The Littlewood–Paley–Stein inequality for Dirichlet space tamed by signed measured curvature lower bounds *

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Abstract

The notion of tamed Dirichlet space by distributional lower Ricci curvature bounds was proposed by Erbar–Rigoni–Sturm–Tamanini [21] as the Dirichlet space having a weak form of Bakry–Émery curvature lower bounds in distribution sense. In this framework, we establish the Littlewood–Paley–Stein inequality for L^p-functions which partially generalizes the result by Kawabi–Miyokawa [34].

Keywords: Strongly local Dirichlet space, tamed Dirichlet space, Bakry–Émery condition, smooth measures of (extended) Kato class, smooth measures of Dynkin class, Littlewood–Paley–Stein inequality, Riesz transform, Wiener space, path space with Gibbs measure, RCD-space, Riemannian manifold with boundary, configuration space,

1 Introduction

The Littlewood–Paley theory is recognized as a theoretical framework to extend some results of harmonic analysis about L^2-functions to L^p-functions for $p \in]\,1, +\infty\,[$, which is stated by Littlewood–Paley inequality in terms of Littlewood–Paley's G-function (see Section 2 below for details of Littlewood–Paley's G-function). This was firstly studied by J. E. Littlewood and R. Paley in the series of papers [41, 42, 43] for one-dimensional Euclidean space. In addition, E. Stein [54] established the theory for higher dimensional Euclidean space or compact Riemmanian manifolds in terms of general symmetric semigroup over measurable space. So nowadays, it is called Littlewood–Paley–Stein inequality after the contribution by E. Stein. On the other hand, P.A. Meyer [47, 48] gave another probabilistic proof for E. Stein [54] in the framework of the abstract Wiener space, which was significantly applied to establish the basics for Malliavin calculus (see I. Shigekawa [52, §3.2]). More precisely, he established the Littlewood–Paley–Stein inequality for Hilbert

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space-valued L^p-functions on the abstract Wiener space, which means the Littlewood–Paley–Stein inequality for k-forms on it ([52, §3.2]), yielding the L^p-boundedness of Riesz transform $\mathscr{R}_\alpha(L) := \nabla(\alpha - L)^{-\frac{1}{2}}$ with $\alpha \geq 0$ acting to Hilbert space valued functions on the abstract Wiener space. Here L is the Ornstein-Uhlenbeck operator on the abstract Wiener space. This means the equivalence of Sobolev norms over the abstract Wiener space (see [52, §4.1]).

To establish the Littlewood–Paley–Stein inequality, it should be noted that the underlying space has a Ricci curvature lower bound in some sense. For example, the abstract Wiener space is a $\mathsf{CD}(1,\infty)$-space, metric measure space having Ricci curvature lower bound 1 in terms of optimal mass transportation (see [23, Theorem 1.5]). On the other hand, in the framework of complete weighted Riemmanian manifolds, D. Bakry [8] established the Littlewood–Paley–Stein inequality for functions and 1-forms under Bakry–Émery Ricci curvature lower bound $K \in \mathbb{R}$. In particular, D. Bakry [8] proved the L^p-boundedness of Riesz transform in terms of weighted Laplacian under Bakry–Émery Ricci curvature lower bound $K \in \mathbb{R}$. After that, Shigekawa–Yoshida [53] also studied the Littlewood–Paley–Stein inequality for symmetric diffusion processes on a general state space. In [53], they assumed the existence of a suitable core $\mathscr{A}$ which is not only a ring but also stable under the operation of the semigroup and the infinitesimal generator to employ Bakry–Emery's Γ_2-method in the proof, and established the Littlewood–Paley–Stein inequality under that Γ_2 is bounded from below by $K\Gamma_1$, i.e., Ricci curvature is lower bounded by $K \in \mathbb{R}$ in Bakry–Émery sense (see [9] for the notions of Γ_2, Γ_1). At the same time, Yoshida [59] gave the Littlewood–Paley–Stein inequality for k-forms over Riemannian manifolds under suitable curvature lower bound conditions and established the equivalence of Sobolev norms over complete Riemanian manifolds. Moreover, he also established the Littlewood–Paley–Stein inequality over infinite dimensional manifolds as the diffusion operator with Ising-type interaction in N. Yoshida [60].

X.-D. Li [39] extended the results of [8] under variable Bakry–Émery Ricci curvature lower bounds $K(\cdot)$ in the framework of weighted Riemannian manifolds under suitable conditions for $K(\cdot)$. On the other hand, T. Coulhon and X.T. Duong [16] gave the Littlewood–Paley–Stein inequality under a gradient estimate condition, which can be understood a generalized version of Bakry–Émery gradient estimate in the framework of complete smooth Riemannian manifolds.

Very recently, Kawabi–Miyokawa [34] established the Littlewood–Paley–Stein inequality extending T. Coulhon and X.T. Duong [16] to the framework of metric measure space. Their condition is satisfied for superprocesses with immigration (see [34, §4.2]) or a class of sub-Riemmanian manifolds (see [46, Theorem 1.5]).

In this paper, we establish the Littlewood–Paley–Stein inequality under a Bakry–Émery condition denoted by $\mathsf{BE}_2(\kappa,\infty)$, a version of Ricci curvature lower bound κ in a weak form of Bochner's inequality, where κ is a signed smooth measure satisfying a suitable condition (see Definition 2.1 below). In a slogan-like way, such a condition is formulated as the notion of tamed Dirichlet space, i.e., Dirichlet forms having signed measured curvature lower bounds in a weak form of Bochner's inequality. The novelty of our results is the curvature lower bound is not necessarily a function nor a constant. But, it should be noted that our result does not cover [34], in particular, superprocesses with immigration and a class of sub-Riemmanian manifolds are not included in our framework. However, examples of $\mathsf{RCD}(K,N)$-spaces (Example 5.1), (non-compact) Riemannian manifolds having Ricci curvature lower bounds with boundary whose second fundamental form on the boundary is bounded below (Example 5.5), and the configuration space over Riemmanian manifolds having Ricci curvature lower bounds (more generally metric measure spaces having Ricci curvature lower bounds) (Example 5.6) are included in our framework. These are not appeared in the past literature.

The organization of this paper is as follows: In Section 2, we introduce the Bakry–Émery condition $\mathsf{BE}(\kappa,\infty)$ for a fixed signed smooth measure κ whose killing part κ^+ is of Dynkin class, and the double of creation part κ^- is of extended Kato class, and formulate the notion of tamed Dirichlet space for Theorem 2.2. We introduce the Littlewood–Paley's G-function for Feynman-Kac semigroup (P_t^{k}) for general signed smooth measure k under a similar suitable condition with κ. In Section 3, to prove Theorem 2.2, we present several related notions and introduce the auxiliary vertical one dimensional Brownian motion, and investigate the stochastic analysis based on the product Dirichlet forms associated with the product diffusion processes along the method by P.A. Meyer [47, 48]. In Section 4, we prove Theorem 2.2 by splitting several subsections. In Section 5, we expose several examples.

2 Statement of Main Theorem

Let (X,τ) be a Hausdorff topological space satisfying $\mathscr{B}(X)=\sigma(C(X))$ (see [45, Chapter IV, Sections 2, 3 and 5]), which is a Lusin space, i.e., a continuous injective image of a Polish space, endowed with a σ-finite Borel measure $\mathfrak{m}$ on X with full topological support. Let $(\mathscr{E},D(\mathscr{E}))$ be a quasi-regular symmetric strongly local Dirichlet space on $L^2(X;\mathfrak{m})$ and $(P_t)_{t\geq 0}$ the associated symmetric sub-Markovian strongly continuous semigroup on $L^2(X;\mathfrak{m})$ (see [45, Chapter IV, Definition 3] for the quasi-regularity and see [35, Theorem 5.1(i)] for the strong locality). Then there exists an $\mathfrak{m}$-symmetric special standard process $\mathbf{X}=(\Omega,X_t,\mathbb{P}_x)$ associated with $(\mathscr{E},D(\mathscr{E}))$, i.e. for $f\in L^2(X;\mathfrak{m})\cap\mathscr{B}(X)$, $P_tf(x)=\mathbb{E}_x[f(X_t)]$ $\mathfrak{m}$-a.e. $x\in X$ (see [45, Chapter IV, Section 3]). Here $\mathscr{B}(X)$ denotes the family of Borel measurable functions on X (the symbol $\mathscr{B}(X)$ is also used for the family of Borel measurable subsets of X in some context). An increasing sequence $\{F_n\}$ of closed subsets is called an *$\mathscr{E}$-nest* if $\bigcup_{n=1}^\infty D(\mathscr{E})_{F_n}$ is $\mathscr{E}_1$-dense in $D(\mathscr{E})$, where $D(\mathscr{E})_{F_n}:=\{u\in D(\mathscr{E})\mid u=0\ \mathfrak{m}\text{-a.e. on } X\setminus F_n\}$. A subset N is said to be *$\mathscr{E}$-exceptional* or *$\mathscr{E}$-polar* if there exists an $\mathscr{E}$-nest $\{F_n\}$ such that $N\subset\bigcap_{n=1}^\infty(X\setminus F_n)$. It is known that any $\mathscr{E}$-exceptional set is $\mathfrak{m}$-negligible and for any $\mathfrak{m}$-negligible $\mathscr{E}$-quasi-open set is $\mathscr{E}$-exceptional, in particular, for an $\mathscr{E}$-quasi-continuous function u, $u\geq 0$ $\mathfrak{m}$-a.e. implies $u\geq 0$ $\mathscr{E}$-q.e. For a statement $P(x)$ with respect to $x\in X$, we say that $P(x)$ holds $\mathscr{E}$-q.e. $x\in X$ (simply P holds $\mathscr{E}$-q.e.) if the set $\{x\in X\mid P(x)\text{ holds }\}$ is $\mathscr{E}$-exceptional. A subset G of X is called an *$\mathscr{E}$-quasi-open set* if there exists an $\mathscr{E}$-nest $\{F_n\}$ such that $G\cap F_n$ is an open set of F_n with respect to the relative topology on F_n for each $n\in\mathbb{N}$. A function u on X is said to be *$\mathscr{E}$-quasi-continuous* if there exists an $\mathscr{E}$-nest $\{F_n\}$ such that $u|_{F_n}$ is continuous on F_n for each $n\in\mathbb{N}$.

It is known that for $u,v\in D(\mathscr{E})\cap L^\infty(X;\mathfrak{m})$ there exists a unique signed finite Borel measure $\mu_{\langle u,v\rangle}$ on X such that

$$2\int_X \tilde{f}\mathrm{d}\mu_{\langle u,v\rangle} = \mathscr{E}(uf,v)+\mathscr{E}(vf,u)-\mathscr{E}(uv,f) \quad \text{for} \quad u,v\in D(\mathscr{E})\cap L^\infty(X;\mathfrak{m}).$$

Here $\tilde{f}$ denotes the $\mathscr{E}$-quasi-continuous $\mathfrak{m}$-version of f (see Fukushima–Oshima–Takeda [25, Theorem 2.1.3], Ma–Röckner [45, Chapter IV, Proposition 3.3(ii)]). We set $\mu_{\langle f\rangle} := \mu_{\langle f,f\rangle}$ for $f\in D(\mathscr{E})\cap L^\infty(X;\mathfrak{m})$. Moreover, for $f,g\in D(\mathscr{E})$, there exists a signed finite measure $\mu_{\langle f,g\rangle}$ on X such that $\mathscr{E}(f,g)=\mu_{\langle f,g\rangle}(X)$, hence $\mathscr{E}(f,f)=\mu_{\langle f\rangle}(X)$. We assume $(\mathscr{E},D(\mathscr{E}))$ admits a carré-du-champ Γ, i.e. $\mu_{\langle f\rangle}\ll\mathfrak{m}$ for all $f\in D(\mathscr{E})$ and set $\Gamma(f):=\mathrm{d}\mu_{\langle f\rangle}/\mathrm{d}\mathfrak{m}$. Then $\mu_{\langle f,g\rangle}\ll\mathfrak{m}$ for all $f,g\in D(\mathscr{E})$ and $\Gamma(f,g):=\mathrm{d}\mu_{\langle f,g\rangle}/\mathrm{d}\mathfrak{m}\in L^1(X;\mathfrak{m})$ is expressed by $\Gamma(f,g)=\frac{1}{4}(\Gamma(f+g)-\Gamma(f-g))$ for $f,g\in D(\mathscr{E})$.

Let k be a signed smooth measure with its Jordan-Hahn decomposition $\mathsf{k}=\mathsf{k}^+-\mathsf{k}^-$. We assume that k^+ is a smooth measure of Dynkin class ($\mathsf{k}^+\in S_D(\mathbf{X})$ in short) and k^- is a smooth measure of extended Kato class ($\mathsf{k}^-\in S_{EK}(\mathbf{X})$ in short). More precisely, $\nu\in S_D(\mathbf{X})$ (resp. $\nu\in S_{EK}(\mathbf{X})$) if and only if $\nu\in S(\mathbf{X})$ and $\mathfrak{m}\text{-}\sup_{x\in X}\mathbb{E}_x[A_t^\nu]<+\infty$ for any/some $t>0$ (resp. $\lim_{t\to 0}\mathfrak{m}\text{-}\sup_{x\in X}\mathbb{E}_x[A_t^\nu]<1$) (see [2]). Here $S(\mathbf{X})$ denotes the family of smooth measures with respect to $\mathbf{X}$ (see [45, Chapter VI, Definition 2.3], [25, p. 83] for the definition of smooth measures) and $\mathfrak{m}\text{-}\sup_{x\in X}f(x)$ denotes the $\mathfrak{m}$-essentially supremum for a function f on X. Then, the quadratic form

$$\mathscr{E}^{\mathsf{k}}(f):=\mathscr{E}(f)+\langle \mathsf{k},\tilde{f}^2\rangle$$

with finiteness domain $D(\mathscr{E}^{\mathsf{k}})=D(\mathscr{E})$ is closed and lower semi-bounded, moreover, there exists $\alpha_0>0$ and $C>0$ such that

$$C^{-1}\mathscr{E}_1(f,f)\le\mathscr{E}^{\mathsf{k}}_{\alpha_0}(f,f)\le C\mathscr{E}_1(f,f) \quad \text{for all} \quad f\in D(\mathscr{E}^{\mathsf{k}})=D(\mathscr{E})$$

(see [14, (3.3)] and [15, Assumption of Theorem 1.1]). Here $\mathscr{E}_\alpha(f,g):=\mathscr{E}(f,g)+\alpha(f,g)_{\mathfrak{m}}$ for $\alpha>0$ and $(f,g)_{\mathfrak{m}}$ denotes the L^2-inner product for $f,g\in L^2(X;\mathfrak{m})$. The Feynman–Kac semigroup $(p_t^{\mathsf{k}})_{t\ge 0}$ defined by

$$p_t^{\mathsf{k}}f(x)=\mathbb{E}_x[e^{-A_t^{\mathsf{k}}}f(X_t)], \quad f\in\mathscr{B}_b(X)$$

is $\mathfrak{m}$-symmetric, i.e.

$$\int_X p_t^{\mathsf{k}}f(x)g(x)\mathfrak{m}(\mathrm{d}x)=\int_X f(x)p_t^{\mathsf{k}}g(x)\mathfrak{m}(\mathrm{d}x) \quad \text{for all} \quad f,g\in\mathscr{B}_+(X)$$

and coincides with the strongly continuous semigroup $(P_t^{\mathsf{k}})_{t\ge0}$ on $L^2(X;\mathfrak{m})$ associated with $(\mathscr{E}^{\mathsf{k}},D(\mathscr{E}^{\mathsf{k}}))$ (see [15, Theorem 1.1]). Here A_t^{k} is a continuous additive functional (CAF in short) associated with the signed smooth measure k under Revuz correspondence. Under $\mathsf{k}^-\in S_{EK}(\mathbf{X})$ and $p\in[2,+\infty]$, $(p_t^{\mathsf{k}})_{t\ge0}$ can be extended to be a bounded operator on $L^p(X;\mathfrak{m})$ denoted by P_t^{k} such that there exist finite constants $C=C(\mathsf{k})>0, C_{\mathsf{k}}\ge 0$ depending only on k^- such that for every $t\ge 0$

$$\|P_t^{\mathsf{k}}\|_{p,p}\le Ce^{C_{\mathsf{k}}t}. \tag{2.1}$$

Here $C=1$ and $C_{\mathsf{k}}\geq 0$ can be taken to be 0 under $\mathsf{k}^-=0$ (cf. [58, Theorem 2.2]). When $\mathsf{k}=-R\mathfrak{m}$ for a constant $R\in\mathbb{R}$, C_{k} (resp. C) can be taken to be $R\vee 0$ (resp. 1). Let Δ^{k} be the L^2-generator associated with $(\mathscr{E}^{\mathsf{k}}, D(\mathscr{E}))$ defined by

$$\begin{cases} D(\Delta^{\mathsf{k}}) & := \{u\in D(\mathscr{E})\mid \text{there exists } w\in L^2(X;\mathfrak{m}) \text{ such that} \\ & \qquad\qquad \mathscr{E}^{\mathsf{k}}(u,v)=-\displaystyle\int_X wv\,\mathrm{d}\mathfrak{m} \quad \text{for any} \quad v\in D(\mathscr{E})\}, \\ \Delta^{\mathsf{k}}u & := w \quad \text{for } w\in L^2(X;\mathfrak{m}) \quad \text{specified as above,} \end{cases} \tag{2.2}$$

called the *Schrödinger operator* with potential k. Formally, Δ^{k} can be understood as "$\Delta^{\mathsf{k}}=\Delta-\mathsf{k}$", where Δ is the L^2-generator associated with $(\mathscr{E}, D(\mathscr{E}))$.

Definition 2.1 (q-Bakry–Émery condition). Suppose that $q\in\{1,2\}$, $\kappa^+\in S_D(\mathbf{X})$, $2\kappa^-\in S_{EK}(\mathbf{X})$ and $N\in[1,+\infty]$. We say that $(X,\mathscr{E},\mathfrak{m})$ or simply X satisfies the q-Bakry–Émery condition, briefly $\mathsf{BE}_q(\kappa,N)$, if for every $f\in D(\Delta)$ with $\Delta f\in D(\mathscr{E})$ and every nonnegative $\phi\in D(\Delta^{q\kappa})$ with $\Delta^{q\kappa}\phi\in L^\infty(X;\mathfrak{m})$ (we further impose $\phi\in L^\infty(X;\mathfrak{m})$ for $q=2$), we have

$$\frac{1}{q}\int_X \Delta^{q\kappa}\phi\Gamma(f)^{\frac{q}{2}}\mathrm{d}\mathfrak{m}-\int_X\phi\Gamma(f)^{\frac{q-2}{2}}\Gamma(f,\Delta f)\mathrm{d}\mathfrak{m}\geq\frac{1}{N}\int_X\phi\Gamma(f)^{\frac{q-2}{2}}(\Delta f)^2\mathrm{d}\mathfrak{m}.$$

The latter term is understood as 0 if $N=+\infty$.

Assumption 1. We assume that X satisfies $\mathsf{BE}_2(\kappa,\infty)$ condition for a given signed smooth measure κ with $\kappa^+\in S_D(\mathbf{X})$ and $2\kappa^-\in S_{EK}(\mathbf{X})$.

Under Assumption 1, we say that $(X,\mathscr{E},\mathfrak{m})$ or simply X is *tamed*. In fact, under $\kappa^+\in S_D(\mathbf{X})$ and $2\kappa^-\in S_{EK}(\mathbf{X})$, the condition $\mathsf{BE}_2(\kappa,\infty)$ is *equivalent* to $\mathsf{BE}_1(\kappa,\infty)$ (see [21, Theorems 3.4 and 3.6, Proposition 3.7 and Theorem 6.10]), in particular, the heat flow $(P_t)_{t\geq 0}$ satisfies

$$\Gamma(P_tf)^{1/2}\leq P_t^\kappa\Gamma(f)^{1/2}\quad \mathfrak{m}\text{-a.e. for any } f\in D(\mathscr{E}) \quad\text{and}\quad t\geq 0, \tag{2.3}$$

which is called the condition $\mathsf{GE}_1(\kappa,\infty)$. This condition also implies $\mathsf{BE}_1(\kappa,\infty)$ (see [21, Definition 3.3 and Theorem 3.4]). The inequality (2.3) plays a crucial role in our paper. Note that our condition $\kappa^+\in S_D(\mathbf{X})$, $\kappa^-\in S_{EK}(\mathbf{X})$ (resp. $\kappa^+\in S_D(\mathbf{X})$, $2\kappa^-\in S_{EK}(\mathbf{X})$) is stronger than the 1-moderate (resp. 2-moderate) condition treated in [21] for the definition of tamed space. The $\mathfrak{m}$-symmetric Markov process $\mathbf{X}$ treated in our paper may not be conservative in general. Under Assumption 1, a sufficient condition (*intrinsic stochastic completeness* called in [21]) for conservativeness of $\mathbf{X}$ is discussed in [21, Section 3.3], in particular, under $1\in D(\mathscr{E})$ with $\mathscr{E}(1)=0$ and Assumption 1, we have the conservativeness of $\mathbf{X}$.

We introduce the Littlewood–Paley G-functions. Before this, we recall the subordination of semigroups. For $t\geq 0$, define a probability measure λ_t on $[0,+\infty[$ whose Laplace transform is given by

$$\int_0^\infty e^{-\gamma s}\lambda_t(\mathrm{d}s)=e^{-\sqrt{\gamma}t},\quad \gamma\geq 0.$$

Then, for $t>0$, λ_t has the following expression

$$\lambda_t(\mathrm{d}s):=\frac{t}{2\sqrt{\pi}}e^{-t^2/4s}s^{-3/2}\mathrm{d}s.$$

Take a signed smooth measure k with $\mathsf{k}^+\in S_D(\mathbf{X})$ and $\mathsf{k}^-\in S_{EK}(\mathbf{X})$. For $\alpha\geq C_{\mathsf{k}}$, we define the subordination $(Q_t^{(\alpha),\mathsf{k}})_{t\geq0}$ of $(P_t^{\mathsf{k}})_{t\geq0}$ by

$$Q_t^{(\alpha),\mathsf{k}}f:=\int_0^\infty e^{-\alpha s}P_s^{\mathsf{k}}f\,\lambda_t(\mathrm{d}s),\quad f\in L^p(X;\mathfrak{m}).$$

When $\mathsf{k}=0$, we write $Q_t^{(\alpha)}$ instead of $Q_t^{(\alpha),0}$. The infinitesimal generator of $(Q_t^{(\alpha),\mathsf{k}})_{t\geq0}$ is denoted by $-\sqrt{\alpha-\Delta_p^{\mathsf{k}}}$. We may omit the subscript p for simplicity. In addition, we can easily see that for $p\in[2,+\infty]$

$$\|Q_t^{(\alpha),\mathsf{k}}f\|_{L^p(X;\mathfrak{m})}\leq Ce^{-\sqrt{\alpha-C_{\mathsf{k}}}t}\|f\|_{L^p(X;\mathfrak{m})},\tag{2.4}$$

where C and C_κ are the same constant as in (2.1). For $f\in L^2(X;\mathfrak{m})\cap L^p(X;\mathfrak{m})$ and $\alpha\geq C_{\mathsf{k}}$, we define the Littlewood–Paley's G-functions by

$$\begin{aligned}
g_f^{\rightarrow\mathsf{k}}(x,t)&:=\left|\frac{\partial}{\partial t}(Q_t^{(\alpha),\mathsf{k}}f)(x)\right|, & G_f^{\rightarrow\mathsf{k}}(x)&:=\left(\int_0^\infty t g_f^{\rightarrow\mathsf{k}}(x,t)^2\mathrm{d}t\right)^{1/2},\\
g_f^{\uparrow\mathsf{k}}(x,t)&:=\left(\Gamma(Q_t^{(\alpha),\mathsf{k}}f)\right)^{1/2}(x), & G_f^{\uparrow\mathsf{k}}(x)&:=\left(\int_0^\infty t g_f^{\uparrow\mathsf{k}}(x,t)^2\mathrm{d}t\right)^{1/2},\\
g_f^{\mathsf{k}}(x,t)&:=\sqrt{g_f^{\rightarrow\mathsf{k}}(x,t)^2+g_f^{\uparrow\mathsf{k}}(x,t)^2}, & G_f^{\mathsf{k}}(x)&:=\left(\int_0^\infty t g_f^{\mathsf{k}}(x,t)^2\mathrm{d}t\right)^{1/2}.
\end{aligned}$$

For notational simplicity, we omit k in G-functions when we replace P_t^{k} with P_t, e.g. we write $g_f^{\rightarrow}(x,t)$ (resp. $G_f^{\rightarrow}(x)$) instead of $g_f^{\rightarrow0}(x,t)$ (resp. $G_f^{\rightarrow0}(x)$) and so on. We say that $(\mathscr{E},D(\mathscr{E}))$ is said to be *irreducible* if any $(P_t)_{t\geq0}$-invariant set A satisfies $\mathfrak{m}(A)=0$ or $\mathfrak{m}(X\setminus A)=0$. Here a measurable set A is said to be $(P_t)_{t\geq0}$-*invariant* if $P_t\mathbf{1}_Au=\mathbf{1}_AP_tu$ for any $u\in L^2(X;\mathfrak{m})$ and $t>0$. The L^2-semigroup $(P_t)_{t\geq0}$ associated with $(\mathscr{E},D(\mathscr{E}))$ is said to be *hypercontractive with* $C>0$ if $P_tf\in L^{q(t)}(X;\mathfrak{m})$ for $f\in L^r(X;\mathfrak{m})$ with $r>1$ and $q(t):=e^{2t/C}(r-1)+1$ and

$$\|P_tf\|_{L^{q(t)}(X;\mathfrak{m})}\leq\|f\|_{L^r(X;\mathfrak{m})}\quad\text{for any}\quad f\in L^r(X;\mathfrak{m})\quad\text{and}\quad t\geq0.$$

Here $C>0$ is a fixed constant. It is known that the hypercontractivity with $C>0$ is equivalent to the following log-Sobolev inequality (see [9, Theorem 5.2.3]):

$$\int_X|f(x)|^2\log\left(\frac{|f(x)|^2}{\|f\|^2_{L^2(X;\mathfrak{m})}}\right)\mathfrak{m}(\mathrm{d}x)\leq2C\mathscr{E}(f,f)\quad\text{for}\quad f\in D(\mathscr{E}).$$

Now we present the Littlewood–Paley–Stein inequality under Assumption 1. Let $(E_\lambda)_{\lambda\in\mathbb{R}}$ be the real resolution of the identity associated to the L^2-generator Δ and set $E_o:=E_0-E_{0-}$ (in another word $E_o:=E_{\{0\}}$ and $E_\lambda:=E_{]-\infty,\lambda]}$) (see [61, Chapter IX, Section 5] for resolution of the identity).

Theorem 2.2. *Suppose Assumption 1. For any $p \in]\,1,+\infty\,[$ and $\alpha \geq C_\kappa$ and $\alpha > 0$, the G-functions $g_f^{\rightarrow}(\cdot,t)$, $g_f^{\uparrow}(\cdot,t)$, $g_f(\cdot,t)$, $G_f^{\rightarrow}$, $G_f^{\uparrow}$ and G_f can be extended to be in $L^p(X;\mathfrak{m})$ for $f \in L^p(X;\mathfrak{m})$ and the following inequalities hold for $f \in L^p(X;\mathfrak{m})$*

$$\|G_f\|_{L^p(X;\mathfrak{m})} \lesssim \|f\|_{L^p(X;\mathfrak{m})}, \tag{2.5}$$

$$\|f\|_{L^p(X;\mathfrak{m})} \lesssim \|G_f^{\rightarrow}\|_{L^p(X;\mathfrak{m})}. \tag{2.6}$$

Moreover, (2.5) *remains valid for $\alpha > 0$ under $p \in]\,1,\,2\,]$. Furthermore, for $p \in]\,1,+\infty\,[$,* (2.5) *remains valid for $\alpha = 0$ under $\kappa^- = 0$ (hence $C_\kappa = 0$). Suppose further that $\alpha = 0$ and $\kappa^- = 0$ with Assumption 1 and $(\mathscr{E}, D(\mathscr{E}))$ is transient, or irreducible, or hypercontractivity of $(P_t)_{t\geq 0}$ with $\mathfrak{m}(X) < +\infty$ is satisfied. Then, for any $p \in]\,1,+\infty\,[$, the operator E_o can be extended to a bounded linear operator on $L^p(X;\mathfrak{m})$ and the following inequalities hold for any $f \in L^p(X;\mathfrak{m})$:*

$$\|G_f\|_{L^p(X;\mathfrak{m})} \lesssim \|f - E_o f\|_{L^p(X;\mathfrak{m})}, \tag{2.7}$$

$$\|f - E_o f\|_{L^p(X;\mathfrak{m})} \lesssim \|G_f^{\rightarrow}\|_{L^p(X;\mathfrak{m})}, \tag{2.8}$$

$$\|f - E_o f\|_{L^p(X;\mathfrak{m})} \lesssim \|G_f^{\uparrow}\|_{L^p(X;\mathfrak{m})}. \tag{2.9}$$

Here, for $u, v \in L^p(X;\mathfrak{m})$, the notation $\|u\|_{L^p(X;\mathfrak{m})} \lesssim \|v\|_{L^p(X;\mathfrak{m})}$ means $\|u\|_{L^p(X;\mathfrak{m})} \leq A(\kappa,p)\|v\|_{L^p(X;\mathfrak{m})}$ for some positive constant $A(\kappa,p)$ depending only on κ and $p \in [\,1,+\infty\,[$.

Remark 2.3. Even if we restrict to the case $\kappa^- = 0$, Theorem 2.2 is new, because we do not assume the existence of a nice core as in the Bakry–Émery framework formulated in [34]. Note that $\mathrm{Test}(X)$ defined below as a subalgebra of $D(\Delta)$ is not necessarily a subset of $C_b(X)$, hence it does not play a nice core as in [34].

Remark 2.4. Our Theorem 2.2 partly generalizes the result in Kawabi–Miyokawa [34] and it can not be covered by [34], because their formulation does not treat the case $\alpha = 0$ (more precisely (2.7),(2.8),(2.9)) as in the framework of abstract Wiener space (See Shigekawa [52, (3.23),(3.24),(3.25)]).

Remark 2.5. In Theorem 2.2, there is no estimate of $G_f^{\uparrow}$ when $\alpha > 0$. If we assume the spectral gap for $(\mathscr{E}, D(\mathscr{E}))$, i.e., there exists small $\varepsilon > 0$ such that $E_{]\,0,\varepsilon\,[} = 0$, then the Green operator G_0 with index 0 is a bounded linear operator on $L^2(X;\mathfrak{m})$ (see [52, (4.42)]), so that we can deduce the same estimate (2.6) for $G_f^{\uparrow}$ holds by way of the argument in [52, Subsection 3.2.11] (see the last statement of Subsection 4.6 below).

Remark 2.6. The Littlewood–Paley–Stein inequalities for functions and 1-forms were firstly proved by X.-D. Li [39] in the framework of weighted complete Riemannian manifolds $(M, g, e^{-\phi}\nu_g)$ with $\phi \in C^2(M)$ if the Bakry–Émery Ricci curvature $\mathrm{Ric}_\phi^\infty := \mathrm{Ric}_g + \mathrm{Hess}\,(\phi)$ has a lower bound of Kato class function $K(\cdot)$ (see [39, Theorem 4.2]). Here ν_g is the Riemannian volume measure. For this, he established gradient type estimates (see [39, (4.8) and (4.9)]).

3 Preliminary

3.1 Test functions

Under Assumption 1, we now introduce $\mathrm{Test}(X)$ the set of test functions:

Definition 3.1. Let $(X,\mathscr{E},\mathfrak{m})$ be a tamed space. Let us define the set of *test functions* by

$$\mathrm{Test}(X):=\{f\in D(\Delta)\cap L^\infty(X;\mathfrak{m})\mid \Gamma(f)\in L^\infty(X;\mathfrak{m}),\Delta f\in D(\mathscr{E})\}.$$

By [21, Proposition 6.8], under Assumption 1, $\mathsf{BE}_2(-\kappa^-,\infty)$ holds. Moreover, for every $f\in L^2(X;\mathfrak{m})\cap L^\infty(X;\mathfrak{m})$ and $t>0$, it holds

$$\Gamma(P_tf)\le\frac{1}{2t}\|P_t^{-2\kappa^-}\|_{\infty,\infty}\cdot\|f\|^2_{L^\infty(X;\mathfrak{m})}. \tag{3.1}$$

In particular, for $f\in L^2(X;\mathfrak{m})\cap L^\infty(X;\mathfrak{m})$, then $P_tf\in\mathrm{Test}(X)$.

By [21, Theorems 3.4 and 3.6, Proposition 3.7 and Theorem 6.10], the condition $\mathsf{BE}_2(\kappa,\infty)$ is equivalent to $\mathsf{BE}_1(\kappa,\infty)$ under $\kappa^+\in S_D(\mathbf{X})$ and $2\kappa^-\in S_{EK}(\mathbf{X})$. In particular, we have (2.3).

Erbar–Rigoni–Sturm–Tamanini [21, Lemma 6.4] proved that for every $f\in\mathrm{Test}(X)$, $\Gamma(f)\in D(\mathscr{E})\cap L^\infty(X;\mathfrak{m})$ and there exists $\mu=\mu^+-\mu^-$ with $\mu^\pm\in D(\mathscr{E})^*$ such that

$$-\mathscr{E}^{2\kappa}(u,\varphi)=\int_X\tilde{\varphi}\,\mathrm{d}\mu\quad\text{ for all }\quad\varphi\in D(\mathscr{E}) \tag{3.2}$$

under Assumption 1 (see also [50, Lemma 3.2] for $\mathsf{RCD}(K,N)$-space). From this, we see that $\Delta(fg)=(\Delta f)g+2\Gamma(f,g)+f(\Delta g)\in D(\mathscr{E})$ for $f,g\in\mathrm{Test}(X)$ under Assumption 1, because $\Gamma(f,g)=\frac14(\Gamma(f+g)-\Gamma(f-g))\in D(\mathscr{E})$ and $\Gamma((\Delta f)g)\le 2(\Delta f)^2\Gamma(g)+2g^2\Gamma(\Delta f)\le 2\|\Gamma(g)\|^2_\infty(\Delta f)^2+2\|g\|^2_\infty\Gamma(\Delta f)\in L^1(X;\mathfrak{m})$, in particular, $\mathrm{Test}(X)$ is an algebra, i.e., for $f,g\in\mathrm{Test}(X)$, $fg\in\mathrm{Test}(X)$, if further $\boldsymbol{f}\in\mathrm{Test}(X)^n$, then $\Phi(\boldsymbol{f})\in\mathrm{Test}(X)$ for every smooth function $\Phi:\mathbb{R}^n\to\mathbb{R}$ with $\Phi(0)=0$ (see [21, Lemma 6.5]).

Remark 3.2. It is also shown in [21, Corollary 6.9] that $\mathrm{Test}(X)$ is dense in $(\mathscr{E},D(\mathscr{E}))$ under Assumption 1. However, $\mathrm{Test}(X)$ is not necessarily a subspace of $C_b(X)$. So, we can not directly apply the same method as in Kawabi–Miyokawa [34], Shigekawa–Yoshida [53], Yoshida [59], though the continuity of test functions is not essential in the proof. When the tamed Dirichlet space comes from RCD-space, the Sobolev-to-Lipschitz property of RCD-spaces ensures $\mathrm{Test}(X)\subset C_b(X)$ (see also [38]).

3.2 The product of diffusion process and auxiliary Brownian motion

Recall that $(\mathbb{P}_x)_{x\in X}$ denotes the diffusion measure of $\mathbf{X}$ associated with the Dirichlet form $(\mathscr{E},D(\mathscr{E}))$. Let $\mathbf{W}:=(B_t,\mathbb{P}_a^{\uparrow})$ be one-dimensional Brownian motion starting at $a\in\mathbb{R}$ with the generator $\frac{\partial^2}{\partial a^2}$. Denote by $(\mathbb{D},D(\mathbb{D}))$ the Dirichlet form on $L^2(\mathbb{R})$ associated to $(B_t,\mathbb{P}_a^{\uparrow})$. We set $\widehat{X}:=X\times\mathbb{R}$, $\hat{x}:=(x,a)\in\widehat{X}$, $\widehat{X}_t:=(X_t,B_t)$, $t\ge0$, $\widehat{\mathfrak{m}}:=\mathfrak{m}\otimes m$ and $\mathbb{P}_{\hat{x}}:=\mathbb{P}_x\otimes\mathbb{P}_a^{\uparrow}$. Then $\widehat{\mathbf{X}}:=(\widehat{X}_t,\mathbb{P}_{\hat{x}})$ is an $\widehat{\mathfrak{m}}$-symmetric diffusion process on $\widehat{X}$ with the (formal) generator $\Delta+\frac{\partial^2}{\partial a^2}$, where m is one-dimensional Lebesgue measure. Note that the life time $\widehat{\zeta}$ of $\widehat{\mathbf{X}}$ coincides with the life time ζ of $\mathbf{X}$. Since $\mathbf{X}$ (hence $\widehat{\mathbf{X}}$) is not necessarily conservative, throughout this paper, all functions defined on $\widehat{X}$ should be regarded as functions on its one point compactification $\widehat{X}_{\widehat{\partial}}:=\widehat{X}\cup\{\widehat{\partial}\}$ vanishing at $\widehat{\partial}$. We put $\mathbb{P}_{\mathfrak{m}}:=\int_X\mathbb{P}_x\mathfrak{m}(\mathrm{d}x)$, $\mathbb{P}_{\mathfrak{m}\otimes\delta_a}:=\int_X\mathbb{P}_{(x,a)}\mathfrak{m}(\mathrm{d}x)$ and denote the integration with respect to $\mathbb{P}_x$, $\mathbb{P}_a^{\uparrow}$, $\mathbb{P}_{(x,a)}$ and $\mathbb{P}_{\mathfrak{m}\otimes\delta_a}$ by $\mathbb{E}_x$, $\mathbb{E}_a^{\uparrow}$, $\mathbb{E}_{(x,a)}$ and $\mathbb{E}_{\mathfrak{m}\otimes\delta_a}$, respectively.

We denote the semigroup on $L^p(\widehat{X};\widehat{\mathfrak{m}})$ associated with the diffusion process $(\widehat{X}_t)_{t\geq 0}$ by $(\widehat{P}_t)_{t\geq 0}$ and its generator by $\widehat{\Delta}_p$. We also denote the Dirichlet form on $L^2(\widehat{X};\widehat{\mathfrak{m}})$ associated with $\widehat{\Delta}_2$ by $(\widehat{\mathscr{E}}, D(\widehat{\mathscr{E}}))$. That is,

$$
\begin{aligned}
D(\widehat{\mathscr{E}}) :&= \left\{ u \in L^2(\widehat{X};\widehat{\mathfrak{m}}) \;\middle|\; \lim_{t\to 0}\frac{1}{t}(u-\widehat{P}_t u, u)_{L^2(\widehat{X};\widehat{\mathfrak{m}})} < +\infty \right\},\\
\widehat{\mathscr{E}}(u,v) :&= \lim_{t\to 0}\frac{1}{t}(u-\widehat{P}_t u, v)_{L^2(\widehat{X};\widehat{\mathfrak{m}})} \qquad \text{for} \quad u,v\in D(\widehat{\mathscr{E}}).
\end{aligned}
$$

Since $(\widehat{\mathscr{E}}, D(\widehat{\mathscr{E}}))$ is associated to the $\widehat{\mathfrak{m}}$-symmetric Borel right process $\widehat{\mathbf{X}}$, it is quasi-regular by Fitzsimmons [24].

Throughout this subsection, we take a signed smooth measure k with $\mathsf{k}^+ \in S_D(\mathbf{X})$ and $\mathsf{k}^- \in S_{EK}(\mathbf{X})$. We define $\widehat{\mathsf{k}}^{\pm} := \mathsf{k}^{\pm}\otimes m$ and $\widehat{\mathsf{k}} := \widehat{\mathsf{k}}^{+} - \widehat{\mathsf{k}}^{-}$. The associated positive continuous additive functional (PCAF in short) $\widehat{A}_t^{\widehat{\mathsf{k}}^{\pm}}$ in Revuz correspondence under $\widehat{\mathbf{X}}$ is given by $\widehat{A}_t^{\widehat{\mathsf{k}}^{\pm}} = A_t^{\mathsf{k}^{\pm}}$, in particular, $\widehat{\mathsf{k}}^+$ (resp. $\widehat{\mathsf{k}}^-$) is a smooth measure of Dynkin (resp. extended Kato) class with respect to $\widehat{\mathbf{X}}$, i.e. $\widehat{\mathsf{k}}^+ \in S_D(\widehat{\mathbf{X}})$ (resp. $\widehat{\mathsf{k}}^- \in S_{EK}(\widehat{\mathbf{X}})$). Then we can define the following quadratic form $(\widehat{\mathscr{E}}^{\widehat{\mathsf{k}}}, D(\widehat{\mathscr{E}}^{\widehat{\mathsf{k}}}))$ on $L^2(\widehat{X};\widehat{\mathfrak{m}})$:

$$
D(\widehat{\mathscr{E}}^{\widehat{\mathsf{k}}}) := D(\widehat{\mathscr{E}}), \qquad \widehat{\mathscr{E}}^{\widehat{\mathsf{k}}}(u,v) := \widehat{\mathscr{E}}(u,v) + \langle \widehat{\mathsf{k}}, \tilde{u}\tilde{v}\rangle \quad \text{for} \quad u,v\in D(\widehat{\mathscr{E}}^{\widehat{\mathsf{k}}}). \tag{3.3}
$$

Here $\tilde{u}$ denotes the $\widehat{\mathscr{E}}$-quasi-continuous $\widehat{\mathfrak{m}}$-version of u with respect to $(\widehat{\mathscr{E}}, D(\widehat{\mathscr{E}}))$. The strongly continuous semigroup $(\widehat{P}_t^{\widehat{\mathsf{k}}})_{t\geq 0}$ on $L^2(\widehat{X};\widehat{\mathfrak{m}})$ associated with the quadratic form $(\widehat{\mathscr{E}}^{\widehat{\mathsf{k}}}, D(\widehat{\mathscr{E}}^{\widehat{\mathsf{k}}}))$ is given by

$$
\widehat{P}_t^{\widehat{\mathsf{k}}} u(\hat{x}) = \mathbb{E}_{\hat{x}}[e^{-A_t^{\mathsf{k}}} u(\widehat{X}_t)] \quad \text{for} \quad u \in L^2(\widehat{X};\widehat{\mathfrak{m}})\cap \mathscr{B}(\widehat{X}). \tag{3.4}
$$

Hereafter, we assume Assumption 1, in particular, $\kappa^+ \in S_D(\mathbf{X})$ and $\kappa^- \in S_{EK}(\mathbf{X})$. In this generality, our framework can not be covered by Kawabi–Miyokawa [34]. We denote by $\widehat{\mathscr{C}} := \mathrm{Test}(X)\otimes C_c^{\infty}(\mathbb{R})$ the totality of all linear combinations of $f\otimes\varphi$, $f\in \mathrm{Test}(X)$, $\varphi\in C_c^\infty(\mathbb{R})$, where $(f\otimes\varphi)(y) := f(x)\varphi(a)$. Under Assumption 1, $\widehat{\mathscr{C}}$ is also an algebra. Meanwhile, the space $L^2(X;\mathfrak{m})\otimes L^2(\mathbb{R})$ and $D(\mathscr{E})\otimes H^{1,2}(\mathbb{R})$ are usual tensor products of Hilbert spaces, where $H^{1,2}(\mathbb{R})$ is the Sobolev space which consists of all functions $\varphi\in L^2(\mathbb{R})$ such that the weak derivative φ' exists and belongs to $L^2(\mathbb{R})$. In the same way of [34, Lemma 2.1], we have the following: $\widehat{\mathscr{C}}$ is dense in $D(\widehat{\mathscr{E}})$. Moreover, for $u,v\in D(\mathscr{E})\otimes H^{1,2}(\mathbb{R})$, we have

$$
\widehat{\mathscr{E}}(u,v) = \int_{\mathbb{R}} \mathscr{E}(u(\cdot,a),v(\cdot,a))m(\mathrm{d}a) + \int_X \mathfrak{m}(\mathrm{d}x)\int_{\mathbb{R}} \frac{\partial u}{\partial a}(x,a)\frac{\partial v}{\partial a}(x,a)m(\mathrm{d}a). \tag{3.5}
$$

Denote by $\mathscr{P}(X)$, the family of all Borel probability measures on X, and by $\mathscr{B}^*(X)$, the family of all universally measurable sets, that is, $\mathscr{B}^*(X) := \bigcap_{\nu\in\mathscr{P}(X)} \overline{\mathscr{B}(X)}^{\nu}$, where $\overline{\mathscr{B}(X)}^{\nu}$ is the ν-completion of $\mathscr{B}(X)$ for $\nu\in\mathscr{P}(X)$. $\mathscr{B}^*(X)$ also denotes the family of universally measurable real valued functions. Moreover, $\mathscr{B}_b^*(X)$ (resp. $\mathscr{B}_+^*(X)$) denotes the family of bounded (resp. non-negative) universally measurable functions. For $f \in \mathscr{B}_b^*(X)$, or $f \in \mathscr{B}_+^*(X)$, we set

$$q_t^{(\alpha),\mathsf{k}} f(x) := \mathbb{E}_x\left[\int_0^\infty e^{-\alpha s - A_s^{\mathsf{k}}} f(X_s)\lambda_t(\mathrm{d}s)\right] \tag{3.6}$$

and write $q_t^{(\alpha)} f(x)$ instead of $q_t^{(\alpha),0} f(x)$. Then $q_t^{(\alpha),\mathsf{k}} f \in \mathscr{B}_b^*(X)$ (resp. $q_t^{(\alpha),\mathsf{k}} f \in \mathscr{B}_+^*(X)$) under $f \in \mathscr{B}_b^*(X)$ (resp. $f \in \mathscr{B}_+^*(X)$). It is easy to see $q_t^{(\alpha),\mathsf{k}} 1(x) \le Ce^{-\sqrt{\alpha - C_{\mathsf{k}}}t}$ ($\alpha \ge C_{\mathsf{k}}$) and $q_t^{(\alpha)} 1(x) \le e^{-\sqrt{\alpha}t}$ ($\alpha \ge 0$). Take $f \in L^2(X;\mathfrak{m}) \cap \mathscr{B}_b^*(X)$ or $f \in L^2(X;\mathfrak{m}) \cap \mathscr{B}_+^*(X)$. According to the equation

$$(q_t^{(\alpha),\mathsf{k}} f, g)_{\mathfrak{m}} = (Q_t^{(\alpha),\mathsf{k}} f, g)_{\mathfrak{m}} \quad \text{for any} \quad g \in L^2(X;\mathfrak{m}), \tag{3.7}$$

$q_t^{(\alpha),\mathsf{k}} f$ is an $\mathfrak{m}$-version of $Q_t^{(\alpha),\mathsf{k}} f$. Here, we fix a function $f \in D(\Delta^{\mathsf{k}}) \cap \mathscr{B}^*(X)$. Let $u(x,a) := q_a^{(\alpha),\mathsf{k}} f(x)$ ($\alpha \ge C_{\mathsf{k}}$). Then it holds that

$$\begin{aligned} &a \mapsto u(\cdot, a) \quad \text{is an } L^2(X;\mathfrak{m})\text{-valued smooth function, and} \\ &\left(\frac{\partial^2}{\partial a^2} + \Delta^{\mathsf{k}} - \alpha\right) u(\cdot, a) = 0 \quad \text{in} \quad L^2(X;\mathfrak{m}). \end{aligned} \tag{3.8}$$

Moreover, for $a \in \mathbb{R}$, we set $v(x,a) := u(x,|a|) = q_{|a|}^{(\alpha),\mathsf{k}} f(x)$ for $\alpha > C_{\mathsf{k}}$. Then by (2.4), we have

$$\|v\|_{L^2(\widehat{X};\widehat{\mathfrak{m}})} \le \left(\int_{\mathbb{R}} Ce^{-2\sqrt{\alpha - C_{\mathsf{k}}}|a|} \|f\|^2_{L^2(X;\mathfrak{m})} \mathrm{d}a\right)^{1/2} = C^{\frac{1}{2}}(\alpha - C_{\mathsf{k}})^{-1/4} \|f\|_{L^2(X;\mathfrak{m})}. \tag{3.9}$$

From now on, we establish the semimartingale decomposition of $\tilde{v}(\widehat{X}_{t\wedge\tau})$, $t \ge 0$, where $\tau := \inf\{t > 0 \mid B_t = 0\}$. By the same way of the proof of [34, Lemma 2.2], we can deduce

$$v \in D(\widehat{\mathscr{E}}) \quad \text{for} \quad v(x,a) := q_{|a|}^{(\alpha),\mathsf{k}} f(x) \quad \text{with} \quad f \in D(\Delta^{\mathsf{k}}) \cap \mathscr{B}^*(X), \quad \alpha > C_{\mathsf{k}}. \tag{3.10}$$

In proving (3.10), we only use $\mathsf{k}^+ \in S_D(\mathbf{X})$ and $\mathsf{k}^- \in S_{EK}(\mathbf{X})$, but we do not impose $\mathsf{BE}_2(\mathsf{k},\infty)$ (we still assume $\mathsf{BE}_2(\kappa,\infty)$ and $\kappa \in S_D(\mathbf{X})$ and $2\kappa^- \in S_{EK}(\mathbf{X})$), moreover, we use the following characterization: $D(\widehat{\mathscr{E}}^{\widehat{\mathsf{k}}}) = D(\widehat{\mathscr{E}})$ and

$$\begin{aligned} D(\widehat{\mathscr{E}}^{\widehat{\mathsf{k}}}) &:= \left\{ u \in L^2(\widehat{X};\widehat{\mathfrak{m}}) \;\middle|\; \lim_{t\to 0} \frac{1}{t}(u - \widehat{P}_t^{\widehat{\mathsf{k}}} u, u)_{L^2(\widehat{X};\widehat{\mathfrak{m}})} < +\infty \right\}, \\ \widehat{\mathscr{E}}^{\widehat{\mathsf{k}}}(u,v) &:= \lim_{t\to 0} \frac{1}{t}(u - \widehat{P}_t^{\widehat{\mathsf{k}}} u, v)_{L^2(\widehat{X};\widehat{\mathfrak{m}})} \qquad \text{for} \quad u,v \in D(\widehat{\mathscr{E}}^{\widehat{\mathsf{k}}}) = D(\widehat{\mathscr{E}}). \end{aligned}$$

By (3.10), we can apply Fukushima's decomposition to v. Then, there are a martingale additive functional of finite energy $\widehat{M}^{[v]}$ and a continuous additive functional of zero energy $\widehat{N}^{[v]}$ such that

$$\tilde{v}(\widehat{X}_t) - \tilde{v}(\widehat{X}_0) = \widehat{M}_t^{[v]} + \widehat{N}_t^{[v]} \quad t \ge 0 \quad \mathbb{P}_{\hat{x}}\text{-a.s. for q.e. } \hat{x}, \tag{3.11}$$

where $\tilde{v}$ is an $\widehat{\mathscr{E}}$-quasi-continuous $\widehat{\mathfrak{m}}$-version of $v \in D(\widehat{\mathscr{E}}^{\mathsf{k}}) = D(\widehat{\mathscr{E}})$. See [25, Theorem 5.2.2] for Fukushima's decomposition theorem. Thanks to [25, Theorem 5.2.3], we know that

$$\langle \widehat{M}^{[v]} \rangle_t = \int_0^t \left\{ \Gamma(v,v)(\widehat{X}_s) + \left(\frac{\partial v}{\partial a}(\widehat{X}_s) \right)^2 \right\} \mathrm{d}s, \quad t \geq 0. \tag{3.12}$$

See also [25, Theorem 5.1.3 and Example 5.2.1] for details.

We now give the explicit expression of $\widehat{N}^{[v]}$. We refer to the idea of Kawabi–Miyokawa [34]. Let us define the signed measure ν on $\widehat{X}$ by

$$\nu(\mathrm{d}x\mathrm{d}a) := 2\sqrt{\alpha - \Delta^{\mathsf{k}}} v(x,a) \mathfrak{m}(\mathrm{d}x) \delta_0(\mathrm{d}a)$$

for $\alpha > C_{\mathsf{k}}$, where δ_0 is Dirac measure on $\mathbb{R}$ with unit mass at origin. The total variation of ν is given by

$$|\nu|(\mathrm{d}x\mathrm{d}a) := 2|\sqrt{\alpha - \Delta^{\mathsf{k}}} v(x,a)| \mathfrak{m}(\mathrm{d}x) \delta_0(\mathrm{d}a).$$

Note here that ν depends on $f \in D(\Delta^{\mathsf{k}}) \cap \mathscr{B}^*(X)$ and $\alpha > C_{\mathsf{k}}$. In the same way of the proof of [34, Lemma 2.3], for $f \in D(\Delta^{\mathsf{k}}) \cap \mathscr{B}^*(X)$ and $\alpha > C_{\mathsf{k}}$, there exists a constant $C > 0$ such that

$$\int_{\widehat{X}} |g \otimes \varphi(x,a)| \cdot |\nu|(\mathrm{d}x\mathrm{d}a) \leq C \sqrt{\widehat{\mathscr{E}}_1(g \otimes \varphi, g \otimes \varphi)} \quad \text{for} \quad g \in \mathrm{Test}(X),\ \varphi \in C_c^\infty(\mathbb{R}). \tag{3.13}$$

In another word, ν is of finite energy integral (see [25, Sections 2.2 and 5.4] for the definitions of measures of finite energy integrals). Then for each $\beta > 0$, there exists a unique $U_\beta \nu \in D(\mathscr{E})$ such that the following relation holds:

$$\widehat{\mathscr{E}}_\beta(U_\beta \nu, g \otimes \varphi) = \int_{\widehat{X}} (g \otimes \varphi)(x,a) \nu(\mathrm{d}x\mathrm{d}a), \quad g \in \mathrm{Test}(X), \varphi \in C_c^\infty(\mathbb{R}). \tag{3.14}$$

Moreover, for $f \in D(\Delta^{\mathsf{k}}) \cap \mathscr{B}^*(X)$, $\alpha > C_{\mathsf{k}}$ and $\beta > 0$, we have

(a) $U_\alpha \nu = v$,

(b) $U_\beta \nu = v - (\beta - \alpha) \widehat{R}_\beta v$ holds, where $(\widehat{R}_\beta)_{\beta > 0}$ is the resolvent of $(\widehat{P}_t)_{t \geq 0}$

in the same way of the proof of [34, Lemma 2.4]. From this with [25, Lemma 5.4.1], we have

$$\widehat{N}_t^{[v]} = \alpha \int_0^t \tilde{v}(\widehat{X}_s) \mathrm{d}s - A_t^\nu, \quad t \geq 0,$$

where $\tilde{v}$ is an $\widehat{\mathscr{E}}$-quasi-continuous $\widehat{\mathfrak{m}}$-version of v and A^ν is the CAF corresponding to ν. Since $X \times \{0\}$ is an $\widehat{\mathscr{E}}$-exceptional set, thanks to [25, Theorem 5.1.5], we have $A_{t \wedge \tau}^\nu = 0$. Thus

$$\widehat{N}_{t \wedge \tau}^{[v]} = \alpha \int_0^{t \wedge \tau} \tilde{v}(\widehat{X}_s) \mathrm{d}s. \tag{3.15}$$

Combining (3.11), (3.12) and (3.15), we have the following semi-martingale decomposition which plays a crucial role in our paper.

Proposition 3.3. *Suppose $\alpha > C_{\mathsf{k}}$, $f \in D(\Delta^{\mathsf{k}}) \cap \mathscr{B}^*(X)$ and set $v(x,a) = q_{|a|}^{(\alpha),\mathsf{k}} f(x)$ for $(x,a) \in \widehat{X}$. Then we have the semi-martingale decomposition*

$$\tilde{v}(\widehat{X}_{t\wedge\tau}) - \tilde{v}(\widehat{X}_0) = \widehat{M}^{[v]}_{t\wedge\tau} + \alpha \int_0^{t\wedge\tau} \tilde{v}(\widehat{X}_s)\mathrm{d}s, \quad t \geq 0, \tag{3.16}$$

under $\mathbb{P}_{\hat{x}}$ for q.e. $\hat{x} = (x,a)$. Moreover it holds

$$\langle \widehat{M}^{[v]} \rangle_{t\wedge\tau} = \int_0^{t\wedge\tau} \left\{ \Gamma(v,v)(\widehat{X}_s) + \left(\frac{\partial v}{\partial a}(\widehat{X}_s) \right)^2 \right\} \mathrm{d}s. \tag{3.17}$$

In particular, by setting $\widehat{M}_t := \widehat{M}^{[v]}_{t\wedge\tau}$,

$$\mathbb{E}_{(x,a)}[\langle \widehat{M} \rangle_\infty] = \mathbb{E}_{(x,a)} \left[\int_0^\tau \left\{ \Gamma(v,v) + \left(\frac{\partial v}{\partial a} \right)^2 \right\} (\widehat{X}_s)\mathrm{d}s \right] < +\infty \tag{3.18}$$

for $\widehat{\mathfrak{m}}$-a.e. $(x,a) \in \widehat{G} := X \times]\,0, +\infty\,[$, because the absorbing process $\widehat{\mathbf{X}}_{\widehat{G}}$ on $\widehat{G}$ is an $\widehat{\mathfrak{m}}$-symmetric transient process and $\Gamma(v) + \left(\frac{\partial v}{\partial a}\right)^2 \in L^1(\widehat{G}; \widehat{\mathfrak{m}})$.

Remark 3.4. The statements of [34, Proposition 2.5] are special cases of (3.16) and (3.17) of Proposition 3.3 under $\mathsf{k} = 0$ and $\kappa = (-R)\mathfrak{m}$ for some $R \in \mathbb{R}$. But [34, Proposition 2.5] does not show (3.18).

This proposition also gives the semi-martingale decomposition of $u(\widehat{X}_{t\wedge\tau})$, because $v(x,a) = u(x,a)$ holds for $a \geq 0$.

We need the following lemma to justify our results in the framework of quasi-regular Dirichlet forms, because the results in Ôkura [49] depends on the framework of regular Dirichlet forms.

Lemma 3.5. *We have the following:*

1. *Let $\{F_n\}$ be an $\widehat{\mathscr{E}}$-nest of closed subsets of $\widehat{X}$. Then $\{(F_n)_a\}$ is an $\mathscr{E}$-nest of closed subsets of X for m-a.e. $a \in \mathbb{R}$, where $(F_n)_a := \{x \in X \mid (x,a) \in F_n\}$.*
2. *Let N be an $\widehat{\mathscr{E}}$-exceptional set. Then, for m-a.e. $a \in \mathbb{R}$, $N_a := \{x \in X \mid (x,a) \in N\}$ is an $\mathscr{E}$-exceptional set. In particular, $\mathfrak{m} \otimes \delta_a$ does not charge any $\widehat{\mathscr{E}}$-exceptional set for m-a.e. $a \in \mathbb{R}$.*
3. *Let $(x,a) \mapsto u(x,a)$ be an $\widehat{\mathscr{E}}$-quasi-continuous function. Then, for m-a.e. $a \in \mathbb{R}$, $x \mapsto u(x,a)$ is an $\mathscr{E}$-quasi-continuous function.*

Proof. (2) and (3) are consequences of (1). We only prove (1). For $\mathfrak{m}$-a.e. strictly positive $\varphi \in L^2(X;\mathfrak{m})$ and m-a.e. strictly positive $\phi \in L^2(\mathbb{R})$, we set $\widehat{h} := G_1\varphi \otimes G_1^w\phi$, where G_1 (resp. G_1^w) is the 1-order resolvent operator on $L^2(X;\mathfrak{m})$ (resp. $L^2(\mathbb{R})$) associated to $(\mathscr{E}, D(\mathscr{E}))$ (resp. $(\mathbb{D}, D(\mathbb{D}))$. Then $\widehat{h}$ is a 2-excessive function of $L^2(\widehat{X};\widehat{\mathfrak{m}})$. Let $\{F_n\}$ be an

$\widehat{\mathscr{E}}$-nest of closed subsets of $\widehat{X}$. Denote the 2-order $\widehat{h}$-weighted capacity by $\widehat{\mathrm{Cap}}_{\widehat{h},2}$ defined by

$$\widehat{\mathrm{Cap}}_{\widehat{h},2}(O):=\inf\left\{\widehat{\mathscr{E}}_2(v,v)\,\middle|\, v\in D(\widehat{\mathscr{E}}),\ v\geq \widehat{h}\ \widehat{\mathfrak{m}}\text{-a.e. on } O\right\}$$

for an open subset O of $\widehat{X}$. Since $\widehat{h}\in D(\widehat{\mathscr{E}})$ is 2-excessive, we can deduce

$$\lim_{n\to\infty}\widehat{\mathrm{Cap}}_{\widehat{h},2}(\widehat{X}\setminus F_n)=0 \tag{3.19}$$

in the same way of the proof of [45, Chapter III, Theorem 2.11(i)] with the help of the 2-order version of [45, Chapter III, Proposition 1.5(iv) with $K=1$]. Note here that the proof of implication "$\{F_n\}$ is an $\widehat{\mathscr{E}}$-nest $\Longrightarrow$ (3.19)" above works for such $\widehat{h}$, though the description in [45, Chapter III, Theorem 2.11(i)] requires that $\widehat{h}$ has the shape $\widehat{h}=\widehat{G}_1\varphi$ for some $\widehat{\mathfrak{m}}$-a.e. strictly positive $\varphi\in L^2(\widehat{X};\widehat{\mathfrak{m}})$. Now we consider the $\widehat{h}$-transformed positivity preserving form $(\widehat{\mathscr{E}}^{\widehat{h}},D(\widehat{\mathscr{E}}^{\widehat{h}}))$ on $L^2(\widehat{X};\widehat{h}^2\widehat{\mathfrak{m}})$ and its usual 2-order capacity $\widehat{\mathrm{Cap}}_2^{\widehat{h}}$. Then

$$\begin{aligned}
\widehat{\mathrm{Cap}}_2^{\widehat{h}}(\widehat{X}\setminus F_n)&=\inf\left\{\widehat{\mathscr{E}}_2^{\widehat{h}}(u,u)\,\middle|\, u\in D(\widehat{\mathscr{E}}^{\widehat{h}}),\ u\geq 1\ \widehat{h}^2\widehat{\mathfrak{m}}\text{-a.e. on } \widehat{X}\setminus F_n\right\}\\
&=\inf\left\{\widehat{\mathscr{E}}(u\widehat{h},u\widehat{h})+2\int_{\widehat{X}}u^2\widehat{h}^2\mathrm{d}\widehat{\mathfrak{m}}\,\middle|\, u\in D(\widehat{\mathscr{E}}^{\widehat{h}}),\ u\geq 1\ \widehat{\mathfrak{m}}\text{-a.e. on } \widehat{X}\setminus F_n\right\}\\
&=\inf\left\{\widehat{\mathscr{E}}_2(v,v)\,\middle|\, v\in D(\widehat{\mathscr{E}}),\ v\geq \widehat{h}\ \widehat{\mathfrak{m}}\text{-a.e. on } \widehat{X}\setminus F_n\right\}\\
&=\widehat{\mathrm{Cap}}_{\widehat{h},2}(\widehat{X}\setminus F_n)\downarrow 0\quad\text{as}\quad n\to\infty.
\end{aligned}$$

In the same way of the proof of Ôkura [49, Theorem 4.1(4)], we have

$$\widehat{\mathscr{E}}_2^{\widehat{h}}(u,u)\geq\int_{\mathbb{R}}\mathscr{E}_2^{\widehat{h}(\cdot,a)}(u(\cdot,a),u(\cdot,a))m(\mathrm{d}a)\geq\int_{\mathbb{R}}\mathscr{E}_1^{\widehat{h}(\cdot,a)}(u(\cdot,a),u(\cdot,a))m(\mathrm{d}a).$$

From this, we can deduce

$$\mathrm{Cap}_{\widehat{h}(\cdot,a)}(X\setminus(F_n)_a)=\mathrm{Cap}_1^{\widehat{h}(\cdot,a)}(X\setminus(F_n)_a)\to 0\quad\text{as}\quad n\to\infty\quad\text{for}\quad m\text{-a.e. } a\in\mathbb{R}.$$

Here $\mathrm{Cap}_{\widehat{h}(\cdot,a)}$ is an $\widehat{h}(\cdot,a)$-weighted capacity for $(\mathscr{E},D(\mathscr{E}))$. Therefore, $\{(F_n)_a\}$ is an $\mathscr{E}$-nest for m-a.e. $a\in\mathbb{R}$ by applying [45, Chapter III, Theorem 2.11(i)]. Note here that $\widehat{h}(\cdot,a)=G_1\varphi G_1^w\phi(a)$ satisfies [45, the condition above in Chapter III, Theorem 2.11(i)]. □

4 Proof of Theorem 2.2

We now prove Theorem 2.2. The original idea of the proof is due to P.A. Meyer [47, 48]. The proofs in Bakry [8], Shigekawa–Yoshida [53] and Yoshida [59, 60] are based on this idea. In these papers, they expanded $\Delta(Q_t^{(\alpha)}f)^p$, $f\in\mathcal{A}$, by employing the usual functional analytic argument in the proof of Littlewood–Paley–Stein inequality. In their calculations, the existence of a good core $\mathcal{A}$ like $\mathrm{Test}(X)$ in this paper is assumed. Though we have a good core Test(X) in the framework of tamed Dirichlet space under Bakry–Émery condition $\mathsf{BE}_2(\kappa,\infty)$, we will not follow their way. We follow the method of the proof of Kawabi–Miyokawa [34] in proving Theorem 2.2. However, our curvature lower bound is not a constant in general, this gives another technical difficulty in proving Theorem 2.2. For this, we should modify the method of the proof in Kawabi–Miyokawa [34]. We prove Theorem 2.2 for $f\in D(\Delta)\cap L^p(X;\mathfrak{m})$ at the beginning. Any $f\in L^p(X;\mathfrak{m})\cap L^2(X;\mathfrak{m})$ with $p\in\,]1,+\infty[$ can be approximated by a sequence in $D(\Delta)\cap L^p(X;\mathfrak{m})$ in L^p-norm and in L^2-norm. Then one can conclude the statement of Theorem 2.2 for general $f\in L^p(X;\mathfrak{m})\cap L^2(X;\mathfrak{m})$ (hence $G_f^{\rightarrow}$, $G_f^{\uparrow}$ and G_f can be extended for general $f\in L^p(X;\mathfrak{m})$) in view of the triangle inequality $|G_{f_1}^{\rightarrow}-G_{f_2}^{\rightarrow}|\le G_{f_1-f_2}^{\rightarrow}$ (resp. $|G_{f_1}^{\uparrow}-G_{f_2}^{\uparrow}|\le G_{f_1-f_2}^{\uparrow}$).

We now split the proof of Theorem 2.2 in various cases in the following subsections.

4.1 Proof of (2.5) for $f\in D(\Delta)\cap L^p(X;\mathfrak{m})\cap\mathscr{B}^*(X)$ under $p\in\,]1,2[$

In this subsection, we return to the proof of the upper estimate (2.5) in Theorem 2.2 in the case of $1<p<2$. Recall the $\widehat{\mathfrak{m}}$-symmetric diffusion process $\widehat{\mathbf{X}}=(\widehat{X}_t,\mathbb{P}_{\hat{x}})$ on $\widehat{X}=X\times\mathbb{R}$ with $\widehat{X}_t=(X_t,B_t)$. We need the following identity (4.1) for our later use, whose proof can be done similarly as in Shigekawa [51, (6.13)], [52, Proposition 3.10]: Let $j:X\times[0,+\infty[\,\to[0,+\infty[$ be a measurable function. Then

$$\mathbb{E}_{\mathfrak{m}\otimes\delta_a}\left[\int_0^\tau j(\widehat{X}_s)\mathrm{d}s\right]=\int_X\mathfrak{m}(\mathrm{d}x)\int_0^\infty(a\wedge t)j(x,t)\mathrm{d}t \tag{4.1}$$

for $a\ge 0$. Here j is understood to be a function on $\widehat{X}_{\widehat{\partial}}$ with $j(\widehat{\partial})=0$ and note that our diffusions $\mathbf{X}$ and $\widehat{\mathbf{X}}$ may not be conservative. Moreover, (4.1) remains valid for general σ-finite measure $\mathfrak{m}$.

Thanks to the independence between $\mathbf{X}$ and $\mathbf{W}$ under $\mathbb{P}_{\mathfrak{m}\otimes\delta_a}$ and the $(P_t)_{t\ge0}$-invariance of $\mathfrak{m}$, we can see the following identity: for any $h\in\mathscr{B}_b(X)$

$$\mathbb{E}_{\mathfrak{m}\otimes\delta_a}[h(X_\tau)]=\int_X h(x)\mathfrak{m}(\mathrm{d}x) \tag{4.2}$$

under $h(\partial)=0$ (see [52, (3.47)] for the proof).

Now, we consider the case $v(x,a):=u(x,|a|):=q_{|a|}^{(\alpha)}f(x)$ for $f\in D(\Delta)\cap L^p(X;\mathfrak{m})\cap\mathscr{B}^*(X)$ and $\alpha>0$. The proof for this case is similar to the proof as in [34, Subsection 2.2]. For readers' convenience, we show the outline of its proof. Applying (3.10) for the case $\mathsf{k}=0$, we have $v\in D(\widehat{\mathscr{E}})$. We abbreviate $M_{t\wedge\tau}^{[v]}$ as M_t for simplicity. Combining Proposition 3.3 and Lemma 3.5(2) with $\mathsf{k}=0$ and $\alpha>0$, there exists a non-negative sequence $\{a_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty}a_n=+\infty$, (3.16) and (3.17) hold under $\mathbb{P}_{\mathfrak{m}\otimes\delta_{a_n}}$ for any $n\in\mathbb{N}$.

Set $V_t:=\tilde{v}(\widehat{X}_{t\wedge\tau})$. Applying Itô's formula to V_t^2, Proposition 3.3 implies

$$\mathrm{d}(V_t)^2=2V_t\mathrm{d}M_t+2(g_f(\widehat{X}_t)^2+\alpha V_t^2)\mathrm{d}t. \tag{4.3}$$

Let $\varepsilon>0$. We also have

$$\mathrm{d}(V_t^2+\varepsilon)^{\frac{p}{2}}\ge p(V_t^2+\varepsilon)^{\frac{p}{2}-1}V_t\mathrm{d}M_t+p(p-1)(V_t^2+\varepsilon)^{\frac{p}{2}-1}g_f(\widehat{X}_t)^2\mathrm{d}t, \tag{4.4}$$

by applying Itô's formula to $(V_t^2+\varepsilon)^{p/2}$ again, where we use $p<2$.

Taking the expectation of the inequality (4.4) and using $u(x,a)=v(x,a)$ for $a\geq 0$, we have

$$\mathbb{E}_{\mathfrak{m}\otimes\delta_a}\left[p(p-1)\int_0^{\tau}(V_t^2+\varepsilon)^{\frac{p}{2}-1}g_f(\widehat{X}_t)^2\mathrm{d}t\right]\leq\int_X(|f(x)|^2+\varepsilon)^{\frac{p}{2}}\mathfrak{m}(\mathrm{d}x), \tag{4.5}$$

where we use (4.2). Here, by recalling (4.1), the left hand side of (4.5) is equal to

$$p(p-1)\int_X\mathfrak{m}(\mathrm{d}x)\int_0^{\infty}(t\wedge a_n)(u(x,t)^2+\varepsilon)^{\frac{p}{2}-1}g_f(x,t)^2\mathrm{d}t.$$

By letting $\varepsilon\to 0$ and $n\to\infty$, we get

$$p(p-1)\int_X\mathfrak{m}(\mathrm{d}x)\int_0^{\infty}tu(x,t)^{p-2}g_f(x,t)^2\mathrm{d}t\leq\int_X|f(x)|^p\mathfrak{m}(\mathrm{d}x). \tag{4.6}$$

From now on, we utilize the following maximal ergodic inequality (see Shigekawa [52, Theorem 3.3] for details)

$$\left\|\sup_{t\geq 0}|P_tf|\right\|_{L^p(X;\mathfrak{m})}\leq\frac{p}{p-1}\|f\|_{L^p(X;\mathfrak{m})},\qquad p>1. \tag{4.7}$$

By (4.6) and (4.7), we obtain

$$\|G_f\|^p_{L^p(X;\mathfrak{m})}\leq\frac{p^{\frac{p(1-p)}{2}}}{(p-1)^{\frac{p(3-p)}{2}}}\|f\|^p_{L^p(X;\mathfrak{m})}. \tag{4.8}$$

Thus (2.5) holds for $f\in D(\Delta)\cap L^p(X;\mathfrak{m})\cap\mathscr{B}^*(X)$ under $p\in\,]1,2[$ and $\alpha>0$.

Next we prove that (2.5) holds under $p\in\,]1,2[$, $\alpha=0$ and $f\in D(\Delta)\cap L^p(X;\mathfrak{m})\cap\mathscr{B}^*(X)$. We note that $\Gamma(Q_\cdot^{(\alpha)}f)^{\frac12}\to\Gamma(Q_\cdot^{(0)}f)^{\frac12}$ in $L^2(X\times[0,+\infty[:\mathfrak{m}\otimes e^{-t}\mathrm{d}t)$ as $\alpha\to 0$ for $f\in D(\mathscr{E})$. Indeed,

$$\begin{aligned}
&\int_0^{\infty}e^{-t}\mathrm{d}t\int_X\left|\Gamma(Q_t^{(\alpha)}f)^{\frac12}-\Gamma(Q_t^{(0)}f)^{\frac12}\right|^2\mathrm{d}\mathfrak{m}\\
&\qquad\leq\int_0^{\infty}e^{-t}\mathrm{d}t\int_X\Gamma(Q_t^{(\alpha)}f-Q_t^{(0)}f)\mathrm{d}\mathfrak{m}\\
&\qquad=\int_0^{\infty}e^{-t}\mathrm{d}t\int_0^{\infty}\lambda\left(e^{-\sqrt{\alpha+\lambda}t}-e^{-\sqrt{\lambda}t}\right)^2\mathrm{d}(E_\lambda f,f)\\
&\qquad=\int_0^{\infty}\lambda\left[\frac{1}{2\sqrt{\alpha+\lambda}+1}-\frac{2}{\sqrt{\alpha+\lambda}+\sqrt{\lambda}+1}+\frac{1}{2\sqrt{\lambda}+1}\right]\mathrm{d}(E_\lambda f,f)\\
&\qquad\to 0\quad\text{as}\quad\alpha\downarrow 0.
\end{aligned}$$

Moreover, $\sqrt{\alpha-\Delta}Q_\cdot^{(\alpha)}f\to\sqrt{-\Delta}Q_\cdot^{(0)}f$ in $L^2(X\times[0,+\infty[\,;\mathfrak{m}\otimes e^{-t}\mathrm{d}t)$ as $\alpha\to 0$ for

$f \in D(\mathscr{E})$. Indeed,

$$
\begin{aligned}
&\int_0^\infty e^{-t}\mathrm{d}t \int_X \left|\sqrt{\alpha-\Delta}Q_t^{(\alpha)}f - \sqrt{-\Delta}Q_t^{(0)}f\right|^2 \mathrm{d}\mathfrak{m} \\
&\quad = \int_0^\infty e^{-t}\mathrm{d}t \int_0^\infty \left|\sqrt{\alpha+\lambda}e^{-\sqrt{\alpha+\lambda}t} - \sqrt{\lambda}e^{-\sqrt{\lambda}t}\right|^2 \mathrm{d}(E_\lambda f, f) \\
&\quad = \int_0^\infty \left(\frac{\alpha+\lambda}{1+2\sqrt{\alpha+\lambda}} - \frac{2\sqrt{\alpha+\lambda}\sqrt{\lambda}}{1+\sqrt{\alpha+\lambda}+\sqrt{\lambda}} + \frac{\lambda}{1+2\sqrt{\lambda}} \right) \mathrm{d}(E_\lambda f, f) \quad \to 0 \quad \text{as} \quad \alpha \downarrow 0.
\end{aligned}
$$

Then there exists a subsequence $\{\alpha_k\}$ tending to 0 as $k \to \infty$ such that $\Gamma(Q_t^{(\alpha_k)}f)(x) \to \Gamma(Q_t^{(0)}f)(x)$ and $\sqrt{\alpha_k - \Delta}Q_t^{(\alpha_k)}f(x) \to \sqrt{-\Delta}Q_t^{(0)}f(x)$ as $k \to \infty$ $\widehat{\mathfrak{m}}$-a.e. (x,t). By Fubini's theorem, the convergences hold $\mathfrak{m}$-a.e. $x \in X$ for a.e. $t \in [0, +\infty[$. Let $G_f^{(\alpha_k)}$ be the Littlewood–Paley G-function defined by $Q_t^{(\alpha_k)}f$. Then we easily see $G_f \le \underline{\lim}_{k\to\infty} G_f^{(\alpha_k)}$ $\mathfrak{m}$-a.e. by Fatou's lemma. We already know that (4.8) holds by replacing G_f with $G_f^{(\alpha_k)}$. Applying Fatou's lemma again, we have (4.8).

Therefore, we can conclude that (2.5) holds under $p \in]\,1, 2\,[$, $\alpha = 0$ and $f \in D(\Delta) \cap L^p(X;\mathfrak{m}) \cap \mathscr{B}^*(X)$ by way of Fatou's lemma and the estimate (4.8) under $\alpha > 0$.

4.2 PROOF OF (2.5) FOR $f \in D(\Delta) \cap L^p(X;\mathfrak{m}) \cap \mathscr{B}^*(X)$ UNDER $p \in]\,2, +\infty\,[$

In this subsection, we assume Assumption 1, consequently, the estimate (2.3) holds for $f \in D(\mathscr{E})$ and $\mathsf{k}^+ \in S_D(\mathbf{X})$ and $\mathsf{k}^- \in S_{EK}(\mathbf{X})$. In the case of $p > 2$, we introduce additional functions, namely H-functions

$$
\begin{aligned}
H_f^{\to\mathsf{k}} &:= \left\{ \int_0^\infty t Q_t^{(\alpha),\mathsf{k}}(g_f^{\to}(\cdot,t)^2)(x)\mathrm{d}t \right\}^{\frac{1}{2}}, \\
H_f^{\uparrow\mathsf{k}} &:= \left\{ \int_0^\infty t Q_t^{(\alpha),\mathsf{k}}(g_f^{\uparrow}(\cdot,t)^2)(x)\mathrm{d}t \right\}^{\frac{1}{2}}, \\
H_f^{\mathsf{k}} &:= \left\{ \int_0^\infty t Q_t^{(\alpha),\mathsf{k}}(g_f(\cdot,t)^2)(x)\mathrm{d}t \right\}^{\frac{1}{2}}
\end{aligned}
$$

under $\alpha \ge C_{\mathsf{k}}$. We write $H_f^{\to} := H_f^{\to 0}$, $H_f^{\uparrow} := H_f^{\uparrow 0}$ and $H_f := H_f^0$. Note that $H_f^{\to\kappa}$, $H_f^{\uparrow\mathsf{k}}$ and $H_f{}^{\mathsf{k}}$ (resp. $H_f^{\to}$, $H_f^{\uparrow}$ and H_f) depend on $\alpha \ge C_{\mathsf{k}}$ (resp. $\alpha \ge 0$ if $\mathsf{k}^- = 0$).

Not only (4.1), we need the following inequality under Assumption 1 extending [52, Proposition 3.11].

Lemma 4.1. *Let $j : X \times [\,0, +\infty\,[\to [\,0, +\infty\,[$ be a measurable function. Then*

$$
\mathbb{E}_{\mathfrak{m}\otimes\delta_a}\left[\int_0^\tau j(\widehat{X}_s)\mathrm{d}s \,\middle|\, X_\tau \right] \ge \int_0^\infty (a \wedge t) Q_t^{(\alpha),\kappa}(j(\cdot,t))(X_\tau)\mathrm{d}t \tag{4.9}
$$

holds for $\alpha \ge C_\kappa$. Here j is understood to be a function on $\widehat{X}_{\widehat{\partial}}$ with $j(\widehat{\partial}) = 0$.

Remark 4.2. Since $\alpha \geq C_\kappa > 0$ for $\kappa^- \neq 0$ under Assumption 1, we can not expect the equality for (4.9) in this case as discussed in [52, Proposition 3.11].

Proof of Lemma 4.1. We may assume $\alpha > C_\kappa$, because the case $\alpha = C_\kappa$ in (4.9) can be deduced by the limit $\alpha \downarrow C_\kappa$. By taking an increasing approximating sequence $\{j_n\} \subset L^1(X \times [0, +\infty[\,; \widehat{\mathfrak{m}})$ of non-negative measurable functions to j, we may assume $j \in L^1(X \times [0, +\infty[\,; \widehat{\mathfrak{m}})$. Thanks to the transience of $\widehat{\mathbf{X}}_{\widehat{G}}$ with the integrability of j, we have

$$\mathbb{E}_{(x,a)}\left[\int_0^\tau j(\widehat{X}_s)\mathrm{d}s\right] < +\infty, \quad \widehat{\mathfrak{m}}\text{-a.e. } (x,a) \tag{4.10}$$

by [25, (1.5.4)]. To prove (4.9), it suffices to show

$$\mathbb{E}_{(x,a)}\left[f(X_\tau)\int_0^\tau j(\widehat{X}_s)\mathrm{d}s\right] \geq \mathbb{E}_{(x,a)}\left[\int_0^\tau \tilde{v}(\widehat{X}_t)j(\widehat{X}_t)\mathrm{d}t\right] \quad \widehat{\mathfrak{m}}\text{-a.e. } (x,a). \tag{4.11}$$

for $v(x,a) := q_{|a|}^{(\alpha),\kappa} f(x)$ with $f \in \mathscr{B}_+(X)$ and $\alpha > C_\kappa$. Indeed, by taking an integration of (4.11) with respect to $\mathfrak{m}$, we have

$$\begin{aligned}
\mathbb{E}_{\mathfrak{m}\otimes\delta_a}\left[f(X_\tau)\int_0^\tau j(\widehat{X}_s)\mathrm{d}s\right] &\geq \mathbb{E}_{\mathfrak{m}\otimes\delta_a}\left[\int_0^\tau \tilde{v}(\widehat{X}_t)j(\widehat{X}_t)\mathrm{d}t\right] \\
&\overset{(4.1)}{=} \int_X \mathfrak{m}(\mathrm{d}x)\int_0^\infty (a\wedge t)v(x,t)j(x,t)\mathrm{d}t \\
&= \int_0^\infty (a\wedge t)\mathbb{E}_{\mathfrak{m}\otimes\delta_a}\left[Q_t^{(\alpha),\kappa}(j(\cdot,t))(X_\tau)f(X_\tau)\right]\mathrm{d}t,
\end{aligned}$$

which implies (4.9). Here we apply (4.2) in the last equality.

From now on, we prove (4.11) for $f \in D(\Delta^\kappa) \cap \mathscr{B}_+^*(X)$. Fix such an f and set $v(x,a) := u(x,|a|) := q_{|a|}^{(\alpha),\kappa} f(x)$ with $\alpha > C_\kappa$. Applying (3.10) to the case $\mathsf{k} = \kappa$, we have $v \in D(\widehat{\mathscr{E}}^{\widehat{\kappa}}) = D(\widehat{\mathscr{E}})$ for $\alpha > C_\kappa$. As proved in Proposition 3.3, we have the semi-martingale decomposition (3.16) with (3.17) and (3.18). We set

$$\begin{aligned}
M_f(t) :&= \widehat{M}_{t\wedge\tau}^{[v]} + \tilde{v}(\widehat{X}_0) \\
&= \tilde{v}(\widehat{X}_{t\wedge\tau}) - \alpha\int_0^{t\wedge\tau} \tilde{v}(\widehat{X}_s)\mathrm{d}s, \quad t \geq 0 \quad \mathbb{P}_{\hat{x}}\text{-a.s. for q.e. } \hat{x}.
\end{aligned}$$

Then by (3.18), $M_f(t)$ is a uniformly integrable martingale with respect to $\mathbb{P}_{(x,a)}$ for q.e. (x,a). As $t \to \infty$, $M_f(t)$ converges to

$$\tilde{v}(\widehat{X}_\tau) - \alpha\int_0^\tau \tilde{v}(\widehat{X}_s)\mathrm{d}s = f(X_\tau) - \alpha\int_0^\tau \tilde{v}(\widehat{X}_s)\mathrm{d}s.$$

As a consequence, $M_f(t)$ is represented by

$$M_f(t) = \mathbb{E}_{(x,a)}\left[f(X_\tau) - \alpha\int_0^\tau \tilde{v}(\widehat{X}_s)\mathrm{d}s \,\middle|\, \mathcal{F}_t\right], \quad \mathbb{P}_{(x,a)}\text{-a.s. for q.e. } (x,a). \tag{4.12}$$

Thus

$$\begin{aligned}\mathbb{E}_{(x,a)}\left[f(X_\tau)\int_0^\tau j(\widehat{X}_s)\mathrm{d}s\right] &\overset{(4.12)}{=} \int_0^\infty \mathbb{E}_{(x,a)}\left[\tilde{v}(\widehat{X}_t)j(\widehat{X}_t)\mathbf{1}_{\{t\le\tau\}}\right]\mathrm{d}t \\ &\quad + \alpha\int_0^\infty \mathbb{E}_{(x,a)}\left[\mathbb{E}_{(x,a)}\left[\int_{t\wedge\tau}^\tau \tilde{v}(\widehat{X}_s)\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] j(\widehat{X}_t)\mathbf{1}_{\{t\le\tau\}}\right]\mathrm{d}t \\ &\ge \int_0^\infty \mathbb{E}_{(x,a)}\left[\tilde{v}(\widehat{X}_t)j(\widehat{X}_t)\mathbf{1}_{\{t\le\tau\}}\right]\mathrm{d}t \\ &= \mathbb{E}_{(x,a)}\left[\int_0^\tau \tilde{v}(\widehat{X}_t)j(\widehat{X}_t)\mathrm{d}t\right].\end{aligned}$$

This shows that (4.11) holds for $f \in D(\Delta^\kappa)\cap\mathscr{B}^*_+(X)$ with $\alpha > C_\kappa$.

Next we prove that (4.11) holds $f \in L^2(X;\mathfrak{m})\cap\mathscr{B}_b(X)_+$ with $\alpha > C_\kappa$. For such an f, we set $f_n := p^\kappa_{1/n}f$. Then $\{f_n\}$ is uniformly bounded and $f_n \in D(\Delta^\kappa)\cap\mathscr{B}^*_b(X)_+$. We already prove that (4.11) holds by replacing f with f_n. Letting $n\to\infty$ with (4.10), we can conclude that (4.11) holds for $f \in L^2(X;\mathfrak{m})\cap\mathscr{B}_b(X)_+$ by way of Lebesgue's dominated convergence theorem. Finally, approximating $f\in\mathscr{B}(X)_+$ by an increasing sequence in $L^2(X;\mathfrak{m})\cap\mathscr{B}_b(X)_+$, we see that (4.11) still holds for any $f\in\mathscr{B}(X)_+$ with $\alpha > C_\kappa$. □

Next, we show the following proposition:

Proposition 4.3. *For $p>2$, the following inequality holds for any $f \in D(\Delta)\cap L^p(X;\mathfrak{m})\cap \mathscr{B}^*(X)$ with $\alpha \ge C_\kappa$ and $\alpha > 0$:*

$$\|H_f{}^\kappa\|_{L^p(X;\mathfrak{m})} \lesssim \|f\|_{L^p(X;\mathfrak{m})}.$$

Proof. By a slight modification, we can prove this proposition as in the proof of Shigekawa–Yoshida [53, Proposition 4.2] or Kawabi–Miyokawa [34, Proposition 2.8]. However we give the proof for readers' convenience, because we treat the Feynman–Kac semigroup (P^κ_t) instead of the given semigroup (P_t). Moreover, the method of the proof of [53, Proposition 4.2] or [34, Proposition 2.8] uses the equality for (4.9) under $\alpha = 0$. But we do not have a similar equality for general $\alpha > 0$.

Let us recall that $v(x,a) := q^{(\alpha)}_{|a|}f(x)$ with $f\in D(\Delta)$ satisfies $v\in D(\widehat{\mathscr{E}})$ under $\alpha>0$ by applying (3.10) with $\mathsf{k}=0$. Owing to (4.3), we have $Z_t := V^2_{t\wedge\tau} - V^2_0$, $t\ge 0$, is a submartingale.

Now we need an inequality for submartingale. Let $\{Z_t\}_{t\ge0}$ be a continuous submartingale with the Doob-Meyer decomposition $Z_t = M_t + A_t$, where $\{M_t\}_{t\ge0}$ is a continuous martingale and $\{A_t\}_{t\ge0}$ is a continuous increasing process with $A_0=0$. Due to Lenglart–Lépingle–Pratelli [36], it holds that

$$\mathbb{E}[A^p_\infty] \le (2p)^p\mathbb{E}\left[\sup_{t\ge0}|Z_t|^p\right],\quad p>1. \tag{4.13}$$

Then by using (4.13) and Doob's inequality with the help of (4.2), we have

$$\mathbb{E}_{\mathfrak{m}\otimes\delta_{a_n}}\left[\left\{2\int_0^\tau\left(\alpha V^2_s + g_f(\widehat{X}_s)^2\right)\mathrm{d}s\right\}^{\frac{p}{2}}\right] \lesssim \|f\|^p_{L^p(X;\mathfrak{m})}. \tag{4.14}$$

On the other hand, by using (4.9), (4.14) and Jensen's inequality, we have

$$\begin{aligned}
\|H_f^\kappa\|_{L^p(X;\mathfrak{m})}^p &= \lim_{n\to\infty}\left\|\left\{\int_0^\infty (a_n\wedge t)Q_t^{(\alpha),\kappa}(g_f(\cdot,t)^2)\mathrm{d}t\right\}^{\frac{p}{2}}\right\|_{L^1(X;\mathfrak{m})} \\
&\overset{(4.2)}{=} \lim_{n\to\infty}\mathbb{E}_{\mathfrak{m}\otimes\delta_{a_n}}\left[\left\{\int_0^\infty (a_n\wedge t)Q_t^{(\alpha),\kappa}(g_f(\cdot,t)^2)(X_\tau)\mathrm{d}t\right\}^{\frac{p}{2}}\right] \\
&\overset{(4.9)}{\leq} \varliminf_{n\to\infty}\mathbb{E}_{\mathfrak{m}\otimes\delta_{a_n}}\left[\mathbb{E}_{\mathfrak{m}\otimes\delta_{a_n}}\left[\int_0^\tau g_f(\widehat{X}_s)^2\mathrm{d}s\,\middle|\, X_\tau\right]^{\frac{p}{2}}\right] \\
&\leq \varliminf_{n\to\infty}\mathbb{E}_{\mathfrak{m}\otimes\delta_{a_n}}\left[\left\{\int_0^\tau \left(\alpha V_s^2+g_f(\widehat{X}_s)^2\right)\mathrm{d}s\right\}^{\frac{p}{2}}\right] \\
&\overset{(4.14)}{\lesssim} \|f\|_{L^p(X;\mathfrak{m})}^p.
\end{aligned}$$

In applying (4.9), we use $\alpha\geq C_\kappa$. We emphasize that the method of the proof as in [53, 34] can not work for $\alpha>0$. This is why we prove (4.9). This completes the proof. □

Next we consider the relationship between G-functions and H-functions.

Proposition 4.4. (1) *For any $f\in D(\mathscr{E})$ and $\alpha\geq C_\kappa$, the following inequality holds:*

$$G_f^{\uparrow}\leq 2\sqrt{C}H_f^{\uparrow\kappa},\quad \mathfrak{m}\text{-}a.e.$$

(2) *For any $f\in D(\mathscr{E})$ and $\alpha\geq 0$, the following inequality holds:*

$$G_f^{\rightarrow}\leq 2H_f^{\rightarrow}.$$

Proof. First we show (1). For this, we prove

$$\Gamma(Q_t^{(\alpha)}f)^{\frac12}\leq\int_0^\infty e^{-\alpha s}\Gamma(P_sf)^{\frac12}\lambda_t(\mathrm{d}s),\quad \mathfrak{m}\text{-a.e.}\tag{4.15}$$

We set a $D(\mathscr{E})$-valued simple function $f_n(s)$ by

$$f_n(s):=\sum_{k=0}^{2^n\cdot n-1}\mathbf{1}_{[\frac{k}{2^n},\frac{k+1}{2^n}[}(s)P_{\frac{k}{2^n}}f+\mathbf{1}_{[n,+\infty[}(s)P_nf.$$

Then for each $s\geq 0$, $\{f_n(s)\}$ converges to P_sf in $(\mathscr{E},D(\mathscr{E}))$ as $n\to\infty$, consequently $\Gamma(f_n(s))^{\frac12}$ converges to $\Gamma(P_sf)^{\frac12}$ in $L^2(X;\mathfrak{m})$ as $n\to\infty$. Similarly, setting $Q_tf_n:=\int_0^\infty e^{-\alpha s}f_n(s)\lambda_t(\mathrm{d}s)$, $\{Q_tf_n\}$ converges to $Q_t^{(\alpha)}f$ in $(\mathscr{E},D(\mathscr{E}))$ as $n\to\infty$, and $\Gamma(Q_tf_n)^{\frac12}$ converges to $\Gamma(Q_t^{(\alpha)}f)^{\frac12}$ in $L^2(X;\mathfrak{m})$ as $n\to\infty$. Since

$$\Gamma(f_n(s))^{\frac12}=\begin{cases}\Gamma(P_{\frac{k}{2^n}}f)^{\frac12}, & \text{if}\quad s\in[\frac{k}{2^n},\frac{k+1}{2^n}[,\quad k\in\{0,1,\cdots,2^n\cdot n-1\},\\ \Gamma(P_nf)^{\frac12}, & \text{if}\quad s\geq n,\end{cases}$$

we can see

$$\Gamma(Q_t f_n)^{\frac{1}{2}} \le \int_0^\infty e^{-\alpha s}\Gamma(f_n(s))^{\frac{1}{2}}\lambda_t(\mathrm{d}s).$$

Letting $n \to \infty$, we obtain (4.15). By (4.15), the upper estimate (2.3) and Schwarz's inequality, we have the following estimate for any $\alpha \ge C_\kappa$ and $f \in D(\mathscr{E})$:

$$\begin{aligned}\Gamma(Q_t^{(\alpha)} f)^{\frac{1}{2}} &\le \int_0^\infty e^{-\alpha s}\Gamma(P_s f)^{\frac{1}{2}}\lambda_t(\mathrm{d}s)\\ &\le \int_0^\infty e^{-\alpha s}P_s^\kappa\Gamma(f)^{\frac{1}{2}}\lambda_t(\mathrm{d}s) \qquad (4.16)\\ &= Q_t^{(\alpha),\kappa}\Gamma(f)^{\frac{1}{2}} = q_t^{(\alpha),\kappa}\Gamma^*(f)^{\frac{1}{2}},\end{aligned}$$

where $\Gamma^*(f)$ is a Borel $\mathfrak{m}$-version of $\Gamma(f)$. Here we use $\mathsf{GE}_1(\kappa,\infty)$ under Assumption 1. Then (4.16) yields that for each $t \ge 0$

$$\begin{aligned}g_f^\uparrow(x,2t)^2 &= \Gamma\left(Q_t^{(\alpha)}(Q_t^{(\alpha)} f)\right)(x)\\ &\le \left(q_t^{(\alpha),\kappa}\Gamma^*(Q_t^{(\alpha)} f)^{\frac{1}{2}}(x)\right)^2\\ &\le q_t^{(\alpha),\kappa}1(x)\cdot Q_t^{(\alpha),\kappa}\Gamma(Q_t^{(\alpha)} f)(x) \qquad (4.17)\\ &\overset{(2.4)}{\le} Ce^{-\sqrt{\alpha - C_\kappa}t}Q_t^{(\alpha),\kappa}(g_f^\uparrow(\cdot,t)^2)(x), \quad \mathfrak{m}\text{-a.e. } x \in X.\end{aligned}$$

Since $t \mapsto Q_t^{(\alpha),\kappa} f$ is a $D(\mathscr{E})$-valued continuous function, $t \mapsto g_f^\uparrow(x,2t)^2 = \Gamma(Q_{2t}^{(\alpha)} f)$ is an $L^1(X;\mathfrak{m})$-valued continuous function, moreover, $t \mapsto Q_t^{(\alpha),\kappa}(g_f^\uparrow(\cdot,2t)^2)$ is an $L^1(X;\mathfrak{m})$-valued continuous function. Thus, $g_f^\uparrow(x,2t)^2 \le Q_t^{(\alpha),\kappa}(g_f^\uparrow(\cdot,t)^2)(x)$ for all $t \ge 0$ $\mathfrak{m}$-a.e. $x \in X$.

Therefore we have

$$\begin{aligned}\left(G_f^\uparrow(x)\right)^2 &= 4\int_0^\infty t g_f^\uparrow(x,2t)^2\mathrm{d}t\\ &\le 4C\int_0^\infty tQ_t^{(\alpha),\kappa}(g_f^\uparrow(\cdot,t)^2)(x)\mathrm{d}t = 4C(H_f^{\uparrow\kappa}(x))^2, \quad \mathfrak{m}\text{-a.e. } x \in X,\end{aligned}$$

where we changed the variable t in $2t$ in the first line and used (4.17) for the second line.

The proof of (2) is similar to the proof of Shigekawa [52, Proposition 3.13 (3.55)]. We omit it. □

Combining Propositions 4.3 and 4.4, (2.5) holds for $f \in D(\Delta) \cap L^p(X;\mathfrak{m}) \cap \mathscr{B}^*(X)$ under $p \in]\,2,+\infty\,[$ and $\alpha \ge C_\kappa$ with $\alpha > 0$. Under $\kappa^- = 0$ and $\alpha = 0$, we can see that (2.5) holds for $f \in D(\Delta) \cap L^p(X;\mathfrak{m}) \cap \mathscr{B}^*(X)$ and $p \in]\,2,+\infty\,[$ by way of the same argument as in the previous subsection.

Remark 4.5. In Kawabi–Miyokawa [34, Proposition 2.9(1)], they prove $G_f^\uparrow \le 2H_f^\uparrow$ for $\alpha > R \vee 0$ with $\kappa = (-R)\mathfrak{m}$ under $\Gamma(P_t f)^{\frac{1}{2}} \le e^{Rt}P_t\Gamma(f)^{\frac{1}{2}}$. In their proof, the estimate $Q_t^{(\alpha),\kappa}\Gamma(f)^{\frac{1}{2}} = Q_t^{(\alpha-R)}\Gamma(f)^{\frac{1}{2}} \le Q_t^{(0)}\Gamma(f)^{\frac{1}{2}}$ by $\alpha > R$ is crucial. In our case, we can not get $Q_t^{(\alpha),\kappa}\Gamma(f)^{\frac{1}{2}} \le Q_t^{(0)}\Gamma(f)^{\frac{1}{2}}$ for $\alpha > C_\kappa$. This is why we prepare the estimate $G_f^\uparrow \le 2\sqrt{C}H_f^{\uparrow\kappa}$ in Proposition 4.4(1) instead of $G_f^\uparrow \le 2H_f^\uparrow$.

4.3 PROOF OF THE UPPER ESTIMATES (2.5) AND (2.7) UNDER $p = 2$

We prove the upper estimates (2.5) and (2.7) under $p = 2$.

Proposition 4.6. *When $\alpha > 0$,*

$$2\|G_f^{\rightarrow}\|_{L^2(X;\mathfrak{m})} = \|f\|_{L^2(X;\mathfrak{m})}, \qquad \textit{for all} \quad f \in L^2(X;\mathfrak{m}), \tag{4.18}$$

$$2\|G_f^{\uparrow}\|_{L^2(X;\mathfrak{m})} \le \|f\|_{L^2(X;\mathfrak{m})}, \qquad \textit{for all} \quad f \in L^2(X;\mathfrak{m}). \tag{4.19}$$

When $\alpha = 0$,

$$2\|G_f^{\rightarrow}\|_{L^2(X;\mathfrak{m})} = \|f - E_o f\|_{L^2(X;\mathfrak{m})}, \qquad \textit{for all} \quad f \in L^2(X;\mathfrak{m}), \tag{4.20}$$

$$2\|G_f^{\uparrow}\|_{L^2(X;\mathfrak{m})} = \|f - E_o f\|_{L^2(X;\mathfrak{m})}, \qquad \textit{for all} \quad f \in L^2(X;\mathfrak{m}). \tag{4.21}$$

In particular, $\sqrt{2}\|G_f\|_{L^2(X;\mathfrak{m})} \le \|f\|_{L^2(X;\mathfrak{m})}$ for all $f \in L^2(X;\mathfrak{m})$ under $\alpha > 0$, and $\sqrt{2}\|G_f\|_{L^2(X;\mathfrak{m})} = \|f - E_o f\|_{L^2(X;\mathfrak{m})}$ for all $f \in L^2(X;\mathfrak{m})$ under $\alpha = 0$.

Proof. The proof is quite same as in the proof of Shigekawa [52, Proposition 3.5]. So we omit the details. □

Lemma 4.7. *Suppose $f \in L^2(X;\mathfrak{m})$. Then $E_o f \in D(\mathscr{E})$ and $\mathscr{E}(E_o f, E_o f) = 0$. In particular, $G_{E_o f} = 0$ for $f \in L^2(X;\mathfrak{m})$.*

Proof. In view of the spectral representation,

$$\begin{aligned}(P_t E_o f, E_o f) &= \int_0^\infty e^{-\lambda t} \mathrm{d}(E_\lambda E_o f, E_o f) \\ &= \lim_{n\to\infty} \sum_{k=0}^{2^n \cdot n-1} e^{-\frac{k}{2^n}t} \int_{[\frac{k}{2^n}, \frac{k+1}{2^n}[} \mathrm{d}(E_\lambda E_o f, E_o f) + e^{-nt} \int_{[n,+\infty[} \mathrm{d}(E_\lambda E_o f, E_o f) \\ &= \lim_{n\to\infty} \sum_{k=0}^{2^n \cdot n-1} e^{-\frac{k}{2^n}t} (E_{[\frac{k}{2^n}, \frac{k+1}{2^n}[} E_{\{0\}} f, E_{\{0\}} f) + e^{-nt} (E_{[n,+\infty[} E_{\{0\}} f, E_{\{0\}} f) \\ &= (E_{[0,\frac{1}{2^n}[} E_{\{0\}} f, E_{\{0\}} f) = (E_{\{0\}} f, E_{\{0\}} f) = \|E_o f\|^2_{L^2(X;\mathfrak{m})},\end{aligned}$$

which yields the first conclusion. In the same way, we can get $\mathscr{E}(P_s E_o f, P_s E_o f) = \mathscr{E}(Q_t^{(0)} E_o f, Q_t^{(0)} E_o f) = 0$ for $f \in L^2(X;\mathfrak{m})$. This implies that for each $t > 0$

$$\int_X g_{E_o f}^{\rightarrow}(\cdot, t)^2 \mathrm{d}\mathfrak{m} = \int_X |\sqrt{-\Delta} Q_t^{(0)} E_o f|^2 \mathrm{d}\mathfrak{m} = \mathscr{E}(Q_t^{(0)} E_o f, Q_t^{(0)} E_o f) = 0,$$

$$\int_X g_{E_o f}^{\uparrow}(\cdot, t)^2 \mathrm{d}\mathfrak{m} = \int_X \Gamma(Q_t^{(0)} E_o f) \mathrm{d}\mathfrak{m} = \mathscr{E}(Q_t^{(0)} E_o f, Q_t^{(0)} E_o f) = 0,$$

which yields the second conclusion. □

4.4 Proof of (2.5) under $p \in]1, +\infty[$ for general $\alpha \geq C_\kappa$ with $\alpha > 0$, and for $\alpha = 0$ and $\kappa^- = 0$

Proof of (2.5). Subsections 4.1, 4.2 and 4.3 conclude the desired upper estimate (2.5) for $f \in D(\Delta) \cap L^p(X;\mathfrak{m})$ under $p \in]1, +\infty[$, and $\alpha \geq C_\kappa$ with $\alpha > 0$ or $\kappa^- = 0$ with $\alpha = 0$. Any $f \in L^p(X;\mathfrak{m}) \cap L^2(X;\mathfrak{m})$ can be approximated by $f_n := P_{1/n}f \in D(\Delta) \cap L^p(X;\mathfrak{m})$ in L^p-norm and in L^2-norm. We can obtain the upper estimate (2.5) for $f \in L^p(X;\mathfrak{m}) \cap L^2(X;\mathfrak{m})$ under $p \in]1, +\infty[$, and $\alpha \geq C_\kappa$ with $\alpha > 0$ or $\kappa^- = 0$ with $\alpha = 0$. Moreover, an L^p-approximation by a sequence in $L^p(X;\mathfrak{m}) \cap L^2(X;\mathfrak{m})$ tells us that (2.5) holds for $f \in L^p(X;\mathfrak{m})$ under $p \in]1, +\infty[$, and $\alpha \geq C_\kappa$ with $\alpha > 0$ or $\kappa^- = 0$ with $\alpha = 0$. □

4.5 Proof of (2.6) under $p \in]1, +\infty[$ for general $\alpha \geq C_\kappa$ with $\alpha > 0$.

Proof of (2.6). The proof of the lower estimate (2.6) under $p \in]1, +\infty[$, $\alpha \geq C_\kappa$ with $\alpha > 0$ follows from the upper estimate (2.5) under $p \in]1, +\infty[$, $\alpha \geq C_\kappa$ with $\alpha > 0$, (4.18) and the argument in [52, Subsection 3.2.11]. Moreover, an L^p-approximation by a sequence in $L^p(X;\mathfrak{m}) \cap L^2(X;\mathfrak{m})$ tells us that (2.6) holds for $f \in L^p(X;\mathfrak{m})$ under $p \in]1, +\infty[$, and $\alpha \geq C_\kappa$ with $\alpha > 0$. □

4.6 Proofs of (2.7), (2.8) and (2.9) under $p \in]1, +\infty[$ for $\alpha = 0$ and $\kappa^- = 0$

We already obtain the estimates (2.7), (2.8) and (2.9) under $p = 2$. To prove (2.7), (2.8) and (2.9) for general $f \in L^p(X;\mathfrak{m})$ and $p \in]1, +\infty[$, we need to show $E_o f$ can be defined as an element in $L^p(X;\mathfrak{m})$ for $f \in L^p(X;\mathfrak{m})$ under the transience, irreducibility or hypercontractivity of $(P_t)_{t\geq 0}$ with $\mathfrak{m}(X) < +\infty$ at first.

Lemma 4.8. *Suppose that $(\mathscr{E}, D(\mathscr{E}))$ is transient or irreducible, or hypercontractivity of $(P_t)_{t\geq 0}$ with $\mathfrak{m}(X) < +\infty$ is satisfied. Then the L^2-operator E_o can be extended to a bounded linear operator on $L^p(X;\mathfrak{m})$.*

Proof. By Lemma 4.7, for $f \in L^2(X;\mathfrak{m})$, we know $E_o f \in D(\mathscr{E})$ and $\mathscr{E}(E_o f, E_o f) = 0$, in particular $P_t E_0 f = E_0 f$ for any $t \geq 0$. Moreover, we can deduce $P_t E_0 f = E_o P_t f$ by way of [61, Chapter IX, Section 5, the end of proof for Theorem 2]. First we suppose that $(\mathscr{E}, D(\mathscr{E}))$ is transient. Then $E_o f = 0 \in L^p(X;\mathfrak{m})$. So, for $f \in L^p(X;\mathfrak{m})$, we can define $E_o f := 0 \in L^p(X;\mathfrak{m})$ in this case. Next we suppose that $(\mathscr{E}, D(\mathscr{E}))$ is irreducible. We may assume that it is also recurrent, because under the irreducibility, $(\mathscr{E}, D(\mathscr{E}))$ is recurrent or transient (see [25, Lemma 1.6.4(iii)]). Then for $f \in L^2(X;\mathfrak{m})$, $E_o f \equiv c_f \in L^2(X;\mathfrak{m})$ for some constant c_f by [25, Exercise 4.7.2(1)]. More precisely, by [25, (4.7.8)], c_f is given by

$$c_f := \begin{cases} \frac{1}{\mathfrak{m}(X)} \int_X f \,\mathrm{d}\mathfrak{m}, & \mathfrak{m}(X) < +\infty, \\ 0, & \mathfrak{m}(X) = +\infty. \end{cases}$$

When $\mathfrak{m}(X) = +\infty$, then $c_f = 0$, hence $E_o f = 0 \in L^p(X;\mathfrak{m})$. In this case, we can define $E_o f := 0 \in L^p(X;\mathfrak{m})$ for $f \in L^p(X;\mathfrak{m})$. When $\mathfrak{m}(X) < +\infty$, then $E_o f \equiv c_f =$

$\frac{1}{\mathfrak{m}(X)}\int_X f\mathrm{d}\mathfrak{m} \in L^p(X;\mathfrak{m})$. In this case, for $f \in L^p(X;\mathfrak{m}) \cap L^2(X;\mathfrak{m})$,

$$\|E_o f\|_{L^p(X;\mathfrak{m})} = \left(\int_X |c_f|^p \mathrm{d}\mathfrak{m}\right)^{\frac{1}{p}} \le \|f\|_{L^p(X;\mathfrak{m})}.$$

This implies that E_o can be extended to a bounded linear operator on $L^p(X;\mathfrak{m})$. Finally we suppose that $\mathfrak{m}(X) < +\infty$ and $(P_t)_{t\ge 0}$ is hypercontractive with $C > 0$. When $1 < p < 2$, for $f \in L^p(X;\mathfrak{m}) \cap L^2(X;\mathfrak{m})$ and $t > 0$ satisfying $2 = e^{2t/C}(p-1)+1$, we have

$$\begin{aligned}\|E_o f\|_{L^p(X;\mathfrak{m})} &= \|P_t E_o f\|_{L^p(X;\mathfrak{m})} = \|E_o P_t f\|_{L^p(X;\mathfrak{m})}\\ &\le \|E_o P_t f\|_{L^2(X;\mathfrak{m})}\mathfrak{m}(X)^{\frac{1}{p}-\frac{1}{2}}\\ &\le \|P_t f\|_{L^2(X;\mathfrak{m})}\mathfrak{m}(X)^{\frac{1}{p}-\frac{1}{2}}\\ &\le \|f\|_{L^p(X;\mathfrak{m})}\mathfrak{m}(X)^{\frac{1}{p}-\frac{1}{2}},\end{aligned}$$

where we use the hypercontractivity of $(P_t)_{t\ge 0}$ from p to 2. This means that E_o can be extended to a bounded linear operator on $L^p(X;\mathfrak{m})$. When $p \ge 2$, for $f \in L^p(X;\mathfrak{m}) \cap L^2(X;\mathfrak{m})$ and $t > 0$ satisfying $p = e^{2t/C}+1$, we have

$$\|E_0 f\|_{L^p(X;\mathfrak{m})} = \|P_t E_0 f\|_{L^p(X;\mathfrak{m})} \le \|E_0 f\|_{L^2(X;\mathfrak{m})} \le \|f\|_{L^2(X;\mathfrak{m})} \le \|f\|_{L^p(X;\mathfrak{m})}\mathfrak{m}(X)^{\frac{1}{2}-\frac{1}{p}},$$

where we use the hypercontractivity of $(P_t)_{t\ge 0}$ from 2 to p. This means that E_o can be extended to a bounded linear operator on $L^p(X;\mathfrak{m})$. □

Proof of (2.7), (2.8) and (2.9). Under $\kappa^- = 0$ and $\alpha = 0$, we already know that (2.5) holds for $f \in L^p(X;\mathfrak{m}) \cap L^2(X;\mathfrak{m})$ with $p \in]1, +\infty[$. Replacing f with $f - E_o f$ in (2.5) and noting $G_{E_o f} = 0$ by Lemma 4.7, we can conclude that (2.7) holds for $f \in L^p(X;\mathfrak{m}) \cap L^2(X;\mathfrak{m})$ with $p \in]1, +\infty[$ under $\kappa^- = 0$, $\alpha = 0$ and the transience, irreducibility or hypercontractivity of $(P_t)_{t\ge 0}$ with $\mathfrak{m}(X) < +\infty$. By Lemma 4.8, we obtain that (2.7) holds for $f \in L^p(X;\mathfrak{m})$ with $p \in]1, +\infty[$ under $\kappa^- = 0$, $\alpha = 0$ and the transience, irreducibility or hypercontractivity of $(P_t)_{t\ge 0}$ with $\mathfrak{m}(X) < +\infty$. Further, under $\kappa^- = 0$ and $\alpha = 0$, we can deduce the lower estimate (2.8) (resp. (2.9)) as discussed in [52, Subsection 3.2.11] from the upper estimate (2.5) by using (4.20), (resp. (4.21)). More precisely, by (4.20) (resp. (4.21)) and (2.5), we can deduce $|(f - E_o f, h)_{\mathfrak{m}}| \le 4(G_f^{\rightarrow}, G_h^{\rightarrow})_{\mathfrak{m}} \le 4\|G_f^{\rightarrow}\|_{L^p(X;\mathfrak{m})}\|G_h^{\rightarrow}\|_{L^q(X;\mathfrak{m})} \lesssim 4\|G_f^{\rightarrow}\|_{L^p(X;\mathfrak{m})}\|h\|_{L^q(X;\mathfrak{m})}$ (resp. $|(f - E_o f, h)_{\mathfrak{m}}| \le 4(G_f^{\uparrow}, G_h^{\uparrow})_{\mathfrak{m}} \le 4\|G_f^{\uparrow}\|_{L^p(X;\mathfrak{m})}\|G_h^{\uparrow}\|_{L^q(X;\mathfrak{m})} \lesssim 4\|G_f^{\uparrow}\|_{L^p(X;\mathfrak{m})}\|h\|_{L^q(X;\mathfrak{m})}$) with the conjugate exponent $q = p/(p-1)$, which implies the lower estimate (2.8) (resp. (2.9)) for $f \in L^p(X;\mathfrak{m}) \cap L^2(X;\mathfrak{m})$ by the denseness of $L^q(X;\mathfrak{m}) \cap L^2(X;\mathfrak{m})$ in $L^q(X;\mathfrak{m})$. By applying Lemma 4.8, we obtain the lower estimates (2.8), (2.9) for $f \in L^p(X;\mathfrak{m})$. □

This completes the whole proof of Theorem 2.2.

5 EXAMPLES

It is well-known that (abstract) Wiener space (B, H, μ) satisfies the Littlewood–Paley–Stein inequality (see Shigekawa [52, Chapter 3]). Though our Theorem 2.2 is not new for

(B, H, μ), we observe the conditions: Let $(\mathscr{E}^{\rm OU}, D(\mathscr{E}^{\rm OU}))$ be the Dirichlet form on $L^2(B;\mu)$ associated to the Ornstein–Uhlenbeck process $\mathbf{X}^{\rm OU}$ and $(T_t^{\rm OU})_{t\geq 0}$ its associated semigroup on $L^2(B;\mu)$. Let D_H be the H-derivative, i.e. $\langle D_H F(z), h\rangle_H = \lim_{\varepsilon\to 0}\frac{F(z+\varepsilon h)-F(z)}{\varepsilon}$, for a cylindrical function F. It is known that $\mathsf{BE}_2(1,\infty)$-condition holds for $(B, \mathscr{E}^{\rm OU}, \mu)$ (see [3, 13.2]), that is, $(B, \mathscr{E}^{\rm OU}, \mu)$ is tamed by $\mu \in S_K(\mathbf{X}^{\rm OU})$. Here $S_K(\mathbf{X}^{\rm OU})$ denotes the family of smooth measures of Kato class. Kawabi–Miyokawa [34] generalized the case of (abstract) Wiener space, but they did not extend [52, (3.23),(3.24),(3.25)]. Our (2.7), (2.8) and (2.9) in Theorem 2.2 generalizes [52, (3.23),(3.24),(3.25)].

Kawabi–Miyokawa gave another example satisfying their conditions in [34]. More concretely, let E be a Hilbert space defined by $E := L^2(\mathbb{R}, \mathbb{R}^d; e^{-2\lambda\chi(x)}\mathrm{d}x)$ with a fixed $\chi \in C^\infty(\mathbb{R})$ satisfying $\chi(x) = |x|$ for $|x| \geq 1$ and another Hilbert space $H := L^2(\mathbb{R}, \mathbb{R}^d; \mathrm{d}x)$. They consider a Dirichlet form $(\mathscr{E}, D(\mathscr{E}))$ on $L^2(E;\mu)$ associated with the diffusion process $\mathbf{X}$ on an infinite volume path space $C(\mathbb{R}, \mathbb{R}^d)$ with (U-)Gibbs measures μ associated with the (formal) Hamiltonian

$$\mathcal{H}(w) := \frac{1}{2}\int_{\mathbb{R}} |w'(x)|^2_{\mathbb{R}^d}\mathrm{d}x + \int_{\mathbb{R}} U(w(x))\mathrm{d}x,$$

where $U : \mathbb{R}^d \to \mathbb{R}$ is an interactions potential satisfying $\nabla^2 U \geq -K_1$ with some $K_1 \in \mathbb{R}$ (see [34, Subsection 4.1]). Then the L^2-semigroup $(P_t)_{t\geq 0}$ associated to $(\mathscr{E}, D(\mathscr{E}))$ satisfies the following gradient estimate

$$|D(P_t f)(w)|_H \leq e^{K_1 t} P_t(|Df|_H)(w) \quad \text{for} \quad \mu\text{-a.e. } w \in E,$$

which is equivalent to $\mathsf{BE}_2(-K_1,\infty)$-condition. Here Df is a closed extension of the Fréchet derivative $Df : E \to H$ for cylindrical function f. Hence $(E, \mathscr{E}, \mu)$ is tamed by $-K_1\mu$ with $|K_1|\mu \in S_K(\mathbf{X})$. Moreover, we have the following log-Sobolev inequality proved in [33, Theorem 2.2]: Under $K_1 < 0$,

$$\int_E f(w)^2 \log\left(\frac{f(w)^2}{\|f\|^2_{L^2(\mu)}}\right)\mu(\mathrm{d}w) \leq \frac{2}{-K_1}\mathscr{E}(f,f) \quad \text{for} \quad f \in D(\mathscr{E}). \tag{5.1}$$

So we have the hypercontractivity of $(P_t)_{t\geq 0}$ with $C = (-K_1)^{-1}$ and $\mu(E) = 1$. Then our inequalities (2.7),(2.8) and (2.9) under $\alpha = 0$ and $K_1 < 0$ are new in this setting.

In the rest, we expose new examples.

Example 5.1 (RCD spaces). A metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is a complete separable metric space (X, d) with a σ-finite Borel measure with $\mathfrak{m}(B) < +\infty$ for any bounded Borel set B. We assume $\mathfrak{m}$ has full topological support, i.e., $\mathfrak{m}(G) > 0$ for non-empty open set G.

Any metric open ball is denoted by $B_r(x) := \{y \in X \mid \mathsf{d}(x,y) < r\}$ for $r > 0$ and $x \in X$. A subset B of X is said to be bounded if it is included in a metric open ball. Denote by $C([0,1], X)$ the space of continuous curve defined on the unit interval $[0,1]$ equipped the distance $\mathsf{d}_\infty(\gamma, \eta) := \sup_{t\in[0,1]} \mathsf{d}(\gamma_t, \eta_t)$ for every $\gamma, \eta \in C([0,1], X)$. This turn $C([0,1], X)$ into complete separable metric space.

Definition 5.2 (Sobolev space $W^{1,2}(X)$). A Borel function $f \in L^2(X;\mathfrak{m})$ belongs to $W^{1,2}(X)$, provided there exists a $G \in L^2(X;\mathfrak{m})$, called 2-*weak upper gradient* of f so that

$$\int_{C([0,1],X)} |f(\gamma_1) - f(\gamma_0)|\boldsymbol{\pi}(\mathrm{d}\gamma) \le \int_{C([0,1],X)} \int_0^1 G(\gamma_t)|\dot{\gamma}_t|\mathrm{d}t\boldsymbol{\pi}(\mathrm{d}\gamma), \quad \forall \boldsymbol{\pi} \text{ 2-test plan.} \tag{5.2}$$

See [26, Subsections 2.1 and 2.3] for the precise definition of metric speed $|\dot{\gamma}_t|$ and 2-test plan $\boldsymbol{\pi}$ on $C([0,1],X)$. The assignment $(t,\gamma) \mapsto G(\gamma_t)|\dot{\gamma}_t|$ is Borel measurable (see [30, Remark 2.1.9]) and the right hand side of (5.2) is finite for $G \in L^2(X;\mathfrak{m})_+$ (see [30, Remark 2.1.10]). These shows not only the finiteness of the right hand side of (5.2) but also the continuity of the assignment $L^2(X;\mathfrak{m}) \ni G \mapsto \int_{C([0,1],X)} \int_0^1 G(\gamma_t)|\dot{\gamma}_t|\mathrm{d}t\boldsymbol{\pi}(\mathrm{d}\gamma)$. This, combined with the closedness of the convex combination of the 2-weak upper gradient, shows that the set of 2-weak upper gradient of a given Borel function f is a closed convex subset of $L^2(X;\mathfrak{m})$. The minimal 2-weak upper gradient, denoted by $|\nabla f|_2$ is then the element of minimal L^2-norm in this class. Also, by making use of the lattice property of the set of 2-weak upper gradient, such a minimality is also in the $\mathfrak{m}$-a.e. sense (see [13, Proposition 2.17 and Theorem 2.18]). Note that we do not define the gradient ∇f, but this can be justified under infinitesimally Hilbertian condition below (see [27, Proposition 2.3.7, Definition 2.3.8 and Proposition 2.3.17]).

Then, $W^{1,2}(X)$ forms a Banach space equipped with the following norm: equipped with the norm

$$\|f\|_{W^{1,2}(X)} := \left(\|f\|^2_{L^2(X;\mathfrak{m})} + \||\nabla f|_2\|^2_{L^2(X;\mathfrak{m})}\right)^{\frac{1}{2}}, \quad f \in W^{1,2}(X).$$

It is in general false that $(W^{1,2}(X), \|\cdot\|_{W^{1,2}(X)})$ is a Hilbert space. When this occurs, we say that $(X,\mathsf{d},\mathfrak{m})$ is *infinitesimally Hilbertian* (see [27, Definition 2.3.16]). Equivalently, we call $(X,\mathsf{d},\mathfrak{m})$ infinitesimally Hilbertian provided the following *parallelogram identity* holds:

$$2|\nabla f|_2^2 + 2|\nabla g|_2^2 = |\nabla(f+g)|_2^2 + |\nabla(f-g)|_2^2, \quad \mathfrak{m}\text{-a.e.} \quad \forall f,g \in W^{1,2}(X). \tag{5.3}$$

For simplicity, we omit the suffix 2 from $|\nabla f|_2$ for $f \in W^{1,2}(X)$, i.e. we write $|\nabla f|$ instead of $|\nabla f|_2$ for $f \in W^{1,2}(X)$. Under (5.3), we can give a bilinear form $\langle \nabla\cdot, \nabla\cdot\rangle$: $W^{1,2}(X) \times W^{1,2}(X) \to L^1(X;\mathfrak{m})$ which is defined by

$$\langle \nabla f, \nabla g\rangle := \frac{1}{4}|\nabla(f+g)|^2 - \frac{1}{4}|\nabla(f-g)|^2, \quad f,g \in W^{1,2}(X).$$

Moreover, under the infinitesimally Hilbertian condition, the bilinear form $(\mathscr{E}, D(\mathscr{E}))$ defined by

$$D(\mathscr{E}) := W^{1,2}(X), \quad \mathscr{E}(f,g) := \frac{1}{2}\int_X \langle \nabla f, \nabla g\rangle \mathrm{d}\mathfrak{m}$$

is a strongly local Dirichlet form on $L^2(X;\mathfrak{m})$. Denote by $(P_t)_{t\ge 0}$ the $\mathfrak{m}$-symmetric semigroup on $L^2(X;\mathfrak{m})$ associated with $(\mathscr{E}, D(\mathscr{E}))$. Under (5.3), let $(\Delta, D(\Delta))$ be the L^2-generator associated with $(\mathscr{E}, D(\mathscr{E}))$ similarly defined as in (2.2) before.

Definition 5.3 (RCD-spaces, [30, Definition 6.1.2]). A metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is said to be an $\mathsf{RCD}(K,\infty)$*-space* if it satisfies the following conditions:

(1) $(X, \mathsf{d}, \mathfrak{m})$ is infinitesimally Hilbertian.

(2) There exist $x_0 \in X$ and constants $c, C > 0$ such that $\mathfrak{m}(B_r(x_0)) \le C\mathrm{e}^{cr^2}$.

(3) If $f \in W^{1,2}(X)$ satisfies $|\nabla f| \le 1$ $\mathfrak{m}$-a.e., then f has a 1-Lipschitz representative.

(4) For any $f \in D(\Delta)$ with $\Delta f \in W^{1,2}(X)$ and $g \in D(\Delta) \cap L^\infty(X;\mathfrak{m})$ with $g \ge 0$ and $\Delta g \in L^\infty(X;\mathfrak{m})$,

$$\frac{1}{2}\int_X |\nabla f|^2 \Delta g \,\mathrm{d}\mathfrak{m} - \int_X \langle \nabla f, \nabla \Delta f\rangle g \,\mathrm{d}\mathfrak{m} \ge K \int_X |\nabla f|^2 g \,\mathrm{d}\mathfrak{m}.$$

Let $N \in [1, +\infty[$. A metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is said to be an $\mathsf{RCD}(K,N)$*-space* if it is an $\mathsf{RCD}(K,\infty)$-space and for any $f \in D(\Delta)$ with $\Delta f \in W^{1,2}(X)$ and $g \in D(\Delta) \cap L^\infty(X;\mathfrak{m})$ with $g \ge 0$ and $\Delta g \in L^\infty(X;\mathfrak{m})$,

$$\frac{1}{2}\int_X |\nabla f|^2 \Delta g \,\mathrm{d}\mathfrak{m} - \int_X \langle \nabla f, \nabla \Delta f\rangle g \,\mathrm{d}\mathfrak{m} \ge K \int_X |\nabla f|^2 g \,\mathrm{d}\mathfrak{m} + \frac{1}{N}\int_X (\Delta f)^2 g \,\mathrm{d}\mathfrak{m}.$$

Remark 5.4.

(i) The $\mathsf{RCD}(K,\infty)$-space (resp. $\mathsf{RCD}(K,N)$-space) in Definition 5.3 is actually called the *metric* $\mathsf{BE}(K,\infty)$*-space* (resp. *metric* $\mathsf{BE}(K,N)$*-space*) in [6, Definition 6.1] and it is proved in [6, Theorem 7.1] that *metric* $\mathsf{BE}(K,\infty)$*-space* (resp. *metric* $\mathsf{BE}(K,N)$*-space*) in [6, Definition 6.1] is equivalent to the original definition of $\mathsf{RCD}(K,\infty)$-space (resp. $\mathsf{RCD}^*(K,N)$-space) in terms of optimal transport, i.e. $\mathsf{CD}(K,\infty)$-space (resp. $\mathsf{CD}^*(K,N)$-space) satisfying infinitesimally Hilbertian condition. As noted in (ii) just below, $\mathsf{RCD}(K,N)$-space coincides with $\mathsf{RCD}^*(K,N)$-space. So we adopt Definition 5.3 for RCD-spaces in this example.

(ii) It is shown in [12, 40] that for $N \in [1, +\infty[$ $\mathsf{RCD}(K,N)$-space is equivalent to $\mathsf{RCD}^*(K,N)$-space. Originally, for $N \in [1, +\infty]$, the notion of $\mathsf{CD}(K,N)$-spaces was defined by Lott–Villani [44] and Sturm [55, 56, 57]. Later, Bacher–Sturm [7] gave the notion of $\mathsf{CD}^*(K,N)$-space as a variant for $N \in [1, +\infty[$. The notion $\mathsf{RCD}(K,\infty)$-space was firstly considered in [4, 5], and the $\mathsf{RCD}(K,N)$-space under $N \in [1, +\infty[$ was also given by [28, 26]. Finally, Erbar–Kuwada–Sturm [20] invented the notion of $\mathsf{CD}^e(K,N)$-space, so-called the metric measure space satisfying entropic curvature dimension condition under $N \in [1, +\infty[$ and prove that $\mathsf{RCD}^*(K,N)$-space coincides with $\mathsf{RCD}^e(K,N)$-space, where $\mathsf{RCD}^e(K,N)$-space is the $\mathsf{CD}^e(K,N)$-space satisfying infinitesimally Hilbertian condition. Moreover, under $N \in [1, +\infty[$, $\mathsf{RCD}(K,N)$-space (or $\mathsf{RCD}^*(K,N)$-space) is a locally compact separable metric space, consequently, $\mathfrak{m}$ becomes a Radon measure.

(iii) If $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathsf{RCD}(K,N)$-space under $N \in [1, +\infty[$, it enjoys the Bishop–Gromov inequality: Let $\kappa := K/(N-1)$ if $N > 1$ and $\kappa := 0$ if $N = 1$. We set $\omega_N := \frac{\pi^{N/2}}{\int_0^\infty t^{N/2}e^{-t}\mathrm{d}t}$ (volume of the unit ball in $\mathbb{R}^N$ under $N \in \mathbb{N}$) and $V_\kappa(r) := \omega_N \int_0^r \mathfrak{s}_\kappa^{N-1}(t)\mathrm{d}t$. Here $\mathfrak{s}_\kappa(s)$ is the solution to Jacobi equation $\mathfrak{s}_\kappa''(s) + \kappa\mathfrak{s}_\kappa(s) = 0$ with $\mathfrak{s}_\kappa(0) = 0$, $\mathfrak{s}_\kappa'(0) = 1$. More concretely, $\mathfrak{s}_\kappa(s)$ is given by

$$\mathfrak{s}_\kappa(s) := \begin{cases} \frac{\sin\sqrt{\kappa}s}{\sqrt{\kappa}} & \kappa > 0, \\ s & \kappa = 0, \\ \frac{\sinh\sqrt{-\kappa}s}{\sqrt{-\kappa}} & \kappa < 0. \end{cases}$$

Then

$$\frac{\mathfrak{m}(B_R(x))}{V_\kappa(R)} \le \frac{\mathfrak{m}(B_r(x))}{V_\kappa(r)}, \qquad x \in X, \qquad 0 < r < R.$$

(iv) If $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathsf{RCD}(K, \infty)$-space, it satisfies the following Bakry–Émery estimate:

$$|\nabla P_t f| \le e^{-Kt} P_t |\nabla f| \quad \mathfrak{m}\text{-a.e.} \quad \text{for} \quad f \in W^{1,2}(X),$$

in particular, $P_t f \in W^{1,2}(X)$ (see [50, Corollary 3.5], [29, Proposition 3.1]).

(v) If $(X, \mathsf{d}, \mathfrak{m})$ is an $\mathsf{RCD}(K, N)$-space with $N \in [\,1, +\infty\,[$, then, for any $f \in \mathrm{Lip}(X)$ $\mathrm{lip}(f) = |\nabla f|$ $\mathfrak{m}$-a.e. holds, because $\mathsf{RCD}(K, N)$ with $N \in [\,1, +\infty\,[$ admits the local volume doubling and a weak local $(1, 2)$-Poincaré inequality (see [13, §5 and §6]).

By definition, for $\mathsf{RCD}(K, N)$-space $(X, \mathsf{d}, \mathfrak{m})$, $(X, \mathscr{E}, \mathfrak{m})$ satisfies $\mathsf{BE}_2(K, N)$-condition and $K^\pm\mathfrak{m}$ is always a Kato class smooth measure, where $K^\pm := \max\{\pm K, 0\}$, hence it is a tamed Dirichlet space by $K\mathfrak{m}$. So we can apply Theorem 2.2 to $\mathsf{RCD}(K, \infty)$-space so that (2.5) and (2.6) hold for $\alpha \ge K^-$ with $\alpha > 0$ under $p \in\,]\,1, +\infty\,[$, (2.5) holds for $\alpha > 0$ under $p \in\,]\,1, 2\,]$, and (2.5) holds for $\alpha = 0$ and $K \ge 0$ under $p \in\,]\,1, +\infty\,[$. Now we consider the case for (2.7), (2.8) and (2.9) in Theorem 2.2. It is known that for $\mathsf{RCD}(K, N)$-space under $N \in [\,1, +\infty\,[$ there exists a heat kernel $p_t(x, y)$ satisfying the Hölder continuity of $x \mapsto p_t(x, y)$ for each $(t, y) \in\,]\,0, \infty\,[\times X$ associated with $(X, \mathscr{E}, \mathfrak{m})$ and it satisfies a global lower and upper heat kernel estimate (see [32, Theorem 1.2]), in particular, $(\mathscr{E}, D(\mathscr{E}))$ is irreducible, hence it is transient or irreducible recurrent. So we can apply Theorem 2.2 to $\mathsf{RCD}(K, N)$-space $(X, \mathsf{d}, \mathfrak{m})$ under $N \in [\,1, +\infty\,[$ so that E_o can be extended to a bounded linear operator on $L^p(X; \mathfrak{m})$ and (2.7), (2.8) and (2.9) holds for $f \in L^p(X; \mathfrak{m})$. For $\mathsf{RCD}(K, \infty)$-space, it satisfies the following log-Sobolev inequality: under $K > 0$

$$\int_X f^2(x) \log\left(\frac{f^2(x)}{\|f\|^2_{L^2(X;\mathfrak{m})}}\right) \mathfrak{m}(\mathrm{d}x) \le \frac{2}{K} \int_X |\nabla f|^2(x)\mathfrak{m}(\mathrm{d}x) \quad \text{ for } f \in D(\mathscr{E}).$$

Indeed, for $\mathsf{CD}(K, \infty)$-space with $K > 0$, a log-Sobolev inequality in terms of mass transport is well-known (see [44, Corollary 6.12]). So we can apply Theorem 2.2 to $\mathsf{RCD}(K, \infty)$-space $(X, \mathsf{d}, \mathfrak{m})$ so that E_o can be extended to a bounded linear operator on $L^p(X; \mathfrak{m})$ and (2.7), (2.8) and (2.9) holds for $f \in L^p(X; \mathfrak{m})$ under $K > 0$ and $\mathfrak{m}(X) < +\infty$.

We remark that there are related works [37, 38] on Littlewood–Paley–Stein inequality for $\mathsf{RCD}(K, \infty)$-space. In particular, [38] proves only (2.5) for $f \in L^2(X; \mathfrak{m})$ over

$\mathsf{RCD}(K,\infty)$-space under $\alpha \geq 0$ with $p \in]1, 2]$ or $\alpha \geq 2K^-$ with $p \in]2, +\infty[$. Their condition $\alpha \geq 2K^-$ is stronger than our condition $\alpha \geq K^-$ with $\alpha > 0$. They do not prove (2.7), (2.8) and (2.9) under $\alpha = 0$ and $K \geq 0$. Moreover, the result of [38] under $\alpha \geq 2K^-$ with $K < 0$ and $p \in]1, +\infty[$ can be deduced from Kawabi–Miyokawa [34], because the space $\mathrm{Test}(X)$ of test functions forms an subalgebra of $C_b(X)$, and hence it satisfies the condition **(A)** in [34] (see Bouleau-Hirsch [10, Corollary 4.2.3] for the validity of **(A)** in [34]).

Example 5.5 (Riemannian manifolds with boundary). Let (M, g) be a smooth Riemannian manifold with boundary ∂M. Denote by $\mathfrak{m} := \nu_g$ the Riemannian volume measure induced by g, and by $\mathfrak{s}$ the surface measure on ∂M (see [11, §1.1.2]). Denote by $D(\mathscr{E}) := W^{1,2}(M^\circ)$ the Sobolev space with respect to $\mathfrak{m}$ on $M^\circ := M \setminus \partial M$. Define $\mathscr{E} : W^{1,2}(M^\circ) \times W^{1,2}(M^\circ) \to [0, +\infty[$ by

$$\mathscr{E}(f, g) := \int_{M^\circ} \langle \nabla f, \nabla g \rangle \mathrm{d}\mathfrak{m}.$$

Then $(\mathscr{E}, D(\mathscr{E}))$ is a strongly local regular Dirichlet form on $L^2(X; \mathfrak{m})$ (see [11, Example 1.18]). Let $k : M \to \mathbb{R}$ and $\ell : \partial M \to \mathbb{R}$ be continuous functions providing lower bounds on the Ricci curvature and the second fundamental form of ∂M, respectively. Suppose that M is compact. Then $(M, \mathscr{E}, \mathfrak{m})$ is tamed in our sense (see [11, Example 2.12]) by

$$\kappa := (\inf_{x\in M} k(x))\mathfrak{m} + (\inf_{x\in\partial M} \ell(x))\mathfrak{s},$$

because $\mathfrak{m}, \mathfrak{s} \in S_K(\mathbf{X})$ (see [21, Theorem 2.36]). Therefore all the statements of Theorem 2.2 to $(M, \mathscr{E}, \mathfrak{m})$ hold. Remark that (2.5) is proved by Shigekawa [51, Propositions 6.2 and 6.4] in the framework of compact smooth Riemannian manifold with boundary.

As in [21, Lemma 2.34, Theorem 2.36, Proof of Theorem 4.6], treating the case of singular boundary, we can construct other concrete examples which satisfy Theorem 2.2.

Example 5.6 (Configuration space over metric measure spaces). Let (M, g) be a complete smooth Riemannian manifold without boundary. The configuration space Υ over M is the space of all locally finite point measures, that is,

$$\Upsilon := \{\gamma \in \mathcal{M}(M) \mid \gamma(K) \in \mathbb{N} \cup \{0\} \quad \text{for all compact sets} \quad K \subset M\},$$

where $\mathcal{M}(M)$ is a family of positive Radon measures on M. In the seminal paper Albeverio–Kondrachev–Röckner [1] identified a natural geometry on Υ by lifting the geometry of M to Υ. It is shown in [1] that the Poisson point measure π on Υ is the unique (up to intensity) measure on Υ under which the gradient ∇^Υ and divergence div^Υ become dual operator in $L^2(\Upsilon; \pi)$. The canonical Dirichlet form

$$\mathscr{E}^\Upsilon(F) = \int_\Upsilon |\nabla^\Upsilon F|^2_\gamma \pi(\mathrm{d}\gamma)$$

constructed in [1, Theorem 6.1] is quasi-regular and strongly local. If $\mathrm{Ric}_g \geq K$ on M with $K \in \mathbb{R}$, then $(\Upsilon, \mathscr{E}^\Upsilon, \pi)$ is tamed by $\kappa := K\pi$ with $|\kappa| \in S_K(\mathbf{X}^\Upsilon)$ (see [19, Theorem 5.10]).

More generally, Dello Schiavo–Suzuki [17] extends the framework of underlying space from complete smooth Riemannian manifold to proper complete separable metric measure space $(X, \mathsf{d}, \mathfrak{m})$ and consider its configuration space Υ and with a reference Borel probability measure μ satisfying [17, Assumption 2.17], commonly understood as the law of a proper point process on X. In [17], they constructed the quasi-regular strongly local Dirichlet form $(\mathscr{E}^{\Upsilon}, D(\mathscr{E}^{\Upsilon}))$ on $L^2(\Upsilon;\mu)$ (see [17, Proposition 3.9 and Theorem 3.45]). Moreover, in Dello Schiavo–Suzuki [18, Theorem 1], for any fixed $K \in \mathbb{R}$ they prove that a Dirichlet form $(\mathscr{E}, D(\mathscr{E}))$ with its carré-du-champ Γ satisfies $\mathsf{BE}_2(K,\infty)$ if and only if the Dirichlet form $(\mathscr{E}^{\Upsilon}, D(\mathscr{E}^{\Upsilon}))$ on $L^2(\Upsilon;\mu)$ with its carré-du-champ Γ^{Υ} satisfies $\mathsf{BE}_2(K,\infty)$. Hence, if $(X, \mathscr{E}, \mathfrak{m})$ is tamed by $K\mathfrak{m}$ with $|K|\mathfrak{m} \in S_K(\mathbf{X})$, then $(\Upsilon, \mathscr{E}^{\Upsilon}, \mu)$ is tamed by $\kappa := K\mu$ with $|\kappa| \in S_K(\mathbf{X}^{\Upsilon})$. Here $\mathbf{X}$ (resp. $\mathbf{X}^{\Upsilon}$) is the $\mathfrak{m}$-symmetric (resp. μ-symmetric) Markov process associated with $(\mathscr{E}, D(\mathscr{E})$ (resp. $(\mathscr{E}^{\Upsilon}, D(\mathscr{E}^{\Upsilon}))$ on $L^2(X;\mathfrak{m})$ (resp. $L^2(\Upsilon;\mu)$). So we can apply Theorem 2.2 to $(\Upsilon, \mathscr{E}^{\Upsilon}, \mu)$ so that (2.5) and (2.6) hold for $\alpha \geq K^-$ with $\alpha > 0$ under $p \in [\,2, +\infty\,[$ or $\alpha > 0$ under $p \in\,]\,1, 2\,[$, and (2.5) holds for $\alpha = 0$ and $K \geq 0$ under $p \in\,]\,1, 2\,[$ under the suitable class of measures μ defined in [17, Assumption 2.17]. It is shown in [1, Theorem 4.3] that if the underlying space $(X, \mathsf{d}, \mathfrak{m})$ is a complete smooth Riemannian manifold (M, d_g, ν_g) $(\mathscr{E}^{\Upsilon}, D(\mathscr{E}^{\Upsilon}))$ on $L^2(\Upsilon;\mu)$, where μ runs over the class of mixed Poisson measures, is irreducible if and only if μ is a Poisson measure (see also [18, paragraph before Example 6.8]). $(\mathscr{E}^{\Upsilon}, D(\mathscr{E}^{\Upsilon}))$ is also recurrent by $\mathscr{E}^{\Upsilon}(1,1) = 0$ and $1 \in D(\mathscr{E}^{\Upsilon})$. Then we can apply Theorem 2.2 to $(\Upsilon, \mathscr{E}^{\Upsilon}, \mu)$ so that E_o can be extended to a bounded linear operator on $L^p(\Upsilon;\mu)$ and (2.7), (2.8) and (2.9) holds for $f \in L^p(\Upsilon;\mu)$ provided the underlying space is a complete smooth Riemannian manifold and μ is a Poisson measure on Υ.

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