POLARS AND ANTIPODAL SETS FOR THE OUTER 3-SYMMETRIC SPACE $\mathbb{S}^7 \times \mathbb{S}^7$

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Abstract. We determine the polar and the maximal antipodal set P for the outer 3-symmetric space $\mathbb{S}^7 \times \mathbb{S}^7 = \mathrm{Spin}_8 / G_2$ where the 3-symmetric structure is given by the triality automorphism τ on Spin_8. It turns out that P has three elements. The 3-symmetric structure extends to a (non-abelian) S_3-structure which we also investigate.

1. Introduction

The Riemannian product $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ of two round $(n-1)$-spheres with equal size is a symmetric space and allows several obvious Γ-symmetric structures. However, for $n = 8$ there is an outer 3-symmetric structure (even a non-abelian S_3-structure) which is not at all obvious. It is related to properties of the octonions, in particular to the triality automorphismus τ making the rotation of the Dynkin diagram D_4 for Spin_8. According to [6, Thm.5.5, p.107] (see also [5]) this is one of the two outer 3-symmetric spaces G/H with G simple, where $G = \mathrm{Spin}_8$ in both cases. We compute the polar and the maximal antipodal set P for this structure; P consists of three points and is preserved by the S_3-structure.

2. Γ-symmetric spaces

Let G be a compact connected Lie group with left-invariant Riemannian metric and $\Gamma \subset \mathrm{Aut}(G)$ a subgroup preserving the metric. Let $H \subset G$ be a closed subgroup with

$$\mathrm{Fix}(\Gamma)^e \subset H \subset \mathrm{Fix}(\Gamma)$$

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where $\mathrm{Fix}(\Gamma) = \{g \in G : \gamma(g) = g \ \forall \gamma \in \Gamma\}$ is the fixed group of Γ and $\mathrm{Fix}(\Gamma)^e$ its identity component. Suppose that the metric on G is H-right-invariant. Then $X := G/H$ inherits a G-invariant submersion metric from G. Further, the action of Γ on G descends to an isometric action on X by setting $\gamma(gH) := \gamma(g)H$. This is well defined since $\gamma(gh) = \gamma(g)h$ for all $h \in H \subset \mathrm{Fix}(\Gamma)$. Further, $o = eH \in X$ is an isolated fixed point of this action. In fact, let $\mathfrak{h} \subset \mathfrak{g}$ be the Lie algebra of H and $\mathfrak{m} = \mathfrak{h}^\perp$ the orthogonal complement which will be identified with the tangent space T_oX. Then Γ acts on $\mathfrak{m}$ without fixed vectors, hence o is an isolated fixed point of Γ on X, and X is called a generalized s-manifold ([4]), in particular, if Γ is finite abelian, then X is called a Γ-symmetric space (cf. [3], [1]).

To generalize symmetric spaces, for every $x \in X$ a conjugate group $\Gamma_x \subset I(X)$ with isolated fixed point x is needed, where $I(X)$ denotes the isometry group of X. In fact, $\Gamma_x = g\Gamma g^{-1}$ when $x = gH$. This makes sense since we consider both G and Γ as subgroups of $I(X)$ where they generate a subgroup $G \cdot \Gamma \subset I(X)$. This is a semidirect product of G and Γ since for all $g \in G$ and $\gamma \in \Gamma$ we have

$$\gamma g = g^\gamma \gamma \tag{1}$$

where $g^\gamma := \gamma(g)$. Note that $(g^\gamma)^\delta = \delta(\gamma(g)) = g^{\delta\gamma}$.

More precisely, for any $x \in X$ we choose some $g_x \in G$ with $x = g_xH$ and define a group homomorphism

$$\phi_x : \Gamma \to I(X) : \gamma \mapsto \gamma_x := g_x \gamma g_x^{-1}. \tag{2}$$

This depends only on x: For all $h \in H$ we have $h\gamma h^{-1} = hh^{-1}\gamma = \gamma$, and replacing g_x by $\tilde{g}_x = g_xh$ we obtain $\tilde{g}_x\gamma\tilde{g}_x^{-1} = g_xh\gamma h^{-1}g_x^{-1} = \gamma_x$.

Using (1) we see the so called kai property

$$\gamma_x \delta_y \gamma_x^{-1} = \phi_{\gamma_x(y)}(\gamma\delta\gamma^{-1}) \tag{3}$$

for all $x, y \in X$ and $\gamma, \delta \in \Gamma$ (see [4]).

3. The automorphism group of the octonions

An algebra structure (with unit) on the Euclidean vector space $\mathbb{A} := \mathbb{R}^n$ is called *normed* if $|xy| = |x||y|$ for all $x, y \in \mathbb{A}$. We call $\mathrm{Re}\ \mathbb{A} = \mathbb{R} \cdot 1$ the real part and $\mathrm{Im}\ \mathbb{A} = (\mathrm{Re}\ \mathbb{A})^\perp$ the imaginary part of $\mathbb{A}$. If $x \in \mathrm{Im}\ \mathbb{A}$ with $|x| = 1$, then $|1 - x^2| = |1 + x||1 - x| = 2$ and $|x^2| = |x|^2 = 1$, thus 1 and x^2 have maximal distance on the unit circle, which means $x^2 = -1$.

An automorphism of $\mathbb{A}$ is a bijective linear map A on $\mathbb{A}$ with

$$A(xy) = A(x)A(y) \tag{4}$$

for all $x, y \in \mathbb{A}$. Clearly, automorphisms preserve the unit 1 and its real multiples, but they preserve also imaginary unit vectors since the equation $x^2 = -1$ is invariant. Therefore the automorphism group of a normed algebra $\mathbb{A}$ is a subgroup of the orthogonal group $O(\mathrm{Im}\ \mathbb{A}) = O_{n-1}$. Consequently automorphisms commute with the conjugation $x = \mathrm{Re}\ x + \mathrm{Im}\ x \mapsto \bar{x} = \mathrm{Re}\ x - \mathrm{Im}\ x$ on $\mathbb{A}$.

The largest normed algebra is the algebra $\mathbb{O}$ of the octonions with dimension $n = 8$. Its automorphism group $\mathrm{Aut}\,\mathbb{A}$ is often named by its Dynkin diagram G_2. This group acts simply transitively on the set of *Cayley triples* (x, y, z), orthonormal triples in $\mathrm{Im}\ \mathbb{O}$ with $z \perp xy$. Therefore G_2 is diffeomorphic to a 3-sphere bundle (z) over the unit tangent bundle (x, y) of $\mathbb{S}^6$; in particular its dimension is 14.

4. The triality automorphism

The only compact simple group G with $|\,\mathrm{Aut}(G)/\,\mathrm{Aut}(G)^e| > 2$ is Spin_8. This is due to the triality automorphism which we want to explain next, following [2].

The Clifford algebra Cl_8 is the algebra of real 16×16 matrices

$$Cl_8 = \left\{ \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) : A, B, C, D \in \mathbb{R}^{8\times 8} \right\}.$$

The even and odd parts $Cl_8^{\pm}$ consist of the block diagonal and antidiagonal matrices $\left(\begin{smallmatrix} A & \\ & D \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} & B \\ C & \end{smallmatrix}\right)$. The Euclidean vector space $\mathbb{R}^8$ can be viewed as the division algebra $\mathbb{O}$ of the octonions where the basis vector $e_1 = (1, 0, \ldots, 0)$ becomes the unit 1. This is embedded in Cl_8^- by the linear isometry

$$\mathbb{R}^8 = \mathbb{O} \ni x \mapsto \hat{x} := \begin{pmatrix} 0 & -L(\overline{x}) \\ L(x) & 0 \end{pmatrix} \tag{5}$$

where $L(x)$ denotes the left translation on $\mathbb{O}$ with $x \in \mathbb{O}$, where $\kappa : x \mapsto \overline{x}$, $\kappa = \mathrm{diag}(1, -1, \ldots, -1)$ is the octonionic conjugation (an anti-automorphism on $\mathbb{O}$) and where the inner product on $\mathbb{R}^{16\times 16}$ is the normalized trace metric:

$$\langle A, B \rangle = \tfrac{1}{16}\,\mathrm{trace}\ A^t B.$$

The group Spin_8 is the subgroup of $SO_8 \times SO_8$ which preserves this set of matrices under conjugation: $(A, B) \in \mathrm{Spin}_8 \iff \mathrm{Ad}\left(\begin{smallmatrix} A & \\ & B \end{smallmatrix}\right)\hat{x} = \hat{w}$ for some $w \in \mathbb{O} \iff$

$$\begin{pmatrix} A & \\ & B \end{pmatrix} \begin{pmatrix} & -L(\overline{x}) \\ L(x) & \end{pmatrix} \begin{pmatrix} A^t & \\ & B^t \end{pmatrix} = \begin{pmatrix} & -L(\overline{w}) \\ L(w) & \end{pmatrix}$$

for some $w \in \mathbb{O}$. This is equivalent to the equation

$$B \cdot L(x) = L(w) \cdot A. \tag{6}$$

From $BL(x)A^t = L(w)$ it follows that the map $C : x \mapsto w$ is linear and isometric, hence $C \in SO_8$ (using connectivity). Applying both sides of (6) to some $y \in \mathbb{O}$ we see that $(A, B) \in \mathrm{Spin}_8$ if and only if there is some $C \in SO_8$ with

$$B(x \cdot y) = (Cx) \cdot (Ay) \tag{7}$$

for all $x, y \in \mathbb{O}$; this property is called *triality*. It is a generalization of the automorphism property (4).

We may view Spin_8 as the set of triples $(A, B, C) \in (SO_8)^3$ with (7).

Lemma 4.1. *These triples form a subgroup of* $(SO_8)^3$.

Proof. We show: When (7) holds for (A, B, C) and (A', B', C'), then it also holds for the product triple $(A'A, B'B, C'C)$. In fact, $B'B(x \cdot y) = B'(Cx \cdot Ay) = C'Cx \cdot A'Ay$. Further, when $(A, B, C) \in \mathrm{Spin}_8$, the same is true for its inverse (A^t, B^t, C^t), that is $B^t(xy) = C^tx \cdot A^ty$. Indeed, $B(C^tx \cdot A^ty) = CC^tx \cdot AA^ty = xy = BB^t(xy)$. □

The *triality automorphism* of Spin_8 is essentially a cyclic change in every triple. More precisely:

Lemma 4.2. *Let* $\kappa = \mathrm{diag}(1, -1, ..., -1)$, $\kappa(x) = \overline{x}$ *be the octonionic conjugation on* $\mathbb{O}$. *Then* $\tilde{D} := \kappa D \kappa \in SO_8$ *for all* $D \in SO_8$, *and*

$$\tau : (A, B, C) \mapsto (\tilde{B}, \tilde{C}, A) \tag{8}$$

is an automorphism of Spin_8 *with order 3.*

Proof. Let $x, y \in \mathbb{O}$ with $|x| = |y| = 1$ and $z = xy$, thus $|z| = 1$. Using octonionic conjugation (an anti-automorphism of $\mathbb{O}$), we obtain

$$(Cx)(Ay) = Bz \iff (\overline{Ay})(\overline{Cx}) = \overline{Bz} \iff \overline{Cx} = (Ay)(\overline{Bz}).$$

Now $\overline{Cx} = \tilde{C}\overline{x}$ with $\tilde{C} = \kappa C \kappa$. Thus (7) $\iff$ $(Ay)(\tilde{B}\overline{z}) = \tilde{C}\overline{x}$, and since $y\overline{z} = \overline{x}$ when z, x have unit length, we have $(\tilde{B}, \tilde{C}, A) \in \mathrm{Spin}_8$.

Clearly, permutations and the maps $B \mapsto \tilde{B}$ etc. are automorphisms.

The iteration of τ gives

$$(A, B, C) \mapsto (\tilde{B}, \tilde{C}, A) \mapsto (C, \tilde{A}, \tilde{B}) \mapsto (A, B, C). \tag{9}$$

□

Lemma 4.3. *There is an outer involution* σ *on* $\mathrm{Spin}_8 \subset (SO_8)^3$,

$$\sigma : (A, B, C) \mapsto (B, A, \tilde{C}). \tag{10}$$

Proof. Instead of $Cx \cdot Ay = B(xy)$ we have to show $\tilde{C}x\, B\tilde{y} = A(x\tilde{y})$ for all $x, \tilde{y} \in \mathbb{O}$. We transform the first equation as follows. Replacing x by $\overline{x}$ and assuming that x, y are unit vectors we obtain

$$C\overline{x}\, Ay = B(\overline{x}y) \iff C\overline{x} = B(\overline{x}y)\, \overline{Ay}$$

$$\begin{aligned} &\iff \quad \overline{C\overline{x}} = Ay\,\overline{B(\overline{x}y)} \\ &\iff \quad \tilde{C}(x) \cdot B(\overline{x}y) = Ay. \end{aligned}$$

Setting $\tilde{y} = \overline{x}y$ we have $x\tilde{y} = x\overline{x}y = y$ and hence $\tilde{C}x\,B\tilde{y} = A(x\tilde{y})$ as required. □

Lemma 4.4. *The two automorphisms σ and τ generate a group $\hat{\Gamma} \subset \mathrm{Aut}(\mathrm{Spin}_8)$ which is isomorphic to S_3.*

Proof. We have to check the relation $\sigma\tau\sigma = \tau^{-1}$. In fact,

$$(A, B, C) \overset{\sigma}{\mapsto} (B, A, \tilde{C}) \overset{\tau}{\mapsto} (\tilde{A}, C, B) \overset{\sigma}{\mapsto} (C, \tilde{A}, \tilde{B}),$$

and $(C, \tilde{A}, \tilde{B})$ is the triple for $\tau^2 = \tau^{-1}$ in (9). □

Lemma 4.5. *The fixed group of $\hat{\Gamma} = \langle\tau, \sigma\rangle_{\mathrm{grp}}$ and of $\Gamma = \langle\tau\rangle_{\mathrm{grp}}$ acting on Spin_8 is $G_2 = \mathrm{Aut}(\mathbb{O})$.*

Proof. Clearly, $(A, B, C) \in \mathrm{Spin}_8$ with $A = B = C$ is fixed by σ and τ, and the triality equation (7) restricts to the automorphism equation (4). Vice versa, if a triple $(A, B, C) \in \mathrm{Spin}_8$ is fixed by τ then $A = \tilde{B} = C$, cf. (8). Then $(A, \tilde{A}, A) \in \mathrm{Spin}_8$. Claim: This implies $\tilde{A} = A$. In fact, from (7) we obtain $\tilde{A}(xy) = A(x)A(y)$, in particular,

$$\tilde{A}(x) = A(x)a = aA(x) \quad \text{with} \quad a = A(1),$$

and for $x = 1$ it follows $\bar{a} = a^2$. But a commutes with the arbitrary octonion $A(x)$, hence $a = 1$ and thus $\tilde{A} = A$. □

5. The outer $\hat{\Gamma}$-symmetric space $\mathbb{S}^7 \times \mathbb{S}^7$.

As a consequence of the last sections, $X = \mathrm{Spin}_8 / G_2$ is an outer $\hat{\Gamma}$-symmetric space. The group Spin_8 can be viewed as a subgroup of $SO_8 \times SO_8$ since C is determined by A, B via (7). As such it acts naturally on $\mathbb{S}^7 \times \mathbb{S}^7 \subset \mathbb{O}^2$.

Lemma 5.1. *The isotropy group of this action at the point $o := (1, 1) \in \mathbb{S}^7 \times \mathbb{S}^7 = X$ is $G_2 = \mathrm{Aut}(\mathbb{O})$. Consequently, $X \cong \mathbb{S}^7 \times \mathbb{S}^7$ (Spin_8-equivariantly diffeomorphic).*

Proof. When $(A, B, C) \in \mathrm{Spin}_8$ with $A(1) = B(1) = 1$, then from (7) we obtain $C(1) \cdot A(1) = B(1)$, hence $C(1) = 1$, further $C(1) \cdot Ay = By$ whence $A = B$, and $Cx \cdot A(1) = Bx$, thus $C = B$. Therefore $(A, B, C) = (A, A, A)$ which implies $A \in G_2$ by (7).
Since $\dim(\mathrm{Spin}_8 / G_2) = 28 - 14 = 14$ is the dimension of $\mathbb{S}^7 \times \mathbb{S}^7$, the action of Spin_8 on $\mathbb{S}^7 \times \mathbb{S}^7$ is transitive. The Spin_8-equivariant diffeomorphism is

$$\mathrm{Spin}_8 / G_2 \ni (A, B) \cdot G_2 \mapsto (A(1), B(1)) \in \mathbb{S}^7 \times \mathbb{S}^7. \tag{11}$$

□

How does the action of $\hat\Gamma$ on Spin_8/G_2 look like on $\mathbb{S}^7\times\mathbb{S}^7$? We have $\tau(A(1),B(1)) = (\tilde B(1),\tilde C(1))$ by (8). From (7) we have $C(1)\cdot A(1) = B(1)$, thus $C(1) = B(1)\cdot\overline{A(1)}$. Setting $x = A(1)$ and $y = B(1)$ we have $\tau(x,y) = (\overline{y},\overline{y\cdot\overline{x}})$, hence

$$\tau(x,y) = (\overline{y}, x\overline{y}). \tag{12}$$

Once more we can see the order 3 of this map:

$$(x,y)\mapsto(\overline{y},x\overline{y})\mapsto(\overline{x\overline{y}},\overline{y}\,\overline{x\overline{y}}) = (y\overline{x},\overline{x})\mapsto(x,y\overline{x}x) = (x,y).$$

5.1. **The fixed points of $\hat\Gamma$.** Thus any $(x,y)\in\mathbb{S}^7\times\mathbb{S}^7$ is fixed by τ $\iff$ $x=\overline{y}$ and $y = x\overline{y}$ $\iff$ $\overline{x}=y=x^2$. In particular, $x^2=\overline{x}$, thus $x^3=1$, and $y=\overline{x}$. Hence the fixed set of τ in $X=\mathbb{S}^7\times\mathbb{S}^7$ is

$$\mathrm{Fix}(\tau) = \{(x,\overline{x}) : x\in\mathbb{O},\ x^3=1\} = \{(1,1)\}\cup Y, \tag{13}$$

$$Y = \{(x,\overline{x}) : x = \tfrac{1}{2}(-1+\sqrt{3}v) : v\in\mathbb{S}^6\}$$

where $\mathbb{S}^6$ is the unit sphere in $\mathbb{R}^7 = \mathrm{Im}\,(\mathbb{O}) = \mathbb{O}\ominus\mathbb{R}$.

Further, σ acts on X by $(A(1),B(1))\mapsto(B(1),A(1))$, cf. (10), hence its fixed set is the diagonal,

$$\mathrm{Fix}(\sigma) = \{(x,x) : x\in\mathbb{S}^7\}.$$

Since $\mathrm{Fix}(\sigma)\cap Y=\emptyset$, we obtain

$$\mathrm{Fix}(\hat\Gamma) = \mathrm{Fix}(\sigma)\cap\mathrm{Fix}(\tau) = \{(1,1)\}. \tag{14}$$

6. Maximal antipodal sets for $\Gamma = \langle\tau\rangle$

Let X be a Γ-symmetric space. An *antipodal set* in X is a subset $P\subset X$ such that $P\subset\mathrm{Fix}(\Gamma_p)$ for all $p\in P$. By translating P if necessary we may always assume $o\in P$. Clearly for the S_3-symmetric space $\mathbb{S}^7\times\mathbb{S}^7$, the only antipodal set is $\{o\}$. In the following we will consider $\mathbb{S}^7\times\mathbb{S}^7$ as a Γ-symmetric space for $\Gamma = \langle\tau\rangle_{\mathrm{grp}}$, an outer 3-symmetric space. We have already seen that Y is a *polar* for o, a positive-dimensional component of $\mathrm{Fix}(\Gamma)$ (mind that $\Gamma=\Gamma_o$).

First let us recall a well known fact on octonions.

Lemma 6.1. *For all $s,x\in\mathbb{O}$ we have*

$$L(s)L(x)L(s) = L(sxs). \tag{15}$$

Proof. Since any two octonions lie in a commom quaternionic subalgebra, we may assume $s,x\in\mathbb{H}\subset\mathbb{O}$. Clearly, (15) holds on $\mathbb{H}$. We have to show it on $\ell\mathbb{H}$ where i, j, ℓ are the usual unit generators of $\mathbb{O}$ (with $\ell \perp i, j, ij$). Recall for every $h \in \mathbb{H}$:

$$L(h)L(\ell) = L(\ell)L(\overline{h}).$$

In fact, let $h = h_o + h'$ with $h_o \in \mathbb{R}$ and $h' \perp \mathbb{R}$. Then

$$h_o\ell + h'\ell = \ell h_o - \ell h' = \ell\overline{h}.$$

Therefore we have on $\mathbb{H}$:

$$L(s)L(x)L(s)L(\ell) = L(\ell)L(\overline{s})L(\overline{x})L(\overline{s}) = L(\ell)L(\overline{sxs}) = L(sxs)L(\ell).$$

□

Theorem 6.2. *Let $\Gamma \subset \mathrm{Aut}(\mathrm{Spin}_8)$ be the group generated by the triality automorphism τ. In the outer Γ-symmetric space $\mathbb{S}^7 \times \mathbb{S}^7 = \mathrm{Spin}_8 / G_2$, every maximal antipodal set containing $o = (1, 1)$ equals*

$$P = \{(1, 1), (s, \overline{s}), (\overline{s}, s)\}$$

with $s = \frac{1}{2}(-1 + \sqrt{3}\, v)$ for some arbitrary $v \in \mathbb{S}^6 \subset \mathrm{Im}\,\mathbb{O}$. The outer involution σ fixes $(1, 1)$ and interchanges the other two elements of P.

Proof. Any $p = (s, \overline{s}) \in \mathrm{Fix}(\tau) \subset \mathbb{S}^7 \times \mathbb{S}^7$ can be written in the form go with $o = (1, 1)$ and $g = (A, B)$ for

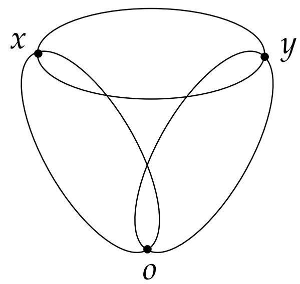
$$A = L(s), \quad B = L(\overline{s}).$$

Then $g = (A, B) \in \mathrm{Spin}_8$ since (7) holds for A, B, C with $Cx = \overline{s}x\overline{s}$:

$$\begin{aligned} (Cx)(Ay) &= (\overline{s}x\overline{s})(sy) = L(\overline{s}x\overline{s})L(s)y \\ &\overset{(15)}{=} L(\overline{s})L(x)L(\overline{s})L(s)y = L(\overline{s})L(x)y = B(xy). \end{aligned}$$

Clearly, both Γ and $\Gamma_p = g\Gamma g^{-1}$ are fixing o and p, and $\mathrm{Fix}(\Gamma_p) = g\,\mathrm{Fix}(\Gamma)$ contains $p = go$ and o. Note that $o = g(q)$ for $q = (\overline{s}, s)$, and $(\overline{s}, s) \in Y \subset \mathrm{Fix}(\Gamma)$, see (13). Therefore $q \in \mathrm{Fix}(\Gamma) \cap \mathrm{Fix}(\Gamma_p)$. Applying σ we see also $p \in \mathrm{Fix}(\Gamma) \cap \mathrm{Fix}(\Gamma_q)$, and Γ fixes $p, q \in Y$. Therefore P is antipodal.

A *maximal* antipodal set through o and p might contain other points $\tilde{q} \in \mathrm{Fix}(\Gamma) \cap g\,\mathrm{Fix}(\Gamma)$. The elements of $g\,\mathrm{Fix}(\Gamma)$ with $g = (L(s), L(\overline{s}))$ are $\tilde{q} = (st, \overline{s}\overline{t})$ for any $t \in \mathbb{O}$ with $t^3 = 1$, see Subsection 5.1. Such $\tilde{q} = (st, \overline{s}\overline{t})$ lies in $\mathrm{Fix}(\Gamma)$ if and only if $\overline{s}\overline{t} = \overline{st}$ which means that s and t commute and hence $(st)^3 = s^3t^3 = 1$. There are three solutions: $t \in \{1, s, \overline{s}\}$, leading to $\tilde{q} = (u, \overline{u})$ for $u \in \{s, \overline{s}, 1\}$. The only new solution is $u = \overline{s}$ with $\tilde{q} = (\overline{s}, s) = q$. □



The fixed space of $\Gamma = \Gamma_o$ consists of the isolated point o and the 6-sphere Y opposite to o, a polar for o, drawn as an ellipse in the preceding figure. The other points p and q in P are antipodal in Y. Likewise, the fixed sets of Γ_p and Γ_q consist p or q (resp.) together with opposite 6-spheres. The three 6-spheres intersect pairwise in antipodal points. These three points o, p, q form a maximal (in fact "great", that is, no other antipodal set contains more points) antipodal set P for the Γ-structure on $\mathbb{S}^7 \times \mathbb{S}^7$.

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