COMPLETE FLAGS ON FLAT VECTOR BUNDLES IN POSITIVE CHARACTERISTIC

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Abstract. Let X be a connected, smooth, and projective curve of genus g over an algebraically closed field of characteristic $p > 0$. This paper investigates a characteristic-p analogue of a well-known fact concerning flat vector bundles in characteristic zero. That is to say, we prove that the inequality $g \leq 1$ holds if and only if any flat vector bundle on X admits a complete flag. We also explore a generalization of this result in a broader setting using Berthelot's differencial operators of level m.

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1. Introduction

Let X be a connected, smooth, and projective curve of genus $g \geq 0$ over an algebraically closed field k. By a flat vector bundle on X, we mean a pair $(\mathcal{E}, \nabla)$, consisting of a vector bundle $\mathcal{E}$ on X and a (flat) connection ∇ on it. When the base field k is the field of complex

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numbers $\mathbb{C}$, a flat vector bundle can be regarded as a $\mathcal{D}$-module, where $\mathcal{D}$ is the sheaf of differential operators on X, and corresponds, via the most basic form of the Riemann-Hilbert correspondence, to a group representation

$$\rho_{(\mathcal{E},\nabla)} : \pi_1(X^{\mathrm{an}}) \to \mathrm{GL}_r(\mathbb{C})$$

uniquely given up to conjugation, where $r := \mathrm{rank}(\mathcal{E})$ and $\pi_1(X^{an})$ denotes the fundamental group of the associated Riemann surface X^{an} (with respect to a fixed base point). Flat vector bundles and their associated group representations have been extensively studied in various contexts, including non-Abelian Hodge theory and the geometric Langlands program.

Now, let us focus on a curve of genus $g = 1$; see [9], [15], [16], [25], and (a part of) [29] for studies on such objects. One of the fundamental facts about these bundles when $k = \mathbb{C}$ is the existence of a *complete flag* (i.e., a *splitting*, in the terminology of [2], [3]) preserved by a given connection. Specifically, for any flat vector bundle $(\mathcal{E}, \nabla)$ on the complex curve $X/\mathbb{C}$ with $g = 1$, there exists a sequence of subbundles

$$0 = \mathcal{E}^0 \subseteq \mathcal{E}^1 \subseteq \cdots \subseteq \mathcal{E}^r = \mathcal{E} \tag{1.1}$$

such that, for every $j = 1, \cdots, r$, the subquotient $\mathcal{E}^j/\mathcal{E}^{j-1}$ is a line bundle and $\mathcal{E}^j$ is closed under ∇. In particular, $(\mathcal{E}, \nabla)$ can be obtained as a successive extension of flat line bundles, which allows for a detailed understanding of the moduli space of flat vector bundles on X.

The proof of this fact is elementary. Indeed, since $\pi_1(X^{\mathrm{an}}) \cong \mathbb{Z}^2$, the image of the group representation $\rho_{(\mathcal{E},\nabla)}$ can be simultaneously upper triangularizable, which implies the desired result. The same statement holds for genus-0 curves, as their fundamental groups are trivial. On the other hand, when the underlying curve is hyperbolic, i.e., $g > 1$, there exist many irreducible group representations $\pi_1(X^{\mathrm{an}}) \to \mathrm{GL}_r(\mathbb{C})$, and the corresponding flat vector bundles do not admit complete flags as in (1.1). It follows that *the existence of such flags depends entirely on the genus of the underlying curve*, and a standard base-change argument shows that this fact generalizes to the case where k is an arbitrary algebraically closed field of characteristic zero.

Then, a natural question is whether a similar result holds in characteristic $p > 0$. To the best of our knowledge, the analogous result in positive characteristic has not been established, and the proof outlined above does not apply in this setting because it fundamentally relies on the analytic nature of the Riemann-Hilbert correspondence. Moreover, some flat vector bundles in characteristic p cannot be obtained by reducing those in characteristic zero modulo p. Therefore, to establish a positive characteristic analogue of the aforementioned result, we need to address phenomena that are specific to characteristic p.

Let us suppose that k has characteristic p. We should note that there are different variations of the sheaf of differential operators "$\mathcal{D}$" on X. On the one hand, the sheaf $\mathcal{D}^{(\infty)}$ of differential operators in the sense of A. Grothendieck was discussed in [21]. A $\mathcal{D}^{(\infty)}$-module is often referred to as a *stratified sheaf* and can be interpreted as an $\mathcal{O}_X$-module admitting infinite Frobenius descent. On the other hand, the sheaves $\mathcal{D}^{(m)}$ of differential operators of *level* $m \in \mathbb{Z}_{\geq 0}$ were introduced by P. Berthelot (cf. [4], [5]) as essential ingredients in defining arithmetic $\mathcal{D}$-modules. A $\mathcal{D}^{(0)}$-module structure amounts essentially to a choice of a flat connection in the usual sense.

As described below, our main result provides a necessary and sufficient condition for the existence of a complete flag on a $\mathcal{D}^{(m)}$-bundle (i.e., a vector bundle equipped with a $\mathcal{D}^{(m)}$-action).

Theorem A (cf. Theorems 3.2, 4.6, and 5.2)**.** *Let m be an element of $\mathbb{Z}_{\geq 0} \sqcup \{\infty\}$ and r a positive integer. Then, the following two conditions (a) and (b) are equivalent:*

(a) *The inequality $g \leq 1$ holds;*

(b) *For any given rank r, any $\mathcal{D}^{(m)}$-bundle $(\mathcal{E}, \nabla)$ of rank r admits a complete flag, i.e., there exists a sequence of subbundles*

$$0 = \mathcal{E}^0 \subset \mathcal{E}^1 \subseteq \cdots \subseteq \mathcal{E}^r = \mathcal{E}$$

of $\mathcal{E}$ such that, for every $j = 1, \cdots, r$, the subquotient $\mathcal{E}^j/\mathcal{E}^{j-1}$ is a line bundle and $\mathcal{E}^j$ is preserved by ∇.

Remark 1.1. As pointed out by an anonymous referee, it is highly expected that a more general reduction theory holds for principal G-bundles, where G is a geometrically connected, reductive algebraic group over an algebraically closed (or perfect) field such that G is of low height with respect to the characteristic $p > 0$. While we focus on vector bundles (i.e., GL_r-bundles) in this paper, developing the precise analogue for such principal G-bundles is an important direction for future research.

2. Preliminaries

In this subsection, we recall some fundamental aspects of connections and $\mathcal{D}$-modules in positive characteristic. For previous studies on these objects, we refer the reader to, e.g., [4], [5], [24], and [29].

2.1. **Notation and Conventions.** Throughout this paper, we fix a prime number p and an algebraically closed field k of characteristic p. We also fix a connected, smooth, and projective curve X over k of genus $g \in \mathbb{Z}_{\geq 0}$. Denote by Ω_X (resp., $\mathcal{T}_X$) the cotangent bundle (resp., the tangent bundle) of X/k and by d the universal derivation $\mathcal{O}_X \to \Omega_X$.

Denote by F_X the absolute Frobenius endomorphism of X. For a positive integer m, the **m-th Frobenius twist** of X over k is, by definition, the base-change $X^{(m)}$ of X along the m-th iterate of the absolute Frobenius automorphism of $\mathrm{Spec}(k)$. The morphism $F^{(m)}_{X/k} : X \to X^{(m)}$ induced naturally from F^m_X is called the **m-th relative Frobenius morphism** of X over k. For convenience, we write $X^{(0)} := X$, $F^{(0)}_{X/k} := \mathrm{id}_X$, and $F_{X/k} := F^{(1)}_{X/k}$.

Let $\mathcal{E}$ be a vector bundle (i.e., a locally free coherent sheaf) on X. We shall say that $\mathcal{E}$ is **decomposable** if there exist vector bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ of positive ranks such that $\mathcal{E} \cong \mathcal{E}_1 \oplus \mathcal{E}_2$. A vector bundle is called **indecomposable** if it is not decomposable.

According to [1, Theorem 2], any such vector bundle $\mathcal{E}$ is isomorphic to a direct sum of indecomposable vector bundles, unique up to reordering. We refer to a vector bundle $\mathcal{E}'$ as an **indecomposable component** of $\mathcal{E}$ if it appears as the direct summand in such a decomposition (cf. [10, Definition 2.3]).

2.2. **Connections on vector bundles.** A **(k-)connection** on an $\mathcal{O}_X$-module $\mathcal{E}$ is defined as a k-linear morphism $\nabla : \mathcal{E} \to \Omega_X \otimes \mathcal{E}$ satisfying the Leibniz rule, i.e, $\nabla(a \cdot v) = a \cdot \nabla(v) + da \otimes v$ for any local sections $a \in \mathcal{O}_X$, $v \in \mathcal{E}$. A **flat vector bundle** X is a pair $(\mathcal{E}, \nabla)$ consisting of a vector bundle $\mathcal{E}$ on X and a k-connection ∇ on $\mathcal{E}$. When $\mathcal{E}$ has rank one, such a pair is called a **flat line bundle**.

If we are given two flat vector bundles $(\mathcal{E}, \nabla)$, $(\mathcal{E}', \nabla')$ on X, then the tensor product $\mathcal{E} \otimes \mathcal{E}'$ $(:= \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E})$ admits a connection

$$\nabla \otimes \nabla' : \mathcal{E} \otimes \mathcal{E}' \to \Omega_X \otimes (\mathcal{E} \otimes \mathcal{E}') \tag{2.1}$$

defined by $(\nabla \otimes \nabla')(a \otimes b) = \nabla(a) \otimes b + a \otimes \nabla'(b)$ for any local sections $a \in \mathcal{E}$, $a' \in \mathcal{E}'$. Also, given a flat vector bundle $(\mathcal{E}, \nabla)$, one can define a connection $\nabla^\vee$ on the dual bundle $\mathcal{E}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ in a natural manner (cf. [24, § 1.1]).

2.3. **$\mathcal{D}^{(m)}$-modules.** For each $m \in \mathbb{Z}_{\geq 0}$, let $\mathcal{D}_X^{(m)}$ denote the sheaf of differential operators of level m on X/k (cf. [4, § 2.2]), where $\mathrm{Spec}(k)$ is equipped with the trivial m-PD structure. It defines a sheaf of (noncommutative) k-algebras and acts naturally on $\mathcal{O}_X$. When there is no fear of confusion, we will write $\mathcal{D}^{(m)} := \mathcal{D}_X^{(m)}$ for simplicity. There exists a natural inductive system of sheaves

$$\mathcal{D}^{(0)} \to \mathcal{D}^{(1)} \to \cdots \to \mathcal{D}^{(m)} \to \cdots$$

(cf. [4, § 2,2]). The limit of this system $\mathcal{D}^{(\infty)} := \varinjlim_{m \in \mathbb{Z}_{\geq 0}} \mathcal{D}^{(m)}$ is called the sheaf of differential operators of *level* ∞ and coincides with the sheaf discussed in [21, § 16.8]. (Note that if k has characteristic zero, then $\mathcal{D}_X^{(1)} = \mathcal{D}_X^{(2)} = \cdots = \mathcal{D}_X$.)

For $m, j \in \mathbb{Z}_{\geq 0} \sqcup \{\infty\}$, we write $\mathcal{D}_{\leq j}^{(m)}$ for the subsheaf of $\mathcal{D}^{(m)}$ consisting of differential operators of order $\leq j$, with the convention that $\mathcal{D}_{\leq \infty}^{(m)} := \mathcal{D}^{(m)}$. Thus, we have $\mathcal{D}^{(m)} = \bigcup_{j \in \mathbb{Z}_{\geq 0}} \mathcal{D}_{\leq j}^{(m)}$. We denote by ${}^L\mathcal{D}_{\leq j}^{(m)}$ (resp., ${}^R\mathcal{D}_{\leq j}^{(m)}$) the sheaf $\mathcal{D}_{\leq j}^{(m)}$ endowed with a structure of $\mathcal{O}_X$-module induced by left (resp., right) multiplication. Given an $\mathcal{O}_X$-module $\mathcal{E}$, we equip the tensor product $\mathcal{D}_{\leq}^{(m)} \otimes \mathcal{E} := {}^R\mathcal{D}_{\leq j}^{(m)} \otimes \mathcal{E}$ with the $\mathcal{O}_X$-module structure given by left multiplication.

Let $\mathcal{E}nd_k(\mathcal{E})$ denote the sheaf of locally defined k-linear endomorphisms of $\mathcal{E}$ endowed with a structure of $\mathcal{O}_X$-module given by left multiplication. Given an $\mathcal{O}_X$-module $\mathcal{E}$, we equip the tensor product $\mathcal{D}_{\leq j}^{(m)} \otimes \mathcal{E} := {}^R\mathcal{D}_{\leq j}^{(m)} \otimes \mathcal{E}$ with the $\mathcal{O}_X$-module structure given by left multiplication. A **$\mathcal{D}^{(m)}$-module structure** on $\mathcal{E}$ is an $\mathcal{O}_X$-linear morphism of sheaves of k-algebras $\nabla : {}^L\mathcal{D}^{(m)} \to \mathcal{E}nd_k(\mathcal{E})$. Equivalently, specifying a $\mathcal{D}^{(m)}$-module structure on $\mathcal{E}$ is a homomorphism ∇ extending the $\mathcal{O}_X$-module structure to a left $\mathcal{D}^{(m)}$-module structure. By a **$\mathcal{D}^{(m)}$-bundle**, we mean a pair $(\mathcal{E}, \nabla)$ consisting of a vector bundle $\mathcal{E}$ on X and a $\mathcal{D}^{(m)}$-module structure ∇ on it. A **morphism of $\mathcal{D}^{(m)}$-bundles** from $(\mathcal{E}, \nabla)$ to $(\mathcal{E}', \nabla')$ is defined as an $\mathcal{O}_X$-linear morphism $f : \mathcal{E} \to \mathcal{E}'$ that is compatible with the respective $\mathcal{D}^{(m)}$-module structures.

A **line $\mathcal{D}^{(m)}$-bundle** is a $\mathcal{D}^{(m)}$-bundle $(\mathcal{E}, \nabla)$ such that $\mathcal{E}$ has rank 1. For example, if

$$\nabla_{\mathrm{triv}}^{(m)} : {}^L\mathcal{D}^{(m)} \to \mathcal{E}nd_k(\mathcal{O}_X)$$

denotes the natural $\mathcal{D}^{(m)}$-action on $\mathcal{O}_X$ mentioned above, then $(\mathcal{O}_X, \nabla_{\mathrm{triv}}^{(m)})$ specifies a line $\mathcal{D}^{(m)}$-bundle.

Let $(\mathcal{E}, \nabla)$ be a $\mathcal{D}^{(m)}$-bundle. A **(line) $\mathcal{D}^{(m)}$-subbundle** of $(\mathcal{E}, \nabla)$ is a (line) subbundle of $\mathcal{E}$ that is stable under the $\mathcal{D}^{(m)}$-action ∇. Each $\mathcal{D}^{(m)}$-subbundle $\mathcal{E}'$ of $\mathcal{E}$ naturally inherits a $\mathcal{D}^{(m)}$-bundle structure, given by restricting ∇.

Finally, it is verified (cf. [6, Theorem 4.8]) that giving a $\mathcal{D}^{(0)}$-module structure on a fixed $\mathcal{O}_X$-module $\mathcal{E}$ amounts to giving a connection on $\mathcal{E}$.

Remark 2.1. Denote by $\mathrm{Str}(X)$ the abelian k-linear rigid tensor category consisting of $\mathcal{D}^{(\infty)}$-bundles. According to [17, Theorem 1.3], $\mathrm{Str}(X)$ is equivalent to the category of F-divided sheaves on X. Here, recall (cf. [13]) that an **F-divided sheaf** on X is a collection

$$\{(\mathcal{E}_m, \alpha_m)\}_{m \in \mathbb{Z}_{\geq 0}},$$

where each pair $(\mathcal{E}_m, \alpha_m)$ consists of a vector bundle $\mathcal{E}_m$ on $X^{(m)}$ and an $\mathcal{O}_{X^{(m)}}$-linear isomorphism $\alpha_m : F^{(1)*}_{X^{(m)}/k}(\mathcal{E}_{m+1}) \xrightarrow{\sim} \mathcal{E}_m$.

Once we choose a k-rational point $x : \mathrm{Spec}(k) \to X$, the assignment $(\mathcal{E}, \nabla) \mapsto x^*(\mathcal{E})$ defines a tensor functor $\omega_x : \mathrm{Str}(X) \to \mathrm{Vec}_k$, where Vec_k denotes the category of finite-dimensional k-vector spaces. The category $\mathrm{Str}(X)$ together with the functor ω_x forms a neutral Tannakian category (cf. [13, § 2.2]). In particular, there exists a pro-algebraic group k-scheme $\pi_1^{\mathrm{str}}(X, x)$ (or simply $\pi_1^{\mathrm{str}}(X)$) such that ω_x induces an equivalence

$$\mathrm{Str}(X) \cong \mathrm{Rep}_k(\pi_1^{\mathrm{str}}(X, x))$$

between $\mathrm{Str}(X)$ and the category $\mathrm{Rep}_k(\pi_1^{\mathrm{str}}(X, x))$ of finite-dimensional k-representations of $\pi_1^{\mathrm{str}}(X, x)$. The pro-algebraic group $\pi_1^{\mathrm{str}}(X, x)$ will be referred to as the **stratified fundamental group** of X (with respect to the base point x).

2.4. **Frobenius pull-back of $\mathcal{D}^{(m)}$-modules.** Suppose that $m \in \mathbb{Z}_{\geq 0}$, and let $(\mathcal{E}, \nabla)$ be a $\mathcal{D}^{(m)}$-bundle. Note that the natural short exact sequence $0 \to \mathcal{D}^{(m)}_{\leq p^{m+1}-1} \to \mathcal{D}^{(m)}_{\leq p^{m+1}} \to \mathcal{T}_X^{\otimes p^{m+1}} \to 0$ admits a canonical split injection $\mathcal{T}_X^{\otimes p^{m+1}} \hookrightarrow \mathcal{D}^{(m)}_{\leq p^{m+1}}$ (cf. [30, § 2.5]). The composite

$$F^{(m+1)*}_{X/k}(\mathcal{T}_{X^{(m+1)}}) \left(= \mathcal{T}_X^{\otimes p^{m+1}}\right) \to {}^L\mathcal{D}^{(m)} \xrightarrow{\nabla} \mathcal{E}nd_k(\mathcal{E}),$$

of this injection with ∇ factors through the inclusion $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}) \hookrightarrow \mathcal{E}nd_k(\mathcal{E})$. The resulting morphism

$$\psi^{m+1}(\nabla) : F^{(m+1)*}_{X/k}(\mathcal{T}_{X^{(m+1)}}) \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$$

is called the **p^{m+1}-curvature** of ∇ (or, of $(\mathcal{E}, \nabla)$).

This description of p^{m+1}-curvature agrees with the definition discussed [19]; see also [27, Definition 3.1.1] for a slightly different formulation of a higher-level generalization of p-curvature. When $m = 0$, this definition coincides with the classical definition of p-curvature (cf. [24, § 5]). That is, the p-curvature $\psi^1(\nabla) : \left(F^*_{X/k}(\mathcal{T}_{X^{(1)}}) =\right) \mathcal{T}_X^{\otimes p} \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{F})$ of a connection ∇ is determined by $\partial^{\otimes p} \mapsto (\nabla_\partial)^p - \nabla_{\partial^p}$ for any local section $\partial \in \mathcal{T}_X$, where ∂^p denotes the section of $\mathcal{T}_X$ corresponding to the p-th iterate $\partial \circ \cdots \circ \partial$ of ∂, and write $\nabla_{(-)} := ((-) \otimes \mathrm{id}_{\mathcal{E}}) \circ \nabla$.

Denote by $\mathcal{S}ol(\nabla)$ the subsheaf of $\mathcal{E}$ on which $\mathcal{D}^{(m)}_+$ acts by zero via ∇, where $\mathcal{D}^{(m)}_+$ denotes the kernel of the canonical projection $\mathcal{D}^{(m)} \twoheadrightarrow \mathcal{O}_X$. This sheaf can be regarded as an $\mathcal{O}_{X^{(m+1)}}$-submodule of $F^{(m+1)}_{X/k*}(\mathcal{E})$ via the underlying homeomorphism of $F^{(m+1)}_{X/k}$.

For a vector bundle $\mathcal{G}$ on $X^{(m+1)}$ (or more generally, an $\mathcal{O}_{X^{(m+1)}}$-module), there exists a canonical $\mathcal{D}^{(m)}$-module structure

$$\nabla^{(m)}_{\mathcal{G},\mathrm{can}} : {}^L\mathcal{D}^{(m)} \to \mathcal{E}nd_k(F^{(m+1)*}_{X/k}(\mathcal{G})) \tag{2.2}$$

on the pull-back $F_{X/k}^{(m+1)*}(\mathcal{G})$ of $\mathcal{G}$ along $F_{X/k}^{(m+1)}$. The resulting assignments $\mathcal{G} \mapsto (F_{X/k}^{(m+1)*}(\mathcal{G}), \nabla_{\mathcal{G},\mathrm{can}}^{(m)})$ and $(\mathcal{E}, \nabla) \mapsto \mathcal{S}ol(\nabla)$ together define an equivalence of categories

$$\begin{pmatrix} \text{the category of} \\ \text{vector bundles on } X^{(m+1)} \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} \text{the category of } \mathcal{D}^{(m)}\text{-bundles} \\ \text{with vanishing } p^{m+1}\text{-curvature} \end{pmatrix}, \tag{2.3}$$

which maps each subbundle of a given vector bundle on $X^{(m+1)}$ to a $\mathcal{D}^{(m)}$-subbundle of the corresponding $\mathcal{D}^{(m)}$-bundle (cf. [27, Corollary 3.2.4]).

Moreover, if s is another nonnegative integer, then the pull-back by $F_{X/k}^{(s)}$ induces an equivalence of categories

$$\begin{pmatrix} \text{the category of} \\ \mathcal{D}_{X^{(s)}}^{(m)}\text{-bundles} \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} \text{the category of} \\ \mathcal{D}_{X}^{(m+s)}\text{-bundles} \end{pmatrix} \tag{2.4}$$

(cf. [5, Théorème 2.3.6]), which preserves the formation of $\mathcal{D}$-subbundles in the above sense. If $(\mathcal{E}, \nabla)$ is a $\mathcal{D}_{X^{(s)}}^{(m)}$-bundle and $(F_{X/k}^{(s)*}(\mathcal{E}), F_{X/k}^{(s)*}(\nabla))$ denotes its image under the above equivalence, then it follows from [5, Proposition 2.2.4] and (the proof of) [19, Proposition 3.6] that the pull-back of $\psi^{m+1}(\nabla)$ via $F_{X/k}^{(s)}$ coincides with $\psi^{m+s+1}(F_{X/k}^{(s)*}(\nabla))$ via the natural inclusion $F_{X/k}^{(s)*}(\mathcal{E}nd_{\mathcal{O}_{X^{(s)}}}(\mathcal{E})) \subseteq \mathcal{E}nd_{\mathcal{O}_X}(F_{X/k}^{(s)*}(\mathcal{E}))$, i.e.,

$$F_{X/k}^{(s)*}(\psi^{m+1}(\nabla)) = \psi^{m+s+1}(F_{X/k}^{(s)*}(\nabla)).$$

In particular, (2.4) restricts to an equivalence between their subcategories

$$\begin{pmatrix} \text{the category of } \mathcal{D}_{X^{(s)}}^{(m)}\text{-bundles} \\ \text{with nilpotent (resp., vanishing)} \\ p^{m+1}\text{-curvature} \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} \text{the category of } \mathcal{D}_{X}^{(m+s)}\text{-bundles} \\ \text{with nilpotent (resp., vanishing)} \\ p^{m+s+1}\text{-curvature} \end{pmatrix}. \tag{2.5}$$

2.5. **Complete flags on $\mathcal{D}^{(m)}$-bundles.** Let m be an element of $\mathbb{Z}_{\geq 0} \sqcup \{\infty\}$ and $\mathcal{E}$ a vector bundle on X of rank $r \in \mathbb{Z}_{>0}$. Recall that a *complete flag* on $\mathcal{E}$ is defined as an r-step increasing filtration

$$0 = \mathcal{E}^0 \subseteq \mathcal{E}^1 \subseteq \cdots \subseteq \mathcal{E}^r = \mathcal{E}$$

such that all the subquotients $\mathcal{E}^j/\mathcal{E}^{j-1}$ are line bundles. The following definition introduces the $\mathcal{D}^{(m)}$-module version of this notion.

Definition 2.2. Suppose that $\mathcal{E}$ is equipped with a $\mathcal{D}^{(m)}$-module structure ∇. A **complete flag** on $(\mathcal{E}, \nabla)$ is a complete flag $\{\mathcal{E}^j\}_{j=0}^r$ on $\mathcal{E}$ such that each subbundle $\mathcal{E}^j$ $(j = 0, \cdots, r)$ is preserved by ∇.

The equivalence of categories (2.4) leads to the following result, which reduces the existence problem of complete flags for $m < \infty$ to the case $m = 0$.

Proposition 2.3. *Let m, s be nonnegative integers and $(\mathcal{G}, \nabla)$ a $\mathcal{D}_{X^{(s)}}^{(m)}$-bundle, and denote by $(F_{X/k}^{(s)*}(\mathcal{G}), F_{X/k}^{(s)*}(\nabla))$ the $\mathcal{D}_X^{(m+s)}$-bundle corresponding to $(\mathcal{G}, \nabla)$ via (2.4). Then, $(\mathcal{G}, \nabla)$ admits a complete flag if and only if $(F_{X/k}^{(s)*}(\mathcal{G}), F_{X/k}^{(s)*}(\nabla))$ admits a complete flag.*

Furthermore, the following assertion is well-known at least for $m = 0$ and follows immediately from the case $m = 0$ by applying Proposition 2.3.

Proposition 2.4. *Let m be a nonnegative integer and $(\mathcal{E}, \nabla)$ a $\mathcal{D}^{(m)}$-bundle such that $\psi^{m+1}(\nabla)$ is nilpotent. Then, $(\mathcal{E}, \nabla)$ admits a complete flag.*

Proof. By the equivalence of categories (2.5), the problem is reduced to the case $m = 0$. We prove the assertion for $m = 0$ by induction on the rank of $\mathcal{E}$. There is nothing to prove when $\mathcal{E}$ has rank 1. For the induction step, suppose that $\mathrm{rank}(\mathcal{E}) > 1$. It follows from [24, Corollary 5.5] that $(\mathcal{E}, \nabla)$ admits a nonzero flat subbundle $(\mathcal{F}, \nabla_{\mathcal{F}})$ with vanishing p-curvature. By the equivalence of categories (2.3) for $m = 0$, $(\mathcal{F}, \nabla_{\mathcal{F}})$ descends to a vector bundle $\mathcal{G}$ on $X^{(1)}$ along $F_{X/k}$. Since the underlying space X is a smooth projective curve, we can find a line subbundle of $\mathcal{G}$. This corresponds to a flat line subbundle $(\mathcal{H}, \nabla_{\mathcal{H}})$ of $(\mathcal{E}, \nabla)$ via (2.3). Let $\overline{\nabla}$ denote the connection on $\overline{\mathcal{E}} := \mathcal{E}/\mathcal{H}$ induced from ∇ via the quotient $\pi : \mathcal{E} \twoheadrightarrow \overline{\mathcal{E}}$. Since $\overline{\nabla}$ has nilpotent p-curvature, the induction hypothesis implies that $(\overline{\mathcal{E}}, \overline{\nabla})$ admits a complete flag $\{\overline{\mathcal{E}}^j\}_j$. Hence, the collection $\{\mathcal{E}^j\}_j$ defined as $\mathcal{E}^0 := 0$ and $\mathcal{E}^{j+1} := \pi^{-1}(\mathcal{E}^j)$ $(j \in \mathbb{Z}_{\geq 0})$ forms the desired complete flag on $(\mathcal{E}, \nabla)$. This completes the proof. □

3. The case of projective line

This section focuses on the case where the underlying curve X has genus 0, i.e., X is isomorphic to the projective line. In particular, we describe vector bundles on X admitting a $\mathcal{D}^{(m)}$-module structure (cf. Proposition 3.1) and prove the existence of complete flags (cf. Theorem 3.2).

3.1. **Vector bundles admitting a $\mathcal{D}^{(m)}$-module structure.** Denote by $\mathbb{P}$ the projective line over k throughout this section. (We use the notation "$\mathbb{P}$" as opposed to the usual notation "$\mathbb{P}^1$" because later on we will need to consider its m-th Frobenius twist for $m \in \mathbb{Z}_{\geq 0}$ and $\mathbb{P}^{(m)}$ is notationally and typographically simpler than $(\mathbb{P}^1)^{(m)}$.) For each $\ell \in \mathbb{Z}$, the unique (up to isomorphism) line bundle on $\mathbb{P}$ of degree ℓ is denoted by $\mathcal{O}_{\mathbb{P}}$. Recall the Birkhoff-Grothendieck theorem (cf. [20]), which states that any vector bundle on $\mathbb{P}$ is isomorphic to a direct sum of line bundles of the form $\mathcal{O}_{\mathbb{P}}$. In particular, every indecomposable vector bundle on $\mathbb{P}$ has rank 1. Applying this theorem, one can prove the following result, generalizing the classification of flat vector bundles in the genus-0 case (cf. [2], [8], [10], and [31]).

Proposition 3.1. *Let m be a nonnegative integer and $\mathcal{E}$ a vector bundle on $X\,(= \mathbb{P})$ of rank $r \in \mathbb{Z}_{>0}$. Then, the following three conditions (a)-(c) are equivalent:*

(a) *$\mathcal{E}$ admits a $\mathcal{D}^{(m)}$-module structure;*
(b) *$\mathcal{E}$ admits a $\mathcal{D}^{(m)}$-module structure with vanishing p^{m+1}-curvature;*
(c) *$\mathcal{E} \cong \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}}(p_i^{m+1})$ for some $\ell_1, \cdots, \ell_r \in \mathbb{Z}$.*

Proof. (b) ⇒ (a) is clear. To prove (a) ⇒ (c), we assume that $\mathcal{E}$ admits a $\mathcal{D}^{(m)}$-module structure $\nabla_{\mathcal{E}}$. By the equivalence of categories (2.4), $(\mathcal{E}, \nabla_{\mathcal{E}})$ corresponds to a flat vector bundle $(\mathcal{G}, \nabla_{\mathcal{G}})$

on $X^{(m)}\left(=\mathbb{P}^{(m)}\right)$ via pull-back by $F^{(m)}_{\mathbb{P}/k}$. It follows from the Birkhoff-Grothendieck theorem that $\mathcal{G}\cong\bigoplus_{i=1}^{r}\mathcal{O}_{\mathbb{P}^{(m)}}(\ell'_i)$ for some $\ell'_1,\cdots,\ell'_r\in\mathbb{Z}$. As proved in the references listed above, there exists a collection $(\ell_1,\cdots,\ell_r)\in\mathbb{Z}^r$ with $\ell'_i=p\ell_i$ for every i. Hence, we have

$$\mathcal{E}\cong F^{(m)*}_{\mathbb{P}/k}(\mathcal{G})\cong F^{(m)*}_{\mathbb{P}/k}(\bigoplus_{i=1}^{r}\mathcal{O}_{\mathbb{P}^{(m)}}(p\ell_i))\cong\bigoplus_{i=1}^{r}F^{(m)*}_{\mathbb{P}/k}(\mathcal{O}_{\mathbb{P}^{(m)}}(p\ell_i))\cong\bigoplus_{i=1}^{r}\mathcal{O}_{\mathbb{P}}(p^{m+1}\ell_i).$$

Thus, $\mathcal{E}$ satisfies the condition (c).

Finally, we shall prove (c) $\Rightarrow$ (b). Assume that $\mathcal{E}=\bigoplus_{i=1}^{r}\mathcal{O}_{\mathbb{P}}(p^{m+1}\ell_i)$ for some $\ell_1,\cdots,\ell_r\in\mathbb{Z}$. By putting $\mathcal{G}:=\bigoplus_{i=1}^{r}\mathcal{O}_{\mathbb{P}^{(m+1)}}(\ell_i)$, we obtain a chain of isomorphisms

$$F^{(m+1)*}_{\mathbb{P}/k}(\mathcal{G})\xrightarrow{\sim}\bigoplus_{i=1}^{r}F^{(m+1)*}_{\mathbb{P}/k}(\mathcal{O}_{\mathbb{P}^{(m+1)}}(\ell_i))\xrightarrow{\sim}\mathcal{E}.$$

Hence, $\nabla^{(m)}_{\mathcal{G},\mathrm{can}}$ (cf. (2.2)) defines a $\mathcal{D}^{(m)}$-module structure on $\mathcal{E}$ (with vanishing p^{m+1}-curvature) via this composite. In particular, $\mathcal{E}$ satisfies the condition (b). This completes the proof. □

3.2. **The existence of complete flags.** Applying Proposition 3.1, we can prove the following assertion, which is a part of Theorem A.

Theorem 3.2. *Suppose that $X=\mathbb{P}$. Let m be an element of $\mathbb{Z}_{\geq 0}\sqcup\{\infty\}$ and $(\mathcal{E},\nabla)$ a (nonzero) $\mathcal{D}^{(m)}$-bundle on X. Then, $(\mathcal{E},\nabla)$ admits a complete flag.*

Proof. In the case of $m=\infty$, we recall from [17, Theorem 2.2] that the stratified fundamental group $\pi_1^{\mathrm{str}}(\mathbb{P})$ of $\mathbb{P}$ (cf. Remark 2.1) is trivial. That is, every $\mathcal{D}^{(\infty)}$-bundle is isomorphic to a direct sum of finitely many copies of $(\mathcal{O}_{\mathbb{P}},\nabla^{(\infty)}_{\mathrm{triv}})$, so the assertion for $m=\infty$ follows immediately from this fact.

For $m\in\mathbb{Z}_{\geq 0}$, the problem is reduced to the case of $m=0$ by Proposition 2.3. Let us prove the assertion for $m=0$ by induction on $r:=\mathrm{rank}(\mathcal{E})$. There is nothing to prove in the base case of rank $r=1$. To consider the induction step, we assume that the assertion for rank $r-1$ flat vector bundles (where $r\geq 2$) has been proved. The equivalence (a) $\Leftrightarrow$ (c) in Proposition 3.1 implies

$$\mathcal{E}\cong\bigoplus_{i=1}^{s}\mathcal{O}_{\mathbb{P}}(p\ell_i)^{\oplus e_i}$$

for $\ell_1>\ell_2>\cdots>\ell_s$ ($s\in\mathbb{Z}_{>0}$) and $e_1,\cdots,e_s\in\mathbb{Z}_{>0}$ (with $\sum_{i=1}^{s}e_i=r$). We write $\mathcal{G}:=\bigoplus_{i=1}^{s}\mathcal{O}_{\mathbb{P}}(\ell_i)^{\oplus e_i}$. Then, there exists a natural isomorphism $\alpha:\mathcal{E}\xrightarrow{\sim}F^{(1)*}_{\mathbb{P}/k}(\mathcal{G})$, which allows us to regard $\nabla^{(0)}_{\mathcal{G},\mathrm{can}}$ as a connection on $\mathcal{E}$. Since the difference $\nabla-\nabla^{(0)}_{\mathcal{G},\mathrm{can}}$ is $\mathcal{O}_{\mathbb{P}}$-linear, ∇ can be expressed as

$$\nabla=\nabla^{(0)}_{\mathcal{G},\mathrm{can}}+\sum_{1\leq i,j\leq s}h_{i,j},$$

where $h_{i,j}\in\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_{\mathbb{P}}(p\ell_i)^{\oplus e_i},\Omega_{\mathbb{P}}\otimes(\mathcal{O}_{\mathbb{P}}(p\ell_j)^{\oplus e_j}))$. Note that $h_{i,j}=0$ whenever $i<j$, in particular, that $h_{1,j}=0$ for every $j=2,\cdots,s$. Since $\mathrm{End}_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_{\mathbb{P}}(p\ell_1))=k$, the endomorphism $h_{1,1}$ can be given by an $e_1\times e_1$ matrix $A\in M_{e_1}(k)$. Choose an automorphism η of $\mathcal{O}_{\mathbb{P}}(p\ell_1)^{\oplus e_1}$

such that, if $P \in M_{e_1}(k)$ is the corresponding matrix, then $P^{-1}AP$ is upper triangular. The gauge transformation by the automorphism $\widetilde{\eta} := \eta \oplus \bigoplus_{i=2}^{s} \mathrm{id}_{\mathcal{O}_{\mathbb{P}}(p\ell_i)^{\oplus e_i}}$ transforms ∇ into another connection ∇'. The 1-st direct summand $\mathcal{L}'$ in $\mathcal{O}_{\mathbb{P}}(p\ell_1)^{\oplus e_1}$, regarded as a line subbundle of $\mathcal{E}$ via α, satisfies $\nabla'(\mathcal{L}') \subseteq \Omega_{\mathbb{P}} \otimes \mathcal{L}'$. It follows that the line subbundle $\mathcal{L}$ of $\mathcal{E}$ corresponding to $\mathcal{L}'$ via $\widetilde{\eta}$ satisfies $\nabla(\mathcal{L}) \subseteq \Omega_{\mathbb{P}^1} \otimes \mathcal{L}$. Write $\overline{\mathcal{E}} := \mathcal{E}/\mathcal{L}$, and let $\overline{\nabla}$ be the connection on $\overline{\mathcal{E}}$ induced from ∇ via the natural quotient $\pi : \mathcal{E} \twoheadrightarrow \overline{\mathcal{E}}$. By the induction hypothesis, $(\overline{\mathcal{E}}, \overline{\nabla})$ admits a complete flag $\{\overline{\mathcal{E}}^j\}_{j=0}^{r-1}$. For each $j = 0, \cdots, r$, we shall set $\mathcal{E}^0 := 0$ and $\mathcal{E}^j := \pi^{-1}(\overline{\mathcal{E}}^{j-1})$ $(j = 1, \cdots, r)$. Then, the resulting filtration $\{\mathcal{E}^j\}_j$ forms the desired complete flag. This completes the proof. □

4. The case of elliptic curves

This section is devoted to studying the case of curves of genus 1. Our discussion relies on Atiyah's classification of indecomposable vector bundles on such curves. As a consequence, we establish a part of our main theorem, proving the existence of complete flags for genus-1 curves (cf. Theorem 4.6).

Throughout this section, this $g = 1$ assumption will be in force. In particular, by our assumption, Ω_X is trivialized. We fix an identification $\mathcal{O}_X = \Omega_X$ by choosing a global generator of Ω_X. Additionally, we fix a k-rational point σ of X, regarded as an effective divisor of degree one.

4.1. **Atiyah's classification of indecomposable bundles.** We begin with reviewing the study of indecomposable vector bundles on a genus-1 curve, discussed in [2].

Let V be a finite-dimensional k-vector space and $\mathcal{E}$ a vector bundle on X. There exists a sequence of natural isomorphisms

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}, V \otimes_k \mathcal{L}) \xrightarrow{\sim} \mathrm{Hom}_k(\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}, \mathcal{L})^\vee, V) \xrightarrow{\sim} \mathrm{Hom}_k(H^0(X, \mathcal{L}^\vee \otimes \mathcal{E}), V), \tag{4.1}$$

where the second isomorphism arises from Serre duality and our fixed identification $\mathcal{O}_X = \Omega_X$. Hence, each extension $\varepsilon : 0 \to V \otimes_k \mathcal{L} \to \mathcal{F} \to \mathcal{E} \to 0$ corresponds to a k-linear morphism

$$\mathrm{ext}_{\mathcal{F}} \ (\text{or } \mathrm{ext}_\varepsilon) : H^0(X, \mathcal{L}^\vee \otimes \mathcal{E}) \to V \tag{4.2}$$

via (4.1). If $V = H^0(X, \mathcal{L}^\vee \otimes \mathcal{E})$, then $\mathrm{ext}_{\mathcal{F}}$ defines a k-linear endomorphism of $H^0(X, \mathcal{L}^\vee \otimes \mathcal{E})$.

Now, recall from [2, Theorem 5, (i)] that, for each positive integer r, there exists a unique (up to isomorphism) indecomposable vector bundle $\mathcal{F}_r$ on X of rank r and degree 0 with $H^0(X, \mathcal{F}_r) \neq 0$. Moreover, $\dim_k(H^0(X, \mathcal{F}_r)) = 1$ and the cokernel of the unique (up to multiplication by an element of $k^\times$) injection $\mathcal{O}_X \hookrightarrow \mathcal{F}_r$ is isomorphic to $\mathcal{F}_{r-1}$ (with the convention that $\mathcal{F}_0 := 0$). In other words, by fixing an isomorphism $\mathrm{Coker}(\mathcal{O}_X \hookrightarrow \mathcal{F}_r) \xrightarrow{\sim} \mathcal{F}_{r-1}$, we obtain a short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{F}_r \to \mathcal{F}_{r-1} \to 0$$

(cf. [2, Lemma 15, (i), and Theorem 5, (i)]). This extension corresponds to the identity morphism of $H^0(X, \mathcal{F}_{r-1})$ via (4.1) in the case where $V = H^0(X, \mathcal{F}_{r-1})$ $(\cong k)$ and $\mathcal{L} = \mathcal{O}_X$.

Denote by

$$\mathrm{Fil}^\bullet(\mathcal{F}_r) := \{\mathrm{Fil}^j(\mathcal{F}_r)\}_{j=0}^{r} \tag{4.3}$$

the filtration on $\mathcal{F}_r$ defined in such a way that for each $j = 0, \cdots, r$, the subsheaf $\mathrm{Fil}^j(\mathcal{F}_r)$ is the kernel of the composite surjection $\mathcal{F}_r \twoheadrightarrow \mathcal{F}_{r-1} \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{F}_{r-j}$. Then, we have $\mathrm{Fil}^j(\mathcal{F}_r)/\mathrm{Fil}^{j-1}(\mathcal{F}_r) \cong \mathcal{O}_X$ for every $j = 1, \cdots, r$.

Next,take an arbitrary indecomposable vector bundle $\mathcal{E}$ on X of rank $r \in \mathbb{Z}_{>0}$ and degree $d \in \mathbb{Z}$. Define $h := \gcd(r, d)$. Following [2, Theorem 6], we shall construct, by induction on j, proper subbundles $\mathrm{Fil}^j(\mathcal{E})$ $(j = 0, 1, 2, \cdots)$ of $\mathcal{E}$, starting with $\mathrm{Fil}^0(\mathcal{E}) := 0$ and satisfying the following conditions:

- $\mathcal{E}/\mathrm{Fil}^j(\mathcal{E})$ is indecomposable;
- The equality $h = \gcd(r_j, d_j)$ holds, where $r_j := \mathrm{rank}(\mathcal{E}/\mathrm{Fil}^j(\mathcal{E}))$ and $d_j := \deg(\mathcal{E}/\mathrm{Fil}^j(\mathcal{E}))$ (hence $(r_0, d_0) = (r, d)$).

To construct $\mathrm{Fil}^j(\mathcal{E})$, assume that a proper subbundle $\mathrm{Fil}^{j-1}(\mathcal{E})$ $(j \geq 1)$ has been constructed, and that $r_{j-1} \nmid d_{j-1}$. Under these assumptions, the vector bundle $\mathcal{L}_j^\vee \otimes (\mathcal{E}/\mathrm{Fil}^{j-1}(\mathcal{E}))$, where $\mathcal{L}_j := \mathcal{O}_X(\left\lfloor \frac{d_{j-1}}{r_{j-1}} \right\rfloor \sigma)$, has positive degree strictly smaller than r_{j-1}. It follows that the natural morphism

$$\alpha_j : H^0(X, \mathcal{L}_j^\vee \otimes (\mathcal{E}/\mathrm{Fil}^{j-1}(\mathcal{E}))) \otimes_k \mathcal{L}_j \to \mathcal{E}/\mathrm{Fil}^{j-1}(\mathcal{E})$$

is nonzero and injective, and its cokernel $\mathrm{Coker}(\alpha_j)$ defines an indecomposable vector bundle (cf. [2, Lemma 15, (ii)]). We define $\mathrm{Fil}^j(\mathcal{E})$ as the inverse image of $\mathrm{Im}(\alpha_j)$ along the quotient $\mathcal{E} \twoheadrightarrow \mathcal{E}/\mathrm{Fil}^{j-1}(\mathcal{E})$. Since $\mathcal{E}/\mathrm{Fil}^j(\mathcal{E})$ coincides with $\mathrm{Coker}(\alpha_j)$, it is indecomposable, and we obtain a short exact sequence

$$0 \to H^0(X, \mathcal{L}_j^\vee \otimes (\mathcal{E}/\mathrm{Fil}^{j-1}(\mathcal{E}))) \otimes_k \mathcal{L}_j \to \mathcal{E}/\mathrm{Fil}^{j-1}(\mathcal{E}) \to \mathcal{E}/\mathrm{Fil}^j(\mathcal{E}) \to 0. \tag{4.4}$$

Equivalently, we obtain an isomorphism of vector bundles

$$H^0(X, \mathcal{L}_j^\vee \otimes (\mathcal{E}/\mathrm{Fil}^{j-1}(\mathcal{E}))) \otimes_k \mathcal{L}_j \xrightarrow{\sim} \mathrm{Fil}^j(\mathcal{E})/\mathrm{Fil}^{j-1}(\mathcal{E}) \,(\neq 0)\,. \tag{4.5}$$

Moreover, the equality

$$\dim_k(H^0(X, \mathcal{L}_j^\vee \otimes (\mathcal{E}/\mathrm{Fil}^{j-1}(\mathcal{E})))) = \dim_k(H^0(X, \mathcal{L}_j^\vee \otimes (\mathcal{E}/\mathrm{Fil}^j(\mathcal{E})))) \tag{4.6}$$

holds (cf. [2, Lemma 15, (ii)]). Under a suitable identification $H^0(X, \mathcal{L}_j^\vee \otimes (\mathcal{E}/\mathrm{Fil}^{j-1}(\mathcal{E}))) = H^0(X, \mathcal{L}_j^\vee \otimes (\mathcal{E}/\mathrm{Fil}^j(\mathcal{E})))$, the extension (4.4) corresponds to the identity morphism of $H^0(X, \mathcal{L}_j^\vee \otimes (\mathcal{E}/\mathrm{Fil}^j(\mathcal{E})))$ via (4.1). This means $\mathrm{ext}_{\mathcal{E}/\mathrm{Fil}^{j-1}(\mathcal{E})} = \mathrm{id}_{H^0(X, \mathcal{L}_j^\vee \otimes (\mathcal{E}/\mathrm{Fil}^j(\mathcal{E})))}$ in the notation of (4.2). Since

$$r_j = r_{j-1} - \left(d_{j-1} - r_{j-1} \cdot \left\lfloor \frac{d_{j-1}}{r_{j-1}} \right\rfloor \right), \quad d_j = d_{j-1} - \left\lfloor \frac{d_{j-1}}{r_{j-1}} \right\rfloor \cdot \left(d_{j-1} - r_{j-1} \cdot \left\lfloor \frac{d_{j-1}}{r_{j-1}} \right\rfloor \right),$$

we have $\mathrm{g.c.d}(r_j, d_j) = \mathrm{g.c.d}(r_{j-1}, d_{j-1}) = h$ and $r_j < r_{j-1}$, $d_j < d_{j-1}$, ensuring that the process terminates.

By repeating this construction, we end up with an increasing filtration $\{\mathrm{Fil}^j(\mathcal{E})\}_{j=0}^m$ of $\mathcal{E}$ for some $m \in \mathbb{Z}_{>0}$ such that $r_m \mid d_m$ (hence $r_m = h$). According to [2, Theorem 5, (ii)], there exists a line bundle $\mathcal{L}_{m+1}$ of degree $\frac{d_m}{r_m}$ along with an isomorphism $\alpha_\mathcal{E} : \mathcal{E}/\mathrm{Fil}^m(\mathcal{E}) \xrightarrow{\sim} \mathcal{L}_{m+1} \otimes \mathcal{F}_{r_m}$. Using this, we extend $\{\mathrm{Fil}^j(\mathcal{E})\}_{j=0}^m$ to a new filtration

$$\mathrm{Fil}^\bullet(\mathcal{E}) := \{\mathrm{Fil}^j(\mathcal{E})\}_{j=0}^\ell, \tag{4.7}$$

where $\ell := m + r_m$, by defining

$$\mathrm{Fil}^j(\mathcal{E}) := \mathrm{Ker}\left(\mathcal{E} \twoheadrightarrow \mathcal{E}/\mathrm{Fil}^m(\mathcal{E}) \xrightarrow{\alpha_\mathcal{E}} \mathcal{L}_{m+1} \otimes \mathcal{F}_{r_m} \twoheadrightarrow \mathcal{L}_{m+1} \otimes (\mathcal{F}_{r_m}/\mathrm{Fil}^{j-m}(\mathcal{F}_{r_m})) \right)$$

for $j = m+1, \cdots, \ell$ (hence $\mathrm{Fil}^{\ell}(\mathcal{E}) = \mathcal{E}$). Setting $\mathcal{L}_j := \mathcal{L}_{m+1}$ for $j = m+1, \cdots, \ell$, we obtain a collection of line bundles

$$L(\mathcal{E}) := (\mathcal{L}_1, \cdots, \mathcal{L}_{m(\mathcal{E})}, \cdots, \mathcal{L}_{\ell}),$$

where $m(\mathcal{E}) := m\,(\geq 0)$. Note that both (4.5) and (4.6) hold even for $j \geq m+1$ (cf. [2, Lemma 16]). Additionally, if $m(\mathcal{E}) > 0$, then we have $\mathcal{L}_1 \cong \mathcal{O}_X(\lfloor \frac{r}{d} \rfloor \sigma)$). Furthermore, in the special case where $\mathcal{E} = \mathcal{F}_r$ ($r \in \mathbb{Z}_{\geq 0}$), the filtration (4.7) coincides with the filtration (4.3).

4.2. **Connections on indecomposable bundles.** As a preparatory step for establishing the main result of this section, we first prove several properties of connections on indecomposable vector bundles. Let us fix an indecomposable vector bundle $\mathcal{E}$ on X, and let $\{\mathrm{Fil}^j(\mathcal{E})\}_{j=0}^{\ell}$ be the increasing filtration on $\mathcal{E}$ constructed as in (4.7). Also, write $L(\mathcal{E}) = (\mathcal{L}_1, \cdots, \mathcal{L}_{\ell})$.

Proposition 4.1. *Suppose that $\mathcal{E}$ admits a connection ∇. Then, for each $j = 1, \cdots, \ell$, the subbundle $\mathrm{Fil}^j(\mathcal{E})$ is stable under ∇, i.e., $\nabla(\mathrm{Fil}^j(\mathcal{E})) \subseteq \Omega_X \otimes \mathrm{Fil}^j(\mathcal{E})$.*

Proof. Suppose, on the contrary, that the filtration $\{\mathrm{Fil}^j(\mathcal{E})\}_j$ is not preserved by ∇. In particular, the integer

$$j_0 := \min\{j \mid \nabla(\mathrm{Fil}^{j_0}(\mathcal{E})) \not\subseteq \Omega_X \otimes \mathrm{Fil}^{j_0}(\mathcal{E})\}\,(> 0)$$

is well-defined. We set $\mathcal{L} := \mathcal{L}_{j_0}$, $\mathcal{M} := \mathcal{E}/\mathrm{Fil}^{j_0-1}(\mathcal{E})$, and $\mathcal{N} := \mathrm{Fil}^{j_0}(\mathcal{E})/\mathrm{Fil}^{j_0-1}(\mathcal{E})$ ($\cong H^0(X, \mathcal{L}_{j_0}^{\vee} \otimes \mathcal{M}) \otimes_k \mathcal{L}_{j_0}$ by (4.5)). Since $\mathrm{Fil}^{j_0-1}(\mathcal{E})$ is closed under ∇, this connection induces a connection $\overline{\nabla}$ on $\mathcal{M}$. It follows that there exists an injection $\iota : \mathcal{L} \hookrightarrow \mathcal{N}\,(\subseteq \mathcal{M})$ with $\overline{\nabla}(\mathrm{Im}(\iota)) \not\subseteq \Omega_X \otimes \mathcal{N}$. Let us consider the $\mathcal{O}_X$-linear composite

$$\mathcal{D}^{(0)}_{\leq 1} \otimes \mathcal{L} \hookrightarrow \mathcal{D}^{(0)} \otimes \mathcal{M} \xrightarrow{\overline{\nabla}} \mathcal{M}, \tag{4.8}$$

where the first arrow denotes the tensor product of the natural inclusion $\mathcal{D}^{(0)}_{\leq 1} \hookrightarrow \mathcal{D}^{(0)}$ and ι. This composite restricts to $\iota : \mathcal{L}\left(= \mathcal{D}^{(0)}_{\leq 0} \otimes \mathcal{L}\right) \to \mathcal{N}$, and induces a *nonzero* $\mathcal{O}_X$-linear morphism $\eta : \mathcal{T}_X \otimes \mathcal{L} \to \mathcal{M}/\mathcal{N}$ via the quotient

$$\mathcal{D}^{(0)}_{\leq 1} \otimes \mathcal{L} \twoheadrightarrow (\mathcal{D}^{(0)}_{\leq 1}/\mathcal{D}^{(0)}_{\leq 0}) \otimes \mathcal{L} = \mathcal{T}_X \otimes \mathcal{L}.$$

Since $\mathcal{T}_X \cong \mathcal{O}_X$, the morphism

$$\eta' : (k \cong)\, H^0(X, \mathcal{T}_X) \to H^0(X, \mathcal{L}^{\vee} \otimes (\mathcal{M}/\mathcal{N}))$$

induced by η is nonzero. Also, we obtain the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{L} & \xrightarrow{\text{inclusion}} & \mathcal{D}^{(0)}_{\leq 1} \otimes \mathcal{L} & \xrightarrow{\text{quotient}} & \mathcal{T}_X \otimes \mathcal{L} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle (4.8)} & & \downarrow{\scriptstyle \wr}\,{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & \mathcal{N} & \xrightarrow{\text{inclusion}} & \pi^{-1}(\mathcal{T}_X \otimes \mathcal{L}) & \xrightarrow{\pi|_{\pi^{-1}(\mathcal{T}_X \otimes \mathcal{L})}} & \mathcal{T}_X \otimes \mathcal{L} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \wr}\,{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{inclusion}} & & \downarrow{\scriptstyle \eta} & & \\
0 & \longrightarrow & \mathcal{N} & \xrightarrow{\text{inclusion}} & \mathcal{M} & \xrightarrow{\pi} & \mathcal{M}/\mathcal{N} & \longrightarrow & 0,
\end{array} \tag{4.9}$$

where π denotes the natural quotient and all horizontal sequences are exact. The extension defined by the lower horizontal sequence in (4.9), denoted by ε, corresponds to the identity morphism of $H^0(X, \mathcal{L}^\vee \otimes (\mathcal{M}/\mathcal{N}))$ via (4.1), i.e., $\mathrm{ext}_\varepsilon = \mathrm{id}_{H^0(X, \mathcal{L}^\vee \otimes (\mathcal{M}/\mathcal{N}))}$. The extension class of the middle horizontal sequence is nontrivial, since it corresponds to η' via the composite isomorphism

$$\begin{aligned}\mathrm{Ext}^1(\mathcal{T}_X \otimes \mathcal{L}, \mathcal{N}) &\xrightarrow{\sim} \mathrm{Ext}^1(\mathcal{T}_X \otimes \mathcal{L}, H^0(X, \mathcal{L}^\vee \otimes \mathcal{M}) \otimes_k \mathcal{L}) \\ &\xrightarrow{\sim} \mathrm{Ext}^1(\mathcal{T}_X \otimes \mathcal{L}, H^0(X, \mathcal{L}^\vee \otimes (\mathcal{M}/\mathcal{N})) \otimes_k \mathcal{L}) \\ &\xrightarrow{\sim} \mathrm{Hom}_k(H^0(X, \mathcal{T}_X), H^0(X, \mathcal{L}^\vee \otimes (\mathcal{M}/\mathcal{N})))\end{aligned}$$

(cf. (4.1), (4.6)). On the other hand, the upper horizontal arrow in (4.9) splits, since the fixed global generator of $\mathcal{T}_X$ provides a splitting of the natural short exact sequence $0 \to \mathcal{O}_X \left(= \mathcal{D}^{(0)}_{\leq 0}\right) \to {}^R\mathcal{D}^{(0)}_{\leq 1} \to \mathcal{T}_X \to 0$. The composite of the resulting splitting $\mathcal{T}_X \otimes \mathcal{L} \hookrightarrow \mathcal{D}^{(0)}_{\leq 1} \otimes \mathcal{L}$ and the inclusion $\mathcal{D}^{(0)}_{\leq 1} \otimes \mathcal{L} \hookrightarrow \pi^{-1}(\mathcal{T}_X \otimes \mathcal{L})$ (i.e., the upper middle vertical arrow in (4.9)) specifies a splitting of the middle horizontal sequence, leading to a contradiction. Thus, the original assumption that ∇ does not preserve the filtration $\{\mathrm{Fil}^j(\mathcal{E})\}_j$ must be false, completing the proof. □

Proposition 4.2. *$\mathcal{E}$ admits a connection if and only if $p \mid \deg(\mathcal{L}_j)$ for every $j = 1, \cdots, \ell$.*

Proof. First, we shall consider the "only if" part of the required equivalence. Suppose that there exists a connection ∇ on $\mathcal{E}$. By Proposition 4.1, ∇ preserve the filtration $\{\mathrm{Fil}^j(\mathcal{E})\}_j$. Hence, for each $j = 1, \cdots, \ell$, ∇ induces a connection on $\mathrm{Gr}^j(\mathcal{E}) := \mathrm{Fil}^j(\mathcal{E})/\mathrm{Fil}^{j-1}(\mathcal{E})$. Since $\mathrm{Gr}^j(\mathcal{E})$ is a direct sum of finitely many copies of $\mathcal{L}_j$, the line bundle $\mathcal{L}_j$ admits a connection (cf. [8, Lemma 2.4]). Then, it follows from [8, Theorem 1.1] or [10, Theorem 1.1] that $p \mid \deg(\mathcal{L}_j)$.

Next, we shall consider the "if" part. Assume that $p \mid \deg(\mathcal{L}_j)$ for every j. By this assumption, each $\mathcal{L}_j$ admits a connection ∇_j (cf. [8, Theorem 1.1] or [10, Theorem 1.1]). We now construct a connection on $\mathcal{E}/\mathrm{Fil}^j(\mathcal{E})$ by descending induction on $j = 0, \cdots, \ell - 1$. In the case of $j = \ell - 1$, we recall that $\mathcal{E}/\mathrm{Fil}^{\ell-1}(\mathcal{E}) \left(= \mathrm{Gr}^\ell(\mathcal{E})\right)$ is isomorphic to a direct sum of copies of $\mathcal{L}_\ell$. Hence, the direct sum of ∇_ℓ defines a connection on $\mathcal{E}/\mathrm{Fil}^{\ell-1}(\mathcal{E})$, establishing the base step of our induction argument. To consider the induction step, we assume that a connection ∇'_j on $\mathcal{E}/\mathrm{Fil}^j(\mathcal{E})$ has been constructed for a fixed integer $j \in \{1, \cdots, \ell - 1\}$. Recall that $\mathrm{Gr}^j(\mathcal{E}) \cong H^0(X, \mathcal{L}_j^\vee \otimes (\mathcal{E}/\mathrm{Fil}^j(\mathcal{E}))) \otimes_k \mathcal{L}_j$ (cf. (4.4) and (4.6)) and the natural extension

$$\varepsilon : 0 \to \mathrm{Gr}^j(\mathcal{E}) \to \mathcal{E}/\mathrm{Fil}^{j-1}(\mathcal{E}) \to \mathcal{E}/\mathrm{Fil}^j(\mathcal{E}) \to 0$$

corresponds to the identity morphism of $H^0(X, \mathcal{L}_j^\vee \otimes (\mathcal{E}/\mathrm{Fil}^j(\mathcal{E})))$ via (4.1). Consider the corresponding morphism

$$\alpha : H^0(X, \mathcal{L}_j^\vee \otimes (\mathcal{E}/\mathrm{Fil}^j(\mathcal{E}))) \otimes_k \mathcal{O}_X \to \mathcal{L}_j^\vee \otimes (\mathcal{E}/\mathrm{Fil}^j(\mathcal{E})).$$

One can find a connection ∇''_j on $H^0(X, \mathcal{L}_j^\vee \otimes (\mathcal{E}/\mathrm{Fil}^j(\mathcal{E}))) \otimes_k \mathcal{O}_X$ such that, under the fixed identification $\mathcal{O}_X = \Omega_X$, the endomorphisms of $H^0(X, \mathcal{L}_j^\vee \otimes (\mathcal{E}/\mathrm{Fil}^j(\mathcal{E})))$ induced from ∇''_j and $\nabla_j^\vee \otimes \nabla'_j$ (cf. (2.1)) coincide via applying the functor $H^0(X, -)$ to α. The connection $\nabla''_j \otimes \nabla_j$ can be transposed into a connection ∇'''_j on $\mathrm{Gr}^j(\mathcal{E})$ by using the isomorphism $H^0(X, \mathcal{L}_j^\vee \otimes (\mathcal{E}/\mathrm{Fil}^j(\mathcal{E}))) \otimes_k \mathcal{L}_j \xrightarrow{\sim} \mathrm{Gr}^j(\mathcal{E})$ (cf. (4.5)). In particular, this yields a connection $\nabla'''_j \otimes (\nabla'_j)^\vee$ on $\mathrm{Gr}^j(\mathcal{E}) \otimes (\mathcal{E}/\mathrm{Fil}^j(\mathcal{E}))^\vee$. Denote by $\mathcal{K}^\bullet[\nabla'''_j \otimes (\nabla'_j)^\vee]$ the complex of sheaves defined to

be $\nabla_j''' \otimes (\nabla_j')^\vee$ concentrated at degrees 0 and 1. The Hodge-to-de Rham spectral sequence associated to this complex gives an exact sequence

$$\begin{aligned}\mathbb{H}^1(X, \mathcal{K}^\bullet[\nabla_j''' \otimes (\nabla_j')^\vee]) \to H^1(X, \mathrm{Gr}^j(\mathcal{E}) \otimes (\mathcal{E}/\mathrm{Fil}^j(\mathcal{E}))^\vee) \\ \xrightarrow{H^1(\nabla_j''' \otimes (\nabla_j')^\vee)} H^1(X, \Omega_X \otimes (\mathrm{Gr}^j(\mathcal{E}) \otimes (\mathcal{E}/\mathrm{Fil}^j(\mathcal{E}))^\vee)).\end{aligned} \tag{4.10}$$

Note that $\mathbb{H}^1(X, \mathcal{K}^\bullet[\nabla_j''' \otimes (\nabla_j')^\vee])$ (resp., $H^1(X, \mathrm{Gr}^j(\mathcal{E}) \otimes (\mathcal{E}/\mathrm{Fil}^j(\mathcal{E}))^\vee)$) classifies extensions $(\mathcal{G}, \nabla_\mathcal{G})$ (resp., $\mathcal{G}$) of $(\mathcal{E}/\mathrm{Fil}^j(\mathcal{E}), \nabla_j')$ (resp., $\mathcal{E}/\mathrm{Fil}^j(\mathcal{E})$) by $(\mathrm{Gr}^j(\mathcal{E}), \nabla_j''')$ (resp., $\mathrm{Gr}^j(\mathcal{E})$) and the first arrow in (4.10) arises from the assignment between extensions $(\mathcal{G}, \nabla_\mathcal{G}) \mapsto \mathcal{G}$. The extension ε specifies an element γ_ε of $H^1(X, \mathrm{Gr}^j(\mathcal{E}) \otimes (\mathcal{E}/\mathrm{Fil}^j(\mathcal{E}))^\vee)$. By the definition of ∇_j'', the image of γ_ε via the second arrow in (4.10) vanishes. Hence, there exists an element of $\mathbb{H}^1(X, \mathcal{K}^\bullet[\nabla_j''' \otimes (\nabla_j')^\vee])$ mapped to γ_ε via the first arrow; this specifies a connection ∇_{j-1}' on $\mathcal{E}/\mathrm{Fil}^{j-1}(\mathcal{E})$ fitting into a short exact sequence

$$0 \to (\mathrm{Gr}^j(\mathcal{E}), \nabla_j''') \to (\mathcal{E}/\mathrm{Fil}^{j-1}(\mathcal{E}), \nabla_{j-1}') \to (\mathcal{E}/\mathrm{Fil}^j(\mathcal{E}), \nabla_j') \to 0.$$

This completes the proof of the induction step. In particular, by considering the case of $j = 0$, we obtain a connection on $\mathcal{E}\left(= \mathcal{E}/\mathrm{Fil}^0(\mathcal{E})\right)$, as desired. □

Remark 4.3 (Finite level vs. Infinite level)**.** Considering the structure of the stratified fundamental group $\pi_1^{\mathrm{str}}(X)$ (cf. Remark 2.1), we find that the only indecomposable vector bundles admitting $\mathcal{D}^{(\infty)}$-module structures are, up to tensoring with line bundles, those isomorphic to $\mathcal{F}_r$'s (cf. [13, Theorem 21] and its proof). This fact is reminiscent of the characteristic 0 setting. The key difference, however, lies in the case of finite-level $\mathcal{D}$-modules, where a wide variety of vector bundles admit such structures, as suggested by the above proposition. This fundamental distinction highlights the necessity of developing specialized arguments for finite-level $\mathcal{D}$-modules, setting them apart from their $\mathcal{D}^{(\infty)}$-module counterparts.

4.3. **The existence of complete flags.** In this subsection, we complete the proof of the remaining part of (a) ⇒ (b) in Theorem A. To begin with, we establish the following assertion.

Lemma 4.4. *Let $\mathcal{E}$ and $\mathcal{E}'$ be indecomposable vector bundles on X, and write $L(\mathcal{E}) = (\mathcal{L}_1, \cdots, \mathcal{L}_\ell)$, $L(\mathcal{E}') = (\mathcal{L}_1', \cdots, \mathcal{L}_{\ell'}')$. Also, let $f : \mathcal{E} \to \mathcal{E}'$ be an $\mathcal{O}_X$-linear morphism with $f(\mathrm{Fil}^1(\mathcal{E})) \not\subseteq \mathrm{Fil}^1(\mathcal{E})$. Then, we have $H^0(X, {\mathcal{L}_1'}^\vee \otimes \mathcal{L}_1) = 0$, or equivalently, one of the following two conditions holds:*

- $\deg(\mathcal{L}_1) < \deg(\mathcal{L}_1')$*;*
- $\deg(\mathcal{L}_1) = \deg(\mathcal{L}_1')$ *and* $\mathcal{L}_1 \ncong \mathcal{L}_1'$.

Proof. Suppose, on the contrary, that $H^0(X, {\mathcal{L}_1'}^\vee \otimes \mathcal{L}_1) \neq 0$. Since $f(\mathrm{Fil}^1(\mathcal{E})) \not\subseteq \mathrm{Fil}^1(\mathcal{E}')$, there exists an injection $\iota : \mathcal{L}_1 \hookrightarrow \mathrm{Fil}^1(\mathcal{E})$ such that $\iota' := \pi \circ f \circ \iota : \mathcal{L}_1 \to \mathcal{E}'/\mathrm{Fil}^1(\mathcal{E}')$ is nonzero, where π denotes the natural quotient $\mathcal{E}' \twoheadrightarrow \mathcal{E}'/\mathrm{Fil}^1(\mathcal{E}')$. In particular, we obtain an extension

$$0 \to \mathrm{Fil}^1(\mathcal{E}') \to \pi^{-1}(\mathrm{Im}(\iota')) \to \mathcal{L}_1 (\cong \mathrm{Im}(\iota')) \to 0. \tag{4.11}$$

Recall that the endomorphism $\mathrm{ext}_{\varepsilon'}$ of $H^0(X, {\mathcal{L}_1'}^\vee \otimes (\mathcal{E}'/\mathrm{Fil}^1(\mathcal{E}')))$ determined, via (4.1), by the extension $\varepsilon' : 0 \to \mathrm{Fil}^1(\mathcal{E}') \to \mathcal{E}' \to \mathcal{E}'/\mathrm{Fil}^1(\mathcal{E}') \to 0$ under a natural identification

$$\mathrm{Fil}^1(\mathcal{E}') \cong H^0(X, {\mathcal{L}_1'}^\vee \otimes (\mathcal{E}'/\mathrm{Fil}^1(\mathcal{E}'))) \otimes_k \mathcal{L}_1'$$

coincides with the identity morphism. Hence, the composite of $\mathrm{ext}_{\varepsilon'}$ with the morphism

$$(0 \neq) H^0(X, {\mathcal{L}'_1}^\vee \otimes \mathcal{L}_1) \to H^0(X, {\mathcal{L}'_1}^\vee \otimes (\mathcal{E}'/\mathrm{Fil}^1(\mathcal{E}')))$$

induced from $\iota' : \mathcal{L}_1 \to \mathcal{E}'/\mathrm{Fil}^1(\mathcal{E}')$ is nonzero. Since (4.11) corresponds to this composite via

$$\begin{aligned}\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{L}_1, \mathrm{Fil}^1(\mathcal{E}')) &\xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{L}_1, H^0(X, {\mathcal{L}'_1}^\vee \otimes \mathcal{E}') \otimes_k \mathcal{L}'_1) \\ &\xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{L}_1, H^0(X, {\mathcal{L}'_1}^\vee \otimes (\mathcal{E}'/\mathrm{Fil}^1(\mathcal{E}'))) \otimes_k \mathcal{L}'_1) \\ &\xrightarrow{\sim} \mathrm{Hom}_k(H^0(X, {\mathcal{L}'_1}^\vee \otimes \mathcal{L}_1), H^0(X, {\mathcal{L}'_1}^\vee \otimes (\mathcal{E}'/\mathrm{Fil}^1(\mathcal{E}'))))\end{aligned}$$

(cf. (4.1), (4.6)), this extension turns out to be nontrivial. But, it contradicts the fact that $f \circ \iota : \mathcal{L}_1 \hookrightarrow \pi^{-1}(\mathrm{Im}(\iota'))\,(\subseteq \mathcal{E}')$ specifies a split injection of (4.11). Thus, our initial assumption must be false, completing the proof of this assertion. □

The above lemma will be applied in the proof of the following proposition.

Proposition 4.5. *Let $\mathcal{E}$ and $\mathcal{E}'$ be indecomposable vector bundles on X, and write $L(\mathcal{E}) = (\mathcal{L}_1, \cdots, \mathcal{L}_\ell)$, $L(\mathcal{E}') = (\mathcal{L}'_1, \cdots, \mathcal{L}'_{\ell'})$. Also, let $f : \mathcal{E} \to \mathcal{E}'$ be an $\mathcal{O}_X$-linear morphism.*

(i) *If any of the following conditions (a)-(c) is satisfied, then $f(\mathrm{Fil}^1(\mathcal{E})) = 0$:*
  (a) $\deg(\mathcal{L}_1) > \deg(\mathcal{L}'_1)$;
  (b) $m(\mathcal{E}) > 0$, $\deg(\mathcal{L}_1) = \deg(\mathcal{L}_2)$, *and* $\mathcal{L}_1 \ncong \mathcal{L}'_1$;
  (c) $m(\mathcal{E}) = m(\mathcal{E}') = 0$, $\deg(\mathcal{L}_1) = \deg(\mathcal{L}_2)$, *and* $\mathcal{L}_1 \ncong \mathcal{L}'_1$.

(ii) *If $\mathcal{L}_1 \cong \mathcal{L}_2$, then $f(\mathrm{Fil}^1(\mathcal{E})) \subseteq \mathrm{Fil}^1(\mathcal{E}')$.*

Proof. We first consider assertion (i). If the condition (a) is fulfilled (which implies $H^0(X, {\mathcal{L}'_1}^\vee \otimes \mathcal{L}_1) \neq 0$), then it follows from Lemma 4.4 that $f(\mathrm{Fil}^1(\mathcal{E})) \subseteq \mathrm{Fil}^1(\mathcal{E}')$. Since

$$\mathrm{Fil}^1(\mathcal{E}) \cong H^0(X, \mathcal{L}_1^\vee \otimes \mathcal{E}) \otimes_k \mathcal{L}_1 \quad \text{and} \quad \mathrm{Fil}^1(\mathcal{E}') \cong H^0(X, {\mathcal{L}'_1}^\vee \otimes \mathcal{E}') \otimes_k \mathcal{L}'_1,$$

the restriction of f to $\mathrm{Fil}^1(\mathcal{E})$ can be regarded as an $\mathcal{O}_X$-linear morphism

$$H^0(X, \mathcal{L}_1^\vee \otimes \mathcal{E}) \otimes_k \mathcal{L}_1 \to H^0(X, {\mathcal{L}'_1}^\vee \otimes \mathcal{E}') \otimes_k \mathcal{L}'_1.$$

However, this must be the zero map due to the assumption $\deg(\mathcal{L}_1) > \deg(\mathcal{L}'_1)$, proving that $f(\mathrm{Fil}^1(\mathcal{E})) = 0$.

Next, suppose that the condition (b) holds. If $m(\mathcal{E}') > 0$, then $\mathcal{L}_1 = \mathcal{L}'_1 = \mathcal{O}_X(s\sigma)$ with $s := \deg(\mathcal{L}_1) = \deg(\mathcal{L}'_1)$, which contradicts the assumption. It follows that $m(\mathcal{E}') = 0$, meaning that $\mathcal{E}' \cong \mathcal{L}'_1 \otimes \mathcal{F}_{r'}$, where $r' := \mathrm{rank}(\mathcal{E}')$. Since each graded piece of the filtration $\mathrm{Fil}^\bullet(\mathcal{E})$ is isomorphic to $\mathcal{L}'_1\,(\ncong \mathcal{L}_1)$, we have $\mathrm{Hom}(\mathcal{L}_1, \mathcal{E}') = 0$, which implies $f(\mathrm{Fil}^1(\mathcal{E})) = 0$, as desired.

We shall impose the assumption (c). This implies that $\mathcal{E}$ and $\mathcal{E}'$ can be identified with $\mathcal{L}_1 \otimes \mathcal{F}_r$ and $\mathcal{L}'_1 \otimes \mathcal{F}_{r'}$ (for some $r, r' \in \mathbb{Z}_{\geq 0}$), respectively. Since the graded pieces of the filtrations $\mathrm{Fil}^\bullet(\mathcal{E})$ and $\mathrm{Fil}^\bullet(\mathcal{E}')$ are isomorphic to $\mathcal{L}_1$ and $\mathcal{L}'_1$, respectively, the latter two conditions in (c) implies $f = 0$. This completes the proof of assertion (i).

Finally, assertion (ii) follows directly from Lemma 4.4. □

By applying the above assertion, we obtain the following theorem.

Theorem 4.6. *Let m be an element of $\mathbb{Z}_{\geq 0} \sqcup \{\infty\}$ and $(\mathcal{E}, \nabla)$ a $\mathcal{D}^{(m)}$-bundle of rank $r \in \mathbb{Z}_{>0}$. Then, there exists a complete flag on $(\mathcal{E}, \nabla)$.*

Proof. First, we shall prove the assertion for $m = 0$ by induction on r. In the base step, i.e., the case where $r = 1$, there is nothing to prove. To consider the induction step, we assume that the assertion for the rank $r - 1$ case (where $r > 1$) has been proved. There exists a collection of line bundles $\mathcal{N}_1, \cdots, \mathcal{N}_a$ on X satisfying the following conditions:

- $\mathcal{N}_1, \cdots, \mathcal{N}_a$ are pairwise non-isomorphic;
- A line bundle $\mathcal{L}$ is isomorphic to $\mathcal{N}_{a'}$ for some $a' \in \{1, \cdots, a\}$ if and only if there exists an indecomposable component $\mathcal{E}'$ of $\mathcal{E}$ with $\mathcal{L} \cong \mathcal{L}_1$, where $L(\mathcal{E}') = (\mathcal{L}_1, \cdots, \mathcal{L}_\ell)$.

After possibly rearranging the order of indices, one can assume that $\deg(\mathcal{N}_1) \geq \deg(\mathcal{N}_2) \geq \cdots \geq \deg(\mathcal{N}_a)$ and $\mathcal{N}_j \ncong \mathcal{O}_X(d\sigma)$ for $j \neq 1$, where $d := \deg(\mathcal{N}_1)$. For each $i = 1, \cdots, a$, let $\mathcal{E}_{i,1}, \cdots, \mathcal{E}_{i,M_i}$ (where $M_i \in \mathbb{Z}_{>0}$) be the indecomposable components of $\mathcal{E}$ whose first line bundles in $L(-)$ are isomorphic to $\mathcal{N}_1$. In particular, $\mathcal{E}$ decomposes as a direct sum

$$\mathcal{E} = \bigoplus_{i=1}^{\ell} \bigoplus_{j=1}^{M_i} \mathcal{E}_{i,j}^{\oplus e_{i,j}}, \tag{4.12}$$

where $e_{i,j} \in \mathbb{Z}_{>0}$. Since $\mathcal{E}$ admits a connection (i.e., ∇), each direct summand $\mathcal{E}_{i,j}$ admits a connection $\nabla_{\mathcal{E}_{i,j}}$ (cf. [8, Lemma 2.4]). The direct sum

$$\nabla' := \bigoplus_{i=1}^{\ell} \bigoplus_{j=1}^{M_i} \nabla_{\mathcal{E}_{i,j}}$$

defines a connection on the right-hand side of (4.12). By regarding ∇' as a connection on $\mathcal{E}$ via (4.12), we obtain the difference $\nabla - \nabla'$, which is $\mathcal{O}_X$-linear. Hence, ∇ can be expressed as

$$\nabla = \nabla' + \sum_{i,i'} \sum_{j,j'} h_{i,i',j,j'}$$

for some $h_{i,i',j,j'} \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_{i,j}^{\oplus e_{i,j}}, \mathcal{E}_{i',j'}^{\oplus e_{i',j'}}) \, (\subseteq \mathrm{End}_{\mathcal{O}_X}(\mathcal{E}))$ under the fixed identification $\mathcal{O}_X = \Omega_X$. According to Proposition 4.5, (i) and (ii), the sum $\sum_{i,i'} \sum_{j,j'} h_{i,i',j,j'}$ restricts to an endomorphism of $\bigoplus_{j=1}^{M_1} \mathrm{Fil}^1(\mathcal{E}_{1,j})$. Combining this fact with Proposition 4.1, we see that $\bigoplus_{j=1}^{M_1} \mathrm{Fil}^1(\mathcal{E}_{1,j})$ is closed under ∇. We denote the resulting connection on $\bigoplus_{j=1}^{M_1} \mathrm{Fil}^1(\mathcal{E}_{1,j})$ by ∇''. It follows from Proposition 4.2 that $p \mid \deg(\mathcal{N}_1)$, and that $\mathcal{N}_1$ admits a connection $\nabla_{\mathcal{N}_1}$ (cf. [8, Theorem 1.1], [10, Theorem 1.1]). Hence, after tensoring $(\mathcal{E}, \nabla)$ with the dual of $(\mathcal{N}_1, \nabla_{\mathcal{N}_1})$, we may assume that $\mathcal{N}_1 = \mathcal{O}_X$. This assumption yields an identification

$$\bigoplus_{j=1}^{M_1} \mathrm{Fil}^1(\mathcal{E}_{1,j}) \cong \mathcal{O}_X^{\oplus s}$$

for some $s \in \mathbb{Z}_{>0}$. Under this identification, the connection ∇'' can be expressed as $\nabla'' = d + A$ for some $s \times s$ matrix $A \in M_s(k)$. By applying a triangulation of A, we find a vector $\vec{v} \in k^{\oplus s} \left(= H^0(X, \mathcal{O}_X^{\oplus s})\right)$ with $\nabla''(\mathcal{O}_X \cdot \vec{v}) \subseteq \Omega_X \otimes (\mathcal{O}_X \cdot \vec{v})$. The line subbundle $\mathcal{L}$ of $\left(\bigoplus_j \mathrm{Fil}^1(\mathcal{E}_{1,j}) \subseteq\right) \mathcal{E}$ determined by $\mathcal{O}_X \cdot \vec{v}$ satisfies $\nabla(\mathcal{L}) \subseteq \Omega_X \otimes \mathcal{L}$, and $\overline{\mathcal{E}} := \mathcal{E}/\mathcal{L}$ admits a connection $\overline{\nabla}$ induced naturally from ∇. By the induction hypothesis, $(\overline{\mathcal{E}}, \overline{\nabla})$ admits a complete flag $\{\overline{\mathcal{E}}^j\}_{j=0}^{r-1}$. Using this filtration, we define $\mathcal{E}^0 := 0$ and $\mathcal{E}^j := \pi^{-1}(\overline{\mathcal{E}}^{j-1})$ $(j = 1, \cdots, r)$, where π denotes the natural quotient $\mathcal{E} \twoheadrightarrow \overline{\mathcal{E}}$. The resulting filtration $\{\mathcal{E}^j\}_{j=0}^{r}$ specifies a complete flag $(\mathcal{E}, \nabla)$, completing the proof for $m = 0$.

Also, by Proposition 2.3 in the case where the pair "(m, s)" is taken to be $(0, m)$, the assertion for $m \in \mathbb{Z}_{>0}$ is induced from the assertion just proved.

Finally, the assertion for $m = \infty$ follows immediately from the structure of $\pi_1^{\mathrm{str}}(X)$ (cf. Remark 2.1) discussed in [13, Theorem 21] and its proof. Indeed, this result states that any $\mathcal{D}^{(\infty)}$-bundle can be written as a direct sum of the form $\bigoplus_i (\mathcal{L}_i, \nabla_{\mathcal{L}_i}) \otimes (\mathcal{E}_i, \nabla_{\mathcal{E}_i})$, where $(\mathcal{L}_i, \nabla_{\mathcal{L}_i})$ has rank 1 and $(\mathcal{E}_i, \nabla_{\mathcal{E}_i})$ is unipotent, i.e., obtained as a successive extension of $(\mathcal{O}_X, \nabla_{\mathrm{triv}}^{(\infty)})$. Since unipotent connections always admit complete flags, the existence of a complete flag for any $\mathcal{D}^{(\infty)}$-bundle follows immediately, completing the proof. □

5. The case of hyperbolic curves

This final section addresses the remaining part of Theorem A, specifically the implication (b) ⇒ (a). To construct a flat vector bundle on a hyperbolic curve that does not admit complete flags, we will leverage a characteristic-p analogue of the Hitchin fibration, known as the p-Hitchin morphism.

5.1. **The p-Hitchin morphism.** Suppose that $g \geq 2$, and let $(\mathcal{E}, \nabla)$ be a flat vector bundle on X of rank $r \in \mathbb{Z}_{>0}$. The p-curvature $\psi(\nabla)$ of ∇ is regarded as a global section of $F_{X/k}^*(\Omega_{X^{(1)}}) \otimes \mathcal{E}nd(\mathcal{E})$, where $\mathcal{E}nd(\mathcal{E}) := \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})$. We associate to $\psi(\nabla)$ its characteristic polynomial

$$\mathrm{Char}(\mathcal{E}, \nabla) := \det(t - \psi(\nabla)) = t^r + a_{r-1}t^{r-1} + \cdots + a_0$$

with coefficients $a_i \in H^0(X, F_{X/k}^*(\Omega_{X^{(1)}}^{\otimes(r-i)}))$ $(0 \leq i \leq r-1)$. For each i, the morphism

$$H^0(X^{(1)}, \Omega_{X^{(1)}}^{\otimes(r-i)}) \to H^0(X, F_{X/k}^*(\Omega_{X^{(1)}}^{\otimes(r-i)}))$$

induced from $\Omega_{X^{(1)}}^{\otimes(r-i)} \to F_{X/k*}(F_{X/k}^*(\Omega_{X^{(1)}}^{\otimes(r-i)}))$ is injective, so we regard $H^0(X^{(1)}, \Omega_{X^{(1)}}^{\otimes(r-i)})$ as a subspace of $H^0(X, F_{X/k}^*(\Omega_{X^{(1)}}^{\otimes(r-i)}))$. Moreover, it is well-known that a_i lies in $H^0(X^{(1)}, \Omega_{X^{(1)}}^{\otimes(r-i)})$ (cf. [26, Proposition 3.2]).

Denote by $\mathcal{M}_{\mathrm{dR}}$ the moduli stack classifying flat vector bundles on X of rank r, and write

$$B^{(1)} := \mathrm{Spec}\left(\mathrm{Sym}\left(\bigoplus_{i=1}^{r} H^0(X^{(1)}, \Omega_{X^{(1)}}^{\otimes i})\right)^{\vee}\right).$$

We identify $B^{(1)}$ with the moduli space classifying polynomials of the form $t^r + a_{r-1}t^{r-1} + \cdots + a_0$ with $a_i \in H^0(X^{(1)}, \Omega_{X^{(1)}}^{\otimes(r-i)})$. The assignment $(\mathcal{E}, \nabla) \mapsto \mathrm{Char}(\mathcal{E}, \nabla)$ determines a k-morphism

$$\mathrm{Hitch} : \mathcal{M}_{\mathrm{dR}} \to B^{(1)},$$

which is called the **p-Hitchin morphism** (cf. [7], [11], [12], [18], [22], [26], [28], and [29]). The image of this morphism contains a dense open subscheme of $B^{(1)}$ (cf. [7, Proposition 4.3, Lemma 4.8]).

Next, we shall write

$$D^{(1)} := \mathrm{Spec}\left(\mathrm{Sym}\left(H^0(X^{(1)}, \Omega_{X^{(1)}})^{\oplus r}\right)^{\vee}\right).$$

This space parametrizes collections of r-tuples $(b_1, \cdots, b_r) \in H^0(X^{(1)}, \Omega_{X^{(1)}})^{\oplus r}$. The assignment $(b_1, \cdots, b_r) \mapsto \prod_{i=1}^{r}(t - b_i)$ determines a k-morphism $\gamma : D^{(1)} \to B^{(1)}$. This morphism captures a special subclass of characteristic polynomials corresponding to cases where the p-curvature is diagonalizable in a suitable basis. Observe that

$$\begin{aligned}\dim_k(\bigoplus_{i=1}^{r} H^0(X^{(1)}, \Omega_{X^{(1)}}^{\otimes i})) &= \sum_{i=1}^{r} \dim_k(H^0(X^{(1)}, \Omega_{X^{(1)}}^{\otimes i})) \\ &= g + \sum_{i=2}^{r}(2i-1)(g-1) \\ &= g + (r^2-1)(g-1) \\ &> rg \\ &= \dim_k(H^0(X^{(1)}, \Omega_{X^{(1)}})^{\oplus r}).\end{aligned}$$

Hence, the morphism γ is not dominant, which implies $\mathrm{Hitch}^{-1}(B^{(1)} \setminus \mathrm{Im}(\gamma)) \neq \emptyset$.

Proposition 5.1. *Let q be a k-rational point of $\mathrm{Hitch}^{-1}(B^{(1)} \setminus \mathrm{Im}(\gamma))$, and denote by $(\mathcal{E}, \nabla)$ the flat vector bundle classified by q. Then, $(\mathcal{E}, \nabla)$ does not admit any complete flag.*

Proof. Suppose, on the contrary, that there exists a complete flag $\{\mathcal{E}^j\}_{j=0}^{r}$ on $(\mathcal{E}, \nabla)$. For each $i \in \{0, 1\}$, let $\mathcal{E}nd(\mathcal{E})^j$ denote the subsheaf of $\mathcal{E}nd(\mathcal{E})$ consisting of endomorphisms h of $\mathcal{E}$ with $h(\mathcal{E}^j) \subseteq \mathcal{E}^{j-i}$ for every i. Since each $\mathcal{E}^j$ is preserved by ∇, the connection $\nabla^\vee \otimes \nabla$ on $\mathcal{E}nd(\mathcal{E})$ restricts to a connection ∇^0 (resp., ∇^1) on $\mathcal{E}nd(\mathcal{E})^0$ (resp., $\mathcal{E}nd(\mathcal{E})^1$). We obtain a composite of natural morphisms

$$\mathcal{E}nd(\mathcal{E})^0 \twoheadrightarrow \mathcal{E}nd(\mathcal{E})^0/\mathcal{E}nd(\mathcal{E})^1 \xrightarrow{\sim} \bigoplus_{j=1}^{r} \mathcal{E}nd(\mathcal{E}^j/\mathcal{E}^{j-1}) \xrightarrow{\sim} \mathcal{O}_X^{\oplus r}, \tag{5.1}$$

which is compatible with the respective connections ∇^0 and $(\nabla_{\mathrm{triv}}^{(0)})^{\oplus r}$. The p-curvature $\psi(\nabla)$ belongs to $H^0(X, F_{X/k}^*(\Omega_{X^{(1)}}) \otimes \mathcal{E}nd(\mathcal{E})^0)$, and its image under the morphism

$$H^0(X, F_{X/k}^*(\Omega_{X^{(1)}}) \otimes \mathcal{E}nd(\mathcal{E})^0) \to H^0(X, F_{X/k}^*(\Omega_{X^{(1)}}))^{\oplus r}$$

induced from the tensor product of (5.1) and the identity morphism of $F_{X/k}^*(\Omega_{X^{(1)}})$ determines an n-tuple $(b_1, \cdots, b_n)$ of elements in $H^0(X^{(1)}, \Omega_{X^{(1)}})$ $\left(\subseteq H^0(X, F_{X/k}^*(\Omega_{X^{(1)}}))\right)$. It follows that the characteristic polynomial $\mathrm{Char}(\mathcal{E}, \nabla)$ is given by $\prod_{i=1}^{n}(t - b_i)$, i.e., coincides with $\gamma(b_1, \cdots, b_n) \in \mathrm{Im}(\gamma)$. This contradicts the assumption $\mathrm{Hitch}^{-1}(B^{(1)} \setminus \mathrm{Im}(\gamma))$, thus completing the proof of this proposition. □

5.2. **The non-existence of complete flags.** The following statement completes the remaining part of Theorem A.

Theorem 5.2. *Let m be an element of $\mathbb{Z}_{\geq 0} \sqcup \{\infty\}$ and r a positive integer. Then, there exists a $\mathcal{D}^{(m)}$-bundle of rank r that does not admit any complete flag.*

Proof. The assertion for $m = 0$ is a direct consequence of Proposition 5.1, and this assertion also induces the case of $m \in \mathbb{Z}_{>0}$ by applying Proposition 2.3 in the case where the pair "(m, s)" is taken to be $(0, m)$.

Next, we shall consider $m = \infty$. It follows from [14, Theorem 1.2] that there exists a stable vector bundle $\mathcal{E}$ on X admitting an isomorphism $\nu : F_X^{s*}(\mathcal{E}) \xrightarrow{\sim} \mathcal{E}$ for some $s \in \mathbb{Z}_{>0}$. In particular, since ν yields an F-divided sheaf structure on $\mathcal{E}$ (cf. Remark 2.1), $\mathcal{E}$ carries a $\mathcal{D}^{(\infty)}$-module structure ∇. Note that $(\mathcal{E}, \nabla)$ is irreducible. Indeed, if $(\mathcal{E}, \nabla)$ has a proper $\mathcal{D}^{(\infty)}$-subbundle $(\mathcal{E}', \nabla')$, then (by considering, e.g., the F-divided structure on its determinant) we see that $\mathcal{E}'$ must have degree 0. However, it contradicts the stability of $\mathcal{E}$. In particular, $(\mathcal{E}, \nabla)$ does not admit any complete flag. This completes the proof for $m = \infty$. □

Remark 5.3 (Other examples using dormant opers)**.** In the proof of the above assertion for $m = \infty$, we constructed an irreducible $\mathcal{D}^{(\infty)}$-bundle whose underlying vector bundle is stable. Here, we provide an example of an *unstable* bundle (of rank 2) that does not admit any complete flag.

To align with the notation in [30], we set $N := m+1$. Fix a theta characteristic Θ of X, i.e., a line bundle Θ equipped with an isomorphism $\eta : \Theta^{\otimes 2} \xrightarrow{\sim} \Omega_X$. We shall regard $\nabla_{\mathrm{triv}}^{(N-1)}$ as a $\mathcal{D}^{(N-1)}$-module structure on $\mathcal{T}_X \otimes \Theta^{\otimes 2}$ via the identification with $\mathcal{O}_X$ arising from η. Thus, the pair $\vartheta := (\Theta, \nabla_{\mathrm{triv}}^{(N-1)})$ defines a $2^{(N)}$-theta characteristic, in the sense of [30, Definition 5.12]. We set $\mathcal{E}_\Theta := \mathcal{D}_{\leq 1}^{(N-1)} \otimes \Theta$ and $\mathcal{E}_\Theta^1 := \mathcal{D}_{\leq 0}^{(N-1)} \otimes \Theta\, (\subseteq \mathcal{E}_\Theta)$.

Following [30, Definition 5.15], a $(\mathrm{GL}_2^{(N)}, \vartheta)$**-oper** on X is defined as a $\mathcal{D}^{(N-1)}$-module structure ∇ on $\mathcal{E}_\Theta$ satisfying the following two conditions:

- The $\mathcal{O}_X$-linear morphism $\mathcal{D}^{(N-1)} \otimes \mathcal{E}_\Theta \to \mathcal{E}_\Theta$ given by ∇ restricts to an isomorphism $\mathcal{D}_{\leq 1}^{(N-1)} \otimes \mathcal{E}_\Theta^1 \xrightarrow{\sim} \mathcal{E}_\Theta$.
- The connection $\deg(\nabla)$ on $\det(\mathcal{E}_\Theta)$ induced naturally from ∇ coincides with $\nabla_{\mathrm{triv}}^{(N-1)}$ via the natural composite

$$\det(\mathcal{E}_\Theta) \xrightarrow{\sim} (\mathcal{E}_\Theta / \mathcal{E}_\Theta^1) \otimes \mathcal{E}_\Theta \xrightarrow{\sim} (\mathcal{T}_X \otimes \Theta) \otimes \Theta \xrightarrow{\sim} \mathcal{T}_X \otimes \Theta^{\otimes 2}.$$

Moreover, a $(\mathrm{GL}_2^{(N)}, \vartheta)$-oper ∇ is said to be **dormant** if it has vanishing p^N-curvature. For the study of opers in positive characteristic, we refer the reader to [22], [23], [28], [29].

The notion of an isomorphism between $(\mathrm{GL}_2^{(N)}, \vartheta)$-opers are defined in a suitable manner (though we here omit the details). In particular, we obtain the set $\mathcal{O}p_N^{\mathrm{Zzz...}}$ consisting of isomorphism classes of dormant $(\mathrm{GL}_2^{(N)}, \vartheta)$-opers on X. Under the assumption $p > 2$, we know that $\mathcal{O}p_N^{\mathrm{Zzz...}}$ is a nonempty finite set for every $N \in \mathbb{Z}_{>0}$ (cf. [30, Eq. (5.8.2), Theorem 6.16, and Proposition 6.26]), and that the limit of the projective system

$$\cdots \to \mathcal{O}p_N^{\mathrm{Zzz...}} \to \cdots \to \mathcal{O}p_2^{\mathrm{Zzz...}} \to \mathcal{O}p_1^{\mathrm{Zzz...}}$$

obtained by reducing levels is nonempty (cf. [30, Corollary 8.28]). By choosing an element of this limit, we obtain a $\mathcal{D}^{(\infty)}\left(= \varinjlim_{N \in \mathbb{Z}_{>0}} \mathcal{D}^{(N)}\right)$-module structure on $\mathcal{E}_\Theta$.

Note that $(\mathcal{E}_\Theta, \nabla)$ is irreducible. Indeed, suppose, on the contrary, that it admits a line $\mathcal{D}^{(\infty)}$-subbundle $(\mathcal{L}, \nabla_{\mathcal{L}})$. Then, $\mathcal{L}$ must have degree 0, so the natural composite $\mathcal{L} \hookrightarrow \mathcal{E}_\Theta \to \mathcal{E}_\Theta / \mathcal{E}_\Theta^1\, (\cong \Theta^\vee)$ coincides with the zero map because $0 = \deg(\mathcal{L}) > \deg(\Theta^\vee) = 1 - g$. This means that $\mathcal{L}$ is contained in $\mathcal{E}_\Theta^1$. Since $0 = \deg(\mathcal{L}) < \deg(\mathcal{E}_\Theta^1) = g - 1$, this contradicts the

assumption that $\mathcal{L}$ is a subbundle of $\mathcal{E}_\Theta$. Consequently, $(\mathcal{E}_\Theta, \nabla)$ is irreducible, and in particular, does not admit any complete flag.

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