REVERSIBLE AND OTHER GENERALISED TORSION ELEMENTS IN SEIFERT FIBERED GROUPS

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ABSTRACT. An element a in a group Γ is called *reversible* if there exists $g \in \Gamma$ such that $gag^{-1} = a^{-1}$. The reversible elements are also known as 'real elements' or 'reciprocal elements' in literature. In this paper, we classify the reversible elements in a Seifert fibered group. We then apply the classification to the braid groups, particularly to the braid group on 3 strands. We further study generalised 3-torsion elements in $\mathrm{PSL}(2,\mathbb{Z})$, and use this to analyse the existence of generalised 3-torsion elements in Seifert fibered groups in general, and braid groups on 3 strands in particular.

1. Introduction

A non-trivial element in a group Γ is said to be 'reversible' if it is conjugate to its own inverse, and the conjugating elements are called 'reverser' or 'reversing symmetry'. If the conjugating element is also an involution, then it is called 'strongly reversible' element. It has also been called by other terms like 'reciprocals' or 'real elements' in other literature, e.g., [2], [9] etc. Reversibility has its own geometric motivations and has been studied widely in literature. For instance, in a Fuchsian group, the collection of conjugacy classes of reversible elements have a 1-1 correspondence to the conjugacy classes of infinite dihedral groups contained in that Fuchsian group. But in general, supposing G is a 3-manifold group (the fundamental group of some 3-manifold), if G is torsion free then a reversible element g in G is represented as a loop which is homotopic to its own inverse. In this paper we are specially interested in Seifert fibered manifolds since the fundamental group of a Seifert fibered manifold is a universal extension of Fuchsian groups or non-orientable 2-orbifold groups.

Furthermore, a non-trivial element g in a group G is said to be a generalised torsion element if it satisfies the following condition

$$h_1 g h_1^{-1} h_2 g h_2^{-1} \dots h_n g h_n^{-1} = Id,$$

for some $h_1, h_2, \dots, h_n \in G$. If n is the minimum number of conjugates yeilding the identity, g is said to be generalised n-torsion element. Generalised n-torsion elements are conjugacy invariant. Reversible elements are therefore also known as generalised 2-torsion elements (see [6]). We also study the existence of generalised 3-torsion elements in $\mathrm{PSL}(2,\mathbb{Z})$ and use it to find such elements in Seifert fibered groups.

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A Seifert fibered manifold is a compact 3-manifold which can be seen as a S^1 bundle over a 2-dimensional orbifold as the base space, such that every fiber has a solid fibered torus neighbourhood. These spaces were first studied by Herbert Seifert in his doctoral thesis [13], and they form an important example of 3-manifolds. In fact, all the compact orientable 3-manifolds in 6 of the 8 possible 3-manifold geometric structures can be seen as Seifert fibered spaces. Seifert fibered spaces have been widely studied, for instance in [7], [10], [8]. If we consider the space of fibers of a Seifert fibered space M with a quotient topology, we get a resulting surface called the orbit surface of M. We denote the orbit surface by G. Then there is a map $\phi : \pi_1(G) \to \{-1, 1\}$ called the classifying homomorphism. A loop α on G has image $\phi(\alpha) = 1$ if the fiber above α is mapped to itself in an orientation preserving way as we follow the loop α, and has image $\phi(\alpha) = -1$ otherwise. The classifying homomorphism plays a crucial role in determining the fiber type of a Seifert fibered space.

Given any such compact, connected Seifert fibered space M with non-empty boundary (i.e., $\pi_1(M)$ is torsion free, see [1, Lemma 2.1.4]), this paper is concerned with checking the reversibility of the elements of $\pi_1(M)$, the fundamental group of M. The fundamental group of M depends on the orientability of the orbit surface G. If G is orientable, it can be shown following Seifert's work and Brin's note in [1], that the fundamental group of M is of the form

$$\begin{aligned} \pi_1(M) = &\langle a_i, b_i, c_i, d_i, h \mid a_i h a_i^{-1} = h^{\phi(a_i)}, \\ &b_i h b_i^{-1} = h^{\phi(b_i)}, c_i h c_i^{-1} = h, d_i h d_i^{-1} = h^{\phi(d_i)}, \\ &c_i^{\mu_i} = h_i^{\beta}, \prod\, [a_i, b_i] \, \prod c_i \, \prod d_i h^b = 1\rangle. \end{aligned} \tag{1.1}$$

If G is not orientable, $\pi_1(M)$ has the presentation

$$\begin{aligned} \pi_1(M) = &\langle x_i, c_i, d_i, h \mid x_i h x_i^{-1} = h^{\phi(x_i)}, \\ &c_i h c_i^{-1} = h, d_i h d_i^{-1} = h^{\phi(d_i)}, c_i^{\mu_i} = h^{\beta_i}, \prod x_i^2 \, \prod c_i \prod d_i h^b = 1\rangle. \end{aligned} \tag{1.2}$$

In both cases the μ_i, β_i and b are the same as used in the representation of M, and h represents an ordinary fiber. The a_i and b_i are the generators of the handlebodies of the orbit surface, the d_i represent the boundaries of the orbit surface, the x_i generate the crosscaps in the orbit surface, and the c_i are the generators of the essential fibers. ϕ is the classifying homomorphism. $\langle h\rangle$ is a normal cyclic subgroup of $\pi_1(M)$, and so we can consider the quotient group $\pi_1(M)/\langle h\rangle$ and every element in $\pi_1(M)/\langle h\rangle$ can be as $[x]$, the projection of $x \in \pi_1(M)$. If the orbit surface G is orientable, this quotient is a Fuchsian group which can be written as

$$\pi_1(M)/\langle h\rangle = \langle [a_i], [b_i], [c_i], [d_i] \mid [c_i]^{\mu_i} = 1, \prod [[a_i], [b_i]] \, \prod [c_i] \, \prod [d_i] = 1\rangle.$$

If G is non-orientable, then

$$\pi_1(M)/\langle h\rangle = \langle [x_i], [c_i], [d_i] \mid [c_i]^{\mu_i} = 1, \prod\, [x_i]^2 \prod\, [c_i] \prod\, [d_i] = 1\rangle.$$

From the representation of $\pi_1(M)$, we can see that the reversibility of the element h is dependent on the classifying homomorphism ϕ. If there is any generator z of the form a_i, b_i, x_i, or d_i such that $\phi(z) = -1$, then the relations in the definition of $\pi_1(M)$ give that h is reversible, i.e.,

$zhz^{-1} = h^{\phi(z)} = h^{-1}$. Our aim is to classify all the reversible elements of $\pi_1(M)$. The main result of our paper is:

Theorem 1.1. *If a Seifert fibered space M has an orientable orbit surface with torsion free $\pi_1(M)$, and if ϕ is nontrivial, then the reversible elements of $\pi_1(M)$ are the following:*

1. *the elements conjugate to any power of h.*
2. *the conjugates of the elements of the form $c_i^{\frac{\mu_i}{2}} k c_j^{\frac{\mu_j}{2}} k^{-1}$ with $\phi(k) = -1$ where μ_i and μ_j are even and k is any element in $\pi_1(M)$, and $\beta_i = \beta_j$.*
3. *the conjugates of the elements of the form $c_i^{\frac{\mu_i}{2}} k c_j^{\frac{-\mu_j}{2}} k^{-1}$ with $\phi(k) = 1$, where μ_i and μ_j are even and k is any element in $\pi_1(M)$, and $\beta_i = \beta_j$.*

If ϕ is trivial, $\pi_1(M)$ has reversible elements as the elements conjugate to $c_i^{\frac{\mu_i}{2}} k c_j^{\frac{-\mu_j}{2}} k^{-1}$.

We also similarly classify the reversible elements in case of Seifert fibered spaces with non-orientable orbit surfaces. Furthermore, we have the following result for generalised 3-torsion elements occurring in Seifert fibered groups.

Proposition 1.2. *Let us consider a Seifert fibered space M with fundamental group* (1) *with some elements c_1 and c_2 such that $c_1^{3p} = h^{\mu_1}$ and $c_2^{3q} = h^{\mu_2}$ for some $p, q \in \mathbb{N}$. Then for some e_i conjugate to c_i, $e_1^p e_2^q h^x$ is a generalised 3-torsion element if $3x + \mu_1 + \mu_2 = 0$.*

In **Section 2**, we study the fundamental groups of two-dimensional orbifolds, and classify the reversible elements of such groups. This classification is then applied to the fundamental groups of any Seifert fibered space in **Section 3**. **Section 4** then discusses an application of the results of **Section 3** on B_3, the braid group on 3 strands. The possibility of using the same methods for any $B_n/\langle h\rangle$ is also discussed there. In **Section 5**, we study the existence of generalised 3-torsion elements in Seifert fibered spaces, and apply it in B_3 to find such elements. We also find conditions for the existence of generalised n-torsion elements.

After completion of the first draft, the authors were informed by Prof. Masakazu Teragaito regarding the article [6] written by Himeno, Motegi, and Teragaito, where they classify 3-manifolds whose fundamental groups admit generalised torsion elements of order two, and further classify such elements. Their proof is topological and completely different from the proof in the **Section 3** of this paper. The two articles were written completely independently and at around the same time. The proof in [6] crucially uses important results by Jaco-Shalen and Hass, while the techniques in this article are simpler in comparison. Himeno in his paper [5], also examines generalised 3-torsion elements in Seifert fibered manifolds with boundary.

2. Reversibility in Fundamental groups of the surfaces

The classification of reversible elements for any finite type Fuchsian groups is studied in [3]. That covers the classification of reversible elements of the fundamental groups of orientable surfaces and orientable 2-dimensional orbifolds. Here we extend the classification for the fundamental groups of non-orientable surfaces and 2-dimensional orbifolds.

2.1. **Reversible classes of non-orientable surface fundamental groups.** Let S be a compact connected non-orientable surface. Due to the classification of surfaces, the fundamental group of S can be presented in the form

$$\langle x_1, x_2, \ldots, x_k, d_1, d_2, \ldots, d_m | \Pi x_i^2 \Pi d_i = 1\rangle \tag{2.1}$$

where k represents the number of cross-caps and the d_i represent the boundary components. In such cases, we have the following lemma.

Lemma 2.1. *If S is a compact, connected non-orientable surface having the fundamental group 2.1, then there are no reversible elements in $\pi_1(S)$ except when S is the Klein bottle or $\mathbb{RP}^2$.*

Proof. By the classification of non-orientable surfaces, we know that if S is closed, then it is homeomorphic to $\underbrace{\mathbb{RP}^2\#\mathbb{RP}^2\#\ldots\#\mathbb{RP}^2}_{k\ times}$ for some $k \in \mathbb{N}$. We also know that a loop l is reversible iff it will freely homotope to its own inverse. So, if S is itself a Klein bottle, then any power of the generating loop is reversible. If S is $\mathbb{RP}^2$, then the fundamental group will be $\mathbb{Z}_2$, and the generator is reversible.

If $k > 2$, there is no finite order element in the fundamental group. Then our claim is that there are no more reversible elements in S. If on the contrary there exists any loop l that homotopes to its own inverse, then $\pi_1(S)$ will no longer be a bi-orderable group. This will lead us to a contradiction.

Now, if S has boundary, then $\pi_1(S)$ is free. In that case, there are no reversible elements, since free groups are bi-orderable. Hence, this proves the lemma.

□

Now we classify the reversible elements in the fundamental group of a non-orientable two-dimensional orbifold Γ' which is neither $\mathbb{RP}^2$ nor Klein bottle.

Lemma 2.2. *Infinite order reversible elements in Γ' are strongly reversible elements, and every reverser is an involution.*

Proof. The group Γ' is a discrete subgroup of $\mathrm{PGL}(2, \mathbb{R})$. Let r in $\mathbf{H}^2/\Gamma'$ be a reversible element in Γ' . Then there is an element b in Γ' such that $b^{-1}rb = r^{-1}$. At first we consider that r is orientation preserving isometry in $\mathbf{H}^2$ (which implies that will not be a parabolic element). Then, b either fixes the fixed points of r or reflects the fixed points to each other. If b fixes the set of fixed points of r then it will either commute with r or reverse r. In the case where it commutes with r it leads us to a contradiction that b is not a reverser, and in the case where it is reversing, b will be an involution. If b interchanges the fixed points, then b must be an involution. So, every reverser is an involution in Γ' .

In r is a orientation reversing element, then r^2 will be orientation preserving element with the same reverser b. From the previous case, $b^2 = 1$. Hence, in any case, r will be strongly reversible. □

Proposition 2.3. *The infinite order reversible elements in Γ' are conjugate to $c_i^{\frac{\mu_i}{2}} k c_j^{\frac{\mu_j}{2}} k^{-1}$ where μ_i and μ_j are even and $k \in \Gamma'$.*

Proof. Since there is no finite order element in Γ' other than the elements conjugate to c_i, this theorem follows from the previous lemma. ☐

3. Seifert fibered space

To find the reversible elements of the fundamental group of a Seifert fibered space, we first focus on the quotient group $\pi_1(M)/\langle h\rangle$ and discuss its reversibility. We first deal with those spaces which have orientable orbit surfaces.

Lemma 3.1. *Any element conjugate to either $[c_i]^{\frac{\mu_i}{2}}$ or $[c_i]^{\frac{\mu_i}{2}}[k][c_j]^{\frac{\mu_j}{2}}[k]^{-1}$ where $[k] \in \pi_1(M)/\langle h\rangle$ is reversible in $\pi_1(M)/\langle h\rangle$ if and only if μ_i and μ_j are even. No other elements in $\pi_1(M)/\langle h\rangle$ are reversible unless it is Klein Bottle or $\mathbb{RP}^2$.*

Proof. Since $\pi_1(M)/\langle h\rangle$ is a Fuchsian group or a fundamental group of 2-dimensional orbifold Γ, all reversible elements in Γ are strongly reversible. By [3, Theorem 1.1] and Proposition 2.3, it is known that an element in Γ is reversible if and only if it is an involution or a product of at most 2 involutions. By the representation of the group, the involutions are the elements conjugate to any $[c_i]^{\frac{\mu_i}{2}}$ where μ_i is even. Hence, the claim follows. ☐

Now we can discuss the reversibility in $\pi_1(M)$. Since there is no finite order element in $\pi_1(M)$, we cannot talk about strongly reversible elements here. But there may be reversible elements, as stated earlier. Note that if an element is reversible, then so are its conjugates.

Lemma 3.2. *Elements conjugate to the power of the fiber generator h is a reversible element if and only if the classifying homomorphism on $\pi_1(G)$ is non-trivial.*

Proof. Since ϕ is a homomorphism, it is determined by its actions on the generators of $\pi_1(G)$. If ϕ is non-trivial, there is some generator x of the form a_i, b_i, x_i or d_i such that $\phi(x) = -1$. Then by the presentation of $\pi_1(M)$, $xhx^{-1} = h^{\phi(x)} = h^{-1}$, and so h is reversible. ☐

Corollary 3.3. *If h is reversible in $\pi_1(M)$, then $c_i^{\mu_i}$ are reversible elements although each c_i is an infinite order element in $\pi_1(M)$.*

Proof. Let h be reversible. Then so is h^{β_i}. The relation $c_i^{\mu_i} = h^{\beta_i}$ now implies that $c_i^{\mu_i}$ is also reversible. ☐

The two following lemmas investigate the reversibility of the other elements of $\pi_1(M)$.

Lemma 3.4. *The powers of $c_i^{\mu_i}$ are reversible, but any other multiple of c_i is not reversible for any $\pi_1(M)$.*

Proof. On the contrary, let a multiple of c_i be a reversible element in $\pi_1(M)$ with reverser ρ. We can consider that ρ is not a power of h, since h is contained in the centralizer of c_i. Then, the reverser ρ is transverse to h, and this implies any power of ρ will always be transverse to h. Hence ρ has a non-trivial image in the group $\pi_1(M)/\langle h\rangle$, denoted by $[\rho]$. Since, from [3], every reversible element in a Fuchsian group is strongly reversible, $[\rho]$ must be of order 2. This implies,

$\rho^2 = h^k$ for some $k \in \mathbb{Z}$, which contradicts our assumption. So, there is no such reverser that reverses any other power of c_i except for the powers of the $c_i^{\mu_i}$. $\square$

Lemma 3.5. *If M has an orientable orbit surface, the only reversible elements in $\pi_1(M)$ can be the conjugates of the powers of h and the conjugates of the elements either of the form $c_i^{\frac{\mu_i}{2}} k c_j^{\frac{\mu_j}{2}} k^{-1}$ with $\phi(k) = -1$ or of the form $c_i^{\frac{\mu_i}{2}} k c_j^{\frac{-\mu_j}{2}} k^{-1}$ with $\phi(k) = 1$, where μ_i and μ_j are even, and k is any element in $\pi_1(M)$, $\beta_i = \beta_j$.*

Proof. Let x be an element in $\pi_1(M)$, other than any power of h. Let x be reversible. Then x is freely homotopic to x^{-1}. Consider the projection of this free homotopy to the Fuchsian group $\pi_1(M)/\langle h\rangle$. This would be a free homotopy of a loop in this Fuchsian complex to its inverse. But this would imply that this is a reversible element in $\pi_1(G)/\langle h\rangle$. But as we have shown earlier, the reversible elements in this group are of the form of conjugates of $c_i^{\frac{\mu_i}{2}} k c_j^{\frac{\mu_j}{2}} k^{-1}$ or $c_i^{\frac{\mu_i}{2}}$ for even μ_i and μ_j. But, for elements that are conjugates of $c_i^{\frac{\mu_i}{2}}$, i.e., $\alpha c_i^{\frac{\mu_i}{2}} \alpha^{-1}$, the reverser of x must be x itself. Then the element in $x \in \pi_1(M)$ will satisfy,

$$\alpha c_i^{\frac{\mu_i}{2}} \alpha^{-1} \alpha c_i^{\frac{\mu_i}{2}} \alpha^{-1} \alpha c_i^{\frac{-\mu_i}{2}} \alpha^{-1} = \alpha c_i^{\frac{-\mu_i}{2}} \alpha^{-1}$$

$$\Rightarrow h^{\beta_i} = 1.$$

Since h is an infinite order element, this is a contradiction. Hence, the lifts of the elements conjugate to $c_i^{\frac{\mu_i}{2}}$ are not reversible in $\pi_1(M)$. So, the only other candidates from $\pi_1(M)/\langle h\rangle$ that may be reversible in $\pi_1(M)$ are the products of the involutions.
Let the element $[x] = [c_i]^{\frac{\mu_i}{2}}[k][c_j]^{\frac{\mu_j}{2}}[k]^{-1}$ be reversible in $\pi_1(M)/\langle h\rangle$ with reverser $[c_i]^{\frac{\mu_i}{2}}$. Then in $\pi_1(M)$,

$$c_i^{\frac{\mu_i}{2}} c_i^{\frac{\mu_i}{2}} k c_j^{\frac{\mu_j}{2}} k^{-1} c_i^{-\frac{\mu_i}{2}} c_i^{\frac{\mu_i}{2}} k c_j^{\frac{\mu_j}{2}} k^{-1} = c_i^{\mu_i} k c_j^{\frac{\mu_j}{2}} k^{-1} k c_j^{\frac{\mu_j}{2}} k^{-1} = h^{\beta_i} k h^{\beta_j} k^{-1}.$$

Then, $h^{\beta_i} k h^{\beta_j} k^{-1}$ will be the identity only if $\beta_i = \beta_j$ and $\phi(k) = -1$. It will be same if we choose any other reverser for that particular reversible element since all the reversers are in the form of $x^l c^{\frac{\mu_i}{2}}$. In the similar way, we can show that $c_i^{\frac{\mu_i}{2}} k c_j^{-\frac{\mu_j}{2}} k^{-1}$ is reversible if $\phi(k) = 1$.

If x is not reversible in $\pi_1(M)/\langle h\rangle$, a reverser of x in $\pi_1(M)$, say ρ, must be a power of h, since the projection to $\pi_1(M)/\langle h\rangle$ must be constant. Then in $\pi_1(M)/\langle h\rangle$, $[x] = [x^{-1}]$. This implies that $[x^2] = 1$. Then, x^2 is a power of h, and x^2 is also reversible with same reverser ρ that x is. Then,

$$h^k x^2 h^{-k} = x^{-2}$$

$$\Rightarrow x^2 = x^{-2}$$

$$\Rightarrow x^4 = Id.$$

This is contradiction since $\pi_1(M)$ is torsion free. This proves our lemma. $\square$

3.1. **Proof of Theorem 1.1.** The previous lemma shows that there are no reversible elements in $\pi_1(M)$ other than those conjugate to h. As shown earlier, the reversibility of h and its conjugates depends on the classifying homomorphism. This is summarised in the statement of Theorem 1.1. $\square$

In the case of non-orientable surfaces, we have certain exceptions to this situation, as shown in the previous section.

The reversible elements of $\pi_1(M)$ in case of a non-orientable orbit surface can be summarised as follows, by the same proofs as above.

Corollary 3.6. *If M has a non-orientable orbit surface other than the Klein bottle or $\mathbb{RP}^2$, the reversible elements of $\pi_1(M)$ are either conjugate to any power of h or either of the form $c_i^{\frac{\mu_i}{2}} k c_j^{\frac{\mu_j}{2}} k^{-1}$ with $\phi(k) = -1$ or of the form $c_i^{\frac{\mu_i}{2}} k c_j^{\frac{-\mu_j}{2}} k^{-1}$ with $\phi(k) = 1$, where μ_i and μ_j are even, and k is any element in $\pi_1(M)$, $\beta_i = \beta_j$.*

When the orbit surface is the Klein bottle or $\mathbb{RP}^2$, we get additional reversible elements which are the generators of the orbit surface. Thus, in the case of a given orbit surface, the reversibility of the elements of the fundamental group of a Seifert fibered space over that orbit surface is solely determined by ϕ.

Remark 3.7. *The classification of the reversible elements in a given Seifert fibered space depends only on the classifying homomorphism.*

4. Application to universal extension of Fuchsian groups

In the previous section, we have classified the reversible elements of the fundamental group of a Seifert fibered space. We can see the fundamental group in the form of an exact sequence:

$$1 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(M) \longrightarrow F \longrightarrow 1 \tag{4.1}$$

Here, $\pi_1(M)$ is the fundamental group of M, and F is a Fuchsian group. So, $\pi_1(M)$ is then a universal extension of F, which can now be seen as $\pi_1(M)/\langle h\rangle$. This gives us a new viewpoint to consider certain groups as fundamental groups of Seifert fibered spaces. Let us consider $F = \mathrm{PSL}(2,\mathbb{Z})$ which is presented by $\langle a, b \mid a^2, b^3\rangle$. We have already mentioned the presentation of $\pi_1(M)$ for a given M in Equation (1).

We can choose an orientable connected 3-manifold M in such a way that the $\pi_1(M) \cong B_3$. The group B_3 has a presentation as $\langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2\rangle$. The center of B_3 is $\mathbb{Z}$, generated by $(\sigma_1\sigma_2)^3$. Thus, $B_3/\langle(\sigma_1\sigma_2)^3\rangle = \langle \sigma_1\sigma_2\sigma_1, \sigma_1\sigma_2 \mid \ (\sigma_1\sigma_2)^3, (\sigma_1\sigma_2\sigma_1)^2\rangle = \mathrm{PSL}(2,\mathbb{Z}) \cong \Delta(2,3,\infty)$. Choosing the c_i and β_i suitably, we can consider the Seifert fibered space M determined by $(O, o, 0 \mid 1, (2,1), (3,1))$. The presentation of the fundamental group of M is

$$\langle c_1, c_2, d, h \mid c_i h c_i^{-1} = h, d h d^{-1} = h^{\phi(d)}, c_i^{\mu_i} = h_i^{\beta}, c_1 c_2 d h = 1\rangle. \tag{4.2}$$

Now, consider the generating elements $c_1 = \sigma_1\sigma_2\sigma_1, c_2 = \sigma_1\sigma_2$, which implies that $h = (\sigma_1\sigma_2)^3$ and $d^{-1} = \sigma_1\sigma_2\sigma_1^2\sigma_2$.

This determines the Fuchsian complex $\pi_1(M)/\langle h\rangle$, and we obtain the resultant orbit surface G

as the open disk D by further quotienting it by the c_i. Then, $\phi : \pi_1(D) \longrightarrow \mathbb{Z}_2$ is the trivial homomorphism. Then, $\pi_1(M)$ can be presented as

$$\langle \sigma_1\sigma_2\sigma_1, \sigma_1\sigma_2, \sigma_1\sigma_2\sigma_1^2\sigma_2, (\sigma_1\sigma_2)^3 \mid (\sigma_1\sigma_2)^3\sigma_1\sigma_2\sigma_1 = \sigma_1\sigma_2\sigma_1(\sigma_1\sigma_2)^3,$$

$$(\sigma_1\sigma_2\sigma_1)^2 = (\sigma_1\sigma_2)^3\rangle$$

$$= \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2\rangle = B_3.$$

So, we can deduce the exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow B_3 \longrightarrow \mathrm{PSL}(2, \mathbb{Z}) \longrightarrow 1 \tag{4.3}$$

Corollary 4.1. *The only reversible elements of B_3 are conjugates to $[\sigma_1\sigma_2\sigma_1, k]$ for any element k in B_3, where $[a, b]$ denotes the commutator of a, b.*

Proof. From the above discussion, ϕ is trivial, and applying Theorem 3.5 we know that reversible elements for the orientable case depend only on ϕ. The claim thus follows from this theorem. □

Note that we cannot use the same technique to find the reversible elements of B_n since B_n is not a Seifert fibered group for any $n > 3$. This follows from the theorem below.

Theorem 4.2. *For any $n > 3$ the braid group B_n is not isomorphic to any Seifert fibered group.*

Proof. On the contrary, let us assume that braid groups are Seifert fibered groups. So, M be a Seifert fibered space such that $B_n \cong \pi_1(M)$, with maximal normal subgroup $\langle h\rangle$ where h represents an ordinary fiber. It is well known that the braid group B_n where $n > 2$ is not virtually abelian.

Let us consider, $x = \sigma_1\sigma_2 \ldots \sigma_{n-1}$ and $y = \sigma_1$. Then, by Tietze transformation, we obtain the following presentation (this is also mentioned in [12, Chapter 2, Section 2, Exercise 2.4]):

$$B_n = \langle x, y \mid x^n = (xy)^{n-1}, x^iyx^{-i}y = yx^iyx^{-i} \text{ where } i = 2, 3, \ldots, m\rangle \tag{4.4}$$

where, $m = \lfloor \frac{n}{2} \rfloor$. Also, $\sigma_i = x^{i-1}yx^{-(i-1)}$ for $i = 1, \ldots, n-1$. Then, the fiber $h = (\sigma_1\sigma_2 \ldots \sigma_{n-1})^n = x^n$ is the generator of $Z(\pi_1(M))$. Then,

$$B_n/\langle h\rangle = \langle [x], [y] \mid [x]^n = ([x][y])^{n-1} = 1, [x]^i[y][x]^{-i}[y] = [y][x]^i[y][x]^{-i},\ i = 2, 3, \ldots, m\rangle$$

is either a Fuchsian group or a fundamental group of a non-orientable 2-dim orbifold, where $[q]$ in $B_n/\langle h\rangle$ is the projection of $q \in B_n$. So, $B_n/\langle h\rangle$ acts as an isometry group in $\mathbb{H}^2$. Since any power of σ_1 is not a member of the center of B_n, then $[y]$ is not a finite order element. This implies, $[x^iyx^{-i}]$ for every i, is not elliptic.

Suppose, $[y]$ is a parabolic element, but x^iyx^{-i} commutes with y for every i. Then, $[x]^i$'s also fix the same fixed point of $[y]$. This implies, therefore, $\langle [x], [y]\rangle$ forms a cyclic group, but any lift of $[y]$ is not contained in $Z(B_n)$; this is the contradiction.

Consider $[y]$ to be a hyperbolic element. Then by similar argument and applying [11, Lemma 4.2.2], we have, $[x^iyx^{-i}]$ and $[y]$ are powers of the same element $\psi \in B_n/\langle h\rangle$, that is, $[x]^i[y][x]^{-i} =$

ψ^u and $[y] = \psi^v$ for some integers u, v. But we know that x^i sends fixed points of $[y]$ to the fixed points of $[x^i y x^{-i}]$. That means either x^i fixes the fixed points of $[y]$ or interchanges them for every i. If it fixes the fixed points, then this is a contradiction, as in the previous case.

So, x^i must interchange the fixed points of y for every $i = 2, 3, \dots, m$, where $m = \lfloor \frac{n}{2} \rfloor$. Then $[x]^2$ must be an involution (orientation preserving or orientation reversing). This implies $n = 4$, otherwise, $[x]^2$ will be trivial. This implies,

$$[x^2 y x^{-2}] = [y^{-1}] \Rightarrow [\sigma_3] = [\sigma_1]^{-1}$$

in B_4, and then $\sigma_1\sigma_3$ is an element in $Z(B_4)$, which is not possible for braid group B_4. So, $B_n/\langle h\rangle$ will not be the fundamental group of a Seifert fibered space for any $n > 3$. □

However, there are always some reversible elements in B_n by the following remark. Here, we are assuming the presentation of B_n as given in (4.4).

Corollary 4.3. *From Corollary 4.1, there are some reversible elements in B_n which are conjugate to the elements of the form $[a, b]$, where the image of a is an involution in the quotient space $B_n/\langle x^n\rangle$ and b is any other element in B_n. This is a sufficient condition to be a reversible element in B_n.*

5. Generalised 3-Torsion elements in $\pi_1(B_3)$

Definition 5.1. *An element $g(\neq Id)$ in the group G is said to be a generalised torsion element if it satisfies the following condition:*

$$k_1 g k_1^{-1} k_2 g k_2^{-1} \dots k_n g k_n^{-1} = Id,$$

for some $k_1, k_2, \dots, k_n \in G$. If n is the minimum number of conjugates yielding the identity, g is said to be a generalised n-torsion element.

Note that, a generalised 2-torsion element is equivalent to a reversible element by definition. This property is invariant over conjugacy and inversion. In this section, we study the generalised 3-torsion elements in the braid group with 3-strands B_3, and provide a sufficient condition for an element to be a 3-torsion element. To do that, we first check the sufficient conditions for the generalised 3-torsion elements in $\mathrm{PSL}(2, \mathbb{Z})$.

Lemma 5.1. *If $g \in \mathrm{PSL}(2, \mathbb{Z})$ is a generalised 3-torsion element, then g is a product of 3-torsion elements.*

Proof. Let us consider the presentation of $\mathrm{PSL}(2, \mathbb{Z}) = \langle a, b \mid a^2, b^3\rangle$. In $\mathrm{PSL}(2, \mathbb{Z})$, let g be a generalised 3-torsion element. If we abelianise $\mathrm{PSL}(2, \mathbb{Z})$, then $[g]^3 = Id$. That means, g is a 3-ordered element or it is a product of 3-ordered elements in $\mathrm{PSL}(2, \mathbb{Z})$. The commutator $[a, b]$ of a, b is also a product of two 3-ordered elements. Then, the reduced form of g is $\prod_{i=1}^{m} k_i[a, b]k_i^{-1} \prod_{j=1}^{n} b_i$ where, $b_i, k_i \in \mathrm{PSL}(2, \mathbb{Z})$ and $|b_j| = 3$. □

If g is an elliptic element in $\mathrm{PSL}(2, \mathbb{Z})$, then it is known to be of order either 2 or 3. Thus, by the previous lemma, a generalised 3-torsion element which is also an elliptic element must be of order 3, since the product of three ordered elements cannot be an involution.

Therefore, we can now assume that $g \in \mathrm{PSL}(2,\mathbb{Z})$ is a non-elliptic generalised 3-torsion element, i.e., there are $h, k \in \mathrm{PSL}(2,\mathbb{Z})$ such that

$$ghgh^{-1}kgk^{-1} = Id,$$

and also g is conjugate to $ab^{k_1}ab^{k_2}\dots ab^{k_n}$, where $k_i = \pm 1$. Since the property of generalised 3-torsion element is invariant over conjugacy, it is sufficient to consider

$$g = ab^{k_1}ab^{k_2}\dots ab^{k_n} \tag{5.1}$$

where, $k_i = \pm 1$ and $1 \leq k_i \leq n$.

Lemma 5.2. *In* $\mathrm{PSL}(2,\mathbb{Z})$*, there are only two parabolic elements which are generalised* 3*-torsion elements upto conjugacy,* $(ab)^{\pm 2}$.

Proof. There is only one type of parabolic conjugacy class in $\mathrm{PSL}(2,\mathbb{Z})$, which is $(ab)^n$, where $n \in \mathbb{Z}$. The matrix form of $(ab)^n$ is $\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$. Let $g = (ab)^n$ be a generalised 3-torsion element, i.e., there exists $h, k \in \mathrm{PSL}(2,\mathbb{Z})$ such that,

$$ghgh^{-1}kgk^{-1} = Id$$
$$\Rightarrow ghgh^{-1} = kg^{-1}k^{-1}.$$

Thus, $ghgh^{-1}$ is a parabolic element, since any conjugate of g or g^{-1} is necessarily parabolic. Therefore, $|\mathrm{trace}(ghgh^{-1})|$ is 2. Suppose h has the matrix form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Let the matrix form of $kg^{-1}k^{-1}$ be denoted by A. Then,

$$\begin{aligned} A &= \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \begin{bmatrix} a+nc & b+nd \\ c & d \end{bmatrix}\begin{bmatrix} d-cn & -b+an \\ -c & a \end{bmatrix} \\ &= \begin{bmatrix} 1-n^2c^2-acn & acn^2+a^2n+n \\ -c^2n & 1+acn \end{bmatrix}. \end{aligned}$$

Since, A is parabolic, this implies, $|\mathrm{trace}(A)| = 2$. The obvious way to get $|\mathrm{trace}(A)| = 2$ is if we consider $c = 0$, which will give us $h = g^i$ for some $i \in \mathbb{Z}$. Then A will be g^2, which cannot be true, since g^{-1} is not conjugate to g^2. Now, we consider $c \neq 0$. That means

$$|\mathrm{trace}(A)| = 2$$
$$\Rightarrow |2 - n^2c| = 2$$
$$\Rightarrow 2 - n^2c = -2$$

since the $2 - n^2c = 2$ case will be reduced to $c = 0$. So,

$$n^2c = 4.$$

That implies either $n = 1$ and $c = 4$, or $n^2 = 4$ and $c = 1$. Since g cannot be seen as a product of 3-torsion elements if we take $n = 1$, therefore we conclude that g may be one of $(ab)^{\pm 2}$. $\square$

Without loss of generality, let us choose g to be $(ab)^2$. This implies, $g = abab$. Also note that, $ababb^{-1}abab^{-1}baba = Id$. Therefore, $ghgh^{-1}kgk^{-1} = Id$ where, $h = b^{-1}$ and $k = b$. This verifies that g is indeed a generalised 3-torsion element. Choosing $(ab)^{-2}$ is similar. This shows that the conditions derived are sufficient.
In the proof of the previous lemma, we have obtained that a parabolic element is the product of two 3-ordered elements. Therefore, let us now check if this is also true for hyperbolic elements.

Lemma 5.3. *Let g be a hyperbolic element in $\mathrm{PSL}(2,\mathbb{Z})$, such that g is a product of two 3-ordered elements, i.e., $g = b_1b_2$, where $|b_1| = |b_2| = 3$. Then g is a generalised 3-ordered torsion element.*

Proof. Consider $g = b_1b_2$ as per the statement of the lemma. This implies,

$$b_1b_2b_2^{-1}b_1b_2^{-1}b_2b_1 = Id$$

$$\Rightarrow gb_2^{-1}gb_2b_2gb_2^{-1} = Id.$$

Therefore $ghgh^{-1}kgk^{-1} = Id$, where $h = b_2^{-1}$ and $k = b_2$. Hence, this proves the lemma. $\square$

Theorem 5.4. *For an element $g \in \mathrm{PSL}(2,\mathbb{Z})$, g is a generalized 3-torsion element if any of the following holds:*

1. *if g is an elliptic element and g is of order 3.*
2. *if g is a parabolic element and g is conjugate to any of $(ab)^{\pm 2}$.*
3. *if g is a hyperbolic element and g is conjugate to an element which is product of two 3-ordered elements.*

Proof. (1) is easy to see from the above discussion. (2) follows from Lemma 5.2 and (3) follows from Lemma 5.3. $\square$

Notice that the condition of the hyperbolic elements obtained in the previous theorem holds true for any other Fuchsian group. This motivates us to see the generalised 3-torsion elements in the universal central extension of Fuchsian groups, i.e., in Seifert fibered groups.

5.1. **Proof of Proposition 1.2.** Consider two elements c_1 and c_2 in M where $c_1^{3p} = h^{\mu_1}$ and $c_2^{3q} = h^{\mu_2}$ for some $p, q \in \mathbb{N}$ and $\mu_1, \mu_2 \in \mathbb{Z}$. We know that in the Fuchsian group $\pi_1(M)/\langle h\rangle$, the product of two 3-ordered elements are generalised 3-torsion elements.

Now we take one of the lifts of $e_1^pe_2^q$ in $\pi_1(M)$, where e_i are conjugates to c_i for $i = 1, 2$, and let $g = e_1^pe_2^qh^x$, where $x \in \mathbb{Z}$. Consider $h_1 = e_2^{-q}$ and $k = e_2^q$. If g satisfies the condition

$$gh_1gh_1^{-1}kgk^{-1} = Id,$$

then

$$e_1^pe_2^qh^xe_2^{-q}e_1^pe_2^qh^xe_2^qe_2^qe_1^pe_2^qh^xe_2^{-q} = Id$$

$$\Rightarrow e_1^p h^x e_1^p h^x e_2^{3q} e_1^p h^x = Id$$
$$\Rightarrow e_1^{3p} h^{3x+\mu_2} = Id.$$

Since, h has infinite order in $\pi_1(M)$, $3x + \mu_1 + \mu_2 = 0$. In that case, g is a generalised 3-torsion element in $\pi_1(M)$. □

Remark 5.5. *We can find some sufficient conditions for the existence of generalised n-torsion elements in a Seifert fibered space following similar arguments as above. In particular, if there exist elements c_i for $i = 1, 2$ such that $c_i^{np_i} = h^{\mu_i}$ in the fundamental group, then any conjugate of $c_1^{p_1} c_2^{p_2} h^x$ is a generalised n-torsion element if $nx + \mu_1 + \mu_2 = 0$.*

In the particular case of the braid group B_3, we now have the following corollary.

Corollary 5.6. *Consider $M = S^3 \setminus K$ where K is the trefoil knot, and hence $\pi_1(M) = B_3 = \langle c_1, c_2, h | c_1^2 = c_2^3 = h, c_i h = h c_i \rangle$. Then the elements which are conjugate to $e_i e_j^2 h^{-1}$, where e_i's are distinct and conjugate to c_2, are generalised 3-torsion elements.*

Proof. Let us choose two 3-ordered elements $[e_1], [e_2]$ in the quotient group $B_3/\langle h \rangle$, conjugate to $[c_2]$. Thar is, in B_3, $e_1^3 = h = e_2^3$. In $\mathrm{PSL}(2, \mathbb{Z})$ we have noticed that the product of two 3-ordered elements are generalised 3-torsion elements.

So, we have two choices of generalised 3-torsion elements. For the first case, we take one of the lifts of $e_1 e_2$, say $g = e_1 e_2 h^x$, where $x \in \mathbb{Z}$. Let g satisfy the condition

$$g h_1 g h_1^{-1} k g k^{-1} = Id,$$

and let us choose $h_1 = e_2^{-1}$ and $k = e_2$. Then,

$$e_1 e_2 h^x e_2^{-1} e_1 e_2 h^x e_2 e_2 e_1 e_2 h^x e_2^{-1} = Id$$
$$\Rightarrow e_1 h^x e_1 h^x e_2^3 e_1 h^x = Id$$
$$\Rightarrow h^{3x+2} = Id.$$

This case is not possible, since h has infinite order and there is no integer x such that $3x + 2 = 0$. For the next case we may consider g as a lift of $e_1 e_2^2$, i.e., $g = e_1 e_2^2 h^x$. Now, if we continue the same process as previous case, we obtain $h^{3x+3} = Id$. Therefore, for $x = -1$, g will be a generalised 3-torsion element. □

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