Polars of Pin^c groups and related compact Lie groups. II

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Abstract

Abstract. We explicitly describe polars of Ss^c groups and covering homomorphisms between related compact Lie groups with Pin^c groups.

1 Introduction

In the previous paper [3] the author explicitely describe polars of $Pin^c(n)$ and related compact Lie groups

$$Spin^c(n), Pin(n), Spin(n), O^c(n), SO^c(n), O(n), SO(n).$$

For a compact Lie group G with identity element e, each connected component of $\{g \in G \mid g^2 = e\}$ is called a *polar* of G. The geometric background of the polar is described in Nagano [1], Tasaki [3] and the references cited in [3]. In this paper we investigate the polars of $Ss^c(4m)$, the quotient groups of $O^c(n)$ and covering homomorphisms between related compact Lie groups. Polars of $Ss(4m)$ and the quotient groups of $O(n)$ are described in [1]. These results seem to be usefull for considering maximal antipodal subgroups of them and maximal antipodal sets in such polars.

We use notions and symbols described in the previous paper [3]. Let $e_1, \dots, e_{4m}$ be the standad orthonormal basis of $\mathbb{R}^{4m}$. In the Cliffod algebra Cl_{4m} we can see that $e_{[4m]} = e_1 e_2 \cdots e_{4m}$ is in the center of Cl_{4m} and $e_{[4m]}^2 = 1$. Thus we can consider the quotient group $Ss(4m) = Spin(4m)/\langle e_{[4m]} \rangle$, which is called *semi-spin group*. Here $\langle x \rangle$ denotes the subgroup generated by an element x in a group. Using $Spin^c(4m)$ we can define $Ss^c(4m) = Spin^c(4m)/\langle e_{[4m]} \rangle$. The natural projection $\sigma : Spin(4m) \to Ss(4m)$ is a double covering homomorphism with kernel $\ker \sigma = \langle e_{[4m]} \rangle$. We can consider a double covering homomorphism $\sigma^c : Spin^c(4m) \to Ss^c(4m)$ with kernel $\ker \sigma^c = \langle e_{[4m]} \rangle$.

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2 Preliminaries

We will prepare some notation and other information to state the main theorem (Theorem 3.1) of this paper.

In order to describe polars of $Ss^c(4m)$ we recall the symbols defined in [3]:

$$\begin{aligned}
&P_k(4m) = \{v_1 \cdots v_k \in Pin(4m) \mid v_i \in \mathbb{R}^{4m},\ \langle v_i, v_j\rangle = \delta_{ij}\ (i, j \in [k])\},\\
&P_0^+(4m) = \{1\},\ P_0^-(4m) = \{-1\},\\
&P_{4m}^+(4m) = \{e_{[4m]}\},\ P_{4m}^-(4m) = \{-e_{[4m]}\},
\end{aligned}$$

where $0 < k < 4m$. $P_k(4m)$ is diffeomorphic to the real oriented Grassmann manifold $\tilde{G}_k(\mathbb{R}^{4m})$ consisting of oriented subspaces of dimension k in $\mathbb{R}^{4m}$.

The transformation of $\tilde{G}_{2m}(\mathbb{R}^{4m})$ corresponding to the transformation of $P_{2m}(4m)$ induced by the product of $e_{[4m]}$ is the transformation that associates the elements of $\tilde{G}_{2m}(\mathbb{R}^{4m})$ with their orthogonal complements with suitable orientations. Let $G^{\#}_{2m}(\mathbb{R}^{4m})$ denote the quotient space obtained by dividing $\tilde{G}_{2m}(\mathbb{R}^{4m})$ by the transformation mentioned above. $P_{2m}(4m))/\langle e_{[4m]}\rangle$ is diffeomorphic to $G^{\#}_{2m}(\mathbb{R}^{4m})$.

Moreover we define

$$\begin{aligned}
&e_{i,j}(\theta) = \cos\theta + \sin\theta e_i e_j \in Spin(4m) \quad (1 \le i, j \le 4m,\ i \ne j),\\
&\tilde{J}_{2m} = e_{1,2}\left(\frac{1}{4}\pi\right) e_{3,4}\left(\frac{1}{4}\pi\right) \cdots e_{4m-1,4m}\left(\frac{1}{4}\pi\right),\\
&\tilde{J}'_{2m} = e_{1,2}\left(\frac{3}{4}\pi\right) e_{3,4}\left(\frac{1}{4}\pi\right) \cdots e_{4m-1,4m}\left(\frac{1}{4}\pi\right),\\
&\tilde{R}^+(4m) = \{x\tilde{J}_{2m}x^{-1} \mid x \in Spin(4m)\},\\
&\tilde{R}^-(4m) = \{x\tilde{J}'_{2m}x^{-1} \mid x \in Spin(4m)\},\\
&R(4m) = \{g \in SO(4m) \mid g^2 = -1_{4m}\}
\end{aligned}$$

in order to describe polars of $Ss^c(4m)$. $R(4m)$ is the manifold consisting of all orthogonal complex structures on $\mathbb{R}^{4m}$. Since any element of $R(4m)$ is alternating, we can consider the Pfaffian Pf of it. Tanaka and the author [2] showed that

$$R(4m) = R^+(4m) \cup R^-(4m)$$

is a decomposition to cnnected components, where

$$\begin{aligned}
R^+(4m) &= \{g \in R(4m) \mid \mathrm{Pf}(g) = 1\},\\
R^-(4m) &= \{g \in R(4m) \mid \mathrm{Pf}(g) = -1\}.
\end{aligned}$$

We can see that $R^+(4m)$ is diffeomorphic to $SO(4m)/U(2m)$, which is the compact symmetric space denoted by $DIII(2m)$. Let $\rho : Spin(4m) \to SO(4m)$ denote the standard covering homomorphism. In the proof of Theorem 3.1 we will show that

$$\rho^{-1}(R^+(4m)) = \tilde{R}^+(4m)\cup(-\tilde{R}^+(4m)), \quad \rho^{-1}(R^-(4m)) = \tilde{R}^-(4m)\cup(-\tilde{R}^-(4m))$$

are decompositions to connected components and $\tilde{R}^+(4m), \tilde{R}^-(4m)$ are diffeomorphic to $DIII(2m)$.

3 Polars of Ss^c groups

In this section we explicitly describe polars of $Ss^c(4m)$.

Theorem 3.1. *Let e denote the identity element of $Ss^c(4m)$ and $Ss(4m)$. The polars of $Ss^c(4m)$ and $Ss(4m)$ are as follows:*

$$\begin{aligned} F(e, Ss^c(4m)) &= \sigma^c(P_0^+(4m)) \cup \sigma^c(P_0^-(4m)) \\ &\cup \bigcup_{1\le k\le (m+1)/2} \sigma^c(\sqrt{-1}P_{4k-2}(4m)) \cup \bigcup_{1\le k\le m/2} \sigma^c(P_{4k}(4m)) \\ &\cup \sigma^c(\tilde{R}^+(4m)) \cup \sigma^c(-\tilde{R}^+(4m)) \cup \sigma^c(\sqrt{-1}\tilde{R}^-(4m)), \\ F(e, Ss(4m)) &= \sigma^c(P_0^+(4m)) \cup \sigma^c(P_0^-(4m)) \\ &\cup \bigcup_{1\le k\le m/2} \sigma^c(P_{4k}(4m)) \\ &\cup \sigma^c(\tilde{R}^+(4m)) \cup \sigma^c(-\tilde{R}^+(4m)). \end{aligned}$$

$\sigma^c(P_0^+(4m))$ and $\sigma^c(P_0^-(4m))$ are poles and the other polars are as follows:

$$\begin{aligned} \sigma^c(\sqrt{-1}P_{4k-2}(4m)) &\cong \tilde{G}_{4k-2}(\mathbb{R}^{4m}) \quad (1 \le k < (m+1)/2), \\ \sigma^c(\sqrt{-1}P_{2m}(4m)) &\cong G^{\#}_{2m}(\mathbb{R}^{4m}) \quad (m\text{:odd}), \\ \sigma^c(P_{4k}(4m)) &\cong \tilde{G}_{4k}(\mathbb{R}^{4m}) \quad (1 \le k < m/2), \\ \sigma^c(P_{2m}(4m)) &\cong G^{\#}_{2m}(\mathbb{R}^{4m}) \quad (m\text{:even}), \\ \sigma^c(\tilde{R}^+(4m)) &\cong \sigma^c(-\tilde{R}^+(4m)) \cong DIII(2m)/\{\pm 1_{4m}\}, \\ \sigma^c(\sqrt{-1}\tilde{R}^-(4m)) &\cong DIII(2m). \end{aligned}$$

Proof. We define tori $T_l\,(l \le 2m)$ in $Spin(4m)$ and $T_l^c\,(l \le 2m)$ in $Spin^c(4m)$ by

$$T_l = \{e_{1,2}(\theta_1)\cdots e_{2l-1,2l}(\theta_l) \mid \theta_1, \dots, \theta_l \in \mathbb{R}\}, \quad T_l^c = U(1)T_l.$$

We can see that T_{2m} is a maximal torus of $Spin(4m)$ and that T_{2m}^c is a maximal torus of $Spin^c(4m)$. Their images $\sigma(T_{2m})$ and $\sigma^c(T_{2m}^c)$ are maximal tori of $Ss(4m)$ and $Ss^c(4m)$ respectively.

We determine $\{\sigma(\xi) \mid \xi \in T_{2m},\, \sigma(\xi)^2 = \sigma(1)\}$ in order to obtain all polars of $Ss(4m)$. We take $\xi \in T_{2m}$. Since $\ker\sigma = \langle e_{[4m]}\rangle$, the condition $\sigma(\xi)^2 = \sigma(1)$ is equivalent to $\xi^2 = 1$ or $e_{[4m]}$. We consider two cases $\xi^2 = 1$ and $\xi^2 = e_{[4m]}$ separately.

(1) The case where $\xi^2 = 1$. We have already considered such ξ in [3]. We obtained

$$\{\xi \in Spin^c(4m) \mid \xi^2 = 1\}$$

$$
\begin{aligned}
&= P_0^+(4m) \cup P_0^-(4m) \cup \bigcup_{k=1}^{m} \sqrt{-1}P_{4k-2}(4m) \cup \bigcup_{k=1}^{m-1} P_{4k}(4m) \\
&\quad \cup P_{4m}^+(4m) \cup P_{4m}^-(4m)
\end{aligned}
$$

in Corollary 4.2 in [3]. Since the image of a polar in $Spin^c(4m)$ under σ^c is a polar in $Ss^c(4m)$, all of polars in $\{\sigma^c(\xi) \mid \xi \in Spin^c(4m), \xi^2 = 1\}$ are

$$
\begin{aligned}
&\{\sigma^c(\sqrt{-1}P_{4k-2}(4m)) \mid 1 \leq k \leq m\} \cup \{\sigma^c(P_{4k}(4m)) \mid 1 \leq k \leq m-1\} \\
&\cup \{\sigma^c(P_0^+(4m)), \sigma^c(P_0^-(4m)), \sigma^c(P_{4m}^+(4m)), \sigma^c(P_{4m}^-(4m))\}.
\end{aligned}
$$

In this description we have

$$
\begin{aligned}
\sigma^c(P_0^+(4m)) &= \sigma^c(e_{[4m]}P_0^+(4m)) = \sigma^c(P_{4m}^+(4m)), \\
\sigma^c(P_0^-(4m)) &= \sigma^c(e_{[4m]}P_0^-(4m)) = \sigma^c(P_{4m}^-(4m)).
\end{aligned}
$$

We need to give $\sigma^c(\sqrt{-1}P_{4k-2}(4m))$ and $\sigma^c(P_{4k}(4m))$ a similar consideration. We note that $e_{[4m]}P_k(4m) = P_{4m-k}(4m)$ for $0 < k < 4m$. It is sufficient to consider $P_k(4m)$ for $k \leq 2m$.

First we consider $\sigma^c(\sqrt{-1}P_{4k-2}(4m))$ for $2 \leq 4k-2 \leq 2m$. The condition where $2 \leq 4k-2 \leq 2m$ is equivalent to that where $1 \leq k \leq (m+1)/2$. If $1 \leq k \leq (m+1)/2$, we have

$$
\begin{aligned}
\sigma^c(\sqrt{-1}P_{4k-2}(4m)) &= \sigma^c(e_{[4m]}\sqrt{-1}P_{4k-2}(4m)) \\
&= \sigma^c(\sqrt{-1}P_{4m-4k+2}(4m)).
\end{aligned}
$$

Moreover, if $1 \leq k < (m+1)/2$, $\sigma^c(\sqrt{-1}P_{4k-2}(4m))$ and $\sigma^c(\sqrt{-1}P_{4m-4k+2}(4m))$ are diffeomorphic to $P_{4k-2}(4m)$. When m is odd, there can be the case where $k = (m+1)/2$. In this case $4m-4k+2 = 4k-2$ and σ^c is a double covering from $\sqrt{-1}P_{2m}(4m)$ onto $\sigma^c(\sqrt{-1}P_{2m}(4m))$. Therefore $\sigma^c(\sqrt{-1}P_{2m}(4m)) \cong P_{2m}(4m))/\langle e_{[4m]}\rangle$.

Second we consider $\sigma^c(P_{4k}(4m))$ for $4 \leq 4k \leq 2m$. The condition where $4 \leq 4k \leq 2m$ is equivalent to that where $1 \leq k \leq m/2$. If $1 \leq k \leq m/2$, we have

$$
\sigma^c(P_{4k}(4m)) = \sigma^c(e_{[4m]}P_{4k}(4m)) = \sigma^c(P_{4m-4k}(4m)).
$$

Moreover, if $1 \leq k < m/2$, $\sigma^c(P_{4k}(4m))$ and $\sigma^c(P_{4m-4k}(4m))$ are diffeomorphic to $P_{4k}(4m)$. When m is even, there can be the case where $k = m/2$. In this case $4m-4k = 4k$ and σ^c is a double covering from $P_{2m}(4m)$ onto $\sigma^c(P_{2m}(4m))$. Therefore $\sigma^c(P_{2m}(4m)) \cong P_{2m}(4m))/\langle e_{[4m]}\rangle \cong G_{2m}^{\#}(\mathbb{R}^{4m})$.

Therefore all of polars in $\{\sigma^c(\xi) \mid \xi \in Spin^c(4m), \xi^2 = 1\}$ are

$$
\begin{aligned}
&\{\sigma^c(\sqrt{-1}P_{4k-2}(4m)) \mid 1 \leq k \leq (m+1)/2\} \\
&\cup \{\sigma^c(P_{4k}(4m)) \mid 1 \leq k \leq m/2\} \\
&\cup \{\sigma^c(P_0^+(4m)), \sigma^c(P_0^-(4m))\}.
\end{aligned}
$$

Here $\sigma^c(P_0^+(4m)), \sigma^c(P_0^-(4m))$ are poles. If $1 \leq k < (m+1)/2$, we have $\sigma^c(\sqrt{-1}P_{4k-2}(4m)) \cong P_{4k-2}(4m) \cong \tilde{G}_{4k-2}(\mathbb{R}^{4m})$. When m is odd,

$$\sigma^c(\sqrt{-1}P_{2m}(4m)) \cong P_{2m}(4m))/\langle e_{[4m]}\rangle \cong G_{2m}^{\#}(\mathbb{R}^{4m}).$$

If $1 \leq k < m/2$, we have $\sigma^c(P_{4k}(4m)) \cong P_{4k}(4m) \cong \tilde{G}_{4k}(\mathbb{R}^{4m})$. When m is even,

$$\sigma^c(P_{2m}(4m)) \cong P_{2m}(4m)/\langle e_{[4m]}\rangle \cong G_{2m}^{\#}(\mathbb{R}^{4m}).$$

(2) The case where $\xi^2 = e_{[4m]}$. We consider $\{\xi \in T_{2m}^c \mid \xi^2 = e_{[4m]}\}$. We set $\xi = ug$ for $u \in U(1)$ and $g \in T_{2m}$. The element $\xi^2 = u^2g^2$ is equal to $e_{[4m]}$ if and only if $u^2 = 1$, $g^2 = e_{[4m]}$ or $u^2 = -1$, $g^2 = -e_{[4m]}$. The equality $u^2 = 1$ holds if and only if $u = \pm 1$. The equality $u^2 = -1$ holds if and only if $u = \pm\sqrt{-1}$. We separately consider the conditions $g^2 = e_{[4m]}$ and $g^2 = -e_{[4m]}$ for $g \in T_{2m}$. We set $g = e_{1,2}(\theta_1)\cdots e_{4m-1,4m}(\theta_{2m})$.

(2.1) The case where $g^2 = e_{[4m]}$ and $u = \pm 1$. The condition $g^2 = e_{[4m]}$ is equivalent to

$$e_{1,2}(2\theta_1)\cdots e_{4m-1,4m}(2\theta_{2m}) = e_{[4m]}.$$

Moreover this is equivalent to

$$(\cos 2\theta_1 + \sin 2\theta_1 e_1e_2)\cdots(\cos 2\theta_{2m} + \sin 2\theta_{2m}e_{4m-1}e_{4m}) = e_{[4m]}.$$

If this holds, then $\sin 2\theta_1 \cdots \sin 2\theta_{2m} = 1$. Conversely if $\sin 2\theta_1 \cdots \sin 2\theta_{2m} = 1$ holds, then $\sin 2\theta_i = \pm 1$ for any $i \in [2m]$. These imply $\cos 2\theta_i = 0$ for any $i \in [2m]$. Hence

$$(\cos 2\theta_1 + \sin 2\theta_1 e_1e_2)\cdots(\cos 2\theta_{2m} + \sin 2\theta_{2m}e_{4m-1}e_{4m}) = e_{[4m]}$$

holds. Therefore we have

$$\begin{aligned}
&\{\xi \in T_{2m} \mid \xi^2 = e_{[4m]}\}\\
&= \{e_{1,2}(\theta_1)\cdots e_{4m-1,4m}(\theta_{2m}) \mid \sin 2\theta_1 \cdots \sin 2\theta_{2m} = 1\}\\
&= \{e_{1,2}(\theta_1)\cdots e_{4m-1,4m}(\theta_{2m}) \mid \sin 2\theta_i = \pm 1, \#\{i \mid \sin 2\theta_i = -1\} : \text{even}\}\\
&= \left\{e_{1,2}(\theta_1)\cdots e_{4m-1,4m}(\theta_{2m}) \,\middle|\, 2\theta_i \in (2\mathbb{Z}+1)\frac{\pi}{2}, \# \left\{i \mid 2\theta_i \in (4\mathbb{Z}+3)\frac{\pi}{2}\right\} : \text{even}\right\}\\
&= \left\{e_{1,2}(\theta_1)\cdots e_{4m-1,4m}(\theta_{2m}) \,\middle|\, \theta_i \in (2\mathbb{Z}+1)\frac{\pi}{4}, \# \left\{i \mid \theta_i \in (4\mathbb{Z}+3)\frac{\pi}{4}\right\} : \text{even}\right\}\\
&= \left\{ \pm\prod_{i\in I} e_{2i-1,2i}\left(\frac{3}{4}\pi\right) \prod_{j\in[2m]\setminus I} e_{2j-1,2j}\left(\frac{1}{4}\pi\right) \,\middle|\, I \subset [2m],\ \#I : \text{even}\right\}\\
&= \left\{ \pm\prod_{i\in I} \frac{1}{\sqrt{2}}(-1 + e_{2i-1}e_{2i}) \prod_{j\in[2m]\setminus I} \frac{1}{\sqrt{2}}(1 + e_{2j-1}e_{2j}) \,\middle|\, I \subset [2m],\ \#I : \text{even}\right\}.
\end{aligned}$$

Here we note that for $a \in \mathbb{Z}$

$$e_{2i-1,2i}\left((2a+1)\frac{\pi}{4}\right)$$

$$= \begin{cases} e_{2i-1,2i}\left(\frac{1}{4}\pi\right) = \frac{1}{\sqrt{2}}(1 + e_{2i-1}e_{2i}) & (a \equiv 0 \ (\mathrm{mod} 4)), \\ e_{2i-1,2i}\left(\frac{3}{4}\pi\right) = \frac{1}{\sqrt{2}}(-1 + e_{2i-1}e_{2i}) & (a \equiv 1 \ (\mathrm{mod} 4)), \\ e_{2i-1,2i}\left(\frac{5}{4}\pi\right) = \frac{1}{\sqrt{2}}(-1 - e_{2i-1}e_{2i}) & (a \equiv 2 \ (\mathrm{mod} 4)), \\ e_{2i-1,2i}\left(\frac{7}{4}\pi\right) = \frac{1}{\sqrt{2}}(1 - e_{2i-1}e_{2i}) & (a \equiv 3 \ (\mathrm{mod} 4)). \end{cases}$$

We set

$$J_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad r(\theta) = \exp \theta J_1 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \in SO(2) \quad (\theta \in \mathbb{R}).$$

We can see

$$\rho\left(e_{2i-1,2i}\left((2a+1)\frac{\pi}{4}\right)\right)\Big|_{\langle e_{2i-1}, e_{2i}\rangle} = \begin{cases} r\left(\frac{1}{2}\pi\right) = J_1 & (a \equiv 0 \ (\mathrm{mod} 4)), \\ r\left(\frac{3}{2}\pi\right) = -J_1 & (a \equiv 1 \ (\mathrm{mod} 4)), \\ r\left(\frac{5}{2}\pi\right) = J_1 & (a \equiv 2 \ (\mathrm{mod} 4)), \\ r\left(\frac{7}{2}\pi\right) = -J_1 & (a \equiv 3 \ (\mathrm{mod} 4)), \end{cases}$$

where $\langle e_{2i-1}, e_{2i}\rangle$ is the subspace sppaned by e_{2i-1}, e_{2i} in $\mathbb{R}^{4m}$. Therefore we obtain

$$\rho(\{\xi \in T_{2m} \mid \xi^2 = e_{[4m]}\}) = \left\{ \pm \begin{bmatrix} \epsilon_1 J_1 & & \\ & \ddots & \\ & & \epsilon_{2m} J_1 \end{bmatrix} \;\middle|\; \begin{matrix} \epsilon_i = \pm 1 \ (i \in [2m]) \\ \#\{i \in [2m] \mid \epsilon_i = -1\} : \text{even} \end{matrix} \right\}.$$

This shows that $\mathrm{Pf}(\rho(\{\xi \in T_{2m} \mid \xi^2 = e_{[4m]}\})) = \{1\}$. The image $\rho(\{\xi \in T_{2m} \mid \xi^2 = e_{[4m]}\})$ and its conjugate orbit by $SO(4m)$ are contained in $R(4m)$. All connected components of $R(4m)$ are $R^+(4m)$ and $R^-(4m)$. Hence $\rho(\{\xi \in T_{2m} \mid \xi^2 = e_{[4m]}\}) \subset R^+(4m)$ and

$$\bigcup_{g \in SO(4m)} g\rho(\{\xi \in T_{2m} \mid \xi^2 = e_{[4m]}\})g^{-1} \subset R^+(4m).$$

Let $J_{2m} = \mathrm{diag}(J_1, \dots, J_1) \in SO(4m)$. We can see $J_{2m} = \rho(\tilde{J}_{2m})$ and

$$\tilde{J}_{2m} \in \{\xi \in T_{2m} \mid \xi^2 = e_{[4m]}\}, \quad J_{2m} \in \rho(\{\xi \in T_{2m} \mid \xi^2 = e_{[4m]}\}).$$

Moreover $J_{2m} \in R^+(4m)$ and $R^+(4m)$ is an $SO(4m)$-conjugate orbit of J_{2m}. Hence we have

$$\bigcup_{g \in SO(4m)} g\rho(\{\xi \in T_{2m} \mid \xi^2 = e_{[4m]}\})g^{-1} = R^+(4m).$$

We consider the left hand side of this equation.

$$\bigcup_{g \in SO(4m)} g\rho(\{\xi \in T_{2m} \mid \xi^2 = e_{[4m]}\})g^{-1}$$

$$
\begin{aligned}
&= \bigcup_{x \in Spin(4m)} \rho(x)\rho(\{\xi \in T_{2m} \mid \xi^2 = e_{[4m]}\})\rho(x)^{-1} \\
&= \bigcup_{x \in Spin(4m)} \rho(x\{\xi \in T_{2m} \mid \xi^2 = e_{[4m]}\}x^{-1}) \\
&= \rho\left(\bigcup_{x \in Spin(4m)} x\{\xi \in T_{2m} \mid \xi^2 = e_{[4m]}\}x^{-1} \right) \\
&= \rho(\{\xi \in Spin(4m) \mid \xi^2 = e_{[4m]}\}).
\end{aligned}
$$

Since $e_{[4m]}$ is in the center of $Spin(4m)$, for any $\xi \in Spin(4m)$ with $\xi^2 = e_{[4m]}$ and any $x \in Spin(4m)$, we have

$$
(x\xi x^{-1})^2 = x\xi^2 x^{-1} = xe_{[4m]}x^{-1} = e_{[4m]}.
$$

Therefore we have

$$
\rho(\{\xi \in Spin(4m) \mid \xi^2 = e_{[4m]}\}) = R^+(4m).
$$

This implies $\{\xi \in Spin(4m) \mid \xi^2 = e_{[4m]}\} \subset \rho^{-1}(R^+(4m))$.

We consider whether $\{\xi \in Spin(4m) \mid \xi^2 = e_{[4m]}\}$ is connected or not. Let $I_1 = \mathrm{diag}(-1, 1) \in O(2)$. Since

$$
\begin{bmatrix} I_1 & \\ & I_1 \end{bmatrix} \begin{bmatrix} J_1 & \\ & J_1 \end{bmatrix} \begin{bmatrix} I_1 & \\ & I_1 \end{bmatrix}^{-1} = \begin{bmatrix} I_1 J_1 I_1^{-1} & \\ & I_1 J_1 I_1^{-1} \end{bmatrix} = \begin{bmatrix} -J_1 & \\ & -J_1 \end{bmatrix},
$$

we can see that all elements of $\rho(\{\xi \in T_{2m} \mid \xi^2 = e_{[4m]}\})$ are $SO(4m)$-conjugate. If we express the above equation as an image of ρ, we get the following equation.

$$
\begin{aligned}
&\rho\left(e_{1,3}\left(\frac{1}{2}\pi\right)\right)\rho\left(e_{1,2}\left(\frac{1}{4}\pi\right)e_{3,4}\left(\frac{1}{4}\pi\right)\right)\rho\left(e_{1,3}\left(\frac{1}{2}\pi\right)\right)^{-1} \\
&= \rho\left(e_{1,2}\left(\frac{3}{4}\pi\right)e_{3,4}\left(\frac{3}{4}\pi\right)\right).
\end{aligned}
$$

A direct calculation shows

$$
e_{1,3}\left(\frac{1}{2}\pi\right)e_{1,2}\left(\frac{1}{4}\pi\right)e_{3,4}\left(\frac{1}{4}\pi\right)e_{1,3}\left(\frac{1}{2}\pi\right)^{-1} = e_{1,2}\left(\frac{3}{4}\pi\right)e_{3,4}\left(\frac{3}{4}\pi\right).
$$

Therefore each element of $\{\xi \in T_{2m} \mid \xi^2 = e_{[4m]}\}$ is conjugate to $\tilde{J}_{2m}$ or $-\tilde{J}_{2m}$. This implies that

$$
\{\xi \in Spin(4m) \mid \xi^2 = e_{[4m]}\} = \tilde{R}^+(4m) \cup (-\tilde{R}^+(4m))
$$

is a decomposition to connected components. From the above argument we have

$$
\rho^{-1}(R^+(4m)) = \{\xi \in Spin(4m) \mid \xi^2 = e_{[4m]}\} = \tilde{R}^+(4m) \cup (-\tilde{R}^+(4m))
$$

and the right hand side is a decomposition to connected components.

$$\rho : \tilde{R}^+(4m) \to R^+(4m), \quad \rho : -\tilde{R}^+(4m) \to R^+(4m)$$

are diffeomorphisms and

$$\tilde{R}^+(4m) \cong -\tilde{R}^+(4m) \cong DIII(2m)$$

holds. In this case polars are

$$\{\sigma^c(\tilde{R}^+(4m)), \sigma^c(-\tilde{R}^+(4m))\}.$$

A direct calculation shows

$$e_{[4m]}\tilde{J}_{2m} = e_{1,2}\left(\frac{3}{4}\pi\right) \cdots e_{4m-1,4m}\left(\frac{3}{4}\pi\right) \in \tilde{R}^+(4m)$$

and we obtain $e_{[4m]}\tilde{R}^+(4m) = \tilde{R}^+(4m)$. Moreover

$$\begin{aligned}\rho(e_{[4m]}\tilde{J}_{2m}) &= \rho\left(e_{1,2}\left(\frac{3}{4}\pi\right) \cdots e_{4m-1,4m}\left(\frac{3}{4}\pi\right)\right) \\ &= \mathrm{diag}(-J_1, \ldots, -J_1) = -J_{2m} = -\rho(\tilde{J}_{2m})\end{aligned}$$

and for any $x \in Spin(4m)$

$$\begin{aligned}\rho(e_{[4m]}x\tilde{J}_{2m}x^{-1}) &= \rho(x)\rho(e_{[4m]}\tilde{J}_{2m})\rho(x)^{-1} = \rho(x)(-\rho(\tilde{J}_{2m}))\rho(x)^{-1} \\ &= -\rho(x\tilde{J}_{2m}x^{-1}).\end{aligned}$$

Thus we obtain

$$\rho(e_{[4m]}\xi) = -\rho(\xi) \quad (\xi \in \tilde{R}^+(4m)).$$

Hence $\sigma^c : \tilde{R}^+(4m) \to \sigma^c(\tilde{R}^+(4m))$ corresponds to the double covering $DIII(2m) \to DIII(2m)/\{\pm 1_{4m}\}$ by the diffeomorphism $\rho : \tilde{R}^+(4m) \to R^+(4m) \cong DIII(2m)$. Therefore $\sigma^c : \tilde{R}^+(4m) \to \sigma^c(\tilde{R}^+(4m))$ is also a double covering. Similarly $\sigma^c : -\tilde{R}^+(4m) \to \sigma^c(-\tilde{R}^+(4m))$ is also a double covering and we have

$$\sigma^c(\tilde{R}^+(4m)) \cong \sigma^c(-\tilde{R}^+(4m)) \cong DIII(2m)/\{\pm 1_{4m}\}.$$

(2.2) The case where $g^2 = -e_{[4m]}$ and $u = \pm\sqrt{-1}$. The condition $g^2 = -e_{[4m]}$ is equivalent to

$$e_{1,2}(2\theta_1) \cdots e_{4m-1,4m}(2\theta_{2m}) = -e_{[4m]}.$$

Moreover this is equivalent to

$$(\cos 2\theta_1 + \sin 2\theta_1 e_1 e_2) \cdots (\cos 2\theta_{2m} + \sin 2\theta_{2m} e_{4m-1} e_{4m}) = -e_{[4m]}.$$

If this holds, then $\sin 2\theta_1 \cdots \sin 2\theta_{2m} = -1$. Conversely if $\sin 2\theta_1 \cdots \sin 2\theta_{2m} = -1$ holds, then $\sin 2\theta_i = \pm 1$ for any $i \in [2m]$. These imply $\cos 2\theta_i = 0$ for any $i \in [2m]$. Hence

$$(\cos 2\theta_1 + \sin 2\theta_1 e_1 e_2) \cdots (\cos 2\theta_{2m} + \sin 2\theta_{2m} e_{4m-1} e_{4m}) = -e_{[4m]}$$

holds. Therefore we have

$$\begin{aligned}
&\{\xi \in T_{2m} \mid \xi^2 = -e_{[4m]}\} \\
&= \{e_{1,2}(\theta_1)\cdots e_{4m-1,4m}(\theta_{2m}) \mid \sin 2\theta_1 \cdots \sin 2\theta_{2m} = -1\} \\
&= \{e_{1,2}(\theta_1)\cdots e_{4m-1,4m}(\theta_{2m}) \mid \sin 2\theta_i = \pm 1, \#\{i \mid \sin 2\theta_i = -1\} : \text{odd}\} \\
&= \left\{e_{1,2}(\theta_1)\cdots e_{4m-1,4m}(\theta_{2m}) \,\middle|\, 2\theta_i \in (2\mathbb{Z}+1)\frac{\pi}{2}, \# \left\{i \mid 2\theta_i \in (4\mathbb{Z}+3)\frac{\pi}{2}\right\} : \text{odd}\right\} \\
&= \left\{e_{1,2}(\theta_1)\cdots e_{4m-1,4m}(\theta_{2m}) \,\middle|\, \theta_i \in (2\mathbb{Z}+1)\frac{\pi}{4}, \# \left\{i \mid \theta_i \in (4\mathbb{Z}+3)\frac{\pi}{4}\right\} : \text{odd}\right\} \\
&= \left\{ \pm \prod_{i \in I} e_{2i-1,2i}\left(\frac{3}{4}\pi\right) \prod_{j \in [2m]\setminus I} e_{2j-1,2j}\left(\frac{1}{4}\pi\right) \,\middle|\, I \subset [2m],\ \#I : \text{odd} \right\} \\
&= \left\{ \pm \prod_{i \in I} \frac{1}{\sqrt{2}}(-1 + e_{2i-1}e_{2i}) \prod_{j \in [2m]\setminus I} \frac{1}{\sqrt{2}}(1 + e_{2j-1}e_{2j}) \,\middle|\, I \subset [2m],\ \#I : \text{odd} \right\}.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&\rho(\{\xi \in T_{2m} \mid \xi^2 = -e_{[4m]}\}) \\
&= \left\{ \pm \begin{bmatrix} \epsilon_1 J_1 & & \\ & \ddots & \\ & & \epsilon_{2m} J_1 \end{bmatrix} \,\middle|\, \begin{array}{c} \epsilon_i = \pm 1 \ (i \in [2m]) \\ \#\{i \in [2m] \mid \epsilon_i = -1\} : \text{odd} \end{array} \right\}.
\end{aligned}$$

This shows that $\mathrm{Pf}(\rho(\{\xi \in T_{2m} \mid \xi^2 = -e_{[4m]}\})) = \{-1\}$. Hence $\rho(\{\xi \in T_{2m} \mid \xi^2 = -e_{[4m]}\}) \subset R^-(4m)$ and

$$\bigcup_{g \in SO(4m)} g\rho(\{\xi \in T_{2m} \mid \xi^2 = -e_{[4m]}\})g^{-1} \subset R^-(4m).$$

Let $J'_{2m} = \mathrm{diag}(-J_1, J_1, \ldots, J_1) \in SO(4m)$. We can see $J'_{2m} = \rho(\tilde{J}'_{2m})$ and

$$\tilde{J}'_{2m} \in \{\xi \in T_{2m} \mid \xi^2 = -e_{[4m]}\}, \quad J'_{2m} \in \rho(\{\xi \in T_{2m} \mid \xi^2 = -e_{[4m]}\}).$$

Moreover $J'_{2m} \in R^-(4m)$ and $R^-(4m)$ is an $SO(4m)$-conjugate orbit of J'_{2m}. Hence we have

$$\bigcup_{g \in SO(4m)} g\rho(\{\xi \in T_{2m} \mid \xi^2 = -e_{[4m]}\})g^{-1} = R^-(4m).$$

We consider the left hand side of this equation.

$$\begin{aligned}
&\bigcup_{g \in SO(4m)} g\rho(\{\xi \in T_{2m} \mid \xi^2 = -e_{[4m]}\})g^{-1} \\
&= \bigcup_{x \in Spin(4m)} \rho(x)\rho(\{\xi \in T_{2m} \mid \xi^2 = -e_{[4m]}\})\rho(x)^{-1}
\end{aligned}$$

$$= \bigcup_{x \in Spin(4m)} \rho(x\{\xi \in T_{2m} \mid \xi^2 = e_{[4m]}\}x^{-1})$$
$$= \rho\left(\bigcup_{x \in Spin(4m)} x\{\xi \in T_{2m} \mid \xi^2 = -e_{[4m]}\}x^{-1}\right)$$
$$= \rho(\{\xi \in Spin(4m) \mid \xi^2 = -e_{[4m]}\}).$$

Since $-e_{[4m]}$ is in the center of $Spin(4m)$, for any $\xi \in Spin(4m)$ with $\xi^2 = -e_{[4m]}$ and any $x \in Spin(4m)$, we have

$$(x\xi x^{-1})^2 = x\xi^2 x^{-1} = x(-e_{[4m]})x^{-1} = -e_{[4m]}.$$

Therefore we have

$$\rho(\{\xi \in Spin(4m) \mid \xi^2 = -e_{[4m]}\}) = R^-(4m).$$

This implies $\{\xi \in Spin(4m) \mid \xi^2 = -e_{[4m]}\} \subset \rho^{-1}(R^-(4m))$.

A similar argument to the previous case shows that each element of $\{\xi \in T_{2m} \mid \xi^2 = -e_{[4m]}\}$ is conjugate to $\tilde{J}'_{2m}$ or $-\tilde{J}'_{2m}$. This implies that

$$\{\xi \in Spin(4m) \mid \xi^2 = -e_{[4m]}\} = \tilde{R}^-(4m) \cup (-\tilde{R}^-(4m))$$

is a decomposition to connected components. From the above argument we have

$$\rho^{-1}(R^-(4m)) = \{\xi \in Spin(4m) \mid \xi^2 = -e_{[4m]}\} = \tilde{R}^-(4m) \cup (-\tilde{R}^-(4m))$$

and the right hand side is a decomposition to connected components.

$$\rho : \tilde{R}^-(4m) \to R^-(4m), \quad \rho : -\tilde{R}^-(4m) \to R^-(4m)$$

are diffeomorphisms and

$$\tilde{R}^-(4m) \cong -\tilde{R}^-(4m) \cong DIII(2m)$$

holds. In this case polars are

$$\{\sigma^c(\sqrt{-1}\tilde{R}^-(4m)), \sigma^c(-\sqrt{-1}\tilde{R}^-(4m))\}.$$

A direct calculation shows

$$e_{[4m]}\tilde{J}'_{2m} = e_{1,2}\left(\frac{5}{4}\pi\right)e_{3,4}\left(\frac{3}{4}\pi\right)\cdots e_{4m-1,4m}\left(\frac{3}{4}\pi\right) \in -\tilde{R}^-(4m)$$

and we obtain $e_{[4m]}\tilde{R}^-(4m) = -\tilde{R}^-(4m)$. Moreover

$$\rho(e_{[4m]}\tilde{J}'_{2m}) = \rho\left(e_{1,2}\left(\frac{5}{4}\pi\right)e_{3,4}\left(\frac{3}{4}\pi\right)\cdots e_{4m-1,4m}\left(\frac{3}{4}\pi\right)\right)$$
$$= \mathrm{diag}(J_1, -J_1, \ldots, -J_1) = -\rho(\tilde{J}'_{2m})$$

and for any $x \in Spin(4m)$

$$\begin{aligned}\rho(e_{[4m]}x\tilde{J}'_{2m}x^{-1}) &= \rho(x)\rho(e_{[4m]}\tilde{J}'_{2m})\rho(x)^{-1} = \rho(x)(-\rho(\tilde{J}'_{2m}))\rho(x)^{-1} \\ &= -\rho(x\tilde{J}'_{2m}x^{-1}).\end{aligned}$$

Thus we obtain

$$\rho(e_{[4m]}\xi) = -\rho(\xi) \quad (\xi \in \tilde{R}^-(4m)).$$

Hence $\sigma^c : \tilde{R}^-(4m) \to \sigma^c(\tilde{R}^-(4m))$ is a diffeomorphism. Similarly $\sigma^c : -\tilde{R}^-(4m) \to \sigma^c(-\tilde{R}^-(4m))$ is also a diffeomorphism.

$$\sigma^c(\tilde{R}^-(4m)) \cong \sigma^c(-\tilde{R}^-(4m)) \cong DIII(2m).$$

□

4 Polars of $O^c(n)/\{\pm 1_n\}$

In this section we explicitly describe polars of $O^c(n)/\{\pm 1_n\}$.

In order to describe polars of $O^c(n)/\{\pm 1_n\}$ w recall the symbols defined in [3]. For $0 \leq k \leq n$, we set $x_k = \mathrm{diag}(-1, \ldots, -1, 1, \ldots, 1) \in O(n)$, where the number of -1's is k, and

$$\begin{aligned}Q_k(n) &= \{g x_k g^{-1} \mid g \in O(n)\} \\ &= \{g \in O(n) \mid g\text{'s eigenvalues and multiplicities} : (-1, k), (1, n-k)\}.\end{aligned}$$

We can see that

$$\{g \in O(n) \mid g^2 = 1_n\} = \bigcup_{0 \leq k \leq n} Q_k(n).$$

Theorem 4.1. *Let $\pi_n : O^c(n) \to O^c(n)/\{\pm 1_n\}$ be a natural projection. The polars of $O^c(n)/\{\pm 1_n\}$ are as follows:*

$$\begin{aligned}&F(s_{\pi_n(1_n)}, \pi_n(O^c(n))) \\ &= \bigcup_{0 \leq k \leq n} \pi_n(Q_k(n)) \cup \bigcup_{0 \leq k \leq n} \pi_n(\sqrt{-1}Q_k(n)) \\ &\quad \cup \pi_n(R^+(n)) \cup \pi_n(R^-(n)) \cup \pi_n(\sqrt{-1}R^+(n)) \cup \pi_n(\sqrt{-1}R^-(n)),\end{aligned}$$

where $R^+(n), R^-(n)$ appear only if n is even.

Proof. We take a element $\pi_n(zg)$ $(z \in U(1), g \in O(n))$ of $F(s_{\pi_n(1_n)}, \pi_n(O^c(n)))$. Since $\pi_n(zg)^2 = \pi_n(1_n)$, we have $z^2 g^2 = \pm 1_n$. This is equivalent to the condition $z^2 = \pm 1$ and $g^2 = \pm 1_n$. Hence we obtain

$$\begin{aligned}&F(s_{\pi_n(1_n)}, \pi_n(O^c(n))) \\ &= \pi_n(\{g \in O(n) \mid g^2 = \pm 1_n\}) \cup \pi_n(\sqrt{-1}\{g \in O(n) \mid g^2 = \pm 1_n\})\end{aligned}$$

and the statement of the theorem. □

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