BRAUER GROUP OF MODULI STACKS OF PARABOLIC PRINCIPAL BUNDLES OVER A CURVE

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Abstract. We prove that the Brauer group of the moduli stack of parabolic stable principal $\mathrm{PGL}(r,\mathbb{C})$-bundles on a curve X, for a generic system of parabolic weights along an arbitrary parabolic divisor, coincides with the Brauer group of the smooth locus of the corresponding coarse moduli space of parabolic stable principal $\mathrm{PGL}(r,\mathbb{C})$-bundles. We also show that for any simple and simply connected complex linear algebraic group G, the analytic and algebraic Brauer groups of the moduli stack of quasi-parabolic principal G-bundles on X vanish.

1. Introduction

The cohomological Brauer group of a quasi-projective variety Y over $\mathbb{C}$, denoted by $\mathrm{Br}(Y)$, is defined to be the torsion part $H^2_{\text{ét}}(Y,\, \mathbb{G}_m)_{\text{torsion}}$. When Y is smooth, it is known that $H^2_{\text{ét}}(Y,\, \mathbb{G}_m)$ is already torsion. Brauer groups are interesting invariants to study, in fact for a number of reasons. For example, it is a stable birational invariant for smooth projective varieties, making it very useful in studying rationality questions. It may be mentioned that the Brauer group has been used in constructing examples of non-rational varieties by many, including Colliot-Thélène, Saltman, Peyre and others. It also plays a central role in Brauer-Manin obstruction theory, which deals with the study of rational points on varieties defined over a number field.

The study of Brauer groups in the context of moduli of parabolic vector bundles over curves has been carried out in recent times by several authors [BB, BCD1, BCD2, BD]. The computation of the Brauer group of the moduli space of stable parabolic bundles for structure groups $\mathrm{SL}(r,\mathbb{C})$, $\mathrm{PGL}(r,\mathbb{C})$ and $\mathrm{Sp}(r,\mathbb{C})$ have been carried out earlier in [BD, BCD1, BCD2]. Our aim here is to generalize an earlier result of [BCD1] for the case of $\mathrm{PGL}(r,\mathbb{C})$-bundles, as well as study the general case for any simple and simply connected linear algebraic group G over $\mathbb{C}$.

More precisely, suppose X is a smooth projective curve of genus g defined over $\mathbb{C}$, with $g \geq 2$. Fix a positive integer $r \geq 2$; and if $r = 2$, we assume $g \geq 3$. Fix a line bundle ξ on X, and systems of multiplicities and weights $(\boldsymbol{m}, \boldsymbol{\alpha})$ along a set of parabolic points on X (see Section 2 for details). Let $\mathfrak{N}^{\boldsymbol{m},\boldsymbol{\alpha}}_X(r,\delta)$ denote the moduli stack of parabolic stable principal $\mathrm{PGL}(r,\mathbb{C})$-bundles on X of topological type $\delta \equiv \deg(\xi) \pmod r$ (see Section 4), and let $\mathcal{N}^{\boldsymbol{m},\boldsymbol{\alpha}}_{\mathrm{X}}(\mathrm{r},\delta)$ denote the corresponding coarse moduli space of stable parabolic principal $\mathrm{PGL}(r,\mathbb{C})$-bundles. Our first main result is the generalization of [BCD1, Theorem 1.1 (1)] for *any* generic system of weights (earlier it was shown only for *full-flag* systems of weights).

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Theorem 1.1 (Theorem 4.2)**.** *Let $\boldsymbol{\alpha}$ be a generic system of weights. Let $\mathcal{N}_{\mathrm{X}}^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{r},\delta)^{sm}$ denote the smooth locus of $\mathcal{N}_{\mathrm{X}}^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{r},\delta)$. Then,*

$$\mathrm{Br}\left(\mathfrak{N}_X^{\boldsymbol{m},\boldsymbol{\alpha}}(r,\delta)\right) \;\simeq\; \mathrm{Br}\left(\mathcal{N}_{\mathrm{X}}^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{r},\delta)^{sm}\right).$$

Let G be a simple and simply connected linear algebraic group defined over $\mathbb{C}$. Fix a sequence $\underline{p} = (p_1, p_2, \cdots, p_n)$ of points on X, as well as a sequence $\underline{P} = (P_1, P_2, \cdots, P_n)$ of parabolic subgroups of G. Let $\mathcal{M}_{\mathrm{G}}^{\mathrm{par}}\left(\underline{p},\ \underline{P}\right)$ denote the moduli stack of quasi-parabolic principal G-bundles on X of type $\underline{P}$ (see [LS] for details).

Theorem 1.2 (Proposition 5.7 and Theorem 5.8)**.** *Both the analytic and the algebraic Brauer group of $\mathcal{M}_{\mathrm{G}}^{\mathrm{par}}\left(\underline{p},\ \underline{P}\right)$ vanish. In other words, if $\mathcal{O}^{an}_{\mathcal{M}_{\mathrm{G}}^{\mathrm{par}}(\underline{p},\ \underline{P})}$ is the sheaf of analytic functions on $\mathcal{M}_{\mathrm{G}}^{\mathrm{par}}\left(\underline{p},\ \underline{P}\right)$, then*

$$H^2\left(\mathcal{M}_{\mathrm{G}}^{\mathrm{par}},\ (\mathcal{O}^{an}_{\mathcal{M}_{\mathrm{G}}^{\mathrm{par}}})^*\right)_{torsion} \;=\; 0 \;=\; \mathrm{Br}\left(\mathcal{M}_{\mathrm{G}}^{\mathrm{par}}\left(\underline{p},\ \underline{P}\right)\right).$$

2. Preliminaries on parabolic vector bundles

Let X be a connected smooth projective curve, defined over $\mathbb{C}$, of genus g, with $g \geq 2$. Fix a positive integer $r \geq 2$; and if $r = 2$, we assume $g \geq 3$. Fix a finite subset $S \subset X$, which will be referred to as the *parabolic points*. The set S shall remain fixed throughout.

Definition 2.1. A *parabolic vector bundle* of rank r on X is an algebraic vector bundle E on X of rank r together with a weighted flag on the fiber E_p of E over every $p \in S$:

$$\begin{gathered} E_p \;=\; E_{p,1} \supsetneq E_{p,2} \supsetneq \cdots \supsetneq E_{p,\ell(p)} \supsetneq E_{p,\ell(p)+1} \;=\; 0 \\ 0 \leq \alpha_{p,1} < \alpha_{p,2} < \cdots < \alpha_{p,\ell(p)} < \alpha_{p,\ell(p)+1} = 1, \end{gathered} \tag{2.1}$$

where $\alpha_{p,i} \in \mathbb{R}$.

- Such a flag is said to be of length $\ell(p)$, and the numbers $m_{p,i} := \dim E_{p,i} - \dim E_{p,i+1}$ are called the *multiplicities* of the flag at p. More precisely, $m_{p,i}$ is the multiplicity associated to the weight $\alpha_{p,i}$.
- The flag at p is said to be *full* if $m_{p,i} = 1$ for every i, in which case clearly we have $\ell(p) = r$.
- The collection of real numbers $\boldsymbol{\alpha} := \{(\alpha_{p,1} < \alpha_{p,2} < \cdots < \alpha_{p,\ell(p)})\}_{p\in S}$ is called a system of weights, and the collection of integers $\boldsymbol{m} := \{(m_{p,1}, m_{p,2}, \cdots, m_{p,\ell(p)})\}_{p\in S}$ is called a system of multiplicities.
- We will often denote a system of multiplicities (respectively, a system of weights) by $\boldsymbol{m}$ (respectively, $\boldsymbol{\alpha}$), when there is no scope of any confusion. Also, a parabolic vector bundle of the above type will also be denoted by E_*.

If we ignore the weights and only consider the data of a vector bundle E together with a given filtration on the fibers $\{E_p\}_{p\in S}$, then such an object will be called a *quasi-parabolic vector bundle*. Thus, a quasi-parabolic vector bundle has an associated system of multiplicities, but no associated system of weights. In Section 5 quasi-parabolic bundles will appear in a more general context.

Definition 2.2. Let E_* and F_* be two parabolic vector bundles with systems of multiplicities and weights $(\boldsymbol{m},\ \boldsymbol{\alpha})$ and $(\boldsymbol{m'},\ \boldsymbol{\alpha'})$ respectively. A *parabolic morphism* $f_* : E_* \longrightarrow F_*$ is an $\mathcal{O}_X$–linear homomorphism $f : E \longrightarrow F$ between the underlying vector bundles such that at each parabolic point p, we have $f_p(E^i_p) \subset F^{j+1}_p$ whenever $\alpha^i_p \geq {\alpha'}^j_p$.

Definition 2.3. Consider a parabolic vector bundle E_* of rank r as in Definition 2.1. The *parabolic degree* of E_* is defined to be the real number

$$\operatorname{par}\deg(E_*) \ := \ \deg(E) + \sum_{p\in S}\sum_{i=1}^{\ell(p)} m_{p,i}\alpha_{p,i}. \tag{2.2}$$

The *parabolic slope* of E_* is defined to be the real number

$$\operatorname{par}\mu(E_*) \ := \ \frac{\operatorname{par}\deg(E_*)}{r}. \tag{2.3}$$

A parabolic vector bundle E_* is called *parabolic semistable* (respectively, *parabolic stable*) if for every proper sub-bundle $0 \neq F \subset E$ we have

$$\operatorname{par}\mu(F_*) \leq \operatorname{par}\mu(E_*) \quad (\text{respectively, } \operatorname{par}\mu(F_*) < \operatorname{par}\mu(E_*)),$$

where F_* denotes the parabolic bundle defined by the parabolic structure on F induced by the parabolic structure of E_* (see [MS] for the details).

Of course, every parabolic stable vector bundle is parabolic semistable, but the converse is not true in general. Systems of weights for which parabolic semistability and parabolic stability coincide are called *generic* (see [BY] for more details).

2.1. **Parabolic push-forward and pull-back.** Let Y be another irreducible smooth projective complex curve and $\gamma : Y \longrightarrow X$ a finite étale Galois covering morphism. If F is a vector bundle on Y of rank n, then the direct image γ_*F is a vector bundle on X of rank nd, where d is the degree of the map γ. Given a parabolic structure on F along parabolic points $S' \subset Y$, there is a natural way to construct a parabolic structure on γ_*F along $\gamma(S') \subset X$ [BM, § 3]. Below, this construction will be briefly recalled.

Let $S \subset X$ be a finite set of points, and suppose $F \longrightarrow Y$ is a vector bundle equipped with a parabolic structure over $\gamma^{-1}(S) \subset Y$. For convenience, consider the special case where $S = \{p\} \subset X$ is a singleton. Let $d = \deg(\gamma)$. Since γ is unramified, the inverse image $\gamma^{-1}(p)$ consists of d distinct points of Y. Suppose the parabolic structure of F_* over $q \in \gamma^{-1}(p)$ is as follows: the parabolic data looks like

$$F_q = F_{q,1} \supsetneq F_{q,2} \supsetneq \cdots \supsetneq F_{q,\ell(q)} \supsetneq F_{q,\ell(q)+1} = 0, \tag{2.4}$$

$$0 \leq \alpha_{q,1} < \alpha_{q,2} < \cdots < \alpha_{q,\ell(q)} < \alpha_{q,\ell(q)+1} = 1. \tag{2.5}$$

We shall construct a parabolic structure on $E = \gamma_*F$ along the point $\{p\}$ from these data. Note that

$$E_p = \bigoplus_{q\in\gamma^{-1}(p)} F_q.$$

Define a positive integer $\ell(p)$ and real numbers $\{\beta_{p,1}, \beta_{p,2}, \cdots, \beta_{p,\ell(p)}\}$ as follows:

$$\{\beta_{p,1}, \beta_{p,2}, \cdots, \beta_{p,\ell(p)}\} = \bigcup_{q\in\gamma^{-1}(p)} \{\alpha_{q,1}, \ \cdots, \ \alpha_{q,\ell(q)}\}.$$

Also assume that the $\beta_{p,i}$'s form an increasing sequence $\left(\beta_{p,1} < \beta_{p,2} < \cdots < \beta_{p,\ell(p)}\right)$. These will be the weights at $\{p\}$.

For each $1 \leq j \leq \ell(p)$, define a filtration of E_p by the subspaces

$$E_{p,j} := \bigoplus_{q\in\gamma^{-1}(p)} F_{q,t(q)},$$

where for each point $q \in \gamma^{-1}(S)$, the integer $t(q) \in [1, \ell(q)]$ (see (2.4)) is the smallest one satisfying $\beta_{p,j} \leq \alpha_{q,t(q)}$. Consequently,

$$E_{p,1} \supsetneq E_{p,2} \supsetneq \cdots \supsetneq E_{p,\ell(p)} \supsetneq E_{p,\ell(p)+1} = 0$$
$$0 \leq \beta_{p,1} < \beta_{p,2} < \cdots < \beta_{p,\ell(p)} < \beta_{p,\ell(p)+1} = 1$$

is a parabolic structure on $E = \gamma_*(F)$ along the point $\{p\}$.

More generally, if $S \subset X$ is a finite subset, and a parabolic vector bundle F_* on Y is given with a parabolic structure along $\gamma^{-1}(S)$, one can perform the above construction for each $p \in S$ to obtain a parabolic push-forward $E_* = \gamma_*(F_*)$ on X with parabolic structure along S.

Let $\gamma : Y \longrightarrow X$ be as above. If E_* is a parabolic vector bundle on X with parabolic structure along S, then $\gamma^* E$ has an induced parabolic structure along $\gamma^{-1}(S)$ by assigning, for any $p \in S$ and $q \in \gamma^{-1}(p)$, the same filtration and weights on the fiber $(\gamma^* E)_q = E_p$ as those on E_p.

3. Fixed points of moduli of parabolic $\mathrm{SL}(r, \mathbb{C})$-bundles

Fix a positive integer r and a line bundle ξ on X. As before, we assume $r \geq 2$; and if $r = 2$, we assume $g \geq 3$. Also, fix a system of weights and multiplicities $(\boldsymbol{m}, \boldsymbol{\alpha})$ along parabolic points $S \subset X$. In this section, we shall assume that $\boldsymbol{\alpha}$ is a *generic* system of weights, so that a parabolic vector bundle is parabolic stable if it is parabolic semistable. Let

$$\mathcal{M}_{\mathrm{X}}^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{r},\ \xi) \tag{3.1}$$

denote the moduli space of stable parabolic vector bundles of rank r on X with determinant ξ and systems of multiplicities and weights $(\boldsymbol{m}, \boldsymbol{\alpha})$ along S. Let

$$\Gamma := \{L \in \mathrm{Pic}(X) \bigm| L^{\otimes r} \simeq \mathcal{O}_X\}$$

denote the group of r-torsion line bundles on X. For any $E_* \in \mathcal{M}_{\mathrm{X}}^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{r},\ \xi)$ (see (3.1)) and $L \in \Gamma$, it is clear that $E_* \otimes L \in \mathcal{M}_{\mathrm{X}}^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{r},\ \xi)$. Thus, the group Γ acts on $\mathcal{M}_{\mathrm{X}}^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{r},\ \xi)$.

If one additionally assumes that the system of multiplicities $\boldsymbol{m}$ is of *full-flag* type, i.e., $m_{p,i} = 1$ for all $1 \leq i \leq \ell(p)$ and every $p \in S$, then it was shown in [BCD1] that the codimension of the fixed point locus of $\mathcal{M}_{\mathrm{X}}^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{r},\ \xi)$ under any non-trivial r-torsion line bundle $L \in \Gamma$ is at least 3. This codimension estimation will be extended for *any* generic system of weights $\boldsymbol{m}$.

Lemma 3.1. *Let V be a finite-dimensional vector space over a field k with a diagonalizable endomorphism $\varphi : V \longrightarrow V$ and a filtration by subspaces*

$$V = V_1 \supsetneq V_2 \supsetneq \cdots \supsetneq V_\ell \supsetneq 0$$

such that $\varphi(V_i) = V_i$ for all $1 \leq i \leq \ell$. There exists a basis $\mathcal{B}_i$ of V_i consisting of eigenvectors for φ, satisfying $\mathcal{B}_i \supset \mathcal{B}_{i+1}$ for all $1 \leq i \leq \ell$.

Proof. By standard theory of semisimple modules, $\varphi|_{V_i}$ is diagonalizable on V_i for each i, so there exists a basis $\mathcal{B}_i$ of V_i consisting of eigenvectors. Moreover, one can write $V_{i-1} = W_{i-1} \oplus V_i$ for some subspace W_{i-1} which is invariant under φ. Thus, one can start with a basis $\mathcal{B}_\ell$ of V_ℓ consisting of eigenvectors, next choose a direct summand $W_{\ell-1}$ of V_ℓ in $V_{\ell-1}$, namely $V_{\ell-1} = W_{\ell-1} \oplus V_\ell$, and choose an eigenbasis of $W_{\ell-1}$; its union with $\mathcal{B}_\ell$ produces a basis $\mathcal{B}_{\ell-1}$ for $V_{\ell-1}$ consisting of eigenvectors and satisfying $\mathcal{B}_{\ell-1} \supset \mathcal{B}_\ell$. Proceeding inductively, one can produce bases $\mathcal{B}_i$ of V_i. This proves the lemma. $\square$

For any $L \in \Gamma \setminus \{\mathcal{O}_X\}$, let

$$\left(\mathcal{M}_{\mathrm{X}}^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{r},\ \xi)\right)^{\mathrm{L}} \ \subset\ \mathcal{M}_{\mathrm{X}}^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{r},\ \xi)$$

(see (3.1)) be the locus of fixed points for the action of L — through tensoring — on $\mathcal{M}_{\mathrm{X}}^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{r},\ \xi)$. If $d = \mathrm{ord}(L)$, choosing a nowhere-vanishing section $s_0 \in H^0(X,\, L^{\otimes d})$, construct the spectral curve

$$Y := \{v \in L \ \mid\ v^{\otimes d} \in s_0(X)\}. \tag{3.2}$$

The natural projection $\gamma : Y \longrightarrow X$ is an étale Galois covering with Galois group $\mathbb{Z}/d\mathbb{Z}$. The isomorphism class of this covering γ does not depend on the choice of the above section s_0.

For simplicity, let us first consider the case where $S = \{p\}$ is a singleton set. Thus we have a system of weights $(\alpha_{_{p,1}} < \alpha_{_{p,2}} < \cdots < \alpha_{_{p,\ell(p)}})$ at $p \in X$. Consider the spectral curve $\gamma : Y \longrightarrow X$ from (3.2). Let

$$\mathrm{Gal}(\gamma) \ =\ \{1,\, \mu,\, \mu^2,\, \cdots,\, \mu^{d-1}\} \subset \mathbb{C}^*.$$

The action of $\mathrm{Gal}(\gamma)$ on $\gamma^{-1}(p)$ is via multiplication. Fix an ordering on the points of $\gamma^{-1}(p)$, say

$$\gamma^{-1}(p) \ =\ \{q_1,\, q_2,\, \cdots,\, q_d\}, \tag{3.3}$$

such that μ^i acts on $\gamma^{-1}(p)$ as the cyclic permutation sending q_j to q_{j+i}, where the subscript $(j+i)$ is to be understood modulo d.

Now, recall that the systems of multiplicities and weights at p are given by

$$\left(m_{_{p,1}},\ m_{_{p,2}},\ \cdots,\ m_{_{p,\ell(p)}}\right) \quad \text{and} \quad \left(\alpha_{_{p,1}},\ \alpha_{_{p,2}},\ \cdots,\ \alpha_{_{p,\ell(p)}}\right)$$

respectively. Let $\boldsymbol{P}$ denote the collection of all nonempty subsets of $\left\{\alpha_{_{p,1}},\ \alpha_{_{p,2}},\ \cdots,\ \alpha_{_{p,\ell(p)}}\right\}$ of cardinality *at most* $\frac{r}{d}$, and let

$$\boldsymbol{T} \ := \ \underbrace{\boldsymbol{P} \times \cdots \times \boldsymbol{P}}_{d \text{ times}}, \tag{3.4}$$

where $d = \deg(\gamma)$.

For each $\mathbf{t} \in \mathbf{T}$, we want to describe a procedure of associating a system of weights, as well as a collection of systems of multiplicities along $\gamma^{-1}(p) = \{q_1,\ q_2,\ \cdots,\ q_d\}$. First associate a system of weights to each $\mathbf{t} \in \mathbf{T}$ as follows: adopt the notation

$$\mathbf{t} \ =\ \left(\lambda_1(\mathbf{t}),\ \lambda_2(\mathbf{t}),\ \cdots,\ \lambda_d(\mathbf{t})\right), \tag{3.5}$$

where each $\lambda_j(\mathbf{t}) \neq \emptyset$ by assumption. Clearly each $\lambda_j(\mathbf{t})$ can be arranged into an increasing sequence. Denote by $\lambda_j(\mathbf{t})$ the set of weights at q_j for each $1 \leq j \leq d$. Thus, each $\mathbf{t} \in \boldsymbol{T}$ prescribes a system of weights on $\gamma^{-1}(p)$. Denote this system of weights determined by $\mathbf{t}$ by $\boldsymbol{\alpha}_{\mathbf{t}}$.

Next, for each fixed $\mathbf{t} \in \mathbf{T}$, denote by

$$A_{\mathbf{t}} \tag{3.6}$$

be the collection of all matrices of size $(d \times \ell(p))$ with *non-negative integer* entries, written in the form

$$\begin{pmatrix} n_{_{q_1,1}} & n_{_{q_1,2}} & \cdots & \cdots & n_{_{q_1,\ell(p)}} \\ n_{_{q_2,1}} & n_{_{q_2,2}} & \cdots & \cdots & n_{_{q_2,\ell(p)}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ n_{_{q_d,1}} & n_{_{q_d,2}} & \cdots & \cdots & n_{_{q_d,\ell(p)}} \end{pmatrix} \tag{3.7}$$

which further satisfy the following two conditions:

(a) The numbers in the j-th row, namely the sequence

$$\left(n_{q_j,1},\ n_{q_j,2},\ \cdots,\ n_{q_j,\ell(p)}\right)$$

must satisfy the condition that for every $1 \leq k \leq \ell(p)$,

$$(n_{q_j,k} \ = \ 0) \quad \Longleftrightarrow \quad (\alpha_{p,k} \ \notin \ \lambda_j(\mathbf{t}))$$

(see (3.5)); note that this makes sense because $\lambda_j(\mathbf{t}) \subset \{\alpha_{p,1},\ \alpha_{p,2},\ \cdots,\ \alpha_{p,\ell(p)}\}$ by construction.

(b) The entries in each row of the matrix must add up to $\frac{r}{d}$, while the entries of the k-th column of it must add up to $m_{p,k}$ for each $k \in [1,\ \ell(p)]$. In other words,

$$\sum_{k=1}^{\ell(p)} n_{q_j,k} \ = \ \frac{r}{d} \quad \forall \ j \in [1,\ d]$$

$$\text{and} \quad \sum_{j=1}^{d} n_{q_j,k} \ = \ m_{p,k} \quad \forall \ k \in [1,\ \ell(p)].$$

Lemma 3.2. *Fix* $\boldsymbol{t} \in \boldsymbol{T}$ *(see (3.4)). Consider the collection of matrices satisfying the conditions (a) and (b) (the collection is denoted by* $A_{\mathbf{t}}$ *in (3.6)). Each such matrix gives rise to a system of multiplicities along* $\gamma^{-1}(p)$ *which is compatible with the system of weights* $\boldsymbol{\alpha}_{\mathbf{t}}$ *described just before (3.6).*

Proof. For any matrix of the form (3.7) coming from the collection $A_{\mathbf{t}}$, its rows are indexed by the points in $\gamma^{-1}(p) = \{q_1,\ q_2,\ \cdots,\ q_d\}$ by construction. For each $j \in [1,\ d]$, discard the zero terms from the sequence $\left(n_{q_j,1}\ , n_{q_j,2},\ \cdots,\ n_{q_j,\ell(p)}\right)$. From condition (b) it follows that $n_{q_j,k}$ is non-zero only when $\alpha_{p,k} \in \lambda_j(\mathbf{t})$; in other words, $n_{p,k} \neq 0$ only when $\alpha_{p,k}$ appears as a weight at $q_j \in \gamma^{-1}(p)$ in the system of weights $\boldsymbol{\alpha}_{\mathbf{t}}$. Thus, whenever $n_{q_j,k} \neq 0$, we can associate $n_{q_j,k}$ as the multiplicity of the weight $\alpha_{p,k}$ at $q_j \in \gamma^{-1}(p)$ (see Definition 2.1). This procedure certainly gives rise to a system of multiplicities along $\gamma^{-1}(p)$ compatible with $\boldsymbol{\alpha}_{\mathbf{t}}$, as in the statement of the lemma. $\square$

To aid the understanding of the above construction, we make the following picture for summarizing the above construction of multiplicities. Below, the vertical arrow denotes the map γ.

$$\begin{array}{cc}
 & \overbrace{\begin{array}{ccccc} \phantom{n_{q_1,1}} & & & & \end{array}}^{\text{numbers in each row add up to } \frac{r}{d} \text{ horizontally}} \\
q_1 & \begin{array}{ccccc} n_{q_1,1} & n_{q_1,2} & \cdots & \cdots & n_{q_1,\ell(p)} \end{array} \\
q_2 & \begin{array}{ccccc} n_{q_2,1} & n_{q_2,2} & \cdots & \cdots & n_{q_2,\ell(p)} \end{array} \\
\vdots & \begin{array}{ccccc} \vdots & \vdots & \cdots & \cdots & \vdots \end{array} \\
q_d & \begin{array}{ccccc} n_{q_d,1} & n_{q_d,2} & \cdots & \cdots & n_{q_d,\ell(p)} \end{array} \\
\Big\downarrow & \\
p & \underbrace{\begin{array}{ccccc} m_{p,1} & m_{p,2} & \cdots & \cdots & m_{p,\ell(p)} \end{array}}_{\text{each } k\text{-th column adds up to } m_{p,k}}
\end{array}$$

For each $\mathbf{t} \in \mathbf{T}$ consider the system of weights $\boldsymbol{\alpha}_{\mathbf{t}}$ along $\gamma^{-1}(p)$ as described just before (3.4). Consider the collection $A_{\mathbf{t}}$ of all possible matrices satisfying conditions (a) and (b). Let $e := \deg(\xi)$, and for each system of multiplicities $\mathbf{n}$ constructed from a matrix in $A_{\mathbf{t}}$ as in Lemma 3.2, let

$$\mathcal{M}_Y^{\mathbf{n},\boldsymbol{\alpha}_{\mathbf{t}}}\left(\frac{r}{d}, e\right) \tag{3.8}$$

denote the moduli space of parabolic stable vector bundles on Y of rank $\frac{r}{d}$ and degree $e = \deg(\xi)$, and having the system of multiplicities and weights given by $(\boldsymbol{n},\, \boldsymbol{\alpha}_{\mathbf{t}})$. Consider the following scheme for each $\mathbf{t} \in \mathbf{T}$:

$$\mathcal{N}_L^{\mathbf{t}} \;:=\; \left\{ F_* \,\in\, \coprod_{\boldsymbol{n}\,\in\, A_{\mathbf{t}}} \mathcal{M}_Y^{\boldsymbol{n},\boldsymbol{\alpha}_{\mathbf{t}}}\left(\frac{r}{d}, e\right) \;\middle|\; \det(\gamma_* F) \,\simeq\, \xi \right\} \tag{3.9}$$

(see (3.8)).

Proposition 3.3. *Assume that $\boldsymbol{\alpha}$ is a generic system of weights. Consider a non-trivial r-torsion line bundle L on X. Then there is a surjective morphism*

$$f \;:\; \left(\coprod_{t\in \boldsymbol{T}} \mathcal{N}_L^{t}\right) \;\longrightarrow\; \left(\mathcal{M}_{\mathrm{X}}^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{r},\ \xi)\right)^{\mathrm{L}}$$

(see (3.8)) given by parabolic push-forward along the spectral curve $\gamma \,:\, Y \,\longrightarrow\, X$ associated to L (see (3.2)).

Proof. For the sake of avoiding complicated notation, we assume that there is only one parabolic point $S = \{p\}$. The general case is very similar. The map f in the statement of the lemma sends any F_* to the parabolic push-forward of F_* constructed in Section 2.1.

It will be shown that $f(F_*)$ is parabolic semistable. Denote $E_* := f(F_*)$. Then

$$\gamma^* E_* \cong \bigoplus_{\sigma\in \mathrm{Gal}(\gamma)} \sigma^* F_*$$

([Se, p. 68]), where $\sigma^* F_*$ has the obvious parabolic structure coming from F_*. Clearly $\mathrm{par}\mu(\sigma^* F_*) = \mathrm{par}\mu(F_*)$ for all σ. Consequently, $\gamma^* E_*$ is a direct sum of parabolic stable bundles of same parabolic slope, which immediately implies that $\gamma^* E_*$ is parabolic semistable. Thus E_* must be parabolic semistable as well, since any sub-bundle $E'_* \,\subset\, E_*$ with strictly larger parabolic slope would give rise to a subbundle $\gamma^* E'_* \,\subset\, \gamma^* E_*$ of strictly larger parabolic slope, contradicting the parabolic semistability of $\gamma^* E_*$. Therefore, $f(F_*)$ is parabolic semistable.

As $\boldsymbol{\alpha}$ is a generic system of weights, it now follows that $E_* \,=\, f(F_*)$ is parabolic stable.

It will be shown that $\mathrm{Im}(f) \,\subseteq\, \left(\mathcal{M}_{\mathrm{X}}^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{r},\ \xi)\right)^{\mathrm{L}}$ (see (3.8)). There is a tautological trivialization of $\gamma^* L$ over Y, which induces an isomorphism

$$\theta \;:\; \mathcal{O}_Y \;\xrightarrow{\;\simeq\;}\; \gamma^* L. \tag{3.10}$$

This induces an isomorphism

$$\mathrm{Id}_F \otimes \theta \;:\; F \;\xrightarrow{\;\simeq\;}\; F \otimes \gamma^* L.$$

Consider

$$\psi \;:=\; \gamma_*(\mathrm{Id}_F \otimes \theta) \,:\, \gamma_* F \;\xrightarrow{\;\simeq\;}\; \gamma_*(F\otimes \gamma^* L) \;=\; (\gamma_* F)\otimes L \;=\; E\otimes L;$$

the above $\gamma_*(F\otimes \gamma^* L) \,=\, (\gamma_* F)\otimes L$ is given by the projection formula. Now $E_p \,=\, \bigoplus_{i=1}^{d} F_{q_i}$ (see (3.3)), and the map $\psi_p \,:\, E_p \,\longrightarrow\, E_p \otimes L_p$ on the fiber takes F_{q_i} to $F_{q_i} \otimes L_p$, which clearly implies that ψ_p preserves the filtration induced on E_p. Thus we have $\mathrm{Im}(f) \,\subseteq\, \left(\mathcal{M}_{\mathrm{X}}^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{r},\ \xi)\right)^{\mathrm{L}}$.

To show that the map f is surjective, take any $E_* \,\in\, \left(\mathcal{M}_{\mathrm{X}}^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{r},\ \xi)\right)^{\mathrm{L}}$, so there is an isomorphism of parabolic bundles

$$\varphi_* \;:\; E_* \;\xrightarrow{\;\simeq\;}\; E_* \otimes L.$$

Since E_* is parabolic stable, it is simple, and hence any parabolic endomorphism of E_* is a constant scalar multiplication. As a consequence, any two parabolic isomorphisms from E_* to $E_* \otimes L$ will differ by a constant scalar multiplication. Thus, we can re-scale φ_* by multiplying with a nonzero scalar, so that the d–fold composition

$$\underbrace{\varphi_* \circ \cdots \circ \varphi_*}_{\text{d-times}} \,:\, E_* \longrightarrow E_* \otimes L^d$$

coincides with $\mathrm{Id}_{E_*} \otimes s_0$, where s_0 is the nowhere vanishing section of L^d used in the construction of Y (see (3.3)). It follows from the proof of [BHog, Lemma 2.1] that there exists a vector bundle F on Y of rank $\frac{r}{d}$ with $\gamma_*(F) \cong E$; the argument from [BHog] will be briefly recalled. Consider the pull-back $\gamma^*\varphi$, and compose it with the tautological trivialization of γ^*L to get a homomorphism

$$\phi \,:\, \gamma^*E \longrightarrow \gamma^*E.$$

Since Y is irreducible, the characteristic polynomial of ϕ_y remains unchanged as $y \in Y$ moves. Consequently, γ^*E decomposes into eigenspace sub-bundles. If F is an eigenspace sub-bundle of γ^*E, then $\gamma_*F \cong E$. Moreover, the decomposition

$$\gamma^*E = \bigoplus_{i=1}^{d} (\mu^i)^*F$$

is precisely the decomposition of γ^*E into eigenspace sub-bundles.

Our next task is to produce a parabolic structure on F along $\gamma^{-1}(p)$ so that the parabolic push-forward $\gamma_*(F_*)$ (in the sense of § 2.1) coincides with E_*. First, recall the description of θ in (3.10), and notice that for any choice of $q \in \gamma^{-1}(p)$, the map ϕ_q is precisely the composition of maps

$$(\gamma^*E)_q = E_p \xrightarrow{\varphi_p} E_p \otimes L_p = (\gamma^*E)_q \otimes (\gamma^*L)_q \xrightarrow{Id\otimes(\theta_q)^{-1}} (\gamma^*E)_q, \tag{3.11}$$

where the map $\theta_q : \mathbb{C} \longrightarrow (\gamma^*L)_q = L_p$ is defined by $z \longmapsto z \cdot q$. Thus, if

$$E_p = E_{p,1} \supsetneq E_{p,2} \supsetneq \cdots \supsetneq E_{p,\ell(p)} \supsetneq 0$$

is the given parabolic filtration of the fiber E_p, then as φ_* is a parabolic isomorphism, the following holds:

$$\forall\, j \in [1,\, \ell(p)],\ \ \big(\varphi_p(E_{p,j}) = E_{p,j} \otimes L_p\big) \implies \big(\phi_q(E_{p,j}) = E_{p,j}\big) \quad [\text{from } (3.11)]. \tag{3.12}$$

We have thus obtained a diagonalizable linear map $\phi_q : E_p \longrightarrow E_p$ with the property that the filtration of E_p given by

$$E_p = E_{p,1} \supsetneq \cdots \supsetneq E_{p,\ell(p)} \supsetneq 0$$

is preserved by ϕ_q. Now Lemma 3.1 guarantees the existence of a basis $\mathcal{B}_j$ of $E_{p,j}$ consisting of eigenvectors for each i, satisfying $\mathcal{B}_j \supset \mathcal{B}_{j+1}$ for all $1 \leq j \leq \ell(p)$.

From these data, a weighted filtration is prescribed on the fiber F_{q_j} of the vector bundle F for each $q_j \in \gamma^{-1}(p)$ in the following manner: for each $k \in [1,\, \ell(p)]$, whenever $\mathcal{B}_k \cap F_{q_j} \neq \emptyset$, define the following subspace of F_{q_j}:

$$F_{q_j,k} := \langle \mathcal{B}_k \cap F_{q_j} \rangle \quad \text{(the subspace generated by the intersection)}.$$

Now, for any two such numbers $k < k'$, we have $\mathcal{B}_k \supset \mathcal{B}_{k'}$, which implies that $F_{q_j,k} \supset F_{q_j,k'}$. This immediately produces a weighted filtration of F_{q_j} by assigning the weight $\alpha_{p,k}$ to $F_{q_j,k}$ whenever $\mathcal{B}_k \cap F_{q_j} \neq \emptyset$.

By applying this process for all $1 \leq j \leq \ell(p)$, a parabolic vector bundle F_* on Y is constructed, for which (see § 2.1)

$$\gamma_* F_* \;=\; E_*.$$

It is not hard to see that the system of multiplicities associated to such filtration do come from a matrix as in (3.7).

Note that F_* must be parabolic stable, because if $F' \subset F$ is any sub-bundle such that $\mathrm{par}\mu(F'_*) \geq \mathrm{par}\mu(F_*)$ (see Definition 2.3), then the equalities

$$\mathrm{par}\deg(\gamma_*(F'_*)) \;=\; \mathrm{par}\deg(F'_*) \quad \text{and} \quad \mathrm{rank}(\gamma_*(F')) \;=\; m \cdot \mathrm{rank}(F')$$

would imply that $\gamma_*(F') \subset E$ violates the parabolic stability condition of E_*. Thus $F_* \in \mathcal{N}_L^{\mathbf{t}}$ (see (3.9)) for some $\mathbf{t} \in \boldsymbol{T}$ (see (3.4)). and the proposition follows. $\square$

Remark 3.4. If the cardinality of the set of parabolic points $S \subset X$ is greater than 1, we repeat this construction for each of the parabolic points in S. Say, for example, $S = \{p_1,\ p_2,\ \cdots,\ p_n\}$; then for each $1 \leq i \leq n$, define the set $\boldsymbol{T}_i$ analogous to $\boldsymbol{T}$ in (3.4), and replace $\boldsymbol{T}$ by $\boldsymbol{T}_1 \times \boldsymbol{T}_2 \times \cdots \boldsymbol{T}_n$. It is not difficult to see that a similar argument as in Proposition 3.3 will prove the same result, albeit the notation will get more cumbersome.

Corollary 3.5. *Let $\boldsymbol{\alpha}$ be a generic system of weights. The codimension of the closed subscheme*

$$Z_{\boldsymbol{\alpha}} \;:=\; \bigcup_{L \in \Gamma \setminus \{\mathcal{O}_X\}} (\mathcal{M}_{\mathrm{X}}^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{r},\ \xi))^{\mathrm{L}} \;\subset\; \mathcal{M}_{\mathrm{X}}^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{r},\ \xi)$$

is at least three.

Proof. Recall the filtration data along the parabolic points S as described in (2.1). Now, for any parabolic point $p \in S$, the flag variety of all filtrations of $\mathbb{C}^r$ of flag type $(m_{p,1},\, m_{p,2},\, \cdots,\, m_{p,\ell(p)})$ (where $m_{p,j}$'s are as in Definition 2.1) is of dimension

$$\sum_{i=1}^{\ell(p)-1} m_{p,i}\left(m_{p,i+1} \;+\; \cdots \;+ m_{p,\ell(p)}\right) \quad \text{(see, e.g. [Bu, 2.1].)} \tag{3.13}$$

Now, from [MS] we get dimension of the moduli space $\mathcal{M}_{\mathrm{X}}^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{r},\ \xi)$ to be

$$\dim\left(\mathcal{M}_{\mathrm{X}}^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{r},\ \xi)\right) \;=\; (r^2-1)(g-1) + \sum_{p \in S} \sum_{i=1}^{\ell(p)-1} m_{p,i}\left(m_{p,i+1} \;+\; \cdots \;+ m_{p,\ell(p)}\right).$$

Moreover, writing $\gamma^{-1}(p) = \{q_{p,1},\ \cdots,\ q_{p,d}\}$ for each $p \in S$, by condition (b) on the collection $A_{\mathbf{t}}$ (see (3.6)),

$$\sum_{j=1}^{d} n_{q_{p,j},i} \;=\; m_{p,i} \quad \text{for each} \;\; 1 \leq i \leq \ell(p) \;\; \text{and} \;\; p \in S.$$

Thus, for each $\mathbf{t} \in \boldsymbol{T}$ (see (3.4)), using the description of $\mathcal{N}_L^{\mathbf{t}}$ in (3.9) and by allowing some of the $n_{q_{p,j},i}$'s to be zero in the following equation,

$$\begin{aligned}
\dim(\mathcal{N}_L^{\mathbf{t}}) &= \left(\frac{r^2}{d^2}(\text{genus}(Y)-1)+1\right) - g + \sum_{p\in S}\left(\sum_{j=1}^{d}\sum_{i=1}^{\ell(p)-1} n_{q_{p,j},i}\left(n_{q_{p,j},i+1} + \cdots + n_{q_{p,j},\ell(p)}\right)\right)\\
&\quad \text{(by (3.13))}\\
&= (g-1)(\frac{r^2}{d}-1) + \sum_{p\in S}\left(\sum_{j=1}^{d}\sum_{i=1}^{\ell(p)-1} n_{q_{p,j},i}\left(n_{q_{p,j},i+1} + \cdots + n_{q_{p,j},\ell(p)}\right)\right)\\
&\quad [\text{because } \text{genus}(Y) = d(g-1)+1]\\
&\le (g-1)(\frac{r^2}{d}-1) + \sum_{p\in S}\left(\sum_{j=1}^{d}\sum_{i=1}^{\ell(p)-1} n_{q_{p,j},i}\left(m_{p,i+1} + \cdots + m_{p,\ell(p)}\right)\right)\\
&\quad [\text{because } n_{q_{p,j},i} \le m_{p,i} \;\; \forall\; i,\, j]
\end{aligned}$$

It follows that, for each $\mathbf{t} \in \boldsymbol{T}$,

$$\begin{aligned}
&\dim\left(\mathcal{M}_{\mathrm{X}}^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{r},\ \xi)\right) - \dim\left(\mathcal{N}_L^{\mathbf{t}}\right)\\
&\ge\; r^2(g-1)\left(1-\frac{1}{d}\right) + \sum_{p\in S}\left(\sum_{i=1}^{\ell(p)-1}\left(m_{p,i+1} + \cdots + m_{p,\ell(p)}\right)\left(m_{p,i} - \sum_{j=1}^{d} n_{q_j,i}\right)\right)\\
&\ge\; r^2(g-1)\left(1-\frac{1}{d}\right) \qquad \left[\text{because } m_{p,i} = \sum_{j=1}^{d} n_{q_j,i} \text{ by condition (b)}\right]\\
&\ge\; 3.
\end{aligned}$$

The last inequality follow from the fact that we always have $d \ge 2$, as well as due to the assumptions on r and g mentioned in the beginning, namely $r \ge 2$ and $g \ge 2$; and if $r = 2$, we assume $g \ge 3$.. The result now follows from Proposition 3.3 and Remark 3.4. □

4. Brauer group of the moduli stack of parabolic $\mathrm{PGL}(r, \mathbb{C})$-bundles

As before, fix a finite set of points $S \subset X$, a positive integer r and line bundle ξ on X. As before, we assume $r \ge 2$; if $r = 2$, we assume $g \ge 3$.

Definition 4.1. A parabolic stable $\mathrm{PGL}(r, \mathbb{C})$-bundle is an equivalence class of parabolic stable vector bundles, where two parabolic stable vector bundles E_* and E'_* are considered equivalent if there exists a line bundle $\mathcal{L}$ such that $E'_* \simeq E_* \otimes \mathcal{L}$ as parabolic vector bundles.

Let $\delta \in [0,\, r-1]$ be the unique integer satisfying the condition $\deg(\xi) \equiv \delta \pmod{r}$. The coarse moduli space of parabolic stable $\mathrm{PGL}(r, \mathbb{C})$–bundles on X of topological type δ and systems of multiplicities and weights $(\boldsymbol{m},\, \boldsymbol{\alpha})$ will be denoted by $\mathcal{N}_X^{\boldsymbol{m},\boldsymbol{\alpha}}(r, \delta)$. Also, denote by $\mathfrak{N}_X^{\boldsymbol{m},\boldsymbol{\alpha}}(r, \delta)$ the moduli *stack* of parabolic stable $\mathrm{PGL}(r, \mathbb{C})$–bundles.

Theorem 4.2. *Let $\boldsymbol{\alpha}$ be a generic system of weights. Let $\mathcal{N}_{\mathrm{X}}^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{r}, \delta)^{sm}$ denote the smooth locus of $\mathcal{N}_{\mathrm{X}}^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{r}, \delta)$. Then,*

$$\mathrm{Br}\left(\mathfrak{N}_{\mathrm{X}}^{\boldsymbol{m},\boldsymbol{\alpha}}(r, \delta)\right) \;\simeq\; \mathrm{Br}\left(\mathcal{N}_{\mathrm{X}}^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{r}, \delta)^{sm}\right).$$

Proof. Recall that the group Γ of r-torsion line bundles on X act on the moduli space $\mathcal{M}_{\mathrm{X}}^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{r},\ \xi)$ of parabolic stable vector bundles. Through this action, $\mathcal{N}_{\mathrm{X}}^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{r},\delta)$ is naturally isomorphic to the quotient variety $\mathcal{M}_{\mathrm{X}}^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{r},\ \xi)\,/\Gamma$ (see [BLS, p. 196, **6**]).

Let $Z_{\boldsymbol{\alpha}}$ be as in Corollary 3.5, and denote $\mathcal{U} := \mathcal{M}_{\mathrm{X}}^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{r},\ \xi)\setminus Z_{\boldsymbol{\alpha}}$. Consider the following diagram:

$$\begin{tikzcd}
{[\mathcal{U}/\Gamma]} \arrow[r, hook] \arrow[d] & {[\mathcal{M}_{\mathrm{X}}^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{r},\ \xi)\,/\Gamma]} \arrow[r, equal] \arrow[d] & \mathfrak{N}_{X}^{\boldsymbol{m},\boldsymbol{\alpha}}(r,\delta) \arrow[d] \\
\mathcal{U}/\Gamma \arrow[r, hook] & \mathcal{M}_{\mathrm{X}}^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{r},\ \xi)\,/\Gamma \arrow[r, equal] & \mathcal{N}_{\mathrm{X}}^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{r},\delta)
\end{tikzcd} \tag{4.1}$$

where each of the top horizontal arrows correspond to an inclusion of open sub-stack, while the bottom horizontal arrows correspond to inclusions of open sub-schemes; the vertical maps correspond to morphisms to coarse moduli spaces. Since taking quotient by the finite group Γ is a finite morphism and hence codimension-preserving, it follows from Corollary 3.5 that the complement of $\mathcal{U}/\Gamma$ in $(\mathcal{N}_{X}^{\boldsymbol{m},\boldsymbol{\alpha}}(r,\delta))^{sm}$ is of codimension at least 3. For a similar reason, the complement of the open sub-stack $[\mathcal{U}/\Gamma]$ in $\mathfrak{N}_{X}^{\boldsymbol{m},\boldsymbol{\alpha}}(r,\delta)$ is of codimension at least 3 as well. As $\mathfrak{N}_{X}^{\boldsymbol{m},\boldsymbol{\alpha}}(r,\delta)$ is a Deligne-Mumford stack, it follows that

$$\mathrm{Br}\,([\mathcal{U}/\Gamma]) \ \simeq\ \mathrm{Br}\,(\mathfrak{N}_{X}^{\boldsymbol{m},\boldsymbol{\alpha}}(r,\delta))$$

(see [BCD1, Proposition 4.2]). Now, as the action of Γ on $\mathcal{U}$ is free, the left-most vertical arrow in the diagram (4.1) is an isomorphism. Hence

$$\mathrm{Br}\,([\mathcal{U}/\Gamma]) \ \simeq\ \mathrm{Br}\,(\mathcal{U}/\Gamma)\,.$$

Also, $\mathcal{M}_{\mathrm{X}}^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{r},\ \xi)$ is a smooth variety, and hence $\mathcal{U}$ is smooth. As Γ acts freely on $\mathcal{U}$, the quotient $\mathcal{U}/\Gamma$ is also smooth. The complement of $\mathcal{U}/\Gamma$ in $(\mathcal{N}_{\mathrm{X}}^{\boldsymbol{m},\boldsymbol{\alpha}}(\mathrm{r},\delta))^{sm}$ clearly has codimension at least 3 as well. Thus we have

$$\mathrm{Br}\,(\mathfrak{N}_{X}^{\boldsymbol{m},\boldsymbol{\alpha}}(r,\delta)) \ \simeq\ Br\,([\mathcal{U}/\Gamma]) \ \simeq\ \mathrm{Br}\,(\mathcal{U}/\Gamma) \ \simeq\ \mathrm{Br}\,(\mathcal{N}_{X}^{\boldsymbol{m},\boldsymbol{\alpha}}(r,\delta)^{sm})\,,$$

where the last isomorphism follows from [Ce, Theorem 1.1]. This completes the proof. $\square$

5. Brauer group of moduli stack of quasi-parabolic G–bundles

Let G be a simple and simply connected affine algebraic group over $\mathbb{C}$. Let X be a smooth projective curve over $\mathbb{C}$ of genus g, with $g \geq 2$. We aim to compute the Brauer group of the moduli stack of quasi-parabolic G–bundles on X. To be more precise, following notation of [LS], let

$$S \ = \ \{p_1,\ \cdots,\ p_n\}$$

be the set of parabolic points on X, and fix a parabolic subgroup $P_i \subset G$ for each p_i. Let $\underline{p}$ and $\underline{P}$ denote the sequences $(p_1,\ p_2,\ \cdots,\ p_n)$ and $(P_1,\ P_2,\ \cdots,\ P_n)$ respectively. For a principal G–bundle E on X, and each $i \in [1,\ n]$, let $E(G/P_i) \longrightarrow X$ denote the fiber bundle associated to E for the left-translation action of G on G/P_i.

Definition 5.1 ([LS, (8.4)]). A *quasi-parabolic principal* G*-bundle* of type $\underline{P}$ is a principal G-bundle E on X, together with an element F_i in the fiber $E(G/P_i)_{p_i}$ of $E(G/P_i)$ over p_i.

Remark 5.2. Consider the case where $G = \mathrm{SL}(r,\mathbb{C})$. A principal $\mathrm{SL}(r,\mathbb{C})$–bundle corresponds to a vector bundle E of rank r with $\det(E) \simeq \mathcal{O}_X$. Parabolic subgroups of $\mathrm{SL}(r,\mathbb{C})$ correspond to block-upper triangular matrices. So the associated fiber bundle $E\,(G/P_i)$ is a flag bundle for E of flag type determined by P_i. Choosing of an element F_i in the fiber $E\,(G/P_i)_{p_i}$ is equivalent to fixing a filtration of the fiber E_{p_i} whose multiplicities (see Definition 2.1) are given by the dimensions of the blocks for the subgroup P_i of block-upper triangular matrices. This way, Definition 5.1 connects to earlier Definition 2.1 in the case when $G = \mathrm{SL}(r, \mathbb{C})$.

Let $\mathcal{M}_G^{par}\left(\underline{p},\ \underline{P}\right)$ denote the moduli stack of quasi-parabolic principal G-bundles on X of type $\underline{P}$ (Definition 5.1). There is a very useful result known as the "uniformization theorem", which expresses the stack $\mathcal{M}_{\mathrm{G}}^{\mathrm{par}}(\underline{p},\ \underline{P})$ as a quotient stack of a certain scheme under the action of a group scheme, which is briefly recalled below (see [LS] for more details).

Consider the ind-Grassmannian $\mathcal{Q}_G$, which is a direct limit of a sequence of integral projective varieties. Define

$$\mathcal{Q}_G^{par}\left(\underline{p},\ \underline{P}\right) := \mathcal{Q}_G \times \prod_{i=1}^{n} G/P_i.$$

Fix a base-point $p_0 \in X \setminus S$. The ind-group scheme $L_X(G) := G(\mathcal{O}(X \setminus \{p_0\}))$ acts on $\mathcal{Q}_G^{par}\left(\underline{p},\ \underline{P}\right)$ on the left. Consider the quotient stack $\left[L_X(G)\backslash \mathcal{Q}_G^{par}\left(\underline{p},\ \underline{P}\right)\right]$.

Theorem 5.3 ([LS, Theorem 8.5])**.** *There is a canonical isomorphism of stacks which is locally trivial in the étale topology:*

$$\left[L_X(G)\backslash \mathcal{Q}_G^{par}\left(\underline{p},\ \underline{P}\right)\right] \ \simeq \ \mathcal{M}_{\mathrm{G}}^{\mathrm{par}}\left(\underline{p},\ \underline{P}\right). \tag{5.1}$$

For convenience, $\mathcal{M}_{\mathrm{G}}^{\mathrm{par}}\left(\underline{p},\ \underline{P}\right)$ and $\mathcal{Q}_G^{par}\left(\underline{p},\ \underline{P}\right)$ will be denoted simply by $\mathcal{M}_G^{par}$ and $\mathcal{Q}_G^{par}$ respectively. Also, denote

$$F \ := \ \prod_{i=1}^{n} G/P_i.$$

For an algebraic stack $\mathscr{X}$ over $\mathbb{C}$, let $H_B^*\left(\mathscr{X},\ \mathbb{Z}\right)$ denote its singular cohomology. The following lemma describes the singular cohomology of the stack $\mathcal{M}_{\mathrm{G}}^{\mathrm{par}}$.

Lemma 5.4. *The following holds:*

$$H_B^*\left(\mathcal{M}_{\mathrm{G}}^{\mathrm{par}},\ \mathbb{Z}\right) \ \simeq \ H_B^*\left(\mathcal{M}_G,\ \mathbb{Z}\right) \otimes H_B^*\left(F,\ \mathbb{Z}\right)$$

Proof. We have the following Cartesian diagram

$$\begin{tikzcd} \mathcal{Q}_G \times F = \mathcal{Q}_G^{par} \arrow[r] \arrow[d] & \mathcal{M}_{\mathrm{G}}^{\mathrm{par}} \arrow[d] \\ \mathcal{Q}_G \arrow[r] & \mathcal{M}_G \end{tikzcd} \tag{5.2}$$

where both the horizontal arrows are principal $L_X(G)$–bundles, and the left vertical arrow is the first projection. Denote by $\mathcal{Q}_{G,\bullet}$ the simplicial space of fibered powers of $\mathcal{Q}_G$ over $\mathcal{M}_G$ (see [Te, p. 21, § 4]); more precisely, for each $n \in \mathbb{Z}_{\geq 0}$, we have

$$\mathcal{Q}_{G,n} \ := \ \mathcal{Q}_G \underbrace{\underset{\mathcal{M}_G}{\times} \mathcal{Q}_G \underset{\mathcal{M}_G}{\times} \cdots \underset{\mathcal{M}_G}{\times}}_{n\ times} \mathcal{Q}_G.$$

Using the fact that $\mathcal{Q}_G \longrightarrow \mathcal{M}_G$ is a principal $L_X(G)$–bundle, it can be shown that

$$H_B^*\left(\mathcal{M}_G,\ \mathbb{Z}\right) \ \simeq \ \mathbb{H}_B^*\left(\mathcal{Q}_{G,\bullet},\ \mathbb{Z}\right); \tag{5.3}$$

see the proof of [Te, Proposition 4.1]. Now, if $\mathcal{Q}^{par}_{G,\bullet}$ analogously denotes the simplicial space of fibered powers of $\mathcal{Q}^{par}_G$ over $\mathcal{M}^{\mathrm{par}}_{\mathrm{G}}$, we observe that for each $n \geq 0$,

$$\begin{aligned}
\mathcal{Q}^{par}_{G,n} &= \mathcal{Q}^{par}_G \underbrace{\underset{\mathcal{M}^{\mathrm{par}}_{\mathrm{G}}}{\times} \mathcal{Q}^{par}_G \underset{\mathcal{M}^{\mathrm{par}}_{\mathrm{G}}}{\times} \cdots \underset{\mathcal{M}^{\mathrm{par}}_{\mathrm{G}}}{\times}}_{n \text{ times}} \mathcal{Q}^{par}_G \\
&\simeq (\mathcal{Q}_G \underbrace{\underset{\mathcal{M}_G}{\times} \mathcal{Q}_G \underset{\mathcal{M}_G}{\times} \cdots \underset{\mathcal{M}_G}{\times}}_{n \text{ times}} \mathcal{Q}_G) \times F \quad \text{(using the Cartesian diagram (5.2))} \\
&\simeq \mathcal{Q}_{G,n} \times F.
\end{aligned}$$

In other words, $\mathcal{Q}^{par}_{G,\bullet} \simeq \mathcal{Q}_{G,\bullet} \times F$ as simplicial spaces, and consequently by Leray–Hirsch theorem it follows that

$$\mathbb{H}^*_B\left(\mathcal{Q}^{par}_{G,\bullet},\ \mathbb{Z}\right) \simeq \mathbb{H}^*_B\left(\mathcal{Q}_{G,\bullet},\ \mathbb{Z}\right) \otimes H^*_B\left(F,\ \mathbb{Z}\right). \tag{5.4}$$

Now, since $\mathcal{Q}^{par}_G \longrightarrow \mathcal{M}^{\mathrm{par}}_{\mathrm{G}}$ is a principal $L_X(G)$–bundle, as in (5.3) we again get

$$H^*_B\left(\mathcal{M}^{\mathrm{par}}_{\mathrm{G}},\ \mathbb{Z}\right) \simeq \mathbb{H}^*_B\left(\mathcal{Q}^{par}_{G,\bullet},\ \mathbb{Z}\right).$$

Using (5.4), we conclude that

$$H^*_B\left(\mathcal{M}^{\mathrm{par}}_{\mathrm{G}},\ \mathbb{Z}\right) \simeq \mathbb{H}^*_B\left(\mathcal{Q}_{G,\bullet},\ \mathbb{Z}\right) \otimes H^*_B\left(F,\ \mathbb{Z}\right) \underset{(5.3)}{\simeq} H^*_B\left(\mathcal{M}_G,\ \mathbb{Z}\right) \otimes H^*_B\left(F,\ \mathbb{Z}\right).$$

This proves the lemma. □

The following result helps in comparing the Brauer group of an algebraic stack with certain Betti cohomology group, under certain assumptions on the stack.

Proposition 5.5 ([BHol, Proposition 2.1]). *Let Y be an algebraic stack satisfying the following two properties:*

(1) Any class in $H^2_B(Y,\ \mathbb{Z})$ is represented by a holomorphic line bundle on Y.
(2) Each holomorphic line bundle on Y admits an algebraic structure.

Then there are isomorphisms

$$\mathrm{Br}\,(Y) \simeq H^2(Y,\ \mathcal{O}^*_{Y,an})_{torsion} \simeq H^3_B(Y,\ \mathbb{Z})_{torsion}.$$

Lemma 5.6. *Let $\mathcal{O}^{an}_{\mathcal{Q}^{par}_G}$ denote the sheaf of analytic functions on $\mathcal{Q}^{par}_G$. Then*

$$\mathrm{Br}\,(\mathcal{Q}^{par}_G) \simeq H^2\left(\mathcal{Q}^{par}_G,\ (\mathcal{O}^{an}_{\mathcal{Q}^{par}_G})^*\right)_{torsion} = 0.$$

Proof. By definition, $\mathcal{Q}^{par}_G = \mathcal{Q}_G \times F$, where $F := \prod_{i=1}^n G/P_i$. It is known that $\mathcal{Q}_G$ is the direct limit of certain projective varieties $\{Q_w\}$, where w varies in the affine Weyl group (see the proof of [BHol, Proposition 3.1] for details). Thus $\mathcal{Q}^{par}_G$ is the direct limit of the projective varieties $\{Q_w \times F\}$. Also,

$$H^1\left(Q_w,\ \mathbb{Z}\right) = H^3\left(Q_w,\ \mathbb{Z}\right) = 0$$

(see the proof of [BHol, Proposition 3.1]). Now, since F is a product of flag varieties, its singular cohomology groups exist only in even degrees. Therefore, the Künneth formula gives the following:

$$\begin{aligned}
H^1_B\left(Q_w \times F,\ \mathbb{Z}\right) &= H^3_B\left(Q_w \times F,\ \mathbb{Z}\right) = 0, \\
\text{and}\quad H^2_B\left(Q_w \times F,\ \mathbb{Z}\right) &\simeq H^2_B(Q_w,\ \mathbb{Z}) \oplus H^2_B(F,\ \mathbb{Z}).
\end{aligned}$$

As $\mathcal{Q}_G^{par}$ is the direct limit of the varieties $\{Q_w \times \ F\}$, it follows that

$$H_B^1\left(\mathcal{Q}_G^{par},\ \mathbb{Z}\right) \ = \ H_B^3\left(\mathcal{Q}_G^{par},\ \mathbb{Z}\right) \ = \ 0, \tag{5.5}$$

$$\text{and} \ \ H_B^2(\mathcal{Q}_G^{par},\ \mathbb{Z}) \ \simeq \ H_B^2(\mathcal{Q}_G,\ \mathbb{Z}) \ \oplus \ H_B^2(F,\ \mathbb{Z}). \tag{5.6}$$

Let $\mathcal{O}_{G/P_i}$ denote the sheaf of regular functions on G/P_i. Since each G/P_i is a rational variety, it follows that $H^1\left(G/P_i,\ \mathcal{O}_{G/P_i}\right) = 0$. Using [Ha, III Ex. 12.6] together with [KN, p. 157, Lemma 2.2], it follows that

$$\operatorname{Pic}\left(\mathcal{Q}_G^{par}\right) \ = \ \operatorname{Pic}\left(\mathcal{Q}_G\right) \oplus \bigoplus_{i=1}^{n} \operatorname{Pic}\left(G/P_i\right). \tag{5.7}$$

We now aim to invoke Proposition 5.5 for the stack $\mathcal{Q}_G^{par}$. It was argued in the proof of [BHol, Proposition 3.1] that $\mathcal{Q}_G$ satisfies the two assumptions of Proposition 5.5. The same is true for the variety F as well. Combining (5.6) and (5.7), we can immediately conclude that $\mathcal{Q}_G^{par}$ also satisfies the two assumptions of Proposition 5.5. Finally, it follows from (5.5) and the comparison isomorphisms in Proposition 5.5 that

$$\operatorname{Br}\left(\mathcal{Q}_G^{par}\right) \ \simeq \ H^2\left(\mathcal{Q}_G^{par},\ (\mathcal{O}^{an}_{\mathcal{Q}_G^{par}})^*\right)_{torsion} \ \simeq \ H_B^3\left(\mathcal{Q}_G^{par},\ \mathbb{Z}\right) \ = \ 0.$$

This proves the lemma. $\square$

Proposition 5.7. *Let G be a simple and simply connected complex affine algebraic group. Let $\mathcal{O}^{an}_{\mathcal{M}_{\mathrm{G}}^{\mathrm{par}}}$ denote the sheaf of analytic functions on $\mathcal{M}_{\mathrm{G}}^{\mathrm{par}}$. The analytic Brauer group of $\mathcal{M}_{\mathrm{G}}^{\mathrm{par}}$ vanishes, namely*

$$H^2\left(\mathcal{M}_{\mathrm{G}}^{\mathrm{par}},\ (\mathcal{O}^{an}_{\mathcal{M}_{\mathrm{G}}^{\mathrm{par}}})^*\right)_{torsion} \ = \ 0.$$

Proof. Let $\mathcal{O}^{an}_{\mathcal{Q}_G^{par}}$ denote the sheaf of analytic functions on $\mathcal{Q}_G^{par}$. Let $BL_X(G)$ denote the classifying space for $L_X(G)$. Using the fact that the quotient morphism $\mathcal{Q}_G^{par} \longrightarrow \mathcal{M}_{\mathrm{G}}^{\mathrm{par}}$ is a principal $L_X(G)$–bundle, the Leray spectral sequence gives rise to the following long exact sequence in analytic topology (cf. [BHol, (4.1)]):

$$\begin{gathered} \cdots \ \longrightarrow \ H^2\left(BL_X(G),\ \mathbb{C}^*\right) \ \longrightarrow \\ \operatorname{Ker}\left[H^2\left(\mathcal{M}_{\mathrm{G}}^{\mathrm{par}},\ (\mathcal{O}^{an}_{\mathcal{M}_{\mathrm{G}}^{\mathrm{par}}})^*\right) \ \longrightarrow \ H^0\left(BL_X(G),\ H^2\left(\mathcal{Q}_G^{par},\ (\mathcal{O}^{an}_{\mathcal{Q}_G^{par}})^*\right)\right)\right] \\ \longrightarrow \ H^1\left(BL_X(G),\ H^1\left(\mathcal{Q}_G^{par},\ (\mathcal{O}^{an}_{\mathcal{Q}_G^{par}})^*\right)\right). \end{gathered} \tag{5.8}$$

In (5.8), we have $H^2\left(BL_X(G),\ \mathbb{C}^*\right) \ = \ 0$ [BHol, Proposition 3.4]. Let us compute the last term in (5.8). Each $\operatorname{Pic}(G/P_i)$ is a free abelian group of finite rank, and the same is true for the group $\operatorname{Pic}(\mathcal{Q}_G)$ [BLS, Lemma 1.4]. Using these, (5.7) implies that

$$H^1\left(\mathcal{Q}_G^{par},\ (\mathcal{O}^{an}_{\mathcal{Q}_G^{par}})^*\right) \ = \ \mathbb{Z}^t$$

for some positive integer t. Since $H^1(BL_X(G),\ \mathbb{Z}) = \ 0$ [BHol, Proposition 3.4], it follows that

$$H^1\left(BL_X(G),\ H^1\left(\mathcal{Q}_G^{par},\ (\mathcal{O}^{an}_{\mathcal{Q}_G^{par}})^*\right)\right) \ = \ H^1\left(BL_X(G),\ \mathbb{Z}^t\right) \ = \ 0.$$

Moreover, as the space $BL_X(G)$ is connected,

$$H^0\left(BL_X(G),\ H^2\left(\mathcal{Q}_G^{par},\ (\mathcal{O}^{an}_{\mathcal{Q}_G^{par}})^*\right)\right) \ \simeq \ H^2\left(\mathcal{Q}_G^{par},\ (\mathcal{O}^{an}_{\mathcal{Q}_G^{par}})^*\right).$$

Therefore, the exact sequence in (5.8) implies the following inclusion of abelian groups:

$$H^2\left(\mathcal{M}_{\mathrm{G}}^{\mathrm{par}},\ (\mathcal{O}^{an}_{\mathcal{M}_{\mathrm{G}}^{\mathrm{par}}})^*\right) \ \hookrightarrow \ H^2\left(\mathcal{Q}_G^{par},\ (\mathcal{O}^{an}_{\mathcal{Q}_G^{par}})^*\right),$$

which implies that

$$H^2\left(\mathcal{M}_{\mathrm{G}}^{\mathrm{par}},\ (\mathcal{O}^{an}_{\mathcal{M}_{\mathrm{G}}^{\mathrm{par}}})^*\right)_{torsion} \ \hookrightarrow \ H^2\left(\mathcal{Q}_G^{par},\ (\mathcal{O}^{an}_{\mathcal{Q}_G^{par}})^*\right)_{torsion}.$$

Now Lemma 5.6 completes the proof. □

Finally, we prove that the algebraic Brauer group of $\mathcal{M}_{\mathrm{G}}^{\mathrm{par}}$ vanishes, by comparing it with the analytic Brauer group of $\mathcal{M}_{\mathrm{G}}^{\mathrm{par}}$.

Theorem 5.8. *Let G be a simple and simply connected complex affine algebraic group. Then*

$$\mathrm{Br}\,(\mathcal{M}_{\mathrm{G}}^{\mathrm{par}}) \;=\; 0.$$

Proof. We have $H^2\left(\mathcal{M}_{\mathrm{G}}^{\mathrm{par}},\, (\mathcal{O}^{an}_{\mathcal{M}_{\mathrm{G}}^{\mathrm{par}}})^*\right)_{torsion} \;=\; 0$ by Proposition 5.7. We would now like to compare this group with the corresponding algebraic Brauer group. This will be done by showing that the two conditions in Proposition 5.5 are satisfied for the stack $\mathcal{M}_{\mathrm{G}}^{\mathrm{par}}$.

(1) Using the facts that $H^1_B(F,\, \mathbb{Z}) \;=\; 0$ as well as that F and $\mathcal{M}_G$ are connected (since G is simply connected), Lemma 5.4 implies that

$$\begin{aligned} H^2_B(\mathcal{M}_{\mathrm{G}}^{\mathrm{par}},\, \mathbb{Z}) \;&\simeq\; \left(H^2_B(\mathcal{M}_G,\, \mathbb{Z}) \otimes H^0_B(F,\, \mathbb{Z})\right) \bigoplus \left(H^0_B(\mathcal{M}_{\mathrm{G}}^{\mathrm{par}},\, \mathbb{Z}) \otimes H^2_B(F,\, \mathbb{Z})\right) \\ &\simeq\; H^2_B(\mathcal{M}_G,\, \mathbb{Z}) \bigoplus H^2_B(F,\, \mathbb{Z})\,. \end{aligned} \tag{5.9}$$

Moreover, by [LS, Theorem 1.1 and Proposition 8.7] we have

$$\mathrm{Pic}\,(\mathcal{M}_{\mathrm{G}}^{\mathrm{par}}) \;\simeq\; \mathrm{Pic}\,(\mathcal{M}_G) \;\oplus\; \mathrm{Pic}(F). \tag{5.10}$$

Now, by (5.9) and (5.10), the first Chern class map

$$\mathrm{Pic}(\mathcal{M}_{\mathrm{G}}^{\mathrm{par}}) \;\longrightarrow\; H^2_B(\mathcal{M}_{\mathrm{G}}^{\mathrm{par}},\, \mathbb{Z})$$

splits as a direct sum of the Chern class maps $\mathrm{Pic}\,(\mathcal{M}_G) \;\longrightarrow\; H^2_B(\mathcal{M}_G,\, \mathbb{Z})$ and $\mathrm{Pic}\,(F) \;\longrightarrow\; H^2_B(F,\, \mathbb{Z})$. As both these maps are surjective (cf. [Te, Proposition 5.1 and Remark 5.2]), we conclude that the Chern class map $\mathrm{Pic}(\mathcal{M}_{\mathrm{G}}^{\mathrm{par}}) \longrightarrow H^2_B(\mathcal{M}_{\mathrm{G}}^{\mathrm{par}},\, \mathbb{Z})$ is also surjective; in other words, the first condition in Proposition 5.5 is satisfied for $\mathcal{M}_{\mathrm{G}}^{\mathrm{par}}$.

(2) As $\mathrm{Pic}\,(\mathcal{M}_{\mathrm{G}}^{\mathrm{par}}) \;\simeq\; \mathrm{Pic}\,(\mathcal{M}_G) \;\oplus\; \mathrm{Pic}(F)$ by (5.10), it is clear that the second condition of Proposition 5.5 is satisfied for $\mathcal{M}_{\mathrm{G}}^{\mathrm{par}}$ as well.

Thus, from the comparison isomorphisms in Proposition 5.5 together with Proposition 5.7, it follows that

$$\mathrm{Br}\,(\mathcal{M}_{\mathrm{G}}^{\mathrm{par}}) \;\simeq\; H^2\left(\mathcal{M}_{\mathrm{G}}^{\mathrm{par}},\, (\mathcal{O}^{an}_{\mathcal{M}_{\mathrm{G}}^{\mathrm{par}}})^*\right)_{torsion} \;=\; 0.$$

This completes the proof. □

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