Best simultaneous Diophantine approximations for a class of Pisot numbers

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Abstract

We compute the sequence of best approximations for the vector $(1/\beta, 1/\beta^2)$ where β is the real dominant root of the polynomial $P_{a,b}(x) = x^3 - ax^2 - bx - 1$, a and b are integers satisfying $-a+1 \leq b \leq -2$. In this case, β is a Pisot number which does not satisfy the finiteness property. To achieve this, we will use the topological properties of a class of Rauzy fractals associated with these Pisot numbers.

1 Introduction

Let v be an element of $\mathbb{R}^d$, where $d \geq 2$, $\|\cdot\|$ be a norm in $\mathbb{R}^d$ and q be an integer number. Let us define

$$\|qv\|_0 = min\left\{\|qv - (p_1, ..., p_d)\|, (p_1, ..., p_d) \in \mathbb{Z}^d\right\}.$$

Let $(q_n)_{n\geq 0}$ be an increasing sequence of integer numbers. We say that $(q_n)_{n\geq 0}$ is the sequence of best approximations of the vector v for the norm $\|\cdot\|$, if for all $n \in \mathbb{Z}^+$ and $0 < q < q_n$, one have $\|q_n v\|_0 < \|qv\|_0$.

This sequence depends on the norm $\|\cdot\|$ and there is no general algorithm that furnishes the sequence of best approximations for all elements of $\mathbb{R}^d$. When $d = 1$ the problem is already solved and the sequence of best approximations is given by the classical algorithm of one-dimensional continued fractions. When $n \geq 2$ the problem of finding sequence of best approximations becomes a very difficult task because there is no algorithm that provides this sequence for all vectors (see [8]). But we can consider this question for classes of vectors, in particular, for vectors which coordinates are cubic Pisot numbers. A Pisot number is an algebraic integer greater than 1 such that its conjugates have modulus less than 1.

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The first results is this sense were obtained in [4], [5], and [7]. These results were obtained by using the topological properties of the so-called Rauzy fractals, in particular, the properties of the tilings that they generate. These fractals can be defined by means of $\beta-$representations (a sequence of integer numbers which we will discuss in Section 2). If a β-representation ends up with infinitely many zeros, it is said to be finite. We say that a Pisot number β has the Finiteness Property (or Property (F)) if the $\beta-$representation of every nonnegative element of $\mathbb{Z}[\beta]$ is finite. The papers mentioned above only considered Pisot numbers having the Property (F).
Assuming that the associated Rauzy fractals has 6 neighbors i.e., the central fractal tile is surrounded by six copies of itself (see Section 2), we shall prove the following:

Theorem 1.1. *Let β be a cubic Pisot unit, with complex conjugates satisfying the equation $X^3 = aX^2 + bX + 1$, $-a + 1 \leq b \leq -2$. Let $(q_n)_{n\geq 0}$ be the Tribonacci sequence defined by*

$$q_0 = 1,\ q_1 = a,\ q_2 = a^2 + b, q_{n+3} = aq_{n+2} + bq_{n+1} + q_n,\ \forall n \geq 0.$$

If β does not have the Property (F) and the associated Rauzy fractals has 6 neighbors, then there exists a norm $\|\cdot\|$ in $\mathbb{R}^2$ (called Rauzy Norm) and $n_0 \in \mathbb{Z}^+$ such that $(q_n)_{n\geq n_0}$ is the sequence of best approximations for the vector $(1/\beta, 1/\beta^2)$ for the norm $\|\cdot\|$.

Our theorem is an extension of that one by Hubert and Messaoudi [7] for a class of vectors which coordinates are Pisot numbers not having the Property (F). Hereafter, we will follow the notations of Rauzy, Messaoudi and Hubert.

2 Rauzy fractals

In this section we briefly describe the $\beta-$numeration system necessary to construct the Rauzy fractal.
Given a real number $\beta > 1$, a β-representation (or β-expansion) of a number $x \in \mathbb{R}^+$ is an infinite sequence $(x_i)_{i\leq k}$, where $k \in \mathbb{Z}$, $x_i \geq 0$ such that $x = \sum_{i=-\infty}^{k} x_i\beta^i$. The digits x_i can be computed using the greedy algorithm (see [12] or [6] for a detailed explanation). As we said it before, if a β-representation ends up with infinitely many zeros, it is said to be finite and the ending zeros can be omitted. Then, the sequence will be denoted by $(x_i)_{n\leq i\leq k}$ or $x_k \cdots x_n$. The digits x_i belong to the set $A = \{0, \cdots, \beta\}$, if β is an integer, or to the set $A = \{0, \cdots, \lfloor\beta\rfloor\}$, otherwise.
Akiyama [1] classified cubic Pisot units as being exactly the set of dominant roots of the polynomial $P_{a,b}(x) = x^3 - ax^2 - bx - 1$, whose coefficients and respective Rényi β-representation of 1, denoted by $d(1, \beta)$(see [16] for the definition), satisfy one of the following conditions:

i) $1 \leq b \leq a$, and $d(1, \beta) = .ab1$;

ii) $b=-1$, $a\geq 2$, and $d(1,\beta)=.(a-1)(a-1)01$;

iii) $b=a+1$, and $d(1,\beta)=.(a+1)00a1$;

iv) $-a+1\leq b\leq -2$, and $d(1,\beta)=.(a-1)(a+b-1)(a+b)^{\infty}$,

where $(a+b)^{\infty}$ is the periodic expansion $(a+b)(a+b)(a+b)\cdots$.
Denote by $\mathrm{Fin}(\beta)$ the set of nonnegative real numbers that have a finite β-representation. A Pisot number β has the Finiteness Property (or Property (F)) if $\mathbb{Z}[\beta]\cap[0,+\infty[\subset \mathrm{Fin}(\beta)$. Notice that the Pisot numbers in the sets **i)**, **ii)** and **iii)** have the Property (F), while the Pisot numbers in **iv)** have not. The authors in [7] studied the sequence of best approximations for the classes **i)**, **ii)**, and **iii)**. In this paper, we study the best approximations for the class **iv)**.
Let us suppose that β is a cubic Pisot unit which does not satisfy the Property (F) and let us denote by α and λ its Galois conjugates. Let $P_{a,b}(x)=x^3-ax^2-bx-1$ be the minimal polynomial of β. Let $(q_n)_{n\geq 0}$ be the Tribonacci sequence defined in the Theorem 1.1. Using the greedy algorithm we have the following.

Proposition 2.1. *Every nonnegative integer n can be uniquely expressed as $n=\sum_{i=0}^{N}d_iq_i$, where $d_i\in\{0,\ldots,a-1\}$ and $d_jd_{j-1}d_{j-2}\cdots d_{j-k}\preceq_{lex}(a-1)(a+b-1)\underbrace{(a+b)\cdots(a+b)}_{k-1\ times}$, for all $j\geq k\geq 0$, where "$\preceq_{lex}$" is the lexicographical order.*

Let $\mathfrak{D}=\{(d_i)_{i\geq k}, k\in\mathbb{Z}, \forall n\geq k, d_nd_{n-1}d_{n-2}\cdots d_{n-k}\preceq_{lex}(a-1)(a+b-1)\underbrace{(a+b)\cdots(a+b)}_{k-1\text{ times}}\}$.

By definition, the Rauzy fractal is the set

$$\mathcal{R}:=\mathcal{R}_{a,b}=\left\{\sum_{i=2}^{+\infty}d_i\theta_i,\ (d_n)_{n\in\mathbb{Z}}\in\mathfrak{D}\right\}$$

where $\theta_i=\alpha^i$, if $\alpha\in\mathbb{C}\setminus\mathbb{R}$ or $\theta_i=(\alpha^i,\lambda^i)$, if $\alpha\in\mathbb{R}$. Notice that $\mathcal{R}\subset\mathbb{C}$ or $\mathcal{R}\subset\mathbb{R}^2$, and the set $\mathcal{R}$ is compact.

Rauzy fractals for the cases where $a=1, b=1$ (called Classic Rauzy Fractal) and $a=3, b=-2$ are shown in Fig.1 **a)** and Fig.1 **b)** respectively. The set $\mathcal{R}_{3,-2}$ is related to Special Pisot numbers, i.e., a Pisot number β such that $\beta/(\beta-1)$ is also a Pisot number (see [19]).
Now we shall give an alternative definition of the Rauzy fractal. Let $N\in\mathbb{Z}^+$ and $(d_j)_{k(N)\geq j\geq 0}$ be a q-representation of N, i.e., $N=\sum_{i=0}^{k(N)}d_iq_i$, where $(d_i)_{i\geq 0}\in\mathfrak{D}$. Let

$$\delta(N)=N(1/\beta,1/\beta^2)-(P_N,Q_N),$$

where

$$P_N=\sum_{j=1}^{k(N)}d_jq_{j-1},\ Q_N=\sum_{j=2}^{k(N)}d_jq_{j-2}.$$

Figure 1: **a)** The set $\mathcal{R}_{1,1}$. **b)** The set $\mathcal{R}_{3,-2}$.

Let us consider the matrix B defined as

$$B = \begin{pmatrix} -b/\beta & -1/\beta \\ 1 - b/\beta^2 & -1/\beta^2 \end{pmatrix}.$$

We have the following property.

Lemma 2.2. *For all $n \geq 2$,*

$$B \begin{pmatrix} q_n/\beta - q_{n-1} \\ q_n/\beta^2 - q_{n-2} \end{pmatrix} = \begin{pmatrix} q_{n+1}/\beta - q_n \\ q_{n+1}/\beta^2 - q_{n-1} \end{pmatrix}$$

where $(q_n)_{n\geq 0}$ is the sequence defined previously.

Proof. The proof is not difficult and it is made by induction. $\square$

Corollary 2.3. *If $N = \sum_{j=0}^{k(N)} d_j q_j$ then $\delta(N) = \sum_{j=0}^{k(N)} d_j B^j \delta(1)$, where $\delta(1) = (1/\beta, 1/\beta^2)$.*

Thus, the Rauzy fractal is the set

$$\mathcal{E} = \overline{\{\delta(N);\, N \in \mathbb{Z}^+\}} \subset \mathbb{R}^2.$$

This set was introduced by G. Rauzy in 1982 [15]. Since then, this set and its generalizations have been extensively studied due to its strong connections with many fields of mathematics such as Dynamical Systems and Number Theory [3, 17, 10, 11, **?**, 18, 9].

As mentioned before, our results highly depends on the topological properties of the class of Rauzy fractals associated with Pisot numbers not having the Property (F). The topological and arithmetical properties of this class of Rauzy fractals were studied in details in [14] (see also [13]). For instance, it was shown that these fractals tile the plane, and there exists an explicit finite state automaton that recognizes the boundaries of these sets, which allows to establish the number of neighbors of $\mathcal{R}_{a,b}$ in the tiling they generate. In particular, it was proved that $\mathcal{R}_{a,b}$ has at least $6+2(K-1)$ neighbors, where $K = \left[\frac{a-1}{a+b+1}\right]$. Let us remind that an element u of a lattice Λ is a neighbor of $\mathcal{R}$ if, and only if, $\mathcal{R} \cap \mathcal{R} + u \neq \emptyset$. We will assume that we are in the case of 6 neighbors. In this case, $u \in \{\pm\alpha, \pm(1+(b+1)\alpha), \pm(1+b\alpha)\}$. Fig.2 **a)** and Fig.2 **b)** illustrate this.

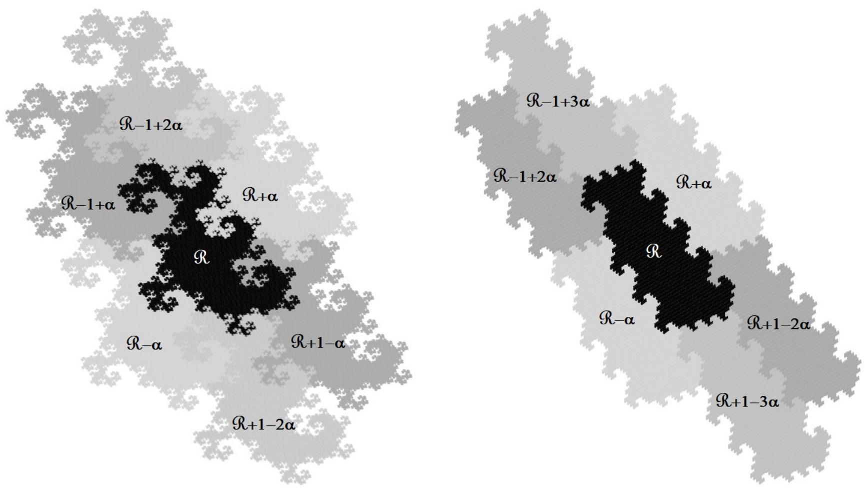

Figure 2: **a)** Tiling the plane by $\mathcal{R}_{3,-2}$. **b)** Tiling the plane by $\mathcal{R}_{5,-3}$.

We will introduce the Rauzy Norm. Let $M = \begin{pmatrix} \lambda + b/\beta & 1/\beta \\ -\alpha - b/\beta & -1/\beta \end{pmatrix}$. One can check that the matrix B defined before is similar to the matrix $\begin{pmatrix} \alpha & 0 \\ 0 & \lambda \end{pmatrix}$ and it satisfies

$$MB = \begin{pmatrix} \alpha & 0 \\ 0 & \lambda \end{pmatrix} M.$$

We will consider the case when β has complex conjugates. In this case, the Rauzy Norm $\|\cdot\|$ is defined by:

$$\|x\| = |(\alpha + b/\beta)x_1 + x_2/\beta|, \forall x = (x_1, x_2) \in \mathbb{R}^2.$$

Remark 2.4. *Notice that $\|x\| = |\pi_1 M x|$, where $\pi_1(x,y) = x$, for all $(x,y) \in \mathbb{R}^2$, and it is not difficult to verify that $\|Bx\| = |\alpha| \, \|x\|$, for all $x \in \mathbb{R}^2$.*

Corollary 2.5. *Let $(d_n)_{n\geq 0}$ be a $q-$representation. Then*

$$\left\| \sum_{n=0}^{+\infty} d_n B^n \delta(1) \right\| = \|\delta(1)\| \, |\sum_{n=0}^{+\infty} d_n \alpha^n|.$$

In particular,

$$\|\delta(q_n)\| = \|\delta(1)\| \, |\alpha^n|, \textit{ for all } n \geq 0.$$

Remark 2.6. *All omitted proofs can be found in [13].*

3 Sequence of best approximations

To prove Theorem 1.1 it will be necessary several auxiliary results. Let us begin with

Theorem 3.1. *There exists a real number $c > 0$ such that for all $q \in \mathbb{Z}^+$ and for all $g \in \mathbb{Z}^2$, if $\|q(1/\beta, 1/\beta^2) - g\| < c$ then $q(1/\beta, 1/\beta^2) - g = \delta(q)$ or $q(1/\beta, 1/\beta^2) - g = \delta(q) - (1,1)$.*

As a corollary we obtain,

Corollary 3.2. *There exists a real number $c > 0$ such that for all $q \in \mathbb{Z}^+$, if $\|q(1/\beta, 1/\beta^2)\|_0 < c$ then $\|q(1/\beta, 1/\beta^2)\|_0 = \|\delta(q)\|$.*

The proof of the following proposition can be adapted from that one found in [7].

Proposition 3.3. *There exists a linear and bijective application from $\mathbb{R}^2$ to $\mathbb{C}$ such that $f(\mathcal{E}) = \mathcal{R}$ and $f(\mathbb{Z}^2) = \mathbb{Z} + \mathbb{Z}\alpha$.*

The next lemma is of crucial importance for the statements that are to follow. The proof is in the Appendix.

Lemma 3.4. $\mathcal{R} \cap (\mathbb{Z} + \mathbb{Z}\alpha) = \{0, -1 - (b+1)\alpha\}$.

Remark 3.5. *The proof of Lemma 3.4 relies on the assumption that we are in the case of 6 neighbors. One could question whether the statement remains true in the case of more neighbors. We conjecture that, assuming that the origin is shared by exactly two tiles, Lemma 3.4 is still valid.*

The next statement is a special case of a theorem of Akiyama[2]:

Proposition 3.6. *$0 \in int(\mathcal{R} \cup (\mathcal{R} + 1 + (b+1)\alpha))$.*

Proof of Theorem 3.1. We have that $\mathcal{R} \cap \mathbb{Z} + \mathbb{Z}\alpha = \{0, -1-(b+1)\alpha)\}$. Hence, $\mathcal{E} \cap \mathbb{Z}^2 = (f^{-1}(0), f^{-1}(-1-(b+1)\alpha)) = ((0,0),(1,1))$. Then there exists a real number $c > 0$ such that for all $g \in \mathbb{Z}^2$, $\inf_{x \in \mathcal{E}} \|g - x\| < c$ implies that $g = (0,0)$ or $g = (1,1)$. Let us suppose that $\|q(1/\beta, 1\beta^2) - g\| < c$. Since $\delta(q) - q(1/\beta, 1/\beta^2) \in \mathbb{Z}^2$ and $\delta(q) \in \mathcal{E}$, then $\|\delta(q) - (\delta(q) - q(1/\beta, 1/\beta^2) - g\| < c$. Therefore $\delta(q) - q(1/\beta, 1/\beta^2) + g = (0,0)$ or $\delta(q) - q(1/\beta, 1/\beta^2) + g = (1,1)$, that is, $\delta(q) = q(1/\beta, 1/\beta^2) - g$ or $q(1/\beta, 1/\beta^2) - g = \delta(q) - (1,1)$. $\square$

Proof of Theorem 1.1. Let us prove that there exists $n_0 \in \mathbb{N}$ such that $(q_n)_{n \geq n_0}$ is the sequence of best approximation for the vector $(1/\beta, 1/\beta^2)$ for the norm $\|\cdot\|$.
Let c be a real number as defined in Theorem 3.1. Let $n_0 \in \mathbb{N}$ such that $\|\delta(q_{n_0})\| = \kappa|\alpha^{n_0}| < c$, where $\kappa = \|\delta(1)\|$. Take $n \geq n_0$ and $0 < q < q_n$. Let us prove that $\|q_n(1/\beta, 1/\beta^2)\|_0 < \|q(1/\beta, 1/\beta^2)\|_0$. Suppose that

$$\|q(1/\beta,1/\beta^2)\|_0 \le \|q_n(1/\beta,1/\beta^2)\|_0 \le \|q_n(1/\beta,1/\beta^2)\| = \kappa|\alpha^n| \le \kappa|\alpha^{n_0}| < c.$$

Hence, $\|q(1/\beta,1/\beta^2)\|_0 < c$. Now, suppose that

$$\left\|q(1/\beta,1/\beta^2)\right\|_0 = \left\|q(1/\beta,1/\beta^2) - g\right\|.$$

Thus, by Theorem 3.1, we obtain that $q(1/\beta,1/\beta^2) - g = \delta(q)$ or $q(1/\beta,1/\beta^2) - g = \delta(q) - (1,1)$. Let us treat these cases separately.

Case 1. $q(1/\beta,1/\beta^2) - g = \delta(q)$. In this case,

$$\left\|q(1/\beta,1/\beta^2)\right\|_0 = \|\delta(q)\| < \|\delta(q_n)\|. \tag{1}$$

Since $q < q_n$, then $q = \sum_{i=0}^{n-1} d_i q_i$. Thus,

$$\|\delta(q)\| = \kappa\left|\sum_{i=0}^{n-1} d_i\alpha^i\right|. \tag{2}$$

From (1) and (2) we obtain that $|\sum_{i=0}^{n-1} d_i\alpha^i| < |\alpha^n|$.
On the other hand, $\sum_{i=0}^{n-1} d_i\beta^i \notin \mathbb{Q}$, otherwise we would have $\sum_{i=0}^{n-1} d_i\beta^i = r/s$, where $r,s \in \mathbb{Z}$. Hence, $N\beta^2 + p_N\beta + q_N = r/s$, that is, $sN\beta^2 + p_Ns\beta + q_Ns = r$. Thus, $sN = p_Ns = 0$ and $q_Ns = r$ and then $s = 0$. Absurd. Therefore, $\lambda = \sum_{i=0}^{n-1} d_i\beta^i$ is an algebraic number of degree 3, since $\mathbb{Q}(\lambda) \subset \mathbb{Q}(\beta)$. Thus $\sum_{i=0}^{n-1} d_i\alpha^i$, $\sum_{i=0}^{n-1} d_i\overline{\alpha}^i$ and $\sum_{i=0}^{n-1} d_i\beta^i$ are Galois conjugates, and hence $|\sum_{i=0}^{n-1} d_i\alpha^i \sum_{i=0}^{n-1} d_i\overline{\alpha}^i \sum_{i=0}^{n-1} d_i\beta^i| \in \mathbb{Z}$. Therefore,

$$\left|\sum_{i=0}^{n-1} d_i\alpha^i\right|^2 \ge \frac{1}{\sum_{i=0}^{n-1} d_i\beta^i} \ge \frac{1}{\beta^n} = |\alpha|^{2n},$$

which implies that

$$\left|\sum_{i=0}^{n-1} d_i\alpha^i\right| \ge |\alpha|^n.$$

Absurd.

Case 2. $q(1/\beta,1/\beta^2) - g = \delta(q) - (1,1)$. In this case, $\|q(1/\beta,1/\beta^2) - g\|_0 = \|q(1/\beta,1/\beta^2) - g\| = \|q(1/\beta,1/\beta^2) - (1,1)\|$. We have that $\|q(1/\beta,1/\beta^2) - g\| = \kappa|\sum_{k=2}^{n-1} d_k\alpha^k|$. Let us suppose that $\|q(1/\beta,1/\beta^2) - (1,1)\| = \kappa|\sum_{i=0}^{n-1} d_i\alpha^i + 1 + (b+1)\alpha|$.
Claim:

$$\|q(1/\beta,1/\beta^2) - (1,1)\| \ge \|\delta(q_n)\| = |\alpha^n|.$$

Indeed,

$$\left|1+(b+1)\alpha+\sum_{i=2}^{n-1}d_i\alpha^i\right|^2\cdot\left|1+(b+1)\beta+\sum_{i=2}^{n-1}d_i\beta^i\right|\geq 1.$$

Hence,

$$\left|1+(b+1)\alpha+\sum_{i=2}^{n-1}d_i\alpha^i\right|\geq\frac{1}{\sqrt{\left|1+(b+1)\beta+\sum_{i=2}^{n-1}d_i\beta^i\right|}}.$$

On the other hand, take $n\geq n_0$ large enough such that $1+(b+1)\beta+\sum_{i=2}^{n-1}d_i\beta^i>0$. Hence, $1+(b+1)\beta+\sum_{i=2}^{n-1}d_i\beta^i\leq\sum_{i=0}^{n-1}d_i\beta^i\leq\beta^n$, because $b+1\leq -1$. Thus,

$$\frac{1}{1+(b+1)\beta+\sum_{i=2}^{n-1}d_i\beta^i}\geq\frac{1}{\beta^n}=|\alpha|^{2n}.$$

Hence, $|1+(b+1)\alpha+\sum_{i=2}^{n-1}d_i\alpha^i|\geq|\alpha|^n$. Absurd.
Therefore, $\|q_n(1/\beta,1/\beta^2)\|_0<\|q(1/\beta,1/\beta^2)\|_0$ for all cases. □

4 Appendix

4.1 Construction of the automaton

This section is dedicated to prove

Lemma 3.4 $\mathcal{R}\cap(\mathbb{Z}+\mathbb{Z}\alpha)=\{0,-1-(b+1)\alpha\}$.

To prove that, we need to construct a finite automaton, denoted by $\mathcal{H}$, that recognizes points with two $\alpha-$representations, that is, a point that can be written in two different ways. These points belong to the boundary of $\mathcal{R}_{a,b}$. Let us begin with a definition.

Definition 4.1. *A finite automaton is a triple (S,A,C), where A is the alphabet of the automaton, S is the set of the states and C is a subset of $S\times A\times S$. We say that a sequence $(a_n)_{n\in\mathbb{N}}$ is recognizable by the automaton (S,A,C) if there exists a sequence $(s_n)\in A^{\mathbb{N}}$ such that $(s_{i-1},a_i,s_i)\in C$, for all $i\in\mathbb{N}$*

Before constructing the automaton we need the following proposition (see [14]):

Proposition 4.2. *Let $x=\sum_{i=l}^{\infty}\varepsilon_i\alpha^i$ and $y=\sum_{i=l}^{\infty}\varepsilon'_i\alpha^i$, where $l\in\mathbb{Z}$ and $(\varepsilon_i)_{i\geq l}$, $(\varepsilon'_i)_{i\geq l}$ belong to $\mathfrak{D}$. Then $x=y$ if, and only if, the set $\{x(k)-y(k),k\geq l\}$ is finite, where $x(k)=\alpha^{-k+2}\sum_{i=l}^{k}\varepsilon_i\alpha^i$ and $y(k)=\alpha^{-k+2}\sum_{i=l}^{k}\varepsilon'_i\alpha^i$, $\forall k\geq l$.*

As a consequence of this proposition, we have the following result.

Theorem 4.3. *Let $(\varepsilon_i)_{i\geq l}$ and $(\varepsilon'_i)_{i\geq l}$ two distinct elements of $\mathfrak{D}$, then $\sum_{i=l}^{\infty}\varepsilon_i\alpha^i = \sum_{i=l}^{\infty}\varepsilon'_i\alpha^i$ if and only if the sequence $((\varepsilon_i,\varepsilon'_i))_{i\geq l}$ is recognizable by the automaton $\mathcal{H}$.*

Now, let us construct the automaton $\mathcal{H}$ such that $n,p\in\mathbb{Z}$ and $(\varepsilon_i)_{i\geq 2},(\varepsilon'_i)_{i\geq 2}\in\mathfrak{D}$, $n+p\alpha+\sum_{i=2}^{+\infty}\varepsilon_i\alpha^i = \sum_{i=2}^{+\infty}\varepsilon'_i\alpha^i$ if, and only if, $(n,0)(p,0)(\varepsilon_2,\varepsilon'_2)\cdots$ is an infinite path in the automaton $\mathcal{H}$.
The idea is (see [14], or [18] for a comprehensive explanation): Let p and q be two states. The set of edges is the set of $(p,(c,d),q)\in S\times\{0,1,...,a-1\}^2\times S$ satisfying $q=\frac{p}{\alpha}+(c-d)\alpha^2$ and the initial state is $\{0\}$. We suppose that we are in the case where $\mathcal{R}_{a,b}$ has 6 neighbors. In this case, it was shown also in [14], Proposition 4.1, that the set of states is $S=\{0,\pm\alpha^2,\pm(\alpha+b\alpha^2),\pm(\alpha+(b+1)\alpha^2),\pm(1+b\alpha+(a-1)\alpha^2),\pm(1+(b+1)\alpha+(a+b)\alpha^2),\pm(1+(b+1)\alpha+(a+b+1)\alpha^2)\}$.
Let $x=\sum_{i=l}^{+\infty}\varepsilon_i\alpha^i$ and $y=\sum_{i=l}^{+\infty}\varepsilon'_i\alpha^i$, where $\varepsilon=(\varepsilon_i)_{i\geq l}$ and $\varepsilon'=(\varepsilon'_i)_{i\geq l}$ belong to $\mathfrak{D}$. Suppose that $x=y$ and for all $k\geq l$ we set $S_k=S_k(\varepsilon,\varepsilon')=x(k)-y(k)$. We have,

$$S_{k+1}=\frac{S_k}{\alpha}+(\varepsilon_{k+1}-\varepsilon_{k+1})\alpha^2. \tag{3}$$

Let t be the smallest integer such that $\varepsilon_t\neq\varepsilon'_t$. Hence $S_i(\varepsilon,\varepsilon')=0$ for all $i\in\{l,...,t-1\}$. Suppose that $(\varepsilon_t,\varepsilon'_t)=(1,0)$. Then, $S_t=\alpha^2$. From (3) we deduce that
$S_{t+1}=\alpha+(\varepsilon_{t+1}-\varepsilon'_{t+1})\alpha^2=\begin{cases}\alpha+b\alpha^2, & \text{if } (\varepsilon_{t+1},\varepsilon_{t+1})=(b,0)\\ \alpha+(b+1)\alpha^2, & \text{if}(\varepsilon_{t+1},\varepsilon'_{t+1})=(b+1,0)\end{cases}$

Then, $(\alpha^2,(\varepsilon+b,\varepsilon),\alpha+b\alpha^2)$ is an edge connecting the state α^2 to state $\alpha+b\alpha^2$, and $(\alpha^2,(\varepsilon+b+1,\varepsilon),\alpha+(b+1)\alpha^2)$ is and edge connecting the state α^2 to state $\alpha+(b+1)\alpha^2$. Continuing with this process, we obtain an infinite path $(S_i,(\varepsilon_i,\varepsilon'_i),S_{i+1})_{i\geq l}$ beginning in the initial state of $\mathcal{H}$. This path will be denoted by $(\varepsilon_i,\varepsilon'_i)_{i\geq l}$. The set of states S is finite, then we obtain a finite automaton (Fig.3).

Proof of Lemma 3.4: Let us notice that 0 and $-1-b\alpha$ belongs to $\mathcal{R}$. Suppose that there exists $n,p\in\mathbb{Z}$ such that $n+p\alpha=\sum_{i=2}^{\infty}\varepsilon_i\alpha^i$, where $(\varepsilon_i)_{i\geq 2}\in\mathfrak{D}$. Suppose that $n>0$. Then,

$$(n,0)(p,0)(0,\varepsilon_2)(0,\varepsilon_2)\cdots$$

is a path in the automaton $\mathcal{H}$. Absurd, because this sequence is not recognizable by $\mathcal{H}$. $\square$

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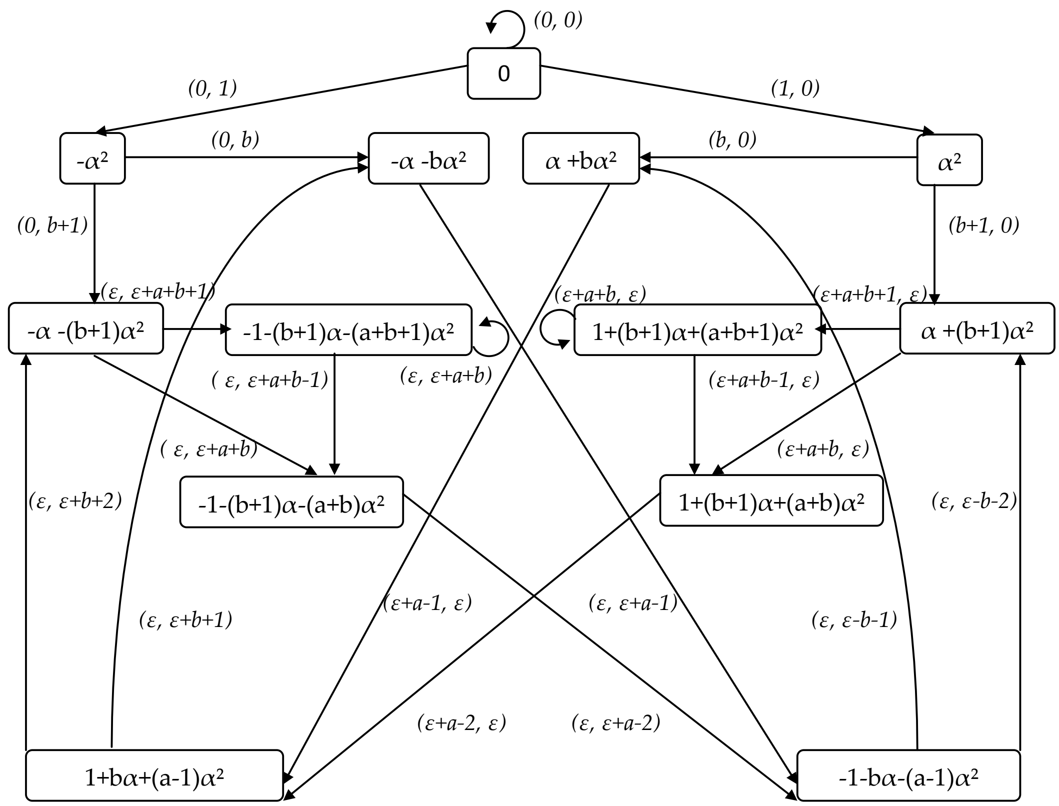

Figure 3: Automaton $\mathcal{H}$.

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