)\varepsilon\big)^p\lesssim M^{\frac{(n-1)p}{2}-n}\Big(\int_0^M\int_{\mathbb{R}^n}|v|^p\theta_M^{2p'}dxdt\Big).
$$

By combining (3.11) and (3.9), we get that

$$
M\lesssim\varepsilon^{-\frac{1}{(p+q^{-1}+2)(qp-1)^{-1}-\frac{n-1}{2}}}.
$$

By combining (3.8) and (3.12), we have

$$
M\lesssim\varepsilon^{-\frac{1}{(q+p^{-1}+2)(qp-1)^{-1}-\frac{n-1}{2}}}.
$$

Similarly, by applying test functions $\eta_M^{2q'}\phi_0$, $\eta_M^{2q'}e^{-t}\phi(x)$, we can also obtain the above two lifespan estimates under the assumption $p\ge q$ and $p>n/(n-1)$. Hence for any $M\in(1,T_\varepsilon)$, we get that

$$
M\lesssim\min\big\{\varepsilon^{-\frac{1}{(p+q^{-1}+2)(qp-1)^{-1}-\frac{n-1}{2}}},\varepsilon^{-\frac{1}{(q+p^{-1}+2)(qp-1)^{-1}-\frac{n-1}{2}}}\big\}.
$$

That is to say

$$
T_\varepsilon\lesssim\min\big\{\varepsilon^{-\frac{1}{(p+q^{-1}+2)(qp-1)^{-1}-\frac{n-1}{2}}},\varepsilon^{-\frac{1}{(q+p^{-1}+2)(qp-1)^{-1}-\frac{n-1}{2}}}\big\}.
$$

So we come to the estimate

$$
T_\varepsilon\lesssim\varepsilon^{-\Gamma_{SS}^{-1}(n,p,q)},\quad \Gamma_{SS}(n,p,q)>0,\quad p,q>n/(n-1)\ .
$$

We obtain the first lifespan estimate in Theorem 1.1.

When p or q less than $n/(n-1)$, we apply Young's inequality to (3.2) with $p \le q$ to get

$$\int_0^M \int_{\mathbb{R}^n} |v|^p \eta_M^{2p'} dxdt \le \frac{1}{2} \int_0^M \int_{\mathbb{R}^n} |u|^q \eta_M^{2p'} dxdt + C \int_{\frac{M}{2}}^M \int_{B_{t+R}} (M^{-2} + |D| M^{-1})^{q'} dxdt\,,$$

$$\int_0^M \int_{\mathbb{R}^n} |u|^q \eta_M^{2p'} dxdt \le \frac{1}{2} \int_0^M \int_{\mathbb{R}^n} |v|^p \eta_M^{2p'} dxdt + C \int_{\frac{M}{2}}^M \int_{B_{t+R}} (M^{-2} + |D| M^{-1})^{p'} dxdt\,.$$

That is

$$\begin{aligned}\int_0^M \int_{\mathbb{R}^n} |v|^p \eta_M^{2p'} dxdt + \int_0^M \int_{\mathbb{R}^n} |u|^q \eta_M^{2p'} dxdt &\lesssim \int_{\frac{M}{2}}^M \int_{B_{t+R}} (M^{-2} + |D| M^{-1})^{q'} dxdt \\ &\lesssim \begin{cases} M^{1-q'}, & 1 < q \leqslant \frac{n}{n-1}, \\ M^{n+1-2q'}, & q \ge \frac{n}{n-1}. \end{cases}\end{aligned} \tag{3.13}$$

Then by (3.12) and (3.13), we can obtain that

$$\varepsilon^p \lesssim \begin{cases} M^{\frac{n-1}{2}p - n + 1 - q'}, & 1 < q \leqslant \frac{n}{n-1}, \\ M^{\frac{n-1}{2}p + 1 - 2q'}, & q \ge \frac{n}{n-1}. \end{cases} \tag{3.14}$$

Then by (3.11) and (3.13), we can obtain that

$$\varepsilon^q \lesssim \begin{cases} M^{\frac{n-1}{2}q - n + 1 - q'}, & 1 < q \leqslant \frac{n}{n-1}, \\ M^{\frac{n-1}{2}q + 1 - 2q'}, & q \ge \frac{n}{n-1}. \end{cases} \tag{3.15}$$

In conclusion, by (3.14) and (3.15), we have

$$T_\varepsilon \lesssim \begin{cases} \varepsilon^{-\frac{2p(q-1)}{2+2q+(n-1)p-(n-1)pq}}, & 1 < p \le q,\ \frac{n}{n-1} \le q < p_S(n)\,, \\ \varepsilon^{-\frac{2p(q-1)}{(n-1)p+2nq-2(n-1)-(n-1)pq}}, & 1 < p \le q \le \frac{n}{n-1}\,. \end{cases} \tag{3.16}$$

In a same way, when $q \le p$, we have

$$T_\varepsilon \lesssim \begin{cases} \varepsilon^{-\frac{2q(p-1)}{2+2p+(n-1)q-(n-1)pq}}, & 1 < q \le p,\ \frac{n}{n-1} \le p < p_S(n)\,, \\ \varepsilon^{-\frac{2q(p-1)}{(n-1)q+2np-2(n-1)-(n-1)pq}}, & 1 < q \le p \leqslant \frac{n}{n-1}\,. \end{cases} \tag{3.17}$$

3.1.2. $\Gamma_{SS}(n,p,q) = 0$.

For nonnegative function $\omega(t,x) \in L^1_{loc}([0,T); L^1(\mathbb{R}^n))$ and $M, T \in (1, T_\varepsilon)$, we set

$$Y[\omega(t,x)](M) = \int_1^M \Big(\int_0^T \int_{\mathbb{R}^n} \omega(t,x) \theta_\sigma^{2p'}(t) dxdt \Big) \sigma^{-1} d\sigma.$$

Then we can get that

$$Y'[\omega(t,x)](M) = \frac{d}{dM} Y[\omega(t,x)](M) = M^{-1} \Big(\int_0^T \int_{\mathbb{R}^n} \omega(t,x) \theta_M^{2p'}(t) dxdt \Big). \tag{3.18}$$

By direct computation, we get that

$$\begin{aligned}
&Y[\omega(t,x)] = \int_1^M \Big(\int_0^T \int_{\mathbb{R}^n} \omega(t,x)\theta_\sigma^{2p'}(t)dxdt\Big)\sigma^{-1}d\sigma \\
&= \int_0^T \int_{\mathbb{R}^n} \omega(t,x) \int_{\frac{t}{M}}^t \theta^{2p'}(s)s^{-1}dsdxdt \\
&\lesssim \int_0^{\frac{M}{2}} \int_{\mathbb{R}^n} \omega(t,x) \int_{\frac{t}{M}}^t \theta^{2p'}(s)s^{-1}dsdxdt + \int_{\frac{M}{2}}^T \int_{\mathbb{R}^n} \omega(t,x) \int_{\frac{t}{M}}^t \theta^{2p'}(s)s^{-1}dsdxdt \\
&\lesssim \int_0^{\frac{M}{2}} \int_{\mathbb{R}^n} \omega(t,x)\eta^{2p'}(\frac{t}{M}) \int_{\frac{1}{2}}^1 s^{-1}dsdxdt + \int_{\frac{M}{2}}^T \int_{\mathbb{R}^n} \omega(t,x)\theta^{2p'}(\frac{t}{M}) \int_{\frac{1}{2}}^1 s^{-1}dsdxdt \\
&\lesssim \ln 2 \int_0^T \int_{\mathbb{R}^n} \omega(t,x)\eta_M^{2p'}(t)dxdt.
\end{aligned}$$

Then we can obtain that

$$Y[\omega(t,x)](M) \lesssim \ln 2 \int_0^T \int_{\mathbb{R}^n} \omega(t,x)\eta_M^{2p'}(t)dxdt. \tag{3.19}$$

(III) Test function 3: $\Psi(t,x) = \eta_M^{2q'}(t)b_a(t,x)$

Let $a = \frac{n-1}{2} - \frac{1}{q} > 0$, by the lower bound of b_a in Lemma 2.3, we have that

$$\int_0^M \int_{\mathbb{R}^n} |v|^p \theta_M^{2q'} b_a dxdt \gtrsim M^{-a} \int_0^M \int_{\mathbb{R}^n} |v|^p \theta_M^{2q'} dxdt \ .$$

By applying (3.7) and (3.11) (by replacing p with q in $\eta_M^{2p'}$), we know that

$$\int_0^M \int_{\mathbb{R}^n} |v|^p \theta_M^{2q'} dxdt \gtrsim \varepsilon^{pq} M^{n+p+1-\frac{n-1}{2}pq} \ .$$

Noting that when $p \geq q$, $\Gamma_{SS} = \frac{p+2+1/q}{pq-1} - \frac{n-1}{2}$, that is to say

$$n+p+1-\frac{n-1}{2}pq - a = \Gamma_{SS}(n,p,q) = 0 \ .$$

Hence we get that

$$\int_0^M \int_{\mathbb{R}^n} |v|^p \theta_M^{2q'} b_a dxdt \gtrsim \varepsilon^{qp} \ . \tag{3.20}$$

Next we choose test function $\Psi = \eta_M^{2q'} b_a$ to replace $\Psi(t,x)$ in (1.5), we get that

$$\begin{aligned}
&\varepsilon C_3(u_0,u_1) + \int_0^M \int_{\mathbb{R}^n} |v|^p \eta_M^{2q'}(t) b_a dxdt \\
&= \int_0^T \int_{\mathbb{R}^n} u\Big((\partial_t^2 \eta_M^{2q'})b_a + 2(\partial_t \eta_M^{2q'})(\partial_t b_a) - D(x)(\partial_t \eta_M^{2q'})b_a\Big)dxdt \\
&= I_6 + I_7 + I_8,
\end{aligned} \tag{3.21}$$

where

$$C_3(u_0,u_1) = \int_{\mathbb{R}^n} u_1(x)b_a(0,x)dx + \int_{\mathbb{R}^n} u_0(x)\Big(D(x)b_a(0,x) + b_{a+1}(0,x)\Big)dx \ .$$

By applying Hölder's inequality, we get that

$$
\begin{aligned}
|I_6| =& \Big| \int_0^T \int_{\mathbb{R}^n} u(\partial_t^2 \eta_M^{2q'}) b_a dxdt \Big| \\
\lesssim& M^{-2} \int_0^M \int_{\mathbb{R}^n} |u(t,x)\theta_M^{2q'-2}(t) b_a| dxdt \\
\lesssim& M^{-2} \Big(\int_0^M \int_{\mathbb{R}^n} |u|^q \theta_M^{2q'} dxdt \Big)^{\frac{1}{q}} \Big(\int_{\frac{M}{2}}^M \int_{B_{t+R}} b_a^{q'} dxdt \Big)^{\frac{1}{q'}} \\
\lesssim& M^{-2} \Big(\int_0^M \int_{\mathbb{R}^n} |u|^q \theta_M^{2q'} dxdt \Big)^{\frac{1}{q}} \Big(\int_{\frac{M}{2}}^M \int_0^{t+R} (t+R)^{(\frac{1}{q} - \frac{n-1}{2})q'} r^{n-1} drdt \Big)^{\frac{1}{q'}} \\
\lesssim& M^{\frac{nq-2n-q}{2q}} \Big(\int_0^M \int_{\mathbb{R}^n} |u|^q \theta_M^{2q'} dxdt \Big)^{\frac{1}{q}}, \\
|I_7| =& \Big| \int_0^T \int_{\mathbb{R}^n} 2u(\partial_t \eta_M^{2q'})(\partial_t b_a) dxdt \Big| \\
\lesssim& M^{-1} \Big(\int_0^M \int_{\mathbb{R}^n} |u|^q \theta_M^{2q'} dxdt \Big)^{\frac{1}{q}} \Big(\int_{\frac{M}{2}}^M \int_{B_{t+R}} b_{a+1}^{q'} dxdt \Big)^{\frac{1}{q'}} \\
\lesssim& M^{\frac{nq-2n-q}{2q}} (\ln M)^{\frac{q-1}{q}} \Big(\int_0^M \int_{\mathbb{R}^n} |u|^q \theta_M^{2q'} dxdt \Big)^{\frac{1}{q}}, \\
|I_8| =& \Big| \int_0^T \int_{\mathbb{R}^n} uD(x)(\partial_t \eta_M^{2q'}) b_a dxdt \Big| \\
\lesssim& M^{-1} \int_0^M \int_{\mathbb{R}^n} |u(t,x)\theta_M^{2q'-1}(t) b_a D(x)| dxdt \\
\lesssim& M^{-1} \Big(\int_0^M \int_{\mathbb{R}^n} |u|^q \theta_M^{2q'}(t) dxdt \Big)^{\frac{1}{q}} \Big(\int_{\frac{M}{2}}^M \int_{B_{t+R}} D^{q'}(x) b_a^{q'} dxdt \Big)^{\frac{1}{q'}} \\
\lesssim& M^{-1} \Big(\int_0^M \int_{\mathbb{R}^n} |u|^q \theta_M^{2q'}(t) dxdt \Big)^{\frac{1}{q}} \\
&\times \Big(\int_{\frac{M}{2}}^M \int_0^{t+R} (R+t)^{(\frac{1}{q} - \frac{n-1}{2})q'} r^{n-1} (1+r)^{-\beta q'} drdt \Big)^{\frac{1}{q'}} \\
\lesssim& \Big(\int_0^M \int_{\mathbb{R}^n} |u|^q \theta_M^{2q'}(t) dxdt \Big) \begin{cases} M^{\frac{nq-2n-q}{2q}}, & q > \frac{n}{n-1}, \\ M^{q' - \frac{n-1}{2}}, & q \le \frac{n}{n-1}. \end{cases}
\end{aligned}
$$

Hence, we get that

$$
\int_0^M \int_{\mathbb{R}^n} |v|^p \eta_M^{2q'}(t) b_a dxdt \lesssim M^{\frac{nq-2n-q}{2q}} (\ln M)^{\frac{q-1}{q}} \Big(\int_0^M \int_{\mathbb{R}^n} |u|^q \theta_M^{2q'} dxdt \Big)^{\frac{1}{q}}, \tag{3.22}
$$

(3.23)

$$
\int_0^M \int_{\mathbb{R}^n} |u|^q \eta_M^{2q'}(t) b_a dxdt \lesssim M^{\frac{np-2n-p}{2p}} (\ln M)^{\frac{p-1}{p}} \Big(\int_0^M \int_{\mathbb{R}^n} |v|^p \theta_M^{2q'} dxdt \Big)^{\frac{1}{p}},
$$

with $\frac{n}{n-1} < q \le p$. Plugging (3.7) into (3.22) and by the lower bound of b_a in Lemma 2.3, we get that

$$
\Big(\int_0^M \int_{\mathbb{R}^n} |v|^p \eta_M^{2q'} b_a dxdt \Big)^{pq} \lesssim (\ln M)^{p(q-1)} \int_0^M \int_{\mathbb{R}^n} |v|^p \theta_M^{2q'} b_a dxdt. \tag{3.24}
$$

Let

$$Y[b_a|v|^p](M)=\int_1^M\Big(\int_0^T\int_{\mathbb{R}^n}|v|^p b_a\theta_\sigma^{2q'}(t)dxdt\Big)\sigma^{-1}d\sigma\ .$$

By combining (3.18), (3.19), (3.20) and (3.24), we have that

$$\begin{aligned}\varepsilon^{qp}\lesssim&MY'[b_a|v|^p](M),\\ \Big(Y[b_a|v|^p](M)\Big)^{pq}\lesssim&(\ln M)^{p(q-1)}MY'[b_a|v|^p](M)\ .\end{aligned}$$

Exploiting Lemma 2.5 with $p_2=p(q-1)+1$, $p_1=pq$, $\delta=\varepsilon^{pq}$, we have that

$$M\lesssim\exp(\varepsilon^{-q(pq-1)}),\quad p\geq q>n/(n-1)\ .$$

Hence for any $M\in(1,T_\varepsilon)$, that is to say

$$T_\varepsilon\lesssim\exp(\varepsilon^{-q(pq-1)}),\quad p\geq q>n/(n-1)\ .$$

Similarly, we could get that

$$T_\varepsilon\lesssim\exp(\varepsilon^{-p(pq-1)}),\quad n/(n-1)<p\leq q\ .$$

Then we have

$$T_\varepsilon\lesssim\exp\big(\varepsilon^{-\min\{p(pq-1),q(pq-1)\}}\big),\quad\Gamma_{SS}(n,p,q)=0\ ,p,q>n/(n-1)\ .$$

In addition, by combining (3.11), (3.12), we have

$$\begin{aligned}&\varepsilon C_2(u_0,u_1)+\varepsilon C_2'(v_0,v_1)+\int_0^M\int_{\mathbb{R}^n}|u+v|^p\eta_M^{2p'}\Phi dxdt\\ \lesssim&\ \varepsilon C_2(u_0,u_1)+\varepsilon C_2'(v_0,v_1)+\int_0^M\int_{\mathbb{R}^n}(|u|^p+|v|^p)\eta_M^{2p'}\Phi dxdt\\ \lesssim&\int_0^M\int_{\mathbb{R}^n}(u+v)\Big((\partial_t^2\eta_M^{2p'})\Phi+2(\partial_t\eta_M^{2p'})(\partial_t\Phi)-D(x)(\partial_t\eta_M^{2p'})\Phi\Big)dxdt\\ \lesssim&\ M^{\frac{(n-1)}{2}-\frac{n}{p}}\Big(\int_0^M\int_{\mathbb{R}^n}|u+v|^p\theta_M^{2p'}dxdt\Big)^{\frac1p}\\ \lesssim&\ M^{\frac{(n-1)}{2}-\frac{n}{p}+\frac{n-1}{2p}-\frac{1}{p^2}}\big(\int_0^M\int_{\mathbb{R}^n}|u+v|^p\theta_M^{2p'}b_adxdt\big)^{\frac1p}\ .\end{aligned}\tag{3.25}$$

Noting that, when $p=q=p_S(n)$, we have that

$$\frac{n-1}{2}-\frac{n}{p}+\frac{n-1}{2p}-\frac{1}{p^2}=0\ .$$

Thus by (3.25), we get

$$\varepsilon^p\lesssim\int_0^M\int_{\mathbb{R}^n}|u+v|^p\theta_M^{2p'}b_adxdt\ .\tag{3.26}$$

Similarly, we can get that

$$\begin{aligned}&\big(C_3(u_0,u_1)+C_3'(v_0,v_1)\big)\varepsilon+\int_0^M\int_{\mathbb{R}^n}|v+u|^p\eta_M^{2p'}(t)b_adxdt\\ \lesssim&\int_0^M\int_{\mathbb{R}^n}(u+v)\Big((\partial_t^2\eta_M^{2p'})b_a+2(\partial_t\eta_M^{2p'})(\partial_tb_a)-D(x)(\partial_t\eta_M^{2p'})b_a\Big)dxdt\\ \lesssim&\ I_9+I_{10}+I_{11}\ .\end{aligned}$$

By Hölder's inequality, we have

$$
\begin{aligned}
|I_9| =& |\int_0^M \int_{\mathbb{R}^n} (u+v)(\partial_t^2 \eta_M^{2p'}) b_a dxdt| \\
\lesssim& \frac{1}{M^2}\Big(\int_0^M \int_{\mathbb{R}^n} |u+v|^p \theta_M^{2p'} b_a dxdt\Big)^{\frac{1}{p}} \Big(\int_{\frac{M}{2}}^M \int_{B_{t+R}} b_a dxdt\Big)^{\frac{p-1}{p}} \\
\lesssim& \frac{1}{M^2}\Big(\int_{\frac{M}{2}}^M \int_0^{t+R} (t+R)^{-(\frac{n-1}{2}-\frac{1}{p})} r^{n-1} drdt\Big)^{\frac{p-1}{p}} \Big(\int_0^M \int_{\mathbb{R}^n} |u+v|^p \theta_M^{2p'} b_a dxdt\Big)^{\frac{1}{p}} \\
\lesssim& \Big(\int_0^M \int_{\mathbb{R}^n} |u+v|^p \theta_M^{2p'} b_a dxdt\Big)^{\frac{1}{p}},
\end{aligned}
$$

$$
\begin{aligned}
|I_{10}| =& |\int_0^M \int_{\mathbb{R}^n} (u+v) 2(\partial_t \eta_M^{2p'})(\partial_t b_a) dxdt| \\
\lesssim& M^{-1}\Big(\int_0^M \int_{\mathbb{R}^n} |u+v|^p \theta_M^{2p'} b_a dxdt\Big)^{\frac{1}{p}} \Big(\int_{\frac{M}{2}}^M \int_{B_{t+R}} b_a^{-(\frac{1}{p-1})} b_a^{\frac{p}{p-1}} dxdt\Big)^{\frac{p-1}{p}} \\
\lesssim& M^{-1}\Big(\int_0^M \int_{\mathbb{R}^n} |u+v|^p \theta_M^{2p'} b_a dxdt\Big)^{\frac{1}{p}} \\
&\times \Big(\int_{\frac{M}{2}}^M \int_0^{t+R} (t+R)^{\frac{n-1}{2}-\frac{1}{p(p-1)}} (t+R+1-r)^{-1} drdt\Big)^{\frac{p-1}{p}} \\
\lesssim& M^{-1}\Big(\int_{\frac{M}{2}}^M (t+R)^{\frac{n-1}{2}-\frac{1}{p(p-1)}} \ln(t+R)\Big)^{\frac{p-1}{p}} \Big(\int_0^M \int_{\mathbb{R}^n} |u+v|^p \theta_M^{2p'} b_a dxdt\Big)^{\frac{1}{p}} \\
\lesssim& (\ln M)^{\frac{p-1}{p}} \Big(\int_0^M \int_{\mathbb{R}^n} |u+v|^p \theta_M^{2p'} b_a dxdt\Big)^{\frac{1}{p}},
\end{aligned}
$$

$$
\begin{aligned}
|I_{11}| =& |\int_0^M \int_{\mathbb{R}^n} (u+v) D(x)(\partial_t \eta_M^{2p'}) b_a dxdt| \\
\lesssim& M^{-1}\Big(\int_0^M \int_{\mathbb{R}^n} |u+v|^p \theta_M^{2p'} b_a dxdt\Big)^{\frac{1}{p}} \Big(\int_{\frac{M}{2}}^M \int_{B_{t+R}} D^{p'}(x) b_a dxdt\Big)^{\frac{p-1}{p}} \\
\lesssim& M^{-1}\Big(\int_0^M \int_{\mathbb{R}^n} |u+v|^p \theta_M^{2p'} b_a dxdt\Big)^{\frac{1}{p}} \\
&\times \Big(\int_{\frac{M}{2}}^M \int_0^{t+R} (t+R)^{-(\frac{n-1}{2}-\frac{1}{p})} (1+r)^{-\beta p'} r^{n-1} drdt\Big)^{\frac{p-1}{p}} \\
\lesssim& \Big(\int_0^M \int_{\mathbb{R}^n} |u+v|^p \theta_M^{2p'} b_a dxdt\Big)^{\frac{1}{p}} \begin{cases} 1, & p > \frac{n}{n-1}, \\ M^{\frac{3p-np+n-3}{2p}-\frac{1}{p^2}}, & p \le \frac{n}{n-1}, \end{cases}
\end{aligned}
$$

where we have used the fact $p = q = p_S(n) > n/(n-1)$ so that $n - p' > 0$. Therefore, by the estimates I_9-I_{11}, we get that

$$
\Big(\int_0^M \int_{\mathbb{R}^n} |v+u|^p \eta_M^{2p'}(t) b_a dxdt\Big)^p \lesssim (\ln M)^{p-1} \int_0^M \int_{\mathbb{R}^n} |u+v|^p \theta_M^{2p'} b_a dxdt \;. \tag{3.27}
$$

Let

$$
Y[b_a|u+v|^p](M) = \int_1^M \Big(\int_0^M \int_{\mathbb{R}^n} |u+v|^p b_a \theta_\sigma^{2q'}(t) dxdt\Big) \sigma^{-1} d\sigma.
$$

By combining (3.18), (3.19), (3.26) and (3.27), we get that

$$\begin{aligned}\varepsilon^p \lesssim & MY'[b_a|u+v|^p](M),\\ \Big(Y[b_a|u+v|^p](M)\Big)^p \lesssim & (\ln M)^{p-1} MY'[b_a|u+v|^p](M)\ .\end{aligned}$$

By exploiting Lemma 2.5 with $p_2 = p_1 = p$, $\delta = \varepsilon^p$, we have that

$$T_\varepsilon \ \lesssim\ \exp(\varepsilon^{-p(p-1)}), \quad p = q = p_S(n).$$

We finish the proof of Theorem 1.1.

3.2. **Proof of Theorem 1.2.**

3.2.1. $\Gamma_{GG}(n,p,q) > 0$.

(I) Test function 1: $\Psi = -\partial_t\big(\eta_M^{2p'}(t)\Phi(t,x)\big)$

We choose test function $\Psi = -\partial_t(\eta_M^{2p'}\Phi)$, where $\Phi(t,x) = e^{-t}\phi(x)$ and ϕ satisfying $\Delta\phi = (1 + D(x) + V(x))\phi$. Besides, it is worth observing that

$$\Psi(t,x) = -\partial_t(\eta_M^{2p'}\Phi) = \eta_M^{2p'}\Phi - 2p'\eta_M^{2p'-1}(\partial_t\eta_M)\Phi \geq \eta_M^{2p'}\Phi > 0\ .$$

Thus, by applying $\Psi = -\partial_t(\eta_M^{2p'}\Phi)$ to (1.5), we have

$$\begin{aligned}
&\varepsilon C_2(u_0,u_1) + \int_0^M\int_{\mathbb{R}^n} |v_t|^p(\eta_M^{2p'}\Phi - 2p'\eta_M^{2p'-1}(\partial_t\eta_M)\Phi)dxdt\\
&\quad= \int_0^M\int_{\mathbb{R}^n} u_t\Big((\partial_t^2\eta_M^{2p'})\Phi + 2(\partial_t\eta_M^{2p'})(\partial_t\Phi) - D(x)(\partial_t\eta_M^{2p'})\Phi\Big)dxdt\\
&\quad\quad+ \int_0^M\int_{\mathbb{R}^n} u_t\eta_M^{2p'}\Big(\partial_t^2\Phi - \Delta\Phi - D(x)\partial_t\Phi + V(x)\Phi\Big)dxdt\\
&\quad= \int_0^M\int_{\mathbb{R}^n} u_t\Big((\partial_t^2\eta_M^{2p'})\Phi + 2(\partial_t\eta_M^{2p'})(\partial_t\Phi) - D(x)(\partial_t\eta_M^{2p'})\Phi\Big)dxdt\\
&\quad\leq \int_0^M\int_{\mathbb{R}^n} |u_t|\Big(|\partial_t^2\eta_M^{2p'}|\Phi + 2|\partial_t\eta_M^{2p'}|\Phi + D(x)|\partial_t\eta_M^{2p'}|\Phi\Big)dxdt\\
&\quad= I_{12} + I_{13} + I_{14}\ .
\end{aligned} \tag{3.28}$$

When $p \leq q$, by Hölder's inequality and Lemma 2.2, we have

$$
\begin{aligned}
I_{12} =& \int_{\mathbb{R}^n} |u_t||\partial_t^2 \eta_M^{2p'}|\Phi dxdt \\
\lesssim& M^{-2}\Big(\int_0^M \int_{\mathbb{R}^n} \theta_M^{2p'}|u_t|^q \Phi dxdt\Big)^{\frac{1}{q}}\Big(\int_{\frac{M}{2}}^M \int_{B_{t+R}} \Phi dxdt\Big)^{\frac{1}{q'}} \\
\lesssim& M^{-2}\Big(\int_0^M \int_{\mathbb{R}^n} \theta_M^{2p'}|u_t|^q \Phi dxdt\Big)^{\frac{1}{q}}\Big(\int_{\frac{M}{2}}^M (t+1)^{\frac{n-1}{2}} dxdt\Big)^{\frac{1}{q'}} \\
\lesssim& M^{\frac{nq-n-3q-1}{2q}}\Big(\int_0^M \int_{\mathbb{R}^n} \theta_M^{2p'}|u_t|^q \Phi dxdt\Big)^{\frac{1}{q}}, \\
I_{13} =& \int_0^M \int_{\mathbb{R}^n} 2|u_t||\partial_t \eta_M^{2p'}|\Phi dxdt \\
\lesssim& M^{-1}\Big(\int_0^M \int_{\mathbb{R}^n} \theta_M^{2p'}|u_t|^q \Phi dxdt\Big)^{\frac{1}{q}}\Big(\int_{\frac{M}{2}}^M \int_{B_{t+R}} \Phi dxdt\Big)^{\frac{1}{q'}} \\
\lesssim& M^{\frac{nq-n-q-1}{2q}}\Big(\int_0^M \int_{\mathbb{R}^n} \theta_M^{2p'}|u_t|^q \Phi dxdt\Big)^{\frac{1}{q}}, \\
I_{14} =& \int_0^M \int_{\mathbb{R}^n} |u_t|D(x)|\partial_t \eta_M^{2p'}|\Phi dxdt \\
\lesssim& I_{13} \lesssim M^{\frac{nq-n-q-1}{2q}}\Big(\int_0^M \int_{\mathbb{R}^n} \theta_M^{2p'}|u_t|^q \Phi dxdt\Big)^{\frac{1}{q}}.
\end{aligned}
$$

Then we get that

(3.29)
$$
\varepsilon C_2(u_0,u_1) + \int_0^M \int_{\mathbb{R}^n} |v_t|^p \eta_M^{2p'} \Phi dxdt \lesssim M^{\frac{nq-n-1-q}{2q}}\Big(\int_0^M \int_{\mathbb{R}^n} |u_t|^q \theta_M^{2p'} \Phi dxdt\Big)^{\frac{1}{q}},
$$

(3.30)
$$
\varepsilon C_2'(v_0,v_1) + \int_0^M \int_{\mathbb{R}^n} |u_t|^q \eta_M^{2p'} \Phi dxdt \lesssim M^{\frac{np-n-1-p}{2p}}\Big(\int_0^M \int_{\mathbb{R}^n} |v_t|^p \theta_M^{2p'} \Phi dxdt\Big)^{\frac{1}{p}}.
$$

Hence we obtain

(3.31)
$$
\varepsilon C_2(u_0,u_1) + \int_0^M \int_{\mathbb{R}^n} |v_t|^p \eta_M^{2p'} \Phi dxdt \lesssim M^{\frac{nqp-2p-n-qp-1}{2qp}}\Big(\int_0^M \int_{\mathbb{R}^n} |v_t|^p \theta_M^{2p'} \Phi dxdt\Big)^{\frac{1}{qp}},
$$

(3.32)
$$
\varepsilon C_2'(v_0,v_1) + \int_0^M \int_{\mathbb{R}^n} |u_t|^q \eta_M^{2p'} \Phi dxdt \lesssim M^{\frac{nqp-2q-n-qp-1}{2qp}}\Big(\int_0^M \int_{\mathbb{R}^n} |u_t|^q \theta_M^{2p'} \Phi dxdt\Big)^{\frac{1}{qp}}.
$$

We set $A = \int_0^M \int_{\mathbb{R}^n} |v_t|^p \eta_M^{2p'} \Phi dxdt$, $B = \int_0^M \int_{\mathbb{R}^n} |u_t|^q \eta_M^{2p'} \Phi dxdt$. By Young's inequality, we have that for some constant $C > 0$,

$$
\varepsilon + A \lesssim M^{\frac{nqp-2p-n-qp-1}{2qp}} A^{\frac{1}{qp}} \leq \frac{A}{pq} + CM^{\frac{nqp-2p-n-qp-1}{2qp}(pq)'}\frac{1}{(pq)'} ,
$$

$$
\varepsilon + B \lesssim M^{\frac{nqp-2q-n-qp-1}{2qp}} B^{\frac{1}{qp}} \leq \frac{B}{pq} + CM^{\frac{nqp-2q-n-qp-1}{2qp}(pq)'}\frac{1}{(pq)'} .
$$

Then we get that

$$\varepsilon+\big(1-\frac{1}{pq}\big)A\lesssim M^{\frac{nqp-2p-n-qp-1}{2qp}(pq)'}\frac{1}{(pq)'}\ ,$$

$$\varepsilon+\big(1-\frac{1}{pq}\big)B\lesssim M^{\frac{nqp-2q-n-qp-1}{2qp}(pq)'}\frac{1}{(pq)'}\ ,$$

which lead to

$$M\le T_\varepsilon\lesssim\begin{cases}\varepsilon^{-\frac{1}{\frac{p+1}{qp-1}-\frac{n-1}{2}}}, & p\le q,\\ \varepsilon^{-\frac{1}{\frac{q+1}{qp-1}-\frac{n-1}{2}}}, & p\le q.\end{cases}\tag{3.33}$$

Similarly, by applying test function $\Psi=-\partial_t\big(\eta_M^{2q'}(t)\Phi(t,x)\big)$ with $q\le p$, we can also obtain (3.33). So we come to the estimate

$$T_\varepsilon\lesssim\varepsilon^{-\Gamma_{GG}^{-1}(n,p,q)},\quad \Gamma_{GG}(n,p,q)>0.$$

Which is first part of lifespan estimate in Theorem 1.2.

3.2.2. $\Gamma_{GG}(n,p,q)=0$.

When $p\le q$ and $\Gamma_{GG}(n,p,q)=\frac{q+1}{pq-1}-\frac{n-1}{2}=0$, by (3.32) and (3.18), (3.19), we have

$$\Big(C_2'(v_0,v_1)\varepsilon+Y[\Phi|u_t|^q](M)\Big)^{qp}\lesssim MY'[\Phi|u_t|^q](M)\ .\tag{3.34}$$

Let $Z(M)=\big(C_2'(v_0,v_1)\varepsilon+Y[\Phi|u_t|^q](M)\big)$, then $Z(1)=\big(C_2'(v_0,v_1)\big)\varepsilon$ and $Z'(M)=Y'(M)$, then we have

$$MZ'(M)\gtrsim Z^{pq}\ ,Z(1)=(C_2'(v_0,v_1))\varepsilon\ .$$

This is a typical ODE inequality which will blow up in finite time. By direct computation, we have that

$$M\le T_\varepsilon\lesssim\exp(\varepsilon^{-(qp-1)})\ ,p\le q\ .$$

Similarly, when $q\le p$ and $\Gamma_{GG}(n,p,q)=\frac{p+1}{pq-1}-\frac{n-1}{2}=0$, we can get the same estimate. This is the second lifespan estimate in Theorem 1.2.

When $p=q$ and $\Gamma_{GG}(n,p,q)=0$, which is equivalent to $p=q=p_G(n)$, we set

$$Y[\Phi|v_t+u_t|^p](M)=\int_1^M\Big(\int_0^T\int_{\mathbb{R}^n}|v_t+u_t|^p\Phi\theta_\sigma^{2p'}(t)dxdt\Big)\sigma^{-1}d\sigma.\tag{3.35}$$

By (3.29) and (3.30), we can get another ODE inequality

$$\Big(C_2(u_0,u_1)\varepsilon+C_2'(v_0,v_1)\varepsilon+Y[\Phi|v_t+u_t|^p]\Big)^p\lesssim MY'[\Phi|v_t+u_t|^p](M)\ ,\tag{3.36}$$

which lead to

$$M\le T_\varepsilon\lesssim\exp(\varepsilon^{-(p-1)})\ ,p=q=p_G(n)\ .$$

3.3. **Proof of Theorem 1.3.**

3.3.1. $\Gamma_{SG}(n,p,q)>0$.

(I)Test function 1: $\Psi(t,x)=\eta_M^{2p'}(t)\phi_0(x)$

Firstly, we choose test function $\Psi=\eta_M^{2p'}(t)\phi_0(x)$, where ϕ_0 satisfies $\Delta\phi_0=V\phi_0$. By replacing $\Psi(t,x)$ in (1.5) and (1.6) with $\eta_M^{2p'}(t)\phi_0(x)$, we get that

$$\begin{aligned}\varepsilon\int u_1(x)\phi_0(x)dx+\int_0^M\int_{\mathbb{R}^n}|v|^q\eta_M^{2p'}\phi_0dxdt\\=\int_0^M\int_{\mathbb{R}^n}u_t\phi_0\Big(-\partial_t\eta_M^{2p'}+D(x)\eta_M^{2p'}\Big)dxdt\\=I_{15}+I_{16},\end{aligned}\tag{3.37}$$

$$\begin{aligned}\varepsilon\int_{\mathbb{R}^n}v_1(x)\phi_0dx+\varepsilon\int_{\mathbb{R}^n}v_0(x)D(x)\phi_0dx+\int_0^M\int_{\mathbb{R}^n}|u_t|^p\eta_M^{2p'}\phi_0dxdt\\=\int_0^M\int_{\mathbb{R}^n}v\phi_0\Big(\partial_t^2\eta_M^{2p'}-D(x)\partial_t\eta_M^{2p'}\Big)dxdt\\=I_{17}+I_{18}\ .\end{aligned}\tag{3.38}$$

Recall that $\phi_0\simeq 1$, then by Hölder's inequality, we have

$$\begin{aligned}|I_{15}|=&|\int_0^M\int_{\mathbb{R}^n}u_t\phi_0\big(\partial_t\eta_M^{2p'}\big)dxdt|\\\lesssim&M^{-1}\Big(\int_0^M\int_{\mathbb{R}^n}|u_t|^p\theta_M^{(2p'-1)p}dxdt\Big)^{\frac1p}\Big(\int_{\frac M2}^M\int_{B_{t+R}}dxdt\Big)^{\frac1{p'}},\\\lesssim&M^{\frac{np-n-1}{p}}\Big(\int_0^M\int_{\mathbb{R}^n}|u_t|^p\theta_M^{2p'}dxdt\Big)^{\frac1p},\end{aligned}\tag{3.39}$$

$$\begin{aligned}|I_{16}|=&|\int_0^M\int_{\mathbb{R}^n}u_t\phi_0D(x)\eta_M^{2p'}dxdt|\\\lesssim&\Big(\int_0^M\int_{\mathbb{R}^n}|u_t|^p\eta_M^{(2p'-1)p}dxdt\Big)^{\frac1p}\Big(\int_0^M\int_{B_{t+R}}(D(x))^{p'}dxdt\Big)^{\frac1{p'}}\\\lesssim&\Big(\int_0^M\int_{\mathbb{R}^n}|u_t|^p\eta_M^{2p'}dxdt\Big)^{\frac1p}\Big(\int_0^M\int_0^{t+R}(1+r)^{-\beta p'}r^{n-1}drdt\Big)^{\frac1{p'}}\\\lesssim&\Big(\int_0^M\int_{\mathbb{R}^n}|u_t|^p\eta_M^{2p'}dxdt\Big)^{\frac1p}\begin{cases}(M\ln(M))^{\frac1{p'}}, & n-p'=0,\\M^{(n+1-p')\frac1{p'}}, & n-p'>0,\\M^{\frac1{p'}}, & n-p'<0.\end{cases}\\\lesssim&\Big(\int_0^M\int_{\mathbb{R}^n}|u_t|^p\eta_M^{2p'}dxdt\Big)^{\frac1q}\begin{cases}(M\ln(M))^{\frac1{p'}}, & p\le\frac{n}{n-1},\\M^{\frac{np-n-1}{p}}, & p>\frac{n}{n-1}\ .\end{cases}\end{aligned}\tag{3.40}$$

Thus by (3.39) (3.40), when $p>n/(n-1)$, we have

$$\int_0^M\int_{\mathbb{R}^n}|v|^q\eta_M^{2p'}dxdt\lesssim M^{\frac{np-n-1}{p}}\Big(\int_0^M\int_{\mathbb{R}^n}|u_t|^p\eta_M^{2p'}dxdt\Big)^{\frac1p}\ .\tag{3.41}$$

The estimates of I_{17} and I_{18} agree with the estimates of I_1 and I_2, so we choose to omit the details here. Same as (3.6), we have

$$\int_0^M \int_{\mathbb{R}^n} |u_t|^p \eta_M^{2p'} dxdt \lesssim M^{\frac{nq-n-q-1}{q}} \Big(\int_0^M \int_{\mathbb{R}^n} |v|^q \theta_M^{2p'} dxdt \Big)^{\frac{1}{q}} , \tag{3.42}$$

when $q > n/(n-1)$ and $p \le q$. Then by (3.41) and (3.42), we obtain

$$\int_0^M \int_{\mathbb{R}^n} |v|^q \eta_M^{2p'} dxdt \lesssim M^{\frac{npq-n-2q-1}{pq-1}}, \ p \le q, \ p,q > n/(n-1) . \tag{3.43}$$

$$\int_0^M \int_{\mathbb{R}^n} |u_t|^p \eta_M^{2p'} dxdt \lesssim M^{\frac{npq-n-pq-p-1}{pq-1}} \ , p \le q, \ p,q > n/(n-1) . \tag{3.44}$$

(II) Test function 2: $\Psi(t,x) = -\partial_t(\eta_M^{2p'}(x)\Phi(t,x)), \eta_M^{2p'}\Phi(t,x)$

Secondly, we choose another test function $\Psi = -\partial_t(\eta_M^{2p'}\Phi)$, where $\Phi = e^{-t}\phi(x)$ and $\phi(x)$ satisfies $\Delta\phi = (1+D+V)\phi$. By replacing $\Psi(t,x)$ in (1.5) with $-\partial_t(\eta_M^{2p'}\Phi)$, we get that

$$\begin{aligned} &\varepsilon C_2(u_0,u_1) + \int_0^M \int_{\mathbb{R}^n} |v|^q \partial_t(-\eta_M^{2p'}\Phi) dxdt \\ &= \int_0^M \int_{\mathbb{R}^n} u_t \Big((\partial_t^2 \eta_M^{2p'})\Phi + 2(\partial_t \eta_M^{2p'})(\partial_t \Phi) - D(x)(\partial_t \eta_M^{2p'})\Phi \Big) dxdt \ . \end{aligned}$$

By the estimates of I_3, I_4, I_5, we can get that

$$\Big(\varepsilon C_2(u_0,u_1)\Big)^p \lesssim M^{\frac{np-p-2n}{2}} \int_0^M \int_{\mathbb{R}^n} |u_t|^p \theta_M^{2p'} dxdt \ . \tag{3.45}$$

By replacing $\Psi(t,x)$ in (1.6) with $\eta_M^{2p'}\Phi(t,x)$, we get that

$$\begin{aligned} &\varepsilon C_2'(v_0,v_1) + \int_0^M \int_{\mathbb{R}^n} |u_t|^p \eta_M^{2p'} \Phi dxdt \\ &= \int_0^M \int_{\mathbb{R}^n} v \Big((\partial_t^2 \eta_M^{2p'})\Phi + 2(\partial_t \eta_M^{2p'})(\partial_t \Phi) - D(x)(\partial_t \eta_M^{2p'})\Phi \Big) dxdt \ . \end{aligned}$$

Similarly, we have

$$\Big(\varepsilon C_2'(v_0,v_1)\Big)^q \lesssim M^{\frac{nq-q-2n}{2}} \int_0^M \int_{\mathbb{R}^n} |v|^q \theta_M^{2p'} dxdt \ . \tag{3.46}$$

By combining (3.44) and (3.45), we obtain that

$$T_\varepsilon \lesssim \varepsilon^{-F_{SG,1}^{-1}(n,p,q)} \ , \ p,q > n/(n-1) \ , F_{SG,1}(n,p,q) > 0 \ .$$

By combining (3.43) and (3.46), we obtain that

$$T_\varepsilon \lesssim \varepsilon^{-F_{SG,2}^{-1}(n,p,q)} \ , \ p,q > n/(n-1) \ , F_{SG,2}(n,p,q) > 0 \ .$$

Recall that $\Gamma_{SG}(n,p,q) = \max\{F_{SG,1}(n,p,q), F_{SG,2}(n,p,q)\}$, so we have

$$T_\varepsilon \lesssim \varepsilon^{-\Gamma_{SG}^{-1}(n,p,q)} \ , \ p,q > n/(n-1) \ , \Gamma_{SG}(n,p,q) > 0 \ .$$

We obtain the fourth lifespan estimate in Theorem 1.3.

In addition, when p or q less than $n/(n-1)$, we apply Young's inequality to (3.37) and (3.38) with $p \le q$ to get

$$\begin{aligned}&\int_0^M \int_{\mathbb{R}^n} |v|^q \eta_M^{2p'} dxdt \\ &\le \frac{1}{2}\int_0^M \int_{\mathbb{R}^n} |u_t|^p \eta_M^{2p'} dxdt + C\int_0^M \int_{B_{t+R}} (M^{-1} + |D(x)|)^{p'} dxdt,\end{aligned}$$

and

$$\begin{aligned}&\int_0^M \int_{\mathbb{R}^n} |u_t|^p \eta_M^{2p'} dxdt \\ &\le \frac{1}{2}\int_0^M \int_{\mathbb{R}^n} |v|^q \eta_M^{2p'} dxdt + C\int_0^M \int_{B_{t+R}} (M^{-2} + |D(x)|M^{-1})^{q'} dxdt.\end{aligned}$$

Then we have

$$\begin{aligned}&\int_0^M \int_{\mathbb{R}^n} |v|^q \eta_M^{2p'} dxdt + \int_0^M \int_{\mathbb{R}^n} |u_t|^p \eta_M^{2p'} dxdt \\ &\lesssim \int_{\frac{M}{2}}^M \int_{B_{t+R}} (M^{-2} + |D(x)|M^{-1})^{q'} dxdt + \int_0^M \int_{B_{t+R}} (M^{-1} + |D(x)|)^{p'} dxdt\ .\end{aligned} \tag{3.47}$$

Note that

$$\int_{\frac{M}{2}}^M \int_{B_{t+R}} (M^{-2} + |D(x)|M^{-1})^{q'} dxdt \lesssim \begin{cases} M^{1-q'}, & 1 < q \le \frac{n}{n-1}\ , \\ M^{n+1-2q'}, & q \ge \frac{n}{n-1}\ . \end{cases} \tag{3.48}$$

$$\int_0^M \int_{B_{t+R}} (M^{-1} + |D(x)|)^{p'} dxdt \lesssim \begin{cases} M, & 1 < p \le \frac{n}{n-1}\ , \\ M^{n+1-p'}, & p \ge \frac{n}{n-1}\ . \end{cases} \tag{3.49}$$

When $1 < p \le q \le \frac{n}{n-1}$, by (3.46) (3.45) (3.47) (3.48) (3.49), we have

$$T_\varepsilon \lesssim \varepsilon^{-\frac{2q}{2n+q-nq-2}},\ T_\varepsilon \lesssim \varepsilon^{-\frac{2p}{2n+p-np-2}}.$$

Similarly, when $1 < q \le\ p \le \frac{n}{n-1}$, we could get that

$$T_\varepsilon \lesssim \varepsilon^{-\frac{2p}{2n+p-np-2}},\ T_\varepsilon \lesssim \varepsilon^{-\frac{2q}{2n+q-nq-2}}.$$

Thus we have

$$T_\varepsilon \lesssim \min\{\varepsilon^{-\frac{2p}{2n+p-np-2}}, \varepsilon^{-\frac{2q}{2n+q-nq-2}}\}\ , 1 < p, q \le n/(n-1)\ .$$

When $1 < p \le \frac{n}{n-1}$, $\frac{n}{n-2} < q < p_S(n)$, it is easy to see $n+1-2q' > 1$, by combining (3.46) (3.47) (3.48) (3.49), we get that

$$T_\varepsilon \lesssim \varepsilon^{-\frac{2q(q-1)}{(n+1)q+2-(n-1)q^2}}\ .$$

When $1 < p \le \frac{n}{n-1}$, $\frac{n}{n-1} \le q \le \frac{n}{n-2}$, we can get

$$T_\varepsilon \lesssim \varepsilon^{-\frac{2p}{2n+p-np-2}}.$$

When $1 < q \le \frac{n}{n-1} \le p \le 1 + \frac{2}{(n-1)q}$, by combining (3.46) (3.47) (3.48) (3.49), we get that

$$T_\varepsilon \lesssim \varepsilon^{-\frac{2q(p-1)}{(n-1)q+2-(n-1)pq}}\ .$$

3.3.2. $\Gamma_{SG}(n,p,q)=0$.

(III)Test function 3: $\Psi(t,x)=-\partial_t(\eta_M^{2p'}(t)b_{a-1}(t,x))$

In this case, we choose test function $\Psi=-\partial_t(\eta_M^{2p'}b_{a-1})$, where b_a is in Lemma 2.3 and $a=\frac{n+1}{2}-\frac{1}{p}$. Noting that

$$\Psi(t,x)=-\partial_t(\eta_M^{2p'}b_{a-1})=\eta_M^{2p'}b_a-\partial_t\eta_M^{2p'}b_{a-1}\geq\eta_M^{2p'}b_a>0\ .$$

By replacing $\Psi(t,x)$ in (1.5) with $-\partial_t(\eta_M^{2p'}b_{a-1})$, we have

$$\begin{aligned}
&\int_0^M\int_{\mathbb{R}^n}|v|^q\eta_M^{2p'}b_a dxdt\\
\leq&\int_0^M\int_{\mathbb{R}^n}u_t\Big((\partial_t^2\eta_M^{2p'})b_{a-1}+2(\partial_t\eta_M^{2p'})(\partial_t b_{a-1})-D(x)(\partial_t\eta_M^{2p'})b_{a-1}\Big)dxdt\\
=&I_{19}+I_{20}+I_{21}\ .
\end{aligned}$$

Note that $a-1=\frac{n-1}{2}-\frac{1}{p}<\frac{n-1}{2}$, by Lemma 2.3, we know that $b_{a-1}\simeq(t+R)^{-a+1}$ and $b_a\gtrsim(t+R)^{-a}$. So we have $b_{a-1}(t,x)\lesssim(t+R)b_a(t,x)$. By Hölder's inequality and Lemma 2.3, we have

$$\begin{aligned}
|I_{19}|=&|\int_0^M\int_{\mathbb{R}^n}u_t\partial_t^2(\eta_M^{2p'})b_{a-1}dxdt|\\
\lesssim&M^{-2}\Big(\int_0^M\int_{\mathbb{R}^n}|u_t|^p\theta_M^{2p'}dxdt\Big)^{\frac{1}{p}}\\
&\times\Big(\int_{\frac{M}{2}}^M\int_0^{t+R}(t+R)^{1-\frac{n-1}{2}p'}(t+R+1-r)^{-1}r^{n-1}drdt\Big)^{\frac{1}{p'}}\\
\lesssim&M^{-2}\Big(\int_0^M\int_{\mathbb{R}^n}|u_t|^p\theta_M^{2p'}dxdt\Big)^{\frac{1}{p}}\Big(\int_{\frac{M}{2}}^M(t+R+1)^{n-\frac{n-1}{2}p'}\ln(t+R)dt\Big)^{\frac{1}{p'}}\\
\lesssim&M^{\frac{np-2n-p-2}{2p}}(\ln M)^{\frac{p-1}{p}}\Big(\int_0^M\int_{\mathbb{R}^n}|u_t|^p\theta_M^{2p'}dxdt\Big)^{\frac{1}{p}},
\end{aligned}$$

$$\begin{aligned}
|I_{20}|=&|\int_0^M\int_{\mathbb{R}^n}u_t2(\partial_t\eta_M^{2p'})(\partial_t b_{a-1})dxdt|\\
\lesssim&M^{-1}\Big(\int_0^M\int_{\mathbb{R}^n}|u_t|^p\theta_M^{2p'}dxdt\Big)^{\frac{1}{p}}\\
&\times\Big(\int_{\frac{M}{2}}^M\int_0^{t+R}(t+R)^{-\frac{n-1}{2}p'}r^{n-1}(t+R+1-r)^{-1}drdt\Big)^{\frac{1}{p'}}\\
\lesssim&M^{-1}\Big(\int_0^M\int_{\mathbb{R}^n}|u_t|^p\theta_M^{2p'}dxdt\Big)^{\frac{1}{p}}\Big(\int_{\frac{M}{2}}^M(t+R)^{n-1-\frac{n-1}{2}p'}\ln(t+R)\Big)^{\frac{1}{p'}}\\
\lesssim&M^{\frac{np-2n-p}{2p}}(\ln M)^{\frac{p-1}{p}}\Big(\int_0^M\int_{\mathbb{R}^n}|u_t|^p\theta_M^{2p'}dxdt\Big)^{\frac{1}{p}},
\end{aligned}$$

$$
\begin{aligned}
|I_{21}| =& \Big|\int_0^M \int_{\mathbb{R}^n} u_t D(x)(\partial_t \eta_M^{2p'}) b_{a-1} dxdt\Big| \\
\lesssim& M^{-1}\Big(\int_0^M \int_{\mathbb{R}^n} |u_t|^p \theta_M^{2p'} dxdt\Big)^{\frac{1}{p}} \\
&\times\Big(\int_{\frac{M}{2}}^M \int_0^{t+R} (t+R)^{1-\frac{n-1}{2}p'}(t+R+1-r)^{-1}(1+r)^{n-1-p'} drdt\Big)^{\frac{1}{p'}} \\
\lesssim& M^{\frac{np-2n-p}{2p}} (\ln M)^{\frac{p-1}{p}} \Big(\int_0^M \int_{\mathbb{R}^n} |u_t|^p \theta_M^{2p'} dxdt\Big)^{\frac{1}{p}} ,
\end{aligned}
$$

where we used the fact that $p' > 1$ and

$$
\int_0^{t+R} (t+R+1-r)^{-1}(1+r)^{n-1-p'} dr \lesssim (t+R)^{n-2} \ln(t+R) .
$$

By the estimates I_{19}-I_{21}, we have

$$
\int_0^M \int_{\mathbb{R}^n} |v|^q \eta_M^{2p'} b_a dxdt \lesssim M^{\frac{np-2n-p}{2p}} (\ln M)^{\frac{p-1}{p}} \Big(\int_0^M \int_{\mathbb{R}^n} |u_t|^p \theta_M^{2p'} dxdt\Big)^{\frac{1}{p}} . \tag{3.50}
$$

When $F_{SG,1}(n,p,q) = 0$, we have

$$
\Big(n - \frac{n-1}{2}p - (\frac{n-1}{2} - \frac{1}{q})\Big)q + \Big(n - \frac{n-1}{2}q - (\frac{n-1}{2} - \frac{1}{p}) - 1\Big) = -F_{SG,1}(n,p,q) = 0.
$$

Taking (3.42) into (3.50) with $F_{SG,1}(n,p,q) = 0$, we have

$$
\Big(\int_0^M \int_{\mathbb{R}^n} |v|^q \eta_M^{2p'} b_a dxdt\Big)^{pq} \lesssim (\ln M)^{q(p-1)} \Big(\int_0^M \int_{\mathbb{R}^n} |v|^q \theta_M^{2p'} b_a dxdt\Big), \tag{3.51}
$$

with $p \le q$ and $q > n/(n-1)$. By combining (3.42), (3.45) and the lower bound of b_a in Lemma 2.3, we acquire that

$$
\varepsilon^{pq} \lesssim \int_0^M \int_{\mathbb{R}^n} |v|^q \theta_M^{2p'} b_a dxdt . \tag{3.52}
$$

From (3.18), (3.19), (3.51) and (3.52), we can get that

$$
\begin{aligned}
\varepsilon^{pq} \lesssim& M Y'[b_a|v|^q](M), \\
\Big(Y[b_a|v|^q](M)\Big)^{pq} \lesssim& (\ln M)^{q(p-1)} M Y'[b_a|v|^q](M).
\end{aligned}
$$

By applying Lemma 2.5 with $p_2 = q(p-1)+1$, $p_1 = pq$ and $\delta = \varepsilon^{pq}$, we can obtain

$$
T_\varepsilon \lesssim \exp(\varepsilon^{-p(pq-1)}) \ , n/(n-1) < p, q \ , F_{SG,1}(n,p,q) = 0 > F_{SG,2}(n,p,q) .
$$

(IV)Test function 4: $\Psi(t,x) = \eta_M^{2q'}(t) b_a(t,x)$

We choose test function $\Psi = \eta_M^{2q'} b_a$, where b_a is defined in Lemma 2.3 and $a = \frac{n-1}{2} - \frac{1}{q}$. By replacing $\Psi(t,x)$ in (1.6) with $\eta_M^{2q'} b_a$, we get that

$$
\begin{aligned}
&\int_0^M \int_{\mathbb{R}^n} |u_t|^p \eta_M^{2q'} b_a dxdt \\
\le& \int_0^M \int_{\mathbb{R}^n} v\Big((\partial_t^2 \eta_M^{2q'}) b_a + 2(\partial_t \eta_M^{2q'})(\partial_t b_a) - D(x)(\partial_t \eta_M^{2q'} b_a)\Big) dxdt .
\end{aligned}
$$

Note that the test function $\eta_M^{2q'} b_a$ is same as in (3.21). Then by the estimates of I_6, I_7, I_8, we can get that

$$\int_0^M \int_{\mathbb{R}^n} |u_t|^p \eta_M^{2q'} b_a dxdt \lesssim M^{\frac{nq-2n-q}{2q}} (\ln M)^{\frac{q-1}{q}} \Big(\int_0^M \int_{\mathbb{R}^n} |v|^q \theta_M^{2q'} dxdt \Big)^{\frac{1}{q}}, \tag{3.53}$$

with $q \le p$.

For the case $F_{SG,2}(n,p,q) = 0$, we observe that this condition yields

$$\Big(n - \frac{n-1}{2}p - (\frac{n-1}{2} - \frac{1}{q})\Big) + \Big(n - \frac{n-1}{2}q - (\frac{n-1}{2} - \frac{1}{p}) - 1\Big)p = -F_{SG,2}(n,p,q) = 0.$$

Notice that, when $D(x) = 0$, we do not have I_{16} term anymore, by the estimate of I_{15}, we have

$$\int_0^M \int_{\mathbb{R}^n} |v|^q \eta_M^{2q'} dxdt \lesssim M^{\frac{np-n-1}{p}} \Big(\int_0^M \int_{\mathbb{R}^n} |u_t|^p \theta_M^{2q'} dxdt \Big)^{\frac{1}{p}} . \tag{3.54}$$

By taking (3.54) into (3.53) with $F_{SG,2}(n,p,q) = 0$, we have

$$\Big(\int_0^M \int_{\mathbb{R}^n} |u_t|^p \eta_M^{2q'} b_a dxdt \Big)^{pq} \lesssim (\ln M)^{p(q-1)} \Big(\int_0^M \int_{\mathbb{R}^n} |u_t|^p \theta_M^{2q'} b_a dxdt \Big). \tag{3.55}$$

By combining (3.54), (3.46) and the lower bound of b_a in Lemma 2.3, we acquire

$$\varepsilon^{pq} \lesssim \int_0^M \int_{\mathbb{R}^n} |u_t|^p \theta_M^{2q'} b_a dxdt. \tag{3.56}$$

From (3.18), (3.19), (3.55) and (3.56), we can get that

$$\begin{aligned} \varepsilon^{pq} \lesssim& M Y'[b_a |u_t|^p](M), \\ \Big(Y[b_a |u_t|^p](M) \Big)^{pq} \lesssim& (\ln M)^{p(q-1)} M Y'[b_a |u_t|^p](M). \end{aligned}$$

By applying Lemma 2.5 with $p_2 = p(q-1)+1$, $p_1 = pq$, $\delta = \varepsilon^{pq}$, we get that

$$T_\varepsilon \lesssim \exp(\varepsilon^{-q(pq-1)}), \quad F_{SG,2}(n,p,q) = 0 > F_{SG,1}(n,p,q) \ .$$

under the condition $D(x) = 0$.

When $F_{SG,2}(n,p,q) = 0 = F_{SG,1}(n,p,q)$, we follow the same way in [8], Section 9. In this case we have

$$n - \frac{n-1}{2}p = \frac{n-1}{2} - \frac{1}{q}, \quad n - \frac{n-1}{2}q = \frac{n+1}{2} - \frac{1}{p} \ .$$

By combining (3.45) and the lower bound of b_a in Lemma 2.3, we have that

$$\varepsilon^p \lesssim \int_0^M \int_{\mathbb{R}^n} |u_t|^p b_a \theta_M^{2q'} dxdt. \tag{3.57}$$

From (3.18), (3.19), (3.55) and (3.57), we can get that

$$\begin{aligned} \varepsilon^{p} \lesssim& M Y'[b_a |u_t|^p](M), \\ \Big(Y[b_a |u_t|^p](M) \Big)^{pq} \lesssim& (\ln M)^{p(q-1)} M Y'[b_a |u_t|^p](M). \end{aligned}$$

By applying Lemma 2.5 with $p_2 = p(q-1)+1$, $p_1 = pq$, $\delta = \varepsilon^p$, which lead to

$$T_\varepsilon \lesssim \exp(\varepsilon^{-(pq-1)}) \ , F_{SG,2}(n,p,q) = 0 = F_{SG,1}(n,p,q) \ .$$

We complete the proof of Theorem 1.3.

References

[1] R. Agemi, Y. Kurokawa, H. Takamura: *Critical curve for p-q systems of nonlinear wave equations in three space dimensions*, J. Differ. Equ. 167(1) (2000), 87–133.

[2] Y. Bai, M. Liu: *Global existence for semilinear damped wave equations in the scattering case*, Differ. Integral Equ. 32(3-4) (2019), 233–248.

[3] K. Deng: *Blow-up of solutions of some nonlinear hyperbolic systems*, Rocky Mt. J. Math. 29 (1999), 807–820.

[4] V. Georgiev, H. Takamura, Y. Zhou: *The lifespan of solutions to nonlinear systems of a high-dimensional wave equation*, Nonlinear Anal. 64(10) (2006), 2215–2250.

[5] R. T. Glassey: *Mathematical Reviews to "Global behavior of solutions to nonlinear wave equations in three space dimensions" of Sideris*, Comm. Partial Differ. Equ. (1983).

[6] K. Hidano, K. Yokoyama: *Life span of small solutions to a system of wave equations*, Nonlinear Anal. 139 (2016), 106–130.

[7] M. Ikeda, M. Sobajima: *Sharp upper bound for lifespan of solutions to some critical semilinear parabolic, dispersive and hyperbolic equations via a test function method*, Nonlinear Anal. 182 (2019), 57–74.

[8] M. Ikeda, M. Sobojima, K. Wakasa: *Blow-up phenomena of semilinear wave equations and their weakly coupled systems*, J. Differ. Equ. 267(9) (2019), 5165–5201.

[9] R. Ikehata, G. Todorova, B. Yordanov: *Critical exponent for semilinear wave equations with space-dependent potential*, Funkcial. Ekvac. 52(3) (2009), 411–435.

[10] T. Imai, M. Kato, H. Takamura, K. Wakasa: *The sharp lower bound of the lifespan of solutions to semilinear wave equations with low powers in two space dimensions*, Adv. Stud. Pure Math. 81 (2019), 31–53.

[11] F. John: *Blow-up of solutions of nonlinear wave equations in three space dimensions*, Manuscripta Math. 28(1-3) (1979), 235–268.

[12] H. Kubo, M. Ohta: *Critical blowup for systems of semilinear wave equations in low space dimensions*, J. Math. Anal. Appl. 240 (1999), 340–360.

[13] H. Kubo, K. Kubota, H. Sunagawa: *Large time behavior of solutions to semi-linear systems of wave equations*, Math. Ann. 335 (2006), 435–478.

[14] Y. Kurokawa: *The lifespan of radially symmetric solutions to nonlinear systems of odd dimensional wave equations*, Nonlinear Anal. 60(7) (2005), 1239–1275.

[15] Y. Kurokawa, H. Takamura: *A weighted pointwise estimate for two dimensional wave equations and its applications to nonlinear systems*, Tsukuba J. Math. 27(2) (2003), 417–448.

[16] Y. Kurokawa, H. Takamura, K. Wakasa: *The blow-up and lifespan of solutions to systems of semilinear wave equation with critical exponents in high dimensions*, Differ. Integral Equ. 25(3/4) (2012), 363–382.

[17] R. Kido, T. Sasaki, S. Takamatsu, H. Takamura: *The generalized combined effect for one dimensional wave equations with semilinear terms including product type*, J. Differ. Equ. 403 (2024), 576–618.

[18] N. Lai, M. Liu, Z. Tu, C. Wang: *Lifespan estimates for semilinear wave equations with space dependent damping and potential*, Calc. Var. Partial Differ. Equ. 62 (2023), Paper No. 44.

[19] N. Lai, H. Takamura: *Nonexistence of global solutions of nonlinear wave equations with weak time-dependent damping related to Glassey's conjecture*, Differ. Integral Equ. 32(1-2) (2019), 37–48.

[20] N. Lai, H. Takamura: *Nonexistence of global solutions of wave equations with weak time-dependent damping and combined nonlinearity*, Nonlinear Anal. Real World Appl. 45 (2019), 83–96.

[21] N. Lai, Z. Tu: *Strauss exponent for semilinear wave equations with scattering space dependent damping*, J. Math. Anal. Appl. 489(2) (2020).

[22] N. Lai, H. Takamura: *Blow-up for semilinear damped wave equations with subcritical exponent in the scattering case*, Nonlinear Anal. 168 (2018), 222–237.

[23] X. Li: *Critical exponent for semilinear wave equation with critical potential*, Nonlinear Differ. Equ. Appl. 20 (2013), 1379–1391.

[24] T. Li, Y. Zhou: *Life-span of classical solutions to nonlinear wave equations in two-space-dimensions. II*, J. Partial Differ. Equ. 6(1) (1993), 17–38.

[25] D. Li, X. Yu, Y. Zhou: *Life-span of classical solutions to one-dimensional nonlinear wave equations*, Chinese Ann. Math. Ser. B. 13(3) (1992), 266–279.

[26] M. Liu, C. Wang: *The blow up of solutions to semilinear wave equations on asymptotically Euclidean manifolds*, Discrete Contin. Dyn. Syst. 43(11) (2023), 3987–4009.

[27] M. Liu, C. Wang: *Blow up for small-amplitude semilinear wave equations with mixed nonlinearities on asymptotically Euclidean manifolds*, J. Differ. Equ. 269(10) (2020), 8573–8596.

[28] M. Liu, C. Wang: *Global existence for semilinear damped wave equations in relation with the Strauss conjecture*, Discrete Contin. Dyn. Syst. 40(2) (2020), 709–724.

[29] H. Lindblad: *Blow-up for solutions of $\Box u = |u|^p$ with small initial data*, Comm. Partial Differ. Equ. 15(6) (1990), 757–821.

[30] S. Ming, S. Lai, X. Fan: *Blow-up for a coupled system of semilinear wave equations with scattering dampings and combined nonlinearities*, Appl. Anal. 101(8) (2022), 2996–3016.

[31] J. Metcalfe, C. Wang: *The Strauss conjecture on asymptotically flat space-time*, SIAM J. Math. Anal. 49 (2017), 4579–4594.

[32] K. Nishihara: *Decay properties for the damped wave equation with space dependent potential and absorbed semilinear term*, Comm. Partial Differ. Equ. 35(8) (2010), 1402–1418.

[33] K. Nishihara, M. Sobajima, Y. Wakasugi: *Critical exponent for the semilinear wave equations with a damping increasing in the far field*, Nonlinear Differ. Equ. Appl. 25(6) (2018), 32–55.

[34] A. Palmieri, H. Takamura: *Blow-up for a weakly coupled system of semilinear damped wave equations in the scattering case with power nonlinearities*, Nonlinear Anal. 187 (2019), 467–492.

[35] A. Palmieri, H. Takamura: *Nonexistence of global solutions for a weakly coupled system of semilinear damped wave equations of derivative type in the scattering case*, Mediterr. J. Math. 17(1) (2020), 13–20.

[36] A. Palmieri, H. Takamura: *Nonexistence of global solutions for a weakly coupled system of semilinear damped wave equations in the scattering case with mixed nonlinear terms*, NoDEA Nonlinear Differential Equations Appl. 27(6) (2020), 39–58.

[37] D. Del Santo: *Global existence and blow-up for a hyperbolic system in three space dimensions*, Rend. Istit. Mat. Univ. Trieste. 29(1-2) (1997), 115–140.

[38] D. Del Santo, V. Georgiev, E. Mitidieri: *Global existence of the solutions and formation of singularities for a class of hyperbolic systems*, Nonlinear Differ. Equ. Appl. 32 (1997), 117–140.

[39] D. Del Santo, E. Mitidieri: *Blow-up of solutions of a hyperbolic system: the critical case*, Differ. Equ. 34(9) (1998), 1157–1163.

[40] W. A. Strauss: *Nonlinear scattering theory at low energy*, J. Funct. Anal. 41(1) (1981), 110–133.

[41] H. Takamura: *Improved Kato's lemma on ordinary differential inequality and its application to semilinear wave equations*, Nonlinear Anal. 125 (2015), 227–240.

[42] S. Takamastsu: *Improvement of the general theory for one dimensional nonlinear wave equations related to the combined effect*, arXiv:2308.02174 (2023).

[43] X. Wei: *Blow-up for systems of semi-linear wave equations with small initial data*, J. Partial Differ. Equ. 17(3) (2004), 198–206.

[44] B. Yordanov, Q. S. Zhang: *Finite time blow up for critical wave equations in high dimensions*, J. Funct. Anal. 231 (2006), 361–374.

[45] Y. Zhou: *Life span of classical solutions to $u_{tt} - u_{xx} = |u|^{1+\alpha}$*, Chinese Ann. Math. Ser. B. 13(2) (1992), 230–243.

[46] Q. S. Zhang: *A blow-up result for a nonlinear wave equation with damping: the critical case*, C. R. Acad. Sci. Paris, t. 333, Série I, p. 109–114, 2001.

Department of Mathematics, Guangxi University, Nanning 530000, P. R. China
Email address: 115077158750@163.com