Periodic Homogenization of Jump-Diffusion Processes on a Bounded Open Set *

Toshihiro Uemura **

June 29, 2026

Abstract

We consider the periodic homogenization problem for jump-diffusion processes with singular potentials on a bounded open set by using the theory of semi-Dirichlet forms and the method of unfolding operators.

1 Introduction

We are concerned with the following classical periodic homogenization:

$$\begin{cases} \mathcal{L}_\delta u_\delta + \lambda u_\delta = f, & \text{in } D, \\ u_\delta = 0, & \text{on } \partial D, \end{cases} \tag{1.1}$$

where $\delta > 0$ is a small parameter used to represent the heterogeneities of the body D, a bounded open set of $\mathbb{R}^d$ $(d \geq 3)$, $\lambda > 0$ and a source $f \in H^{-1}(D)$, the dual space of $H_0^1(D)$, is understood as an external force. Here $\mathcal{L}_\delta$ is formally defined as the sum of a second order elliptic differential operator, a first order differential (drift) operator, an integro-differential operator and a potential (killing) operator:

$$\begin{aligned} \mathcal{L}_\delta u &:= \mathcal{L}^c_\delta u + \mathcal{L}^d_\delta u + \mathcal{L}^j_\delta u + \mathcal{L}^k_\delta u \\ &:= -\mathrm{div}\Big(A\Big(\frac{\cdot}{\delta}\Big)\nabla u\Big) + B\Big(\frac{\cdot}{\delta}\Big)\cdot\nabla u - \lim_{n\to\infty}\int_{|h|>1/n}\big(u(\,\cdot\,+h) - u(h)\big)\tilde{J}^\delta(\cdot,h)dh + k\Big(\frac{\cdot}{\delta}\Big)u, \end{aligned} \tag{1.2}$$

where $A(x) = (a_{ij}(x))$ is a symmetric $d \times d$-matrix-valued measurable function on $\mathbb{R}^d$, $B(x) = (b_i(x))$ is a d-dimensional vector-valued measurable function on $\mathbb{R}^d$ and $k(x)$ is a nonnegative measurable function on $\mathbb{R}^d$. For the function $\tilde{J}^\delta$, it is given by $\tilde{J}^\delta(x,h) = \frac{1}{2}(J^\delta(x,h) + J^\delta(x+h,h))$, where $J^\delta(x,h) := J(x/\delta, h)$ for some measurable function $J(x,h) = J(x,-h) \geq 0,\ x,h \in \mathbb{R}^d$ with $h \neq 0$. We assume that

$$x \mapsto A(x), \quad x \mapsto B(x), \quad x \mapsto J(x,h) \quad \text{and} \quad x \mapsto k(x)$$

*2020 Mathematics Subject Classification: Primary 31C25; Secondary 60J46, 60J25

Keywords and Phrases: Homogenization, Jump-diffusion, Semi-Dirichlet form, Periodic unfolding method

**Department of Mathematics, Faculty of Engineering Science, Kansai University, Suita, Osaka Japan, t-uemura@kansai-u.ac.jp

are all periodic (for each $h \neq 0$ in $J(x,h)$). In the present work, we employ a symmetrized jump kernel $\tilde{J}^\delta(x,h)$ defined as the average of the kernel at the starting and ending points of a jump. While the overall system (1.2) is non-symmetric due to the presence of the drift term, this construction ensures that the non-local operator remains a self-adjoint principal part. This separation allows for a clear distinction between the diffusive effects of the jumps and the convective effects of the drift in the homogenization limit.

To provide a rigorous variational formulation, the problem (1.1) is interpreted in the weak sense: find $u_\lambda^\delta \in H_0^1(D)$ for given $f \in H^{-1}(D)$, $\delta > 0$ and $\lambda > 0$ so that

$$\mathcal{E}_\lambda^\delta(u_\lambda^\delta, \varphi) := \mathcal{E}^\delta(u_\lambda^\delta, \varphi) + \lambda(u_\lambda^\delta, \varphi) = \langle f, \varphi \rangle_{H^{-1},H_0^1}, \quad \varphi \in H_0^1(D), \tag{1.3}$$

where

$$\begin{aligned}\mathcal{E}^\delta(u,v) &:= \mathcal{E}^{c,\delta}(u,v) + \mathcal{E}^{d,\delta}(u,v) + \mathcal{E}^{j,\delta}(u,v) + \mathcal{E}^{k,\delta}(u,v) \\ &:= \int_D A(x/\delta)\nabla u(x) \cdot v(x)dx + \int_D B(x/\delta) \cdot \nabla u(x)\, v(x)dx \\ &\quad + \iint_{\substack{x \in D, x+h \in D \\ h \neq 0}} \big(u(x+h) - u(x)\big)\big(v(x+h) - v(x)\big) J(x/\delta, h)dhdx + \int_D u(x)v(x)k(x/\delta)dx.\end{aligned} \tag{1.4}$$

Here $(u,v) := \int_D u(x)v(x)dx$ denotes the inner product of u, v, $\|u\| := \sqrt{(u,u)}$ is the norm of u in $L^2(D) := L^2(D; dx)$ and $\langle f, \varphi \rangle_{H^{-1},H_0^1}$ means the duality pairing of $f \in H^{-1}(D)$ and $\varphi \in H_0^1(D)$. It is worth noting that while the jump term in (1.4) is written using $J(x/\delta, h)$, it is equivalent to the one using the symmetrized kernel $\tilde{J}^\delta$ due to the change of variables and the symmetry $J(x,h) = J(x,-h)$. Consequently, the symmetric part of the bilinear form $\mathcal{E}^\delta$ consists of the second-order diffusion, the jump part and potential (killing) part, while the anti-symmetric part arises solely from the drift term B.

The basic homogenization procedure involves the following three steps:

(1) The existence of solutions $\{u_\delta\}_{\delta>0} \subset H_0^1(D)$ of the equations (1.1) or (1.3) for $f \in H^{-1}(D)$ and $\delta > 0$;

(2) The convergence of $\{u_\delta\}$ as $\delta \to 0$;

(3) The construction of the *homogenized equation* which the limit satisfies.

To carry out the program, we again employ the *unfolding method* and the techniques of the theory of *(semi-)Dirichlet forms* (see: [15], *c.f.*, [17] for the symmetric case). Of course various methods, such as G-convergence, H-convergence, two-scale convergence and so on, have been developed in the theory of the partial differential equations (*e.g.,* [1, 2, 3, 5, 4, 11, 12]).

For the step (1), we will show in the next section that the quadratic forms $\mathcal{E}^\delta$ induce regular lower bounded semi-Dirichlet forms on $L^2(D)$ with a uniform constant $\beta_0 > 0$ (independent of $\delta > 0$) and the Sobolev space $H_0^1(D)$ is the (uniform) domain of the forms under appropriate conditions on the data $\{A, B, J, k\}$ in addition to the periodicity. This ensures the well-posedness of the problem and provides uniform a priori estimates for the solutions for $f \in H^{-1}(D)$, $\delta > 0$ and $\lambda > \beta_0$. In particular, when the external force f belongs to $L^2(D)$, the solutions $\{u_\lambda^\delta\}$ of homogenization problems (1.1) are given as the (L^2-)resolvents $G^\delta_\lambda f(= u^\delta_\lambda)$ of the semi-Dirichlet forms. By using the Banach-Alaoglu theorem and the Rellich theorem, we can find a converging subsequence which partially justifies step (2). Finally using the *unfolding operators* in conjunction with the *corrector functions*, the limit is uniquely characterized (which, in turn, implies the convergence of the entire sequence) and then the step (3) is shown in the section 3.

As a by-product, the resolvents of the semi-Dirichlet forms $(\mathcal{E}^\delta, H^1_0(D))$ on $L^2(D)$ converge to the resolvent of a semi-Dirichlet form $(\mathcal{E}^0, H^1_0(D))$ on $L^2(D)$ *strongly* on $L^2(D)$ as $\delta \to 0$, or equivalently, $(\mathcal{E}^\delta, H^1_0(D))$ converges to $(\mathcal{E}^0, H^1_0(D))$ on $L^2(D)$ in the sense of Mosco (see, [10, 13]). Since the semi-Dirichlet forms are regular and so there are strong Markov processes (in fact, which are Hunt processes) on D associated with $(\mathcal{E}^\delta, H^1_0(D))$, also called, jump-diffusion processes, we can conclude that the jump-diffusion processes converge in the sense of finite dimensional distributions as $\delta \to 0$.

2 Construction of lower bounded semi-Dirichlet forms

Let $Y = (0,1]^d$ be the d-dimensional unit cube ($d \geq 3$). Suppose that $A(x) = (a_{ij}(x))$ is a symmetric $d \times d$-matrix valued measurable function on $\mathbb{R}^d$, $B(x) = (b_i(x))$ is a d-dimensional vector-valued measurable function on $\mathbb{R}^d$ and $k(x)$ is a nonnegative measurable function on $\mathbb{R}^d$ so that

$$x \mapsto A(x), \quad x \mapsto B(x) \quad \text{and} \quad x \mapsto k(x)$$

are all Y-*periodic* in the sense that, for a function $f(x)$ on $\mathbb{R}^d$,

$$f(x + k\boldsymbol{e}_i) = f(x), \quad \text{a.e. } x \in \mathbb{R}^d, \quad k \in \mathbb{Z},\ i = 1, 2, \cdots, d$$

hold, where $\boldsymbol{e}_i$ is the i-th directional unit vector on $\mathbb{R}^d$. Suppose also that $J(x,h)$ is a measurable function such that $J(x,h) = J(x,-h) \geq 0$ for $x, h \in \mathbb{R}^d$ with $h \neq 0$ and $x \mapsto J(x,h)$ is Y-periodic for each $h \neq 0$. We impose the following conditions:

(A) (uniform ellipticity and boundedness) $\exists\, \alpha, \beta > 0$ s.t. $\alpha|\xi|^2 \leq A(x)\xi \cdot \xi \leq \beta|\xi|^2,\ x,\ \xi \in \mathbb{R}^d$;

(B) (integrability-1) $\exists\, q_1 \in (d, \infty]$ s.t. $b_i \in L^{q_1}_{\mathsf{loc}}(\mathbb{R}^d)$ for $i = 1, 2, \cdots, d$;

(K) (integrability-2) $\exists\, q_2 \in (d/2, \infty]$ s.t. $k \in L^{q_2}_{\mathsf{loc}}(\mathbb{R}^d)$ and $k \geq 0$;

(J) (Lévy condition) $\exists\, \varphi : \mathbb{R}^d \setminus \{0\} \to [0, \infty)$ s.t.

$$\operatorname*{ess\,sup}_{x \in \bar{Y}} J(x,h) \leq \varphi(h),\ h \neq 0 \quad \text{and} \quad M := \int_{h \neq 0} (1 \wedge |h|^2)\varphi(h)dh < \infty.$$

Let D be a bounded open set of $\mathbb{R}^d$. Recall that the Dirichlet integral is given by

$$\mathbb{D}(u,u) = \int_D \nabla u(x) \cdot \nabla v(x) dx$$

and the Sobolev space

$$H^1(D) := \left\{ u \in L^2(D) : \frac{\partial u}{\partial x_i} \in L^2(D),\ i = 1, 2, \cdots, d \right\}$$

equipped with the scalar product

$$\mathbb{D}_1(u,v) := \mathbb{D}(u,v) + (u,v)$$

is a Hilbert space. The space $H_0^1(D)$ is the closure of $C_0^\infty(D)$ with respect to $H^1(D)$-norm, $\sqrt{\mathbb{D}_1(\cdot,\cdot)}$. Recall also that the following Poincaré inequality holds for a bounded open set D:

$$\int_D u(x)^2 dx \le c_p \mathbb{D}(u,u), \quad u \in H_0^1(D).$$

for some $c_p > 0$ (see *e.g.,* [6, Chapter 5]). To show that $(\mathcal{E}^\delta, H_0^1(D))$ are lower-bounded semi-Dirichlet forms on $L^2(D)$ for $\delta > 0$, we introduce the following proposition:

Proposition 2.1 *Assume* (A), (B), (J) *and* (K) *are satisfied. Then the following inequalities hold*:

(1) $\alpha \mathbb{D}(u,u) \le \mathcal{E}^{c,\delta}(u,u) = \int_D A(x/\delta)\nabla u(x)\cdot\nabla u(x)dx \le \beta\mathbb{D}(u,u), \quad u \in C_0^\infty(D),\ \delta > 0;$

(2) $\exists\, c_1 > 0$ s.t. $u \in C_0^\infty(D)$ *and* $\delta > 0$,

$$\big(0 \le\big)\ \mathcal{E}^{j,\delta}(u,u) = \iint_{\substack{x\in D, x+h\in D\\ h\neq 0}} \big(u(x+h)-u(x)\big)^2 J(x/\delta,h)dhdx \le c_1\mathbb{D}_1(u,u); \tag{2.1}$$

(3) $\forall\,\lambda > 0,\ \exists\, c_2 > 0$ s.t. *for* $u \in C_0^\infty(D)$ *and* $\delta > 0$,

$$\big|\mathcal{E}^{d,\delta}(u,u)\big| = \Big|\int_D B(x/\delta)\cdot\nabla u(x)u(x)dx\Big| \le \lambda\mathbb{D}(u,u) + c_2\|u\|^2, \tag{2.2}$$

(4) $\exists\, c_3 > 0$ s.t. $u \in C_0^\infty(D)$ *and* $\delta > 0$,

$$\int_D u^2(x)k(x/\delta)dx \le c_3\mathbb{D}_1(u,u). \tag{2.3}$$

Proof: (1) follows from (A). (3) and (4) are well-known (*e.g.,* [15, Lemma 2.1]). So we omit their proofs. We only show (2). From (J1), we see that for $\delta > 0$ and $u \in C_0^\infty(D)$,

$$\begin{aligned}\mathcal{E}^{j,\delta}(u,u) &\le \iint_{\substack{x\in D, x+h\in D\\ h\neq 0}} \big(u(x+h)-u(x)\big)^2\varphi(h)dhdx\\ &= \Big(\iint_{\substack{x\in D, x+h\in D\\ |h|\ge 1}} + \iint_{\substack{x\in D, x+h\in D\\ 0<|h|<1}}\Big)\big(u(x+h)-u(x)\big)^2\varphi(h)dhdx =: (\mathsf{I}) + (\mathsf{II}).\end{aligned}$$

$$(\mathsf{I}) \le 2\iint_{\substack{x\in D, x+h\in D\\ |h|\ge 1}} \big(u(x+h)^2 + u(x)^2\big)\varphi(h)dhdx \le 4M_1\int_D u(x)^2dx = 4M_1\|u\|^2.$$

As for (II), by using the mean value theorem and then the Jensen inequality, we see that

$$(\mathsf{II}) = \iint_{\substack{x\in D, x+h\in D\\ 0<|h|<1}} \Big(\int_0^1 h\cdot\nabla u(x+th)dt\Big)^2\varphi(h)dhdx$$

$$\leq \iint_{\substack{x\in D, x+h\in D\\ 0<|h|<1}} \Big(\int_0^1 |h|^2|\nabla u(x+th)|^2 dt\Big)\varphi(h)dhdx$$
$$= \int_0^1 \int_{0<|h|<1} \Big(\int_D 1_D(x+h)|\nabla u(x+th)|^2 dx\Big)|h|^2\varphi(h)dhdt.$$

Since $u \in C_0^\infty(D)$, it can be naturally extended to $C_0^\infty(\mathbb{R}^d)$ by setting $u = 0$ outside D. Under this extension, we have $|\nabla u(z)| = 0$ for $z \notin D$. Thus, by the change of variables $z = x + th$, the inner integral is estimated as follows:

$$\int_D 1_D(x+h)|\nabla u(x+th)|^2 dx \leq \int_{\mathbb{R}^d} |\nabla u(z)|^2 dz = \int_D |\nabla u(z)|^2 dz.$$

Therefore, the last inequality is estimated by

$$(\mathsf{II}) \leq \int_{0<|h|<1} \Big(\int_D |\nabla u(z)|^2 dz\Big)|h|^2\varphi(h)dh \leq M_1\mathbb{D}(u,u).$$

Hence, setting $c_1 = 4M_1$,

$$\mathcal{E}^{j,\delta}(u,u) \leq M_1\mathbb{D}(u,u) + 4M_1\|u\|^2 \leq c_1\mathbb{D}_1(u,u), \quad u \in C_0^\infty(D)$$

holds. $\square$

In the following, we briefly introduce the periodic Sobolev spaces and the unfolding operators following [5] and [4].

Periodic Sobolev Spaces

Recall that $Y = (0,1]^d$ is the unit cube in $\mathbb{R}^d$. Define the periodic Sobolev space $H_\#(Y)$ by the closure of $C_\#^\infty(Y)$, the restriction of the smooth functions defined on $\mathbb{R}^d$ which are Y-periodic, with respect to $H^1(Y)$-norm:

$$\|u\|^2_{H^1(Y)} := \int_Y \nabla u(x)\cdot\nabla u(x)dx + \int_Y u(x)^2 dx, \quad u \in H^1(Y),$$

and $H_\#(Y)$ is a Hilbert space with the same $H^1(Y)$-inner product.

Lemma 2.1 ([5, Proposition3.49]) *If $u \in H_\#(Y)$, then the trace of the "opposite site of Y" are equal.*

More specifically, this lemma means that for a function $u \in H_\#(Y)$ defined on the semi-open unit cell $Y = (0,1]^d$, the trace on the boundary where $x_i = 1$ (the "included" face) coincides with the trace on the opposite boundary where $x_i = 0$ (the "missing" face) for each $i \in \{1,\dots,d\}$. This ensures that u can be extended to a Y-periodic function on $\mathbb{R}^d$ without any discontinuity at the cell interfaces.

We also define the quotient space $\mathcal{H}_\#(Y) := H_\#(Y)/\mathbb{R}$ of $H_\#(Y)$ under the equivalent relation "$\sim$":

$$u \sim v \iff u - v = \text{constant} \quad \text{a.e. on } Y.$$

In other words, $\mathcal{H}_\#(Y)$ is the collections of all equivalence classes $\tilde{u}$ of functions u in $H_\#(Y)$:

$$\tilde{u} := \Big\{v \in H_\#(Y):\ u \sim v\Big\}.$$

Lemma 2.2 ([5, Proposition3.52]) *The space* $\mathcal{H}_{\#}(Y)$ *is also a Hilbert space under the following inner product*:

$$((\tilde{u},\tilde{v})) := \int_Y \nabla f(x)\cdot\nabla g(x)dx, \quad f\in\tilde{u},\ g\in\tilde{v}, \quad \tilde{u},\tilde{v}\in\mathcal{H}_{\#}(Y).$$

Moreover the dual space $(\mathcal{H}_{\#}(Y))'$ *of* $\mathcal{H}_{\#}(Y)$ *is identified with the set*

$$\Big\{\ell\in\big(H_{\#}(Y)\big)' : \ \ell(c)=0, \quad c\in\mathbb{R}\Big\},$$

where

$$\ell(\tilde{u}) := \langle \ell, f\rangle_{(H_{\#}(Y))', H_{\#}(Y)}, \quad f\in\tilde{u}.$$

For a normed space $(\mathcal{A},\|\cdot\|_{\mathcal{A}})$, the space $L^2(D;\mathcal{A})$ means the set of all functions $u: D\to\mathcal{A}$ so that the map $x\mapsto\|u(x)\|_{\mathcal{A}}$ is measurable and satisfies $\int_D\|u(x)\|^2_{\mathcal{A}}dx<\infty$. In particular, a function $u\in L^2(D;\mathcal{H}_{\#}(Y))$ is identified with a function $\tilde{u}: D\times\mathbb{R}^d\to\mathbb{R}$ so that $u(x)(y)=\tilde{u}(x,y)$, where $y\mapsto\tilde{u}(x,y)$ belongs to $\mathcal{H}_{\#}(Y)$ for each x and $x\mapsto\|\tilde{u}(x,\cdot)\|_{\mathcal{H}_{\#}(Y)}$ belongs to $L^2(D)$.

Unfolding Operators

For each $x\in\mathbb{R}^d$, one has $x=\lfloor x\rfloor+\{x\}$, where $\lfloor x\rfloor=(m_1,m_2,\dots,m_d)^{\mathsf{T}}\in\mathbb{Z}^d$ denotes the unique integer combination $\sum_{j=1}^d m_j\boldsymbol{e}_j$ such that $\{x\}=x-\lfloor x\rfloor$ belongs to $Y=(0,1]^d$ a.e. Then for a.e. $x\in\mathbb{R}^d$ and $\delta>0$,

$$x=\delta\Big(\Big\lfloor\frac{x}{\delta}\Big\rfloor+\Big\{\frac{x}{\delta}\Big\}\Big).$$

Following [4], define

$$\Xi_\delta := \Big\{\xi\in\mathbb{Z}^d : \ \delta(\xi+Y)\subset D\Big\}, \qquad D_\delta := \text{interior}\Big\{\bigcup_{\xi\in\Xi_\delta}\delta\big(\xi+\bar{Y}\big)\Big\}, \quad \bar{Y}=[0,1]^d,$$

$$\text{and} \quad \Lambda_\delta := D\setminus D_\delta.$$

The set D_δ is the largest union of $\delta\big(\xi+\bar{Y}\big)$ cells for $\xi\in\mathbb{Z}^d$ included in D.

Definition 2.1 *For a measurable function* ϕ *on* D_δ, *the unfolding operator* $\mathcal{T}_\delta$ *is defined by*

$$\mathcal{T}_\delta(\phi)(x,y)=\begin{cases}\phi\Big(\delta\Big\lfloor\dfrac{x}{\delta}\Big\rfloor+\delta y\Big), & \text{a.e. } (x,y)\in D_\delta\times Y,\\[2ex] 0, & \text{a.e. } (x,y)\in\Lambda_\delta\times Y.\end{cases} \tag{2.4}$$

Note that the unfolding operator $\mathcal{T}_\delta$ satisfies the following properties for measurable functions u and v:

$$\mathcal{T}_\delta(uv)=\mathfrak{T}_\delta(u)\mathcal{T}_\delta(v), \qquad \mathcal{T}_\delta\big(f(u)\big)=f(\mathcal{T}_\delta(u)),$$

where f is a continuous function on $\mathbb{R}$ with $f(0)=0$.

Proposition 2.2 ([4, Proposition 1.5]) *Let* f *be a measurable function on* Y *and extend it by* Y*-periodicity to the whole space* $\mathbb{R}^d$ *as* $f(x)=f(\{x\})$, a.e. $x\in\mathbb{R}^d$. *Define the sequence* $\{f_\delta\}_{\delta>0}$ *by*

$$f_\delta(x)=f\Big(\frac{x}{\delta}\Big), \quad \text{a.e. } x\in\mathbb{R}^d.$$

Then

$$\mathcal{T}_\delta(f_\delta|_D)(x,y) = \begin{cases} f(y), & \text{for a.e. } (x,y) \in D_\delta \times Y, \\ 0, & \text{for a.e. } (x,y) \in \Lambda_\delta \times Y \end{cases}$$

Moreover, if $f \in L^p(Y)$ $(1 \le p < \infty)$, $\mathcal{T}_\delta(f_\delta|_D)$ converges to f strongly in $L^p(D \times Y)$.

Proposition 2.3 ([4, Proposition 1.8]) *Suppose $1 \le p \le \infty$. The operator $\mathcal{T}_\delta$ is a linear and continuous from $L^p(D_\delta)$ into $L^p(D \times Y)$. For $u \in L^1(D_\delta)$, $v \in L^1(D)$ and $w \in L^p(D)$,*

(i) $\displaystyle\int_{D_\delta} u(x)dx = \iint_{D\times Y} \mathcal{T}_\delta(u)(x,y)dxdy;$

(ii) $\displaystyle\Big|\int_D v(x)dx - \iint_{D\times Y} \mathcal{T}_\delta(v)(x,y)dxdy\Big| \le \int_{\Lambda_\delta} |v(x)|dx;$

(iii) $\|\mathcal{T}_\delta(w)\|_{L^p(D\times Y)} \le \|w\|_{L^p(D)}$.

Remark 2.1 If $\{u_\delta\}_{\delta>0}$ is bounded in $L^2(D)$, then $\lim_{\delta\to 0}\int_{\Lambda_\delta}|u_\delta(x)|dx = 0$ holds. In fact, by means of the Schwarz inequality, we see

$$\int_{\Lambda_\delta} |u_\delta(x)|dx = \int_D |u_\delta(x)| 1_{\Lambda_\delta}(x)dx \le \sqrt{\mathsf{vol}(\Lambda_\delta)}\sqrt{\int_D u_\delta(x)^2 dx} \to 0 \quad as \quad \delta \to 0.$$

□

Proposition 2.4 ([4, Proposition 1.9]) *Suppose $1 \le p < \infty$.*

(i) *For $u \in L^p(D)$, $\mathcal{T}_\delta(u)$ converges to u strongly in $L^p(D \times Y)$ as $\delta \to 0$.*

(ii) *Let $\{u_\delta\}$ be a sequence in $L^p(D)$ such that u_δ converges to some u strongly in $L^p(D)$. Then $\mathcal{T}_\delta(u_\delta)$ converges to u strongly in $L^p(D \times Y)$.*

Proposition 2.5 ([4, Theorem 1.41]) *Let $\{u_\delta\}$ be a sequence in $H^1(D)$ such that u_δ converges to some u weakly in $H^1(D)$. Then for a subsequence, there exists some $\hat{u} \in L^2(D; \mathcal{H}_\#(Y))$ such that*

$$\mathcal{T}_\delta(\nabla u_\delta) \text{ converges to } \nabla u + \nabla_y \hat{u} \quad \text{weakly in } \big(L^2(D\times Y)\big)^d.$$

Based on these concepts, we now state one of our main theorems:

Theorem 2.1 *Let D be a bounded open set of $\mathbb{R}^d$ $(d \ge 3)$. Suppose that the functions*

$$x \mapsto A(x), \quad x \mapsto B(x), \quad x \mapsto J(x,h) \quad \text{and} \quad x \mapsto k(x)$$

are Y-periodic (for each $h \ne 0$ in $J(x,h)$). Suppose also that the conditions (A), (B), (J) *and* (K) *are satisfied by $A(x), B(x), J(x,h)$ and $k(x)$, respectively.*

Then, for $\delta > 0$, the quadratic forms $\mathcal{E}^\delta$ defined by (1.4) induce regular lower-bounded semi-Dirichlet forms on $L^2(D)$ having the Sobolev space $H^1_0(D)$ as their domains (uniformly in $\delta > 0$) with some index $\beta_0 > 0$: for $u, v \in H^1_0(D)$,

$$\mathcal{E}^\delta(u,v) := \mathcal{E}^{c,\delta}(u,v) + \mathcal{E}^{d,\delta}(u,v) + \mathcal{E}^{j,\delta}(u,v) + \mathcal{E}^{k,\delta}(u,v)$$

$$:= \int_D A(x/\delta)\nabla u(x)\cdot\nabla v(x)dx + \int_D B(x/\delta)\cdot\nabla u(x)v(x)dx$$
$$+ \iint_{\substack{x\in D, x+h\in D\\ h\neq 0}} \big(u(x+h)-u(x)\big)\big(v(x+h)-v(x)\big)J(x/\delta,h)dhdx + \int_D u(x)v(x)k(x/\delta)dx.$$

That is, the following are satisfied:

($\mathcal{E}$.1) (lower boundedness) $\mathcal{E}^\delta_{\beta_0}(u,u)\geq 0,\ u\in H^1_0(D)$;

($\mathcal{E}$.2) (weak sector condition) $\exists\ K>0$ *(independent of $\delta>0$)* s.t.

$$\big|\mathcal{E}^\delta(u,v)\big| \leq K\sqrt{\mathcal{E}^\delta_{\beta_0}(u,u)}\sqrt{\mathcal{E}^\delta_{\beta_0}(v,v)}, \quad u,v\in H^1_0(D);$$

($\mathcal{E}$.3) (closedness) $(H^1_0(D)(\gamma),\mathcal{E}^{\delta,(s)}_\gamma)$ *is a real Hilbert space for any/some* $\gamma>\beta_0$*. Here* $\mathcal{E}^{\delta,(s)}_\gamma(u,v):=\frac{1}{2}\big(\mathcal{E}^\delta(u,v)+\mathcal{E}^\delta(v,u)\big)+\gamma(u,v)$.

($\mathcal{E}$.4) (Markov property) $\mathcal{E}^\delta(u^\#,u-u^\#)\geq 0$ *for any* $u\in H^1_0(D)$*, where* $u^\#=(0\wedge u)\vee 1$ *is the unit contraction of a function* u.

($\mathcal{E}$.5) (regularity) $H^1_0(D)\cap C_0(D)$ *is dense both in* $C_0(D)$ *with respect to the uniform norm* $\|\cdot\|_\infty$ *and in* $H^1_0(D)$ *with respect to the norm* $\sqrt{\mathcal{E}^{\delta,(s)}_\gamma(\cdot,\cdot)}$ *for any/some* $\gamma>\beta_0$*, where* $C_0(D)$ *is the set of all continuous functions on* D *with compact support.*

Moreover the following inequality also holds: there exist $c_i>0$ $(i=4,5)$ s.t.

$$c_4\mathbb{D}_1(u,u)\leq \mathcal{E}^\delta_{\beta_0}(u,u)\leq c_5\mathbb{D}_1(u,u), \quad u\in H^1_0(D) \text{ and } \delta>0. \tag{2.5}$$

Proof: We first show the existence of a constant β_0 (uniformly in $\delta>0$). By (A),

$$\alpha\mathbb{D}(u,u)\leq \mathcal{E}^{c,\delta}(u,u)=\int_D A(x/\delta)\nabla u(x)\cdot\nabla u(x)dx\leq\beta\mathbb{D}(u,u), \quad u\in C^\infty_0(D).$$

Setting $\lambda=\alpha/2$ in (2.2), there exists a $c_2>0$ such that,

$$\big|\mathcal{E}^{d,\delta}(u,u)\big|\leq\frac{\alpha}{2}\mathbb{D}(u,u)+c_2\|u\|^2, \quad u\in C^\infty_0(D),\ \delta>0.$$

Therefore it follows that

$$\begin{aligned}\mathcal{E}^\delta(u,u) &= \mathcal{E}^{c,\delta}(u,u)+\mathcal{E}^{d,\delta}(u,u)+\underline{\mathcal{E}^{j,\delta}(u,u)}_{\geq 0}+\underline{\mathcal{E}^{k,\delta}(u,u)}_{\geq 0}\\ &\geq \alpha\mathbb{D}(u,u)-\big|\mathcal{E}^{d,\delta}(u,u)\big|\\ &\geq \frac{\alpha}{2}\mathbb{D}(u,,u)-c_2(u,u), \quad u\in C^\infty_0(D) \text{ and } \delta>0,\end{aligned}$$

and then, putting

$$\beta_0:=c_2>0,$$

we find that the bilinear forms $\mathcal{E}^\delta_{\beta_0}(\cdot,\cdot)$ are not just nonnegative, but also satisfy the following inequality:

$$\mathcal{E}^\delta_{\beta_0}(u,u)\geq\frac{\alpha}{2}\mathbb{D}(u,u)\geq 0, \quad u\in C^\infty_0(D),\ \delta>0.$$

On the other hand, by Proposition 2.1 (3)(4),

$$\mathcal{E}^{j,\delta}(u,u) \leq c_2\mathbb{D}_1(u,u), \quad u \in C_0^\infty(D),\ \delta > 0$$

and

$$\int_D u(x)^2 k(x/\delta)dx \leq c_3\mathbb{D}_1(u,u), \quad u \in C_0^\infty(D),\ \delta > 0$$

hold for some $c_2 > 0$ and $c_3 > 0$. Therefore it follows that

$$\frac{\alpha}{2}\mathbb{D}(u,u) \leq \mathcal{E}^\delta(u,u) + c_2\|u\|^2 \leq \Big(\beta + \frac{\alpha}{2} + c_2 + c_3\Big)\mathbb{D}(u,u) + \Big(c_1 + 2c_2 + c_3\Big)\|u\|^2$$

for $u \in C_0^\infty(D)$ and $\delta > 0$. By noting the Poincaré inequality and then taking closures of $C_0^\infty(D)$ with respect to $\mathcal{E}_\lambda^\delta(\cdot,\cdot)$-norms for $\lambda \geq \beta_0$, the Sobolev space $H_0^1(D)$ is the (uniform) domain of the bilinear forms $\mathcal{E}_\lambda^\delta$ for $\delta > 0$. Moreover we see that $(\mathcal{E}_{\beta_0}^\delta, H_0^1(D))$ satisfy

$$c_4\mathbb{D}_1(u,u) \leq \mathcal{E}_{\beta_0}^\delta(u,u) \leq c_5\mathbb{D}_1(u,u), \quad u \in H_0^1(D),\ \delta > 0. \tag{2.6}$$

for some positive constants c_4, c_5 independent of $\delta > 0$. Then the closedness $(\mathcal{E}.3)$ is clear from the inequality (2.6) and the Markov property $(\mathcal{E}.4)$ is shown as in [14, §1.5] (see also [8, 18]).

We finally show the weak sector condition $(\mathcal{E}.2)$. Note that the diffusion part $\mathcal{E}^{c,\delta}$, the symmetric jump part $\mathcal{E}^{j,\delta}$ and the potential part $\mathcal{E}^{k,\delta}$ are nonnegative symmetric forms on $L^2(D)$. So it is enough to estimate the term $\mathcal{E}^{d,\delta}$ only. Similar arguments in the proof of the proposition 2.1 tell us that the drift term is estimated as

$$\begin{aligned}\big|\mathcal{E}^{d,\delta}(u,v)\big| &= \Big|\int_D B(x/\delta)\nabla u(x)v(x)dx\Big| \leq \sum_{i=1}^d \int_D |b_i(x/\delta)\partial_i u(x)v(x)|dx \\ &\leq c_6\sqrt{\mathbb{D}_1(u,u)}\sqrt{\mathbb{D}_1(v,v)}, \quad u,v \in H_0^1(D)\end{aligned}$$

for some constant $c_6 > 0$ again independent of $\delta > 0$. Hence it follows that

$$\begin{aligned}\big|\mathcal{E}^\delta(u,v)\big| &\leq \big|\mathcal{E}^{c,\delta}(u,v)\big| + \big|\mathcal{E}^{d,\delta}(u,v)\big| + \big|\mathcal{E}^{j,\delta}(u,v)\big| + \big|\mathcal{E}^{k,\delta}(u,v)\big| \\ &\leq \sqrt{\mathcal{E}^{c,\delta}(u,u)}\sqrt{\mathcal{E}^{c,\delta}(v,v)} + \big|\mathcal{E}^{d,\delta}(u,v)\big| + \sqrt{\mathcal{E}^{j,\delta}(u,u)}\sqrt{\mathcal{E}^{j,\delta}(v,v)} + \sqrt{\mathcal{E}^{k,\delta}(u,u)}\sqrt{\mathcal{E}^{k,\delta}(v,v)} \\ &\leq \beta\sqrt{\mathbb{D}(u,u)}\sqrt{\mathbb{D}(v,v)} + c_6\sqrt{\mathbb{D}_1(u,u)}\sqrt{\mathbb{D}_1(v,v)} + c_2\sqrt{\mathbb{D}_1(u,u)}\sqrt{\mathbb{D}_1(v,v)} \\ &\quad + c_3\sqrt{\mathbb{D}_1(u,u)}\sqrt{\mathbb{D}_1(v,v)}\end{aligned}$$

for $u, v \in H_0^1(D)$, and then, using (2.6), the weak sector condition $(\mathcal{E}.2)$ follows for some $K > 0$, independent of $\delta > 0$:

$$\big|\mathcal{E}^\delta(u,v)\big| \leq K\sqrt{\mathcal{E}_{\beta_0}^\delta(u,u)}\sqrt{\mathcal{E}_{\beta_0}^\delta(v,v)}, \quad u,v \in H_0^1(D),\ \delta > 0.$$

□

We now show the existence of the solutions of (1.1) and the "a priori estimate":

Theorem 2.2 *Assume the conditions hold in the previous theorem. Then for each $f \in H^{-1}(D)$, $\lambda > \beta_0$ and $\delta > 0$, there exists a unique $u_\lambda^\delta \in H_0^1(D)$ such that*

$$\mathcal{E}_\lambda^\delta(u_\lambda^\delta, \varphi) = \langle f, \varphi\rangle_{H^{-1},H_0^1}, \quad \varphi \in H_0^1(D) \tag{2.7}$$

and, the sequence $\{u_\lambda^\delta\}_{\delta>0}$ *satisfies that*

$$\mathbb{D}_1(u_\lambda^\delta, u_\lambda^\delta) \le c_8 \|f\|^2_{H^{-1}(D)}, \quad 0 < \delta \tag{2.8}$$

for some $c_8 > 0$.

Proof: The Lax-Milgram theorem implies the following: Since $\mathcal{E}_\lambda^\delta$ are bilinear functionals on $H_0^1(D)$ which are bounded and coercive for $\delta > 0$ and $\lambda > \beta_0$, there exists a unique $u_\lambda^\delta \in H_0^1(D)$ for each $f \in H^{-1}(D)$ such that

$$\mathcal{E}_\lambda^\delta(u_\lambda^\delta, \varphi) = \langle f, \varphi \rangle_{H^{-1}, H_0^1}, \quad \varphi \in H_0^1(D).$$

By means of (2.6) and plugging $\varphi = u_\lambda^\delta$ in the above equation, we see

$$\mathbb{D}_1(u_\lambda^\delta, u_\lambda^\delta) \le \frac{1}{c_1}\mathcal{E}_\lambda^\delta(u_\lambda^\delta, u_\lambda^\delta) = \frac{1}{c_1}\langle f, u_\lambda^\delta\rangle_{H^{-1},H_0^1} \le \frac{1}{c_1}\|f\|_{H^{-1}}\|u_\lambda^\delta\|_{H_0^1(D)} = \frac{1}{c_1}\|f\|_{H^{-1}}\sqrt{\mathbb{D}_1(u_\lambda^\delta, u_\lambda^\delta)}$$

and, dividing the both sides by $\sqrt{\mathbb{D}_1(u_\lambda^\delta, u_\lambda^\delta)}$ we have

$$\sqrt{\mathbb{D}_1(u_\lambda^\delta, u_\lambda^\delta)} \le \frac{1}{c_1}\|f\|_{H^{-1}}; \qquad \mathbb{D}_1(u_\lambda^\delta, u_\lambda^\delta) \le \frac{1}{c_1^2}\|f\|^2_{H^{-1}}.$$

□

Remark 2.1. The inequality (2.8) tells us that the sequence $\{u_\lambda^\delta\}_{\delta>0}$ is uniformly bounded in $H_0^1(D)$. Then by the Rellich theorem, there exists a subsequence (denoted by the same symbol $\{u_\lambda^\delta\}$) and a limit point $u_0 \in H_0^1(D)$ such that

$$u_\lambda^\delta \ \to \ u_0 \textit{ weakly in } H_0^1(D) \quad \textit{and} \quad u_\lambda^\delta \ \to \ u_0 \textit{ strongly in } L^2(D) \textit{ as } \delta \to 0.$$

3 Periodic homogenization of jump-diffusion processes

In this section, we show that the full sequence of the solutions $\{u_\lambda^\delta\}_{\delta>0}$ in (2.7) converges to the limit point u_0 *weakly in* $H_0^1(D)$ and *strongly in* $L^2(D)$. Before this, we show the existence of the "correctors".

Proposition 3.1 (Correctors) *For each* $i = 1, 2, \cdots, d$, *there exists a unique* $\omega_i \in \mathcal{H}_\#(Y)$ *such that*

$$\int_Y A(y)\nabla\omega_i(y)\cdot\nabla\varphi(y)dy = -\int_Y A(y)\boldsymbol{e}_i\cdot\nabla\varphi(y)dy, \quad \varphi \in \mathcal{H}_\#(Y). \tag{3.1}$$

Proof: Since $\int_Y A(y)\nabla u(y)\cdot\nabla v(y)dy$ is a bounded and coercive bilinear form on $\mathcal{H}_\#(Y)$, applying the Lax-Milgram theorem to the linear functional $\ell_i(u) := -\int_Y A(y)\boldsymbol{e}_i\cdot\nabla u(y)dy$ for each $i = 1, 2, \cdots, d$ it follows that there exist a unique $\omega_i \in \mathcal{H}_\#(Y)$ such that

$$\int_Y A(y)\nabla\omega_i(y)\cdot\nabla\varphi(y)dy = \ell_i(\varphi) = -\int_Y A(y)\boldsymbol{e}_i\cdot\nabla\varphi(y)dy, \quad \varphi \in \mathcal{H}_\#(Y).$$

□

Theorem 3.1 *Assume the conditions in Theorem 2.1. Then for each* $f \in H^{-1}(D)$ *and* $\lambda > \beta_0$, *there exist* $u_0 \in H_0^1(D)$ *and* $u_1 \in L^2(D; \mathcal{H}_\#(Y))$ *such that, as* $\delta \to 0$,

(i) u_λ^δ *converges to* u_0 *weakly in* $H_0^1(D)$;

(ii) u_λ^δ *converges to* u_0 *strongly in* $L^2(D)$;

(iii) $\mathcal{T}_\delta(\nabla u_\lambda^\delta)$ *converges to* $\nabla u_0 + \nabla_y u_1$ *strongly in* $\big(L^2(D\times Y)\big)^d$

and the pair (u_0, u_1) *is the unique solution of the following problem:*

$$\begin{aligned}
\langle f, v\rangle_{H^{-1},H_0^1} = &\iint_{D\times Y} A(y)\big(\nabla u_0(x) + \nabla_y u_1(x,y)\big)\cdot\big(\nabla v(x) + \nabla_y w(x,y)\big)dxdy \\
&+ \iint_{D\times Y} B(y)\cdot\big(\nabla u_0(x) + \nabla_y u_1(x,y)\big)v(x)dxdy \\
&+ \iint_{\substack{x\in D, x+h\in D\\ h\neq 0}} \big(u_0(x+h) - u_0(x)\big)\big(v(x+h) - v(x)\big)J^0(h)dhdx \\
&+ k^0\int_D u_0(x)v(x)dx + \lambda(u_0, v), \quad v\in H_0^1(D),\ w\in L^2(D;\mathcal{H}_\#(Y)).
\end{aligned} \tag{3.2}$$

Moreover the following also hold:

$$\begin{aligned}
\lim_{\delta\to 0}\mathcal{E}_\lambda^\delta(u_\lambda^\delta, u_\lambda^\delta) = &\int_D A^0\nabla u_0(x)\cdot u_0(x)dx + \int_D B^0\cdot\nabla u_0(x)u_0(x)dx \\
&+ \iint_{\substack{x\in D, x+h\in D\\ h\neq 0}} \big(u_0(x+h) - u_0(x)\big)^2 J^0(h)dhdx + k^0\int_D u_0(x)^2dx + \lambda\int_D u_0(x)^2dx \\
=:& \ \mathcal{E}^0(u_0,u_0) + \lambda(u_0,u_0).
\end{aligned} \tag{3.3}$$

Here $A^0 = (a_{ij}^0)$ *is a* $d\times d$*-symmetric matrix,* $B^0 = (b_i^0)$ *is a* d*-dimensional vector,* $J^0(h)$ *is a Lévy density and* k^0 *is a nonnegative constant given by:*

- $a_{ij}^0 := \displaystyle\int_Y \Big(a_{ij}(y) + \sum_{k=1}^d a_{ik}(y)\frac{\partial\omega_j}{\partial y_k}(y)\Big)dy, \quad b_i^0 := \int_Y \Big(b_i(y) + \sum_{k=1}^d b_i(y)\frac{\partial\omega_i}{\partial y_k}(y)\Big)dy$
- $J^0(h) := \displaystyle\int_Y J(y,h)dy, \quad k^0 := \int_Y k(y)dy,$

where $\{\omega_i\}_{i=1}^d$ *are the correctors.*

The proof of this theorem is quite long (*c.f.*, [15, Theorem 4.1]). So, in the rest of this section, we will show this by dividing into several lemmas and propositions. First of all, according to Proposition 2.5, (2.8) in Theorem 2.2 and the Rellich theorem, there exists a subsequence (denote by the same symbol $\{\delta\}$ for simplicity) and a pair $(u_0, u_1)\in H_0^1(D)\times L^2(D;\mathcal{H}_\#(Y))$ such that, as $\delta\to 0$,

$$u_\lambda^\delta \text{ converges to } u_0 \textit{ weakly in } H_0^1(D), \qquad u_\lambda^\delta \text{ converges to } u_0 \textit{ strongly in } L^2(D) \quad \text{and}$$

$$\mathcal{T}_\delta(\nabla u_0) \text{ converges to } \nabla u_0 + \nabla_y u_1 \textit{ weakly in } \big(L^2(D\times Y)\big)^d.$$

Keeping these facts in mind at this moment, we now show that:

For each $v \in C_0^\infty(D)$, each term of the left hand side in (2.7) has a limit: "$\lim_{\delta\to 0} \mathcal{E}_\lambda^\delta(u_\lambda^\delta, v)$"

for the subsequence. Note that, from (2.7), it follows that

$$\begin{aligned}
\langle f, v\rangle_{H^{-1},H_0^1} &= \mathcal{E}_\lambda^\delta(u_\lambda^\delta, v) \\
&= \mathcal{E}^{c,\delta}(u_\lambda^\delta, v) + \mathcal{E}^{d,\delta}(u_\lambda^\delta, v) + \mathcal{E}^{j,\delta}(u_\lambda^\delta, v) + \mathcal{E}^{k,\delta}(u_\lambda^\delta, v) + \lambda(u_\lambda^\delta, v) \qquad (3.4) \\
&= \int_D A(x/\delta)\nabla u_\lambda^\delta(x)\cdot\nabla v(x)dx + \int_D B(x/\delta)\cdot\nabla u_\lambda^\delta(x)v(x)dx \\
&\quad + \iint_{\substack{x\in D, x+h\in D\\ h\neq 0}} \big(u_\lambda^\delta(x+h) - u_\lambda^\delta(x)\big)\big(v(x+h)-v(x)\big)J(x/\delta, h)dhdx \\
&\quad + \int_D u_\lambda^\delta(x)v(x)k(x/\delta)dx + \lambda\int_D u_\lambda^\delta(x)v(x)dx.
\end{aligned}$$

We show that all terms of the right hand side converge when $\delta \to 0$, respectively. To this end, the unfolding operators in the first two terms in the right hand side are used since $\mathsf{vol}(\Lambda_\delta) \to 0$ as $\delta \to 0$ (see Proposition 2.3).

Lemma 3.1 *The limit "$\lim_{\delta\to 0} \mathcal{E}^{c,\delta}(u_\lambda^\delta, v)$" exists.*

Proof: By using the unfolding operators,

$$\begin{aligned}
\lim_{\delta\to 0}\mathcal{E}^{c,\delta}(u_\lambda^\delta, v) &= \lim_{\delta\to 0}\int_D A(x/\delta)\nabla u_\lambda^\delta(x)\cdot\nabla v(x)dx = \lim_{\delta\to 0}\iint_{D\times Y}\mathcal{T}_\delta\big(A(\cdot/\delta)\nabla u_\lambda^\delta\cdot\nabla v\big)(x,y)dxdy \\
&= \lim_{\delta\to 0}\iint_{D\times Y}\underline{\mathcal{T}_\delta(A(\cdot/\delta)(x,y)}\mathcal{T}_\delta(\nabla u_\lambda^\delta)(x,y)\cdot\mathcal{T}_\delta(\nabla v)(x,y)dxdy \\
&= \lim_{\delta\to 0}\iint_{D\times Y}\underline{A(y)}\mathcal{T}_\delta(\nabla u_\lambda^\delta)(x,y)\cdot\mathcal{T}_\delta(\nabla v)(x,y)dxdy.
\end{aligned}$$

In the last term, the double integral converges to

$$\iint_{D\times Y} A(y)\big(\nabla u_0(x) + \nabla_y u_1(x,y)\big)\cdot\nabla v(x)dxdy,$$

since $A(y)$ is a bounded (matrix-valued) function, $\mathcal{T}_\delta(\nabla v)$ converges to ∇v *strongly in* $\big(L^2(D\times Y)\big)^d$ by Proposition 2.2 and $\mathcal{T}_\delta(\nabla u_\lambda^\delta)(x,y)$ converges to $\nabla u_0(x) + \nabla_y\nabla u_1(x,y)$ *weakly in* $\big(L^2(D\times Y)\big)^d$ by Proposition 2.5. $\square$

Lemma 3.2 *The limit "$\lim_{\delta\to 0} \mathcal{E}^{d,\delta}(u_\lambda^\delta, v)$" exists.*

Proof: This is shown similarly:

$$\lim_{\delta\to 0}\mathcal{E}^{d,\delta}(u_\lambda^\delta, v) = \lim_{\delta\to 0}\int_D B(x/\delta)\cdot\nabla u_\lambda^\delta(x)v(x)dx = \lim_{\delta\to 0}\iint_{D\times Y} B(y)\cdot\mathcal{T}_\delta(\nabla u_\lambda^\delta)(x,y)\mathcal{T}_\delta(v)(x,y)dxdy.$$

Note that $b_i(y)\mathcal{T}_\delta(v)(x,y)$ converges *strongly in* $L^2(D\times Y)$ as $\delta\to 0$ for each $i = 1, 2, \cdots, d$. In fact, since $v \in C_0^\infty(D) \subset L^{2q_1/(q_1-2)}(D)$, it follows that $\mathcal{T}_\delta(v)(x,y)$ converges to v *strongly in*

$L^{2q_1/(q_1-2)}(D)$ by Proposition 2.3. So, by the Hölder inequality ($1/s+1/t=1$ for $s=q_1/2>1$ and $t=q_1/(q_1-2)$),

$$\Big(\iint_{D\times Y}\Big(b_i(y)\mathcal{T}_\delta(v)(x,y)-b_i(y)v(x)\Big)^2dxdy\Big)^{1/2}=\Big(\iint_{D\times Y}b_i(y)^2\big(\mathcal{T}_\delta(v)(x,y)-v(x)\big)^2dxdy\Big)^{1/2}$$
$$\le\Big(\iint_{D\times Y}|b_i(y)|^{q_1}dxdy\Big)^{1/q_1}\Big(\iint_{D\times Y}\big|\mathcal{T}_\delta(v)(x,y)-v(x)\big|^{2q_1/(q_1-2)}dxdy\Big)^{(q_1-2)/(2q_1)}$$

and the right hand side goes to 0 as $\delta\to 0$. Therefore, noting that $\mathcal{T}_\delta(\nabla u^\delta_\lambda)(x,y)$ converges to $\nabla u_0(x)+\nabla_y u_1(x,y)$ *weakly in* $\big(L^2(D\times Y)\big)^d$, we see that

$$\begin{aligned}\lim_{\delta\to0}\mathcal{E}^{d,\delta}(u^\delta_\lambda,v)&=\lim_{\delta\to0}\iint_{D\times Y}B(y)\cdot\mathcal{T}_\delta(\nabla u^\delta_\lambda)(x,y)\mathcal{T}_\delta(v)(x,y)dxdy\\&=\iint_{D\times Y}B(y)\cdot\big(\nabla u_0(x)+\nabla_y u_1(x,y)\big)v(x)dxdy.\end{aligned}$$

□

Lemma 3.3 *The limit "*$\lim_{\delta\to0}\mathcal{E}^{j,\delta}(u^\delta_\lambda,v)$*" exists.*

Proof: By using the Fubini theorem, we see that

$$\begin{aligned}\mathcal{E}^{j,\delta}(u^\delta_\lambda,v)&=\iint_{\substack{x\in D,x+h\in D\\h\neq0}}\big(u^\delta_\lambda(x+h)-u^\delta_\lambda(x)\big)\big(v(x+h)-v(x)\big)J(x/\delta,h)dhdx\\&=\int_{h\neq0}\Big(\int_D 1_D(x+h)\big(u^\delta_\lambda(x+h)-u^\delta_\lambda(x)\big)\big(v(x+h)-v(x)\big)J(x/\delta,h)dx\Big)dh\\&=:\int_{h\neq0}\Phi_\delta(h)dh,\end{aligned}$$

where

$$\Phi_\delta(h)=\int_D 1_D(x+h)\big(u^\delta_\lambda(x+h)-u^\delta_\lambda(x)\big)\big(v(x+h)-v(x)\big)J(x/\delta,h)dx,\quad h\neq0.$$

We estimate Φ_δ. Since $\{u^\delta_\lambda\}$ converges to u_0 *strongly in* $L^2(D)$, it follows that the sequence $\{u^\delta_\lambda(\cdot+h)-u^\delta_\lambda(\cdot)\}$ converges to $u_0(x+h)-u(x)$ strongly in $L^2(D)$ for $h\neq0$. So the sequence of functions

$$x\mapsto 1_D(x+h)\big(u^\delta_\lambda(x+h)-u^\delta_\lambda(x)\big)\big(v(x+h)-v(x)\big)$$

converges to

$$x\mapsto 1_D(x+h)\big(u_0(x+h)-u_0(x)\big)\big(v(x+h)-v(x)\big)$$

in $L^1(D)$ *strongly.* Then noting that $x\mapsto J(x,h)$ is Y-periodic and bounded in for each x for $h\neq0$, we see that $\Phi_\delta(h)$ converge to

$$\int_D 1_D(x+h)\big(u_0(x+h)-u_0(x)\big)\big(v(x+h)-v(x)\big)J^0(h)dx$$

as $\delta \to 0$ for each $h \neq 0$, where $J^0(h) = \int_Y J(y,h)dy$. Therefore we need to check that $h \mapsto \Phi_\delta(h)$ is integrable (uniformly with respect to δ) in order that the exchange of the limit and the integral (in h) to apply the dominated convergence theorem. By (J), $J(y,h) \le \varphi(h)$, $h \neq 0$. So, for $|h| \ge 1$,

$$\begin{aligned}
|\Phi_\delta(h)| &\le \varphi(h) \int_D 1_D(x+h)\big|\big(u_\lambda^\delta(x+h) - u_\lambda^\delta(x)\big)\big(v(x+h)-v(x)\big)\big|dx \\
&\le \varphi(h)\sqrt{\int_D 1_D(x+h)\big(u_\lambda^\delta(x+h) - u_\lambda^\delta(x)\big)^2 dx}\sqrt{\int_D 1_D(x+h)\big(v(x+h)-v(x)\big)^2 dx} \\
&\le 2\varphi(h)\sqrt{\int_D 1_D(x+h)\big(u_\lambda^\delta(x+h)^2 + u_\lambda^\delta(x)^2\big)dx}\sqrt{\int_D 1_D(x+h)\big(v(x+h)^2+v(x)^2\big)dx} \\
&\le 4\varphi(h)\|u_\lambda^\delta\|\|v\|
\end{aligned}$$

For $0 < |h| < 1$, by making use of the mean value theorem,

$$\begin{aligned}
|\Phi_\delta(h)| &\le \varphi(h)\int_D 1_D(x+h)\Big|\Big(\int_0^1 h\cdot\nabla u_\lambda^\delta(x+th)dt\Big)\Big(\int_0^1 h\cdot\nabla v(x+sh)ds\Big)\Big|dx \\
&\le |h|^2\varphi(h)\int_0^1\int_0^1\Big(\int_D 1_D(x+h)|\nabla u_\lambda^\delta(x+th)||\nabla v(x+st)|dx\Big)dtds \\
&\le |h|^2\varphi(h)\int_0^1\int_0^1\sqrt{\int_D 1_D(x+h)|\nabla u_\lambda^\delta(x+th)|^2dx}\sqrt{\int_D 1_D(x+h)|\nabla v(x+st)|^2dx}\,dtds \\
&= |h|^2\varphi(h)\int_0^1\int_0^1\sqrt{\int_{x-th\in D} 1_D(x+(1-t)h)|\nabla u_\lambda^\delta(x)|^2dx} \\
&\qquad\qquad\times\sqrt{\int_{x-sh\in D} 1_D(x+(1-s)h)|\nabla v(x)|^2dx}\,dtds
\end{aligned}$$

Since $u_\lambda^\delta \in H_0^1(D)$ and $v \in C_0^\infty(D)$, they are naturally extended by zero outside D, so that their gradients vanish on $\mathbb{R}^d \setminus D$. Therefore, the following inequality holds for any $t, s \in [0,1]$ and $h \in \mathbb{R}^d$:

$$\int_{x-th\in D} 1_D(x+(1-t)h)|\nabla u_\lambda^\delta(x)|^2dx \le \int_{\mathbb{R}^d}|\nabla u_\lambda^\delta(x)|^2dx = \int_D |\nabla u_\lambda^\delta(x)|^2dx = \mathbb{D}(u_\lambda^\delta, u_\lambda^\delta)$$

and

$$\int_{x-sh\in D} 1_D(x+(1-s)h)|\nabla v(x)|^2dx \le \int_{\mathbb{R}^d}|\nabla v(x)|^2dx = \int_D |\nabla v(x)|^2dx = \mathbb{D}(v,v).$$

Consequently, we have

$$\big|\Phi_\delta(h)\big| \le |h|^2\varphi(h)\sqrt{\mathbb{D}(u_\lambda^\delta,u_\lambda^\delta)}\sqrt{\mathbb{D}(v,v)} \le C|h|^2\varphi(h), \quad 0<|h|<1.$$

Combined with the estimation for $|h| \ge 1$, we obtain for some constant $C > 0$,

$$\big|\Phi_\delta(h)\big| \le C\big(1\wedge|h|^2\big)\varphi(h), \quad h\neq 0,\ 0<\delta$$

and the right hand side is integrable in h by (J). Finally we can conclude that

$$\lim_{\delta\to 0}\mathcal{E}^{j,\delta}(u_\lambda^\delta,v) = \int\limits_{h\neq 0}\Big(\iint\limits_{D\times Y} 1_D(x+h)\big(u_0(x+h)-u_0(x)\big)\big(v(x+h)-v(x)\big)J(y,h)dxdy\Big)dh$$
$$= \iint\limits_{\substack{x\in D, x+h\in D\\ h\neq 0}} \big(u_0(x+h)-u_0(x)\big)\big(v(x+h)-v(x)\big)J^0(h)dhdx,$$

where $J^0(h) = \displaystyle\int_Y J(y,h)dy$. □

Lemma 3.4 *The following hold:*

$$\lim_{\delta\to 0}\mathcal{E}^{k,\delta}(u_\lambda^\delta,v) = \lim_{\delta\to 0}\int_D u_\lambda^\delta(x)v(x)k(x/\delta)dx = k^0\int_D u_0(x)v(x)dx \quad\text{and}\quad \lim_{\delta\to 0}\lambda(u_\lambda^\delta,v) = \lambda(u_0,v),$$

where $k^0 = \int_Y k(y)dy$.

Proof: The second limit is clear since u_λ^δ converges to u_0 *strongly in* $L^2(D)$. So we only show the first limit. Since $k(x/\delta)$ converges to $\int_D k(y)dy$ *weakly in* $L^2(D)$ and u_λ^δ converges to u_0 *strongly in* $L^2(D)$, the limit

$$\lim_{\delta\to 0}\mathcal{E}^{k,\delta}(u_\lambda^\delta,v) = \lim_{\delta\to 0}\int_D u_\lambda^\delta(x)v(x)k(x/\delta)dx = k^0\int_D u_0(x)v(x)dx.$$

□

From these four lemmas (Lemma 3.1 -Lemma 3.4), we have the following proposition.

Proposition 3.2 *For the limit point* (u_0,u_1),

$$\begin{aligned}\langle f,v\rangle_{H^{-1},H_0^1} &= \lim_{\delta\to 0}\mathcal{E}_\lambda^\delta(u_\lambda^\delta,v)\\ &= \iint\limits_{D\times Y} A(y)\big(\nabla u_0(x)+\nabla_y u_1(x,y)\big)\cdot\nabla v(x)dxdy\\ &\quad+\iint\limits_{D\times Y} B(y)\cdot\big(\nabla u_0(x)+\nabla_y u_1(x,y)\big)v(x)dxdy\\ &\quad+\iint\limits_{\substack{x\in D, x+h\in D\\ h\neq 0}}\big(u_0(x+h)-u_0(x)\big)\big(v(x+h)-v(x)\big)J^0(h)dhdx\\ &\quad+k^0\int_D u_0(x)v(x)dx+\lambda(u_0,v),\quad v\in C_0^\infty(D).\end{aligned}\tag{3.5}$$

□

We now turn to show that the pair of the limits (u_0,u_1) is unique and the full sequence $\{u_\lambda^\delta\}_{\delta>0}$ converges to the limit. To this end, let $v\in C_0^\infty(D)$ and $w\in C_\#^\infty(Y)$ and set

$$v_\delta(x) = \delta v(x)w\Big(\frac{x}{\delta}\Big),\quad x\in D,\ \delta>0.$$

Then $\{v_\delta\}_{\delta>0} \subset C_0^\infty(D)$ and v_δ converges to 0 *uniformly* in D, and hence, *strongly* in $L^2(D)$ as $\delta \to 0$. Moreover

$$\nabla v_\delta(x) = \delta \nabla v(x) w\Big(\frac{x}{\delta}\Big) + v(x)\Big(\nabla w\Big(\frac{x}{\delta}\Big)\Big), \quad x \in D$$

are bounded in $\big(L^2(D)\big)^d$ with respect to $\delta > 0$. This means that v_δ converges to 0 *weakly* in $H_0^1(D)$.

Lemma 3.5 *The limit "$\lim\limits_{\delta\to 0} \mathcal{E}^\delta(u_\lambda^\delta, v_\delta)$" exists (for the subsequence).*

Proof: By the definition of the unfolding operators, we see that

$$\mathcal{T}_\delta(\nabla v_\delta)(x,y) = \delta \mathcal{T}_\delta(\nabla v)(x,y) w(y) + \mathcal{T}_\delta(v)(x,y)(\nabla_y w)(y)$$

converges to $v(x)(\nabla_y w)(y)$ strongly in $\big(L^2(D\times Y)\big)^d$. We take $v_\delta(x) = \delta v(x) w(x/\delta)$ as a test function in (2.7). Since $v_\delta \rightharpoonup 0$ weakly in $H_0^1(D)$, the term $\langle f, v_\delta\rangle_{H^{-1},H_0^1} \to 0$. Passing to the limit $\delta \to 0$ in $\mathcal{E}_\lambda^\delta(u_\lambda^\delta, v_\delta)$, we obtain

$$\begin{aligned}
0 &= \lim_{\delta\to 0}\langle f, v_\delta\rangle_{H^{-1},H_0^1} = \lim_{\delta\to 0}\mathcal{E}_\lambda^\delta(u_\lambda^\delta, v_\delta)\\
&= \lim_{\delta\to 0}\Bigg\{\int_D A(x/\delta)\nabla u_\lambda^\delta(x)\cdot\nabla v_\delta(x)dx + \int_D B(x/\delta)\cdot\nabla u_\lambda^\delta(x) v_\delta(x)dx\\
&\qquad + \iint_{\substack{x\in D, x+h\in D\\ h\neq 0}} \big(u_\lambda^\delta(x+h) - u_\lambda^\delta(x)\big)\big(v_\delta(x+h) - v_\delta(x)\big)J(x/\delta)dhdx\\
&\qquad + \int_D u_\lambda^\delta(x) v_\delta(x) k(x/\delta)dx + \lambda(u_\lambda^\delta, v_\delta)\Bigg\}\\
&= \lim_{\delta\to 0}\Bigg\{\iint_{D\times Y} A(y)\mathcal{T}_\delta(\nabla u_\lambda^\delta)(x,y)\cdot\Big(\delta\mathcal{T}_\delta(\nabla v)(x,y)w(y) + \mathcal{T}_\delta(v)(x,y)(\nabla_y w)(y)\Big)dxdy\\
&\qquad\quad \underbrace{+\,\delta\iint_{D\times Y} B(y)\cdot\mathcal{T}_\delta(\nabla u_\lambda^\delta)(x,y)\mathcal{T}_\delta(v)(x,y)w(y)dxdy}_{=:(a)}\\
&\qquad \underbrace{+\iint_{\substack{x\in D, x+h\in D\\ h\neq 0}} \big(u_\lambda^\delta(x+h) - u_\lambda^\delta(x)\big)\big(v_\delta(x+h) - v_\delta(x)\big)J(x/\delta)dhdx}_{=:(b)}\\
&\qquad \underbrace{+\int_D u_\lambda^\delta(x) v_\delta(x) k(x/\delta)dx + \lambda(u_\lambda^\delta, v_\delta)}_{=:(c)}\Bigg\}\\
&= \iint_{D\times Y} A(y)\Big(\nabla u_0(x) + \nabla_y u_1(x,y)\Big)\cdot\nabla_y w(y) v(x)dxdy + \lim_{\delta\to 0}\big((a)+(b)+(c)\big).
\end{aligned}$$

Here, the terms (a) and (c) vanish as $\delta \to 0$, as they are scaled by δ (stemming from v_δ) while all other factors in their respective integrands remain bounded independently of δ.

Next we show that the term (b) also vanishes as $\delta \to 0$. To this end, we decompose the difference $v_\delta(x+h) - v_\delta(x)$ as follows:

$$v_\delta(x+h) - v_\delta(x) = \delta w\Big(\frac{x+h}{\delta}\Big)\big(v(x+h) - v(x)\big) + \delta v(x)\left(w\Big(\frac{x+h}{\delta}\Big) - w\Big(\frac{x}{\delta}\Big)\right).$$

Substituting this into (b), we have $(b) = (b_1) + (b_2)$, where:

$$(b_1) = \delta \iint_{\substack{x\in D, x+h\in D\\ h\neq 0}} \big(u_\lambda^\delta(x+h) - u_\lambda^\delta(x)\big) w\Big(\frac{x+h}{\delta}\Big)\big(v(x+h) - v(x)\big) J(x/\delta, h) dh dx$$

and

$$(b_2) = \delta \iint_{\substack{x\in D, x+h\in D\\ h\neq 0}} \big(u_\lambda^\delta(x+h) - u_\lambda^\delta(x)\big) v(x) \left(w\Big(\frac{x+h}{\delta}\Big) - w\Big(\frac{x}{\delta}\Big)\right) J(x/\delta, h) dh dx.$$

As for the term (b_1), using the Schwarz inequality and Proposition 2.1 (2),

$$\begin{aligned}
|(b_1)| &\le \delta \|w\|_\infty \sqrt{\iint_{\substack{x\in D, x+h\in D\\ h\neq 0}} \big(u_\lambda^\delta(x+h) - u_\lambda^\delta(x)\big)^2 J(x/\delta, h) dh dx} \\
&\qquad \times \sqrt{\iint_{\substack{x\in D, x+h\in D\\ h\neq 0}} \big(v(x+h) - v(x)\big)^2 J(x/\delta, h) dh dx} \\
&\le \delta c_1 \|w\|_\infty \sqrt{\mathbb{D}_1(u_\lambda^\delta, u_\lambda^\delta)} \sqrt{\mathbb{D}_1(v, v)} \\
&\to 0 \quad \text{as } \delta \to 0,
\end{aligned}$$

where we used the boundedness of the sequence $\{u_\lambda^\delta\}_{\delta>0}$ in $H_0^1(D)$.

For the term (b_2), we follow the same argument as in the proof of Lemma 3.3 to show that it vanishes as $\delta \to 0$. More precisely, we first rewrite (b_2) as follows:

$$\begin{aligned}
(b_2) &= \int_{h\neq 0} \left\{ \delta \int_D 1_D(x+h) \big(u_\lambda^\delta(x+h) - u_\lambda^\delta(x)\big) v(x) \left(w\Big(\frac{x+h}{\delta}\Big) - w\Big(\frac{x}{\delta}\Big)\right) J(x/\delta, h) dx \right\} dh \\
&= \int_{h\neq 0} \Phi_\delta(h) dh,
\end{aligned}$$

where

$$\Phi_\delta(h) := \int_D \delta 1_D(x+h) \big(u_\lambda^\delta(x+h) - u_\lambda^\delta(x)\big) v(x) \left(w\Big(\frac{x+h}{\delta}\Big) - w\Big(\frac{x}{\delta}\Big)\right) J(x/\delta, h) dx, \ \ h \neq 0.$$

For each $h \neq 0$, the part of the integrand of $\Phi_\delta(h)$,

$$\delta 1_D(x+h) v(x) \left(w\Big(\frac{x+h}{\delta}\Big) - w\Big(\frac{x}{\delta}\Big)\right) J(x/\delta, h)$$

goes to 0 uniformly in x as $\delta \to 0$ because w is continuous on the compact set $\bar{Y}$ and the kernel $J(\cdot, h)$ is bounded for each $h \neq 0$ by the assumption (J). Then noting that $\{u_\lambda^\delta\}_{\delta>0}$ is bounded in $L^2(D)$, we conclude that $\Phi_\delta(h) \to 0$ as $\delta \to 0$. On the other hand, by the Lipschitz continuity of w and the assumption $J(x/\delta, h) \le \varphi(h)$, we have

$$\begin{aligned}
|\Phi_\delta(h)| &\le \int_D \big|(u_\lambda^\delta(x+h) - u_\lambda^\delta(x)) v(x)\big| \cdot \underbrace{\left|\delta \left(w\Big(\frac{x+h}{\delta}\Big) - w\Big(\frac{x}{\delta}\Big)\right)\right|}_{\le \left(\|\nabla w\|_\infty + 2\|w\|_\infty\right)\left(|h|\wedge 1\right)} \cdot \underbrace{J(x/\delta, h)}_{\le \varphi(h)} dx \\
&\le \Big(\|\nabla w\|_\infty + 2\|w\|_\infty\Big)\big(|h| \wedge 1\big)\varphi(h) \int_D |u_\lambda^\delta(x+h) - u_\lambda^\delta(x)||v(x)| dx \\
&\le \Big(\|\nabla w\|_\infty + 2\|w\|_\infty\Big)\big(|h| \wedge 1\big)\varphi(h) \|u_\lambda^\delta(\cdot + h) - u_\lambda^\delta\|_{L^2} \|v\|_{L^2}.
\end{aligned}$$

Since $\{u^\delta_\lambda\}$ is bounded in $H^1_0(D)$, we have $\|u^\delta_\lambda(\cdot+h)-u^\delta_\lambda\|_{L^2(D)} \le C(|h|\wedge 1)$. Consequently,

$$|\Phi_\delta(h)| \le C'(|h|\wedge 1)^2\varphi(h) = C'(|h|^2\wedge 1)\varphi(h)$$

and the right-hand side is integrable from the assumption (J), which justifies the limit $(b_2)\to 0$ by the dominated convergence theorem. Collecting these results, (a), (b) and (c) vanish as $\delta\to 0$. Consequently, passing to the limit in the weak formulation, it follows that

$$0 = \lim_{\delta\to 0}\mathcal{E}^\delta_\lambda(u^\delta_\lambda, v_\delta) = \iint_{D\times Y} A(y)\Big(\nabla u_0(x)+\nabla_y u_1(x,y)\Big)\cdot\nabla_y w(y)v(x)dxdy,$$

for any $v\in C^\infty_0(D)$ and $w\in C^\infty_\#(Y)$. □

Since $C^\infty_0(D)\times C^\infty_\#(Y)$ is dense in $L^2(D;\mathcal{H}_\#(Y))$, we conclude that

$$\iint_{D\times Y} A(y)\Big(\nabla u_0(x)+\nabla_y u_1(x,y)\Big)\cdot\nabla_y w(x,y)dxdy = 0, \quad w\in L^2(D;\mathcal{H}_\#(Y)). \tag{3.6}$$

By combining this with (3.5), it follows that

$$\begin{aligned}\langle f,v\rangle_{H^{-1},H^1_0} = &\iint_{D\times Y} A(y)\big(\nabla u_0(x)+\nabla_y u_1(x,y)\big)\cdot\big(\nabla v(x)+\nabla_y w(x,y)\big)dxdy\\ &+\iint_{D\times Y} B(y)\cdot\big(\nabla u_0(x)+\nabla_y u_1(x,y)\big)v(x)dxdy\\ &+\iint_{\substack{x\in D, x+h\in D\\ h\neq 0}}\big(u_0(x+h)-u_0(x)\big)\big(v(x+h)-v(x)\big)J^0(h)dhdx\\ &+k^0\int_D u_0(x)v(x)dx+\lambda(u_0,v),\end{aligned}\tag{3.7}$$

for any $v\in H^1_0(D)$ and $w\in L^2(D;\mathcal{H}_\#(Y))$.

Proposition 3.3 *The pair (u_0,u_1) is the unique solution of the following problem:*

Find a pair $(u_0,u_1)\in H^1_0(D)\times L^2(D;\mathcal{H}_\#(Y))$ so that

$$\begin{aligned}\langle f,v\rangle_{H^{-1},H^1_0} = &\iint_{D\times Y} A(y)\big(\nabla u_0(x)+\nabla_y u_1(x,y)\big)\cdot\big(\nabla v(x)+\nabla_y w(x,y)\big)dxdy\\ &+\iint_{D\times Y} B(y)\cdot\big(\nabla u_0(x)+\nabla_y u_1(x,y)\big)v(x)dxdy\\ &+\iint_{\substack{x\in D, x+h\in D\\ h\neq 0}}\big(u_0(x+h)-u_0(x)\big)\big(v(x+h)-v(x)\big)J^0(h)dhdx\\ &+k^0\int_D u_0(x)v(x)dx+\lambda(u_0,v)\end{aligned}\tag{3.8}$$

hold for any $(v,w)\in H^1_0(D)\times L^2(D;\mathcal{H}_\#(Y))$.

Proof: This is proved by using the variational form in the space: $\mathfrak{H} := H_0^1(D) \times L^2(D; \mathcal{H}_\#(Y))$ endowed with the norm

$$\|(v,w)\|_{\mathfrak{H}} := \Big(\mathbb{D}(v,v) + \iint_{D\times Y} |\nabla_y w(x,y)|^2 dxdy\Big)^{1/2}.$$

To show the existence, define a bilinear functional η on $\mathfrak{H} \times \mathfrak{H}$ by

$$\begin{aligned}\eta\big((v_1,w_1),(v_2,w_2)\big) := &\iint\limits_{D\times Y} A(y)\big(\nabla v_1(x) + \nabla_y w_1(x,y)\big)\cdot\big(\nabla v_2(x) + \nabla_y w_2(x,y)\big)dxdy \\ &+ \iint\limits_{D\times Y} B(y)\cdot\big(\nabla v_1(x) + \nabla_y w_1(x,y)\big)v_2(x)dxdy \\ &+ \iint\limits_{\substack{x\in D, x+h\in D\\ h\neq 0}} \big(v_1(x+h)-v_1(x)\big)\big(v_2(x+h)-v_2(x)\big)J^0(h)dhdx \\ &+ \iint_{D\times Y} v_1(x)v_2(x)k(y)dxdy + \lambda(v_1,v_2).\end{aligned}$$

Then η is a coercive bounded functional. In fact, for $(v_1,w_1),(v_2,w_2)\in\mathfrak{H}$,

$$\begin{aligned} &\big|\eta\big((v_1,w_1),(v_2,w_2)\big)\big| \\ &\le \beta\sqrt{\iint\limits_{D\times Y}|\nabla v_1(x)+\nabla_y w_1(x,y)|^2dxdy}\sqrt{\iint\limits_{D\times Y}|\nabla v_2(x)+\nabla_y w_2(x,y)|^2dxdy} \\ &\quad + \sqrt{\iint\limits_{D\times Y}|\nabla v_1(x)+\nabla_y w_1(x,y)|^2dxdy}\sqrt{\iint\limits_{D\times Y}|B(y)|^2(v_2(x))^2dxdy} \\ &\quad + \Big(\iint\limits_{\substack{x\in D,x+h\in D\\ 0<|h|<1}} + \iint\limits_{\substack{x\in D,x+h\in D\\ |h|\ge 1}}\Big)\Big|\big(v_1(x+h)-v_1(x)\big)\big(v_2(x+h)-v_2(x)\big)\Big|J^0(h)dhdx \\ &\quad + k^0\|v_1\|_{L^2(D)}\|v_2\|_{L^2(D)} + \lambda\|v_1\|_{L^2(D)}\|v_2\|_{L^2(D)}, \end{aligned} \tag{3.9}$$

where $k^0 = \int_Y k(y)dy$. Note that $\int_Y \nabla_y w_i(x,y)dy = 0$ a.e. $x\in D$ since $w_i \in L^2(D;\mathcal{H}_\#(Y))$ for $i=1,2$. So it follows that

$$\iint\limits_{D\times Y}|\nabla v_i(x)+\nabla_y w_i(x,y)|^2dxdy = \iint\limits_{D\times Y}\Big(\big|\nabla v_i(x)\big|^2 + \big|\nabla_y w_i(x,y)\big|^2\Big)dxdy = \|(v_i,w_i)\|_{\mathfrak{H}}^2$$

for $i=1,2$. By using the fact that $B\in\big(L^{q_1}(D)\big)^d \subset \big(L^2(D)\big)^d$ $(3\le d<q_1\le\infty)$ and the Poincaré inequality, we see that

$$\begin{aligned}\iint_{D\times Y}|B(y)|^2v_2(x)^2dxdy &= \Big(\int_Y |B(y)|^2dy\Big)\|v_2\|_{L^2(D)}^2 \le C\big\||B|\big\|_{L^2(D)}^2\mathbb{D}(v_2,v_2) \\ &\le C\big\||B|\big\|_{L^2(D)}^2\|(v_2,w_2)\|_{\mathfrak{H}}^2\end{aligned}$$

for some constant $C>0$. As in the proof of Lemma 3.3, the big jumps part is estimated as follows.

$$\iint\limits_{\substack{x\in D,x+h\in D\\ |h|\ge 1}}\Big|\big(v_1(x+h)-v_1(x)\big)\big(v_2(x+h)-v_2(x)\big)\Big|J^0(h)dhdx$$

$$\leq \sqrt{\iint\limits_{\substack{x\in D, x+h\in D\\ |h|\geq 1}} \big(v_1(x+h)-v_1(x)\big)^2 J^0(h)dhdx}\sqrt{\iint\limits_{\substack{x\in D, x+h\in D\\ |h|\geq 1}} \big(v_2(x+h)-v_2(x)\big)^2 J^0(h)dhdx}$$
$$\leq 2\sqrt{\iint\limits_{\substack{x\in D, x+h\in D\\ |h|\geq 1}} \big(v_1(x+h)^2+v_1(x)^2\big) J^0(h)dhdx}\sqrt{\iint\limits_{\substack{x\in D, x+h\in D\\ |h|\geq 1}} \big(v_2(x+h)^2+v_2(x)^2\big) J^0(h)dhdx}$$
$$\leq 4M_1\|v_1\|_{L^2(D)}\|v_2\|_{L^2(D)} \leq 4M_1 C\sqrt{\mathbb{D}(v_1,v_1)}\sqrt{\mathbb{D}(v_2,v_2)}$$
$$\leq 4M_1 C\|(v_1,w_1)\|_{\mathfrak{H}}\|(v_2,w_2)\|_{\mathfrak{H}}.$$

Similarly,

$$\iint\limits_{\substack{x\in D, x+h\in D\\ 0<|h|<1}} \Big|\big(v_1(x+h)-v_1(x)\big)\big(v_2(x+h)-v_2(x)\big)\Big| J^0(h)dhdx$$
$$\leq \sqrt{\iint\limits_{\substack{x\in D, x+h\in D\\ 0<|h|<1}} \big(v_1(x+h)-v_1(x)\big)^2 J^0(h)dhdx}\sqrt{\iint\limits_{\substack{x\in D, x+h\in D\\ 0<|h|<1}} \big(v_2(x+h)-v_2(x)\big)^2 J^0(h)dhdx}.$$

It is enough to consider the first term of the right hand side only. By means of the mean value theorem, we see that

$$\iint\limits_{\substack{x\in D, x+h\in D\\ 0<|h|<1}} \big(v_1(x+h)-v_1(x)\big)^2 J^0(h)dhdx = \iint\limits_{\substack{x\in D, x+h\in D\\ 0<|h|<1}} \Big(\int_0^1 \nabla v_1(x+th)\cdot h dt\Big)^2 J^0(h)dxdh$$
$$\leq \iint\limits_{\substack{x\in D, x+h\in D\\ 0<|h|<1}} \Big(\int_0^1 |\nabla v_1(x+th)|^2\,|h|^2 dt\Big) J^0(h)dhdx$$
$$= \int_0^1 \Big(\int_{0<|h|<1}\Big(\int_D 1_D(x+h)\big|\nabla v_1(x+th)|^2 dx\Big)|h|^2 J^0(h)dh\Big)dt$$
$$\leq M_2\mathbb{D}(v_1,v_1) \leq M_2\|(v_1,w_1)\|_{\mathfrak{H}}^2.$$

For the last two terms in (3.9), using again the Poincaré inequality, we have

$$(k^0+\lambda)\|v_1\|_{L^2(D)}\|v_2\|_{L^2(D)} \leq (k^0+\lambda)C\sqrt{\mathbb{D}(v_1,v_1)}\sqrt{\mathbb{D}(v_2,v_2)}.$$

Therefore it follows that

$$\big|\eta((v_1,w_1),(v_2,w_2))\big| \leq C'\|(v_1,w_1)\|_{\mathfrak{H}}\|(v_2,w_2)\|_{\mathfrak{H}}, \quad (v_1,w_1),(v_2,w_2)\in\mathfrak{H}$$

for some constant $C'>0$. As for the coerciveness, by Proposition 2.1,

$$\eta\big((v,w),(v,w)\big) := \iint\limits_{D\times Y} A(y)\big(\nabla v(x)+\nabla_y w(x,y)\big)\cdot\big(\nabla v(x)+\nabla_y w(x,y)\big)dxdy$$
$$+\iint\limits_{D\times Y} B(y)\cdot\big(\nabla v(x)+\nabla_y w(x,y)\big)v(x)dxdy$$

$$
\begin{aligned}
&+\iint\limits_{\substack{x\in D, x+h\in D\\ h\neq 0}} \big(v(x+h)-v(x)\big)^2 J^0(h)dhdx + (k^0+\lambda)\|v\|^2_{L^2(D)}\\
&\geq \alpha \iint_{D\times Y} \big|\nabla v(x)+\nabla_y w(x,y)\big|^2 dxdy\\
&\quad - \Big| \iint_{D\times Y} B(y)\cdot\big(\nabla v(x)+\nabla_y w(x,y)\big)v(x)dxdy\Big| + \lambda\|v\|^2_{L^2(D)}\\
&\geq \alpha \iint_{D\times Y} \big(|\nabla v(x)|^2 + |\nabla_y w(x,y)|^2\big)dxdy\\
&\quad - \frac{\alpha}{2}\|(v,w)\|^2_{\mathfrak{H}} - \beta_0\|v\|^2_{L^2(D)} + (k^0+\lambda)\|v\|^2_{L^2(D)}\\
&\geq \frac{\alpha}{2}\|(v,w)\|^2_{\mathfrak{H}}
\end{aligned}
$$

holds for putting $\lambda = \alpha/2$. Therefore, for any $f \in H^{-1}(D)$, a linear functional

$$
\ell((v,w)) := \langle f, v\rangle_{H^{-1},H^1_0}, \quad (v,w)\in\mathfrak{H}
$$

satisfies that

$$
|\ell((v,w))| \leq \|f\|_{H^{-1}}\|v\|_{H^1} \leq \|f\|_{H^{-1}}\|(v,w)\|_{\mathfrak{H}},
$$

namely, ℓ is continuous, by virtue of the Lax-Milgram theorem, there exists a unique $(u_0,u_1)\in\mathfrak{H}$ such that

$$
\begin{aligned}
\langle f, v\rangle_{H^{-1},H^1_0} &= \ell((v,w)) = \eta((u_0,u_1),(v,w)), \quad (v,w)\in\mathfrak{H}\\
&= \iint_{D\times Y} A(y)\big(\nabla u_0(x)+\nabla_y u_1(x,y)\big)\cdot\big(\nabla v(x)+\nabla_y w(x,y)\big)dxdy\\
&\quad + \iint_{D\times Y} B(y)\cdot\big(\nabla u_0(x)+\nabla_y u_1(x,y)\big)v(x)dxdy\\
&\quad + \iint\limits_{\substack{x\in D, x+h\in D\\ h\neq 0}} \big(u_0(x+h)-u_0(x)\big)\big(v(x+h)-v(x)\big)J^0(h)dhdx\\
&\quad + k^0\int_D u_0(x)v(x)dx + \lambda\int_D u_0(x)v(x)dx, \quad (v,w)\in\mathfrak{H}.
\end{aligned}
$$

This shows that the pair (u_0,u_1) is the unique solution of (3.2). □

In the next lemma, we characterize the function $u_1 \in L^2(D;\mathcal{H}_\#(Y))$.

Lemma 3.6 *The function u_1 satisfies*

$$
u_1(x,y) = \sum_{i=1}^d \frac{\partial u_0}{\partial x_i}(x)\omega_i(y), \tag{3.10}
$$

where $\{\omega_i\}$ are the correctors defined in Proposition 3.1.

Proof: In the equation (3.2), plugging $(0,w)\in\mathfrak{H} = H^1_0(D)\times L^2(D;\mathcal{H}_\#(Y))$, it follows that

$$
\iint_{D\times Y} A(y)\Big(\nabla u_0(x)+\nabla_y u_1(x,y)\Big)\cdot\nabla_y w(x,y)dxdy = 0. \tag{3.11}
$$

Then inserting $w(x,y) = \theta_1(x)\theta_2(y)$ for $\theta_1 \in C_0^\infty(D)$ and $\theta_2 \in \mathcal{H}_\#(Y)$, we see that

$$\int_D \theta_1(x)\Big(\int_Y A(y)\Big(\nabla u_0(x) + \nabla_y u_1(x,y)\Big)\cdot \nabla_y \theta_2(y)dy\Big)dx = 0.$$

Thus the equation

$$\int_Y A(y)\Big(\nabla u_0(x) + \nabla_y u_1(x,y)\Big)\cdot \nabla_y \theta_2(y)dy = 0 \quad \text{a.e. } x \in D$$

holds for any $\theta_2 \in \mathcal{H}_\#(Y)$. This is exactly the variational form of the cell problem.

On the other hand, the correctors $\omega_i \in \mathcal{H}_\#(Y)$ are uniquely defined as the solution to:

$$\int_Y A(y)(\mathbf{e}_i + \nabla \omega_i(y)) \cdot \nabla \theta(y)dy = 0, \quad \forall \theta \in \mathcal{H}_\#(Y).$$

By the linearity of this equation with respect to the forcing term $\mathbf{e}_i$, for any constant vector $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$, the function $\phi_\xi(y) := \sum_{i=1}^d \xi_i \omega_i(y)$ satisfies:

$$\int_Y A(y)(\xi + \nabla_y \phi_\xi(y)) \cdot \nabla \theta(y)dy = 0, \quad \forall \theta \in \mathcal{H}_\#(Y).$$

Now, for almost every $x \in D$, we fix $\xi = \nabla u_0(x)$. Then the function $\phi(x,y) := \sum_{i=1}^d \frac{\partial u_0}{\partial x_i}(x)\omega_i(y)$ satisfies:

$$\int_Y A(y)(\nabla u_0(x) + \nabla_y \phi(x,y)) \cdot \nabla \theta(y)dy = 0, \quad \forall \theta \in \mathcal{H}_\#(Y).$$

Comparing this with the equation for u_1:

$$\int_Y A(y)(\nabla u_0(x) + \nabla_y u_1(x,y)) \cdot \nabla \theta(y)dy = 0, \quad \forall \theta \in \mathcal{H}_\#(Y),$$

we see that both $u_1(x,\cdot)$ and $\phi(x,\cdot)$ solve the same cell problem for a fixed x.

By the uniqueness of the solution to the cell problem in $\mathcal{H}_\#(Y)$ (up to an additive constant, which is fixed to zero mean in $\mathcal{H}_\#(Y)$), we conclude that $u_1(x,y) = \phi(x,y)$, i.e.,

$$u_1(x,y) = \sum_{i=1}^d \frac{\partial u_0}{\partial x_i}(x)\omega_i(y).$$

$\square$

The next lemma proves that the limit "$\lim_{\delta\to 0} \mathcal{E}_\lambda^\delta(u_\lambda^\delta, u_\lambda^\delta)$" in (3.3) exists.

Lemma 3.7 *The limit "$\lim\limits_{\delta\to 0} \mathcal{E}_\lambda^\delta(u_\lambda^\delta, u_\lambda^\delta)$" exists and the limit has the expression* (3.3)*:*

$$\begin{aligned}\lim_{\delta\to 0} \mathcal{E}_\lambda^\delta(u_\lambda^\delta, u_\lambda^\delta) &= \int_D A^0 \nabla u_0(x) \cdot \nabla u_0(x)dx + \int_D B^0 \cdot \nabla u_0(x) u_0(x)dx \\ &\quad + \iint\limits_{\substack{x\in D, x+h\in D\\ h\neq 0}} \big(u_0(x+h) - u_0(x)\big)^2 J^0(h)dhdx + k^0 \int_D u_0(x)^2dx + \lambda \int_D u_0(x)^2 dx \qquad (3.12)\\ &=: \mathcal{E}^0(u_0,u_0) + \lambda(u_0,u_0) =: \mathcal{E}_\lambda^0(u_0,u_0),\end{aligned}$$

where

$$a^0_{ij} = \int_Y \Big(a_{ij}(y) + \sum_{k=1}^d a_{ik}(y)\frac{\partial \omega_j}{\partial y_k}(y)\Big)dy, \quad b^0_i = \int_Y \Big(b_i(y) + \sum_{k=1}^d b_k(y)\frac{\partial \omega_i}{\partial y_k}(y)\Big)dy$$

$$J^0(h) = \int_Y J(y,h)dy \quad \text{and} \quad k^0 = \int_Y k(y)dy.$$

Proof: We first show the following limit.

$$\begin{aligned}\lim_{\delta\to 0}&\int_D A(x/\delta)\nabla u^\delta_\lambda(x)\cdot\nabla u^\delta_\lambda(x)dx \\ &= \iint\limits_{D\times Y} A(y)\big(\nabla u_0(x) + \nabla_y u_1(x,y)\big)\cdot\big(\nabla u_0(x) + \nabla_y u_1(x,y)\big)dxdy.\end{aligned}\tag{3.13}$$

Note that, since u^δ_λ converges to u_0 *weakly in* $H^1_0(D)$ as $\delta \to 0$,

$$\lim_{\delta\to 0}\langle f, u^\delta_\lambda\rangle_{H^{-1},H^1_0} = \langle f, u_0\rangle_{H^{-1},H^1_0}$$

holds and the right hand side is expressed as follows by (3.5) for putting $v = u_0$:

$$\begin{aligned}\langle f, u_0\rangle_{H^{-1},H^1_0} &= \iint\limits_{D\times Y} A(y)\big(\nabla u_0(x) + \nabla_y u_1(x,y)\big)\cdot\nabla u_0(x)dxdy \\ &\quad + \iint\limits_{D\times Y} B(y)\cdot\big(\nabla u_0(x) + \nabla_y u_1(x,y)\big)u_0(x)dxdy \\ &\quad + \iint\limits_{\substack{x\in D, x+h\in D\\ h\neq 0}} \big(u_0(x+h) - u_0(x)\big)^2 J^0(h)dhdx + k^0\int_D u_0(x)^2dx + \lambda(u_0,u_0).\end{aligned}$$

On the other hand, u^δ_λ is the solution to (2.7), it follows that

$$\begin{aligned}\langle f, u^\delta_\lambda\rangle_{H^{-1},H^1_0} &= \mathcal{E}^\delta_\lambda(u^\delta_\lambda, u^\delta_\lambda) \\ &= \mathcal{E}^{c,\delta}(u^\delta_\lambda, u^\delta_\lambda) + \mathcal{E}^{d,\delta}(u^\delta_\lambda, u^\delta_\lambda) + \mathcal{E}^{j,\delta}(u^\delta_\lambda, u^\delta_\lambda) + \mathcal{E}^{k,\delta}(u^\delta_\lambda, u^\delta_\lambda) + \lambda(u^\delta_\lambda, u^\delta_\lambda) \\ &= \int_D A(x/\delta)\nabla u^\delta_\lambda(x)\cdot\nabla u^\delta_\lambda(x)dx + \int_D B(x/\delta)\cdot\nabla u^\delta_\lambda(x)\,u^\delta_\lambda(x)dx \\ &\quad + \iint\limits_{\substack{x\in D, x+h\in D\\ h\neq 0}} (u^\delta_\lambda(x+h) - u^\delta_\lambda(x))^2 J(x/\delta,h)dxdh + \int_D u^\delta_\lambda(x)^2 k(x/\delta)dx + \lambda(u^\delta_\lambda, u^\delta_\lambda).\end{aligned}$$

Now we consider the convergences of all terms except the diffusion part of the right hand side. Since the sequence $\{u^\delta_\lambda\}$ converges to u_0 *strongly in* $L^2(D)$ and *weakly in* $H^1_0(D)$ as $\delta \to 0$, using the unfolding operators, we see

$$\lim_{\delta\to 0}\int_D B(x/\delta)\cdot\nabla u^\delta_\lambda(x)\,u^\delta_\lambda(x)dx = \lim_{\delta\to 0}\iint_{D\times Y} B(y)\cdot\mathcal{T}_\delta(\nabla u^\delta_\lambda)(x,y)\mathcal{T}_\delta(u^\delta_\lambda)(x,y)dxdy.$$

Since $\mathcal{T}_\delta(\nabla u^\delta_\lambda)$ converges to $\nabla u_0 + \nabla_y u_1$ *weakly in* $\big(L^2(D\times Y)\big)^d$ and $\mathcal{T}_\delta(u^\delta_\lambda)$ converges to u_0 *strongly in* $L^2(D\times Y)$, the right hand side goes to

$$\iint_{D\times Y} B(y)\cdot\big(\nabla u_0(x) + \nabla_y u_1(x,y)\big)u_0(x)dxdy$$

as $\delta\to 0$. For the jump part, note that $1_D(\cdot+h)(u^\delta_\lambda(\cdot+h)-u^\delta_\lambda(\cdot))$ converges to $1_D(\cdot+h)(u_0(\cdot+h)-u(\cdot))$ *strongly in* $L^2(D)$ for each $h\neq 0$. Then, set

$$\begin{aligned}&\iint\limits_{\substack{x\in D, x+h\in D\\ h\neq 0}} (u^\delta_\lambda(x+h)-u^\delta_\lambda(x))^2 J(x/\delta,h)dxdh\\ &\quad= \int\limits_{h\neq 0}\Big(\int_D 1_D(x+h)(u^\delta_\lambda(x+h)-u^\delta_\lambda(x))^2J(x/\delta,h)dx\Big)dh =: \int\limits_{h\neq 0}\Phi_\delta(h)dh,\end{aligned}$$

where

$$\Phi_\delta(h) = \int_D 1_D(x+h)(u^\delta_\lambda(x+h)-u^\delta_\lambda(x))^2J(x/\delta,h)dx, \quad h\neq 0.$$

We now show that $\Phi_\delta(h)$ converges to

$$\Phi_0(h) := \int_D 1_D(x+h)(u_0(x+h)-u_0(x))^2J(h)dx, \quad J^0(h) := \int_Y J(y,h)dy.$$

as $\delta\to 0$ for each $h\neq 0$. In fact,

$$\begin{aligned}\big|\Phi_\delta(h)-\Phi_0(h)\big| &\le \Big|\int_D 1_D(x+h)\Big((u^\delta_\lambda(x+h)-u^\delta_\lambda(x))^2-(u_0(x+h)-u_0(x))^2\Big)J(x/\delta,h)dx\Big|\\ &\quad+\Big|\int_D 1_D(x+h)(u_0(x+h)-u_0(x))^2J(x/h,h)dx\\ &\qquad-\int_D 1_D(x+h)(u_0(x+h)-u_0(x))^2J^0(h)dx\Big|\end{aligned}$$

and the first term of the right hand side is estimated as

$$\begin{aligned}&\Big|\int_D 1_D(x+h)\Big((u^\delta_\lambda(x+h)-u^\delta_\lambda(x))^2-(u_0(x+h)-u_0(x))^2\Big)J(x/\delta,h)dx\Big|\\ &\le\sqrt{\int_D 1_D(x+h)\Big((u^\delta_\lambda(x+h)-u^\delta_\lambda(x))+(u_0(x+h)-u_0(x))\Big)^2J(x/\delta,h)dx}\\ &\quad\times\sqrt{\int_D 1_D(x+h)\Big((u^\delta_\lambda(x+h)-u^\delta_\lambda(x))-(u_0(x+h)-u_0(x))\Big)^2J(x/\delta,h)dx}\\ &\le 4\varphi(h)\sqrt{\|u^\delta_\lambda\|^2_{L^2(D)}+\|u_0\|^2_{L^2(D)}}\\ &\quad\times\sqrt{\int_D 1_D(x+h)\Big(\big(u^\delta_\lambda(x+h)-u^\delta_\lambda(x)\big)-\big(u_0(x+h)-u_0(x)\big)\Big)^2dx}\end{aligned}$$

which goes to 0 as $\delta\to 0$, while the second term also goes to 0 as $\delta\to 0$ since the function $x\mapsto J(x,h)$ is Y-periodic for each $h\neq 0$ so that it is bounded (in x) by (J) and then $J(x/\delta,h)$

converges to $J^0(h)$ *weakly in* $L^1(D)$ as $\delta \to 0$ for each $h \neq 0$. So $\Phi_\delta(h)$ converges to $\Phi_0(h)$ for each $h \neq 0$. On the other hand, as in the proof of Lemma 3.3, it follows that

$$0 \le \Phi_\delta(h) \le \big(\mathbb{D}(u_\lambda^\delta, u_\lambda^\delta) + 4\|u_\lambda^\delta\|_{L^2(D)}^2\big)\big(1 \wedge |h|^2\big)\varphi(h), \quad h \neq 0$$

hold so that the right hand side is uniformly bounded in δ and $(1 \wedge |h|^2)\varphi(h)$ is integrable. Hence Lebesgue's dominated convergence theorem implies that

$$\begin{aligned}
&\lim_{\delta\to 0} \iint\limits_{\substack{x\in D, x+h\in D\\ h\neq 0}} (u_\lambda^\delta(x+h) - u_\lambda^\delta(x))^2 J(x/\delta, h)dxdh \\
&= \lim_{\delta\to 0}\int_{h\neq 0} \Phi_\delta(h)dh = \int_{h\neq 0}\Phi_0(h)dh = \iint\limits_{\substack{x\in D, x+h\in D\\ h\neq 0}} \big(u_0(x+h) - u_0(x)\big)^2 J^0(h)dxdh.
\end{aligned}$$

where $J^0(h) = \int_Y J(y,h)dy, h \neq 0$. This result implies that the microscopic energy correctly captures the macroscopic energy, including the non-local effects of the jumps, which are averaged out as J^0. Next, the potential (killing) part is also estimated as

$$\lim_{\delta\to 0}\int_D u_\lambda^\delta(x)^2 k(x/\delta)dx = \lim_{\delta\to 0}\iint_{D\times Y} \big(\mathcal{T}_\delta(u_\lambda^\delta)(x,y)\big)^2 k(y)dxdy$$

and, then noting that $\mathcal{T}_\delta(u_\lambda^\delta)$ converges to u_0 *strongly in* $L^2(D \times Y)$, the right hand side goes to

$$\iint_{D\times Y} u_0(x)^2 k(y)dxdy = k^0\int_D u_0(x)^2 dx,$$

where $k^0 = \int_Y k(y)dy$. The last term is easily seen to hold

$$\lim_{\delta\to 0}\int_D \lambda(u_\lambda^\delta, u_\lambda^\delta) = \lambda(u_0, u_0),$$

since u_λ^δ converges to u_0 *strongly in* $L^2(D)$.

Therefore, summarizing these estimates, it follows that

$$\begin{aligned}
&\lim_{\delta\to 0}\int_D A(x/\delta)\nabla u_\lambda^\delta(x)\cdot\nabla u_\lambda^\delta(x)dx \\
&= \lim_{\delta\to 0}\Big\{\langle f, u_\lambda^\delta\rangle_{H^{-1},H_0^1} - \int_D B(x/\delta)\cdot\nabla u_\lambda^\delta(x)\, u_\lambda^\delta(x)dx \\
&\quad - \iint\limits_{\substack{x\in D, x+h\in D\\ h\neq 0}} (u_\lambda^\delta(x+h) - u_\lambda^\delta(x))^2 J(x/\delta, h)dxdh - \int_D u_\lambda^\delta(x)^2 k(x/\delta)dx - \lambda(u_\lambda^\delta, u_\lambda^\delta)\Big\} \\
&= \langle f, u_0\rangle_{H^{-1},H_0^1} - \iint\limits_{D\times Y} B(y)\cdot\big(\nabla u_0(x) + \nabla_y u_1(x,y)\big)u_0(x)dxdy \\
&\quad - \iint\limits_{\substack{x\in D, x+h\in D\\ h\neq 0}} \big(u_0(x+h) - u_0(x)\big)^2 J^0(h)dhdx - k^0\int_D u_0(x)^2dx - \lambda(u_0,u_0) \\
&= \iint_{D\times Y} A(y)\big(\nabla u_0(x) + \nabla_y u_1(x,y)\big)\cdot\nabla u_0(x)dxdy. \qquad (3.14)
\end{aligned}$$

According to the fact that $u_1(x,y)=\sum_{i=1}^d \frac{\partial u_0}{\partial x_i}(x)\omega_i(y)$ and

$$\int_Y A(y)\Big(\nabla u_0(x)+\nabla u_1(x,y)\Big)\cdot\nabla_y\theta(y)dy=0 \quad \text{a.e. } x\in D \quad \text{for} \quad \theta\in\mathcal{H}_\#(Y),$$

the right hand side of (3.14) is equal to

$$\iint_{D\times Y} A(y)\big(\nabla u_0(x)+\nabla_y u_1(x,y)\big)\cdot\big(\nabla u_0(x)+\nabla_y u_1(x,y)\big)dxdy. \tag{3.15}$$

We now characterize the term (3.15). To this end, we first note that $A^0=(a^0_{ij})$ defined by

$$a^0_{ij}:=\int_Y\Big(a_{ij}(y)+\sum_{k=1}^d a_{ik}(y)\frac{\partial\omega_j}{\partial y_k}(y)\Big)dy, \quad i,j=1,2,\cdots,d,$$

is a symmetric matrix. In fact, since $\omega_i\in\mathcal{H}_\#(Y)$ for $i=1,2,\cdots,d$, it follows from (3.11) that

$$\int_Y A(y)\big(\boldsymbol{e}_j+\nabla\omega_j(y)\big)\cdot\nabla\omega_i(y)dy=0, \quad i,j=1,2,\cdots,d$$

and then

$$\begin{aligned}a^0_{ij}&=\int_Y\Big(a_{ij}(y)+\sum_{k=1}^d a_{ik}(y)\frac{\partial\omega_j}{\partial y_k}(y)\Big)dy=\int_Y A(y)\Big(\boldsymbol{e}_j+\nabla\omega_j(y)\Big)\cdot\boldsymbol{e}_i dy\\&=\int_Y A(y)\Big(\boldsymbol{e}_j+\nabla\omega_j(y)\Big)\cdot\Big(\boldsymbol{e}_i+\nabla\omega_i(y)\Big)dy=a^0_{ji}.\end{aligned}$$

So using the relation (3.10) again and the symmetry of $A^0=(a^0_{ij})=(a^0_{ji})$, we find that

$$\iint_{D\times Y} A(y)\big(u_0(x)+\nabla_y u_1(x,y)\big)\cdot\nabla u_0(x)dxdy=\int_D A^0\nabla u_0(x)\cdot\nabla u_0(x)dx. \tag{3.16}$$

Hence we obtain (3.12). $\square$

Finally we complete the proof of Theorem 3.1 by showing the following lemma.

Lemma 3.8 *$\mathcal{T}_\delta(\nabla u^\delta_\lambda)(x,y)$ converges to $\nabla u_0(x)+\nabla_y u_1(x,y)$ strongly in $\big(L^2(D\times Y)\big)^d$ as $\delta\to 0$.*

Proof: The uniform ellipticity of A implies that the following inequality holds:

$$\begin{aligned}&\alpha\iint_{D\times Y}\big|\mathcal{T}_\delta(\nabla u^\delta_\lambda)(x,y)-\big(\nabla u_0(x)+\nabla_y u_1(x,y)\big)\big|^2dxdy\\&\le\iint_{D\times Y}A(y)\Big(\mathcal{T}_\delta(\nabla u^\delta_\lambda)(x,y)-\big(\nabla u_0(x)+\nabla_y u_1(x,y)\big)\Big)\cdot\\&\qquad\qquad\cdot\Big(\mathcal{T}_\delta(\nabla u^\delta_\lambda)(x,y)-\big(\nabla u_0(x)+\nabla_y u_1(x,y)\big)\Big)dxdy\\&=\iint_{D\times Y}\mathcal{T}_\delta\big(A(\cdot/\delta)\nabla u^\delta_\lambda\cdot\nabla u^\delta_\lambda\big)(x,y)dxdy-\iint_{D\times Y}A(y)\mathcal{T}_\delta(\nabla u^\delta_\lambda)(x,y)\cdot\big(\nabla u_0(x)+\nabla_y u_1(x,y)\big)dxdy\end{aligned}$$

$$
\begin{aligned}
&-\iint\limits_{D\times Y} A(y)\big(\nabla u_0(x)+\nabla_y u_1(x,y)\big)\cdot \mathcal{T}_\delta(\nabla u_\lambda^\delta)(x,y)dxdy\\
&+\iint\limits_{D\times Y} A(y)\big(\nabla u_0(x)+\nabla_y u_1(x,y)\big)\cdot \big(\nabla u_0(x)+\nabla_y u_1(x,y)\big)dxdy.
\end{aligned}
$$

Then, using the unfolding operators and then the estimates (3.13) and (3.16), the first term is estimated as

$$\lim_{\delta\to 0}\int_D A(x/\delta)\nabla u_\lambda^\delta(x)\cdot\nabla u_\lambda^\delta dx=\int_D A^0\nabla u_0(x)\cdot\nabla u_0(x)dx$$

as $\delta\to 0$, while the rest terms of the right hand side go to

$$-\int_D A^0\nabla u_0(x)\cdot\nabla u_0(x)dx$$

and, thus we conclude that $\mathcal{T}_\delta(\nabla u_\lambda^\delta)(x,y)$ converges to $\nabla u_0(x)+\nabla_y u_1(x,y)$ *strongly in* $\big(L^2(D\times Y)\big)^d$ as $\delta\to 0$. □

Corollary 3.1 *The quadratic form* $(\mathcal{E}^0,H_0^1(D))$ *defined by*

$$
\begin{aligned}
&\mathcal{E}^0(u,v)\\
&=\int_D A^0\nabla u(x)\cdot\nabla v(x)dx+\int_D B^0\cdot\nabla u(x)v(x)dx\\
&+\iint\limits_{\substack{x\in D,x+h\in D\\ h\neq 0}}\big(u(x+h)-u(x)\big)\big(v(x+h)-v(x)\big)J^0(h)dhdx+k^0\int_D u(x)v(x)dx,\quad u,v\in H_0^1(D)
\end{aligned}
$$

becomes a lower bounded semi-Dirichlet form on $L^2(D)$ *with some* $\beta_1>0$ *and* $\{(\mathcal{E}^\delta,H_0^1(D)),\delta>0\}$ *converges to* $(\mathcal{E}^0,H_0^1(D))$ *on* $L^2(D)$ *in the sense of Mosco as* $\delta\to 0$.

Finally we give an example.

Example 3.1. (Symmetric Stable-type Jump-diffusion Processes)

Let D be a bounded open set of $\mathbb{R}^d$ and $A(x)=(a_{ij}(x))$ a symmetric matrix-valued measurable function on $\mathbb{R}^d$ such that

$$c_1|\xi|^2\le A(x)\xi\cdot\xi\le c_2|\xi|^2,\quad x,\xi\in\mathbb{R}^d$$

for some positive constants c_1,c_2. Let $\alpha(x)$ be a measurable function on $\mathbb{R}^d$ and $n(x,h)$ a positive measurable function on $\mathbb{R}^d\times(\mathbb{R}^d\setminus\{0\})$ such that

(α) $0<\underline{\alpha}\le\alpha(x)\le\overline{\alpha}<2,\quad x\in\mathbb{R}^d$;

(n) $0<c_1\le n(x,-h)=n(x,h)\le c_2(1\vee|\log|h||)^\kappa,\quad x,h\in\mathbb{R}^d,\ h\neq 0$;

for some constants $0<c_i<\infty$ $(i=1,2)$ and $\kappa\in\mathbb{R}$. Put

$$J(x,h):=\frac{n(x,h)}{|h|^{d+\alpha(x)}},\quad x,h\in\mathbb{R}^d,h\neq 0.$$

Then the function $J(x,h)$ satisfies (J). In fact, since

$$J(x,h) = \frac{n(x,h)}{|h|^{d+\alpha(x)}} \le c_2\big(1 \vee |\log|h||\big)^{\kappa}\big(|h|^{-d-\underline{\alpha}} \vee |h|^{-d-\overline{\alpha}}\big), \quad x,h \in \mathbb{R}^d,\ h \neq 0,$$

the condition (J) is satisfied if we put $\varphi(h) := c_2\big(1 \vee |\log|h||\big)^{\kappa}\big(|h|^{-d-\underline{\alpha}} \vee |h|^{-d-\overline{\alpha}}\big)$. For simplicity we set $B(x) = \mathbf{0}$ and $k(x) = 0$. Assuming further that the functions $x \mapsto A(x)$, $x \mapsto \alpha(x)$ and $x \mapsto n(x,h)$ are all Y-periodic (for each $h \neq 0$), then for $\delta > 0$, the quadratic forms

$$\begin{aligned}\mathcal{E}^{\delta}(u,v) &:= \mathcal{E}^{c,\delta}(u,v) + \mathcal{E}^{j,\delta}(u,v)\\ &:= \int_D A(x/\delta)\nabla u(x)\cdot\nabla v(x)dx + \iint_{\substack{x\in D, x+h\in D\\ h\neq 0}} \big(u(x+h)-u(x)\big)\big(v(x+h)-v(x)\big)\frac{n(x/\delta,h)}{|h|^{d+\alpha(x/\delta)}}dhdx\end{aligned}$$

with $H_0^1(D)$ produce regular symmetric Dirichlet forms on $L^2(D)$ and then taking $\delta \to 0$, they converge to the following regular symmetric Dirichlet form on $L^2(D)$ in the sense of Mosco:

$$\mathcal{E}^0(u,v) = \int_D A^0\nabla u(x)\cdot\nabla v(x)dx + \iint_{\substack{x\in D, x+h\in D\\ h\neq 0}} \big(u(x+h)-u(x)\big)\big(v(x+h)-v(x)\big)J^0(h)dhdx,\ u,v \in H_0^1(D),$$

where $A^0 = (a^0_{ij})$ and $J^0(h)$ are defined as in Theorem 3.1:

$$a_{ij} = \int_Y \Big(a_{ij}(y) + \sum_{k=1}^d a_{ik}(y)\frac{\partial\omega_j}{\partial y_k}(y)\Big)dy,\ i,j = 1,2,\cdots,d \ \text{ and }\ J^0(h) = \int_Y \frac{n(y,h)}{|h|^{d+\alpha(y)}}dy, \quad h \neq 0.$$

□

Remark 3.1. The homogenization of jump processes was investigated in Tomisaki [16] through the convergence of cádlág processes. A key difference from [16] is the scaling of the jumping kernel: while [16] considers $n(x/\delta, h/\delta)$, which leads to different limit processes, we focus on the scaling $n(x/\delta, h)$. In our setting, the diffusion term remains in the limit, and the periodic scaling in x leads to a homogenized operator of jump-diffusion type, where a stable-like Lévy jump appears. Thus, the result in Example 3.1 provides a different perspective by capturing the coexistence of diffusion and jump effects in the homogenized limit.

Acknowledgment. The author would like to express his sincere gratitude to the anonymous referee for the careful reading of the manuscript and for providing insightful comments and valuable suggestions, which have significantly improved the clarity and quality of the paper. This work was supported in part by JSPS KAKENHI Grant Number 25K07056.

References

[1] G. Allaire, Shape Optimization by the Homogenization Method, Springer, 2002

[2] H. Attouch, Variational Convergence for Functions and Operators, Pitman, 1984

[3] A. Bensoussan, J.-L. Lions and G. Papanicolaou, Asymptotic Analysis for Periodic Structures, AMS Chelsea Publishing, 2011 (originally published from North-Holland Pub. Co., 1978).

[4] D. Cioranescu, A. Damlamian and G. Griso, The Periodic Unfolding Method – Theory and Applications to Partial Differential Problems –, Springer, 2018

[5] D. Cioranescu and P. Donato, An Introduction to Homogenization, Oxford University Press, 1999 (Reprinted, 2010)

[6] L.C. Evans, Partial Differential Equations, Second Edition, American Mathematical Society, 2010

[7] L.C. Evans and R.F. Gariepy, Measure Theory and Fine Properties of Functions, Revised Edition, CRC Press, 2015

[8] M. Fukushima, Y. Ōshima and M. Takeda, Dirichlet Forms and Symmetric Markov Processes, 2nd revised and extended ed., De Gruyter, 2011

[9] M. Fukushima and T. Uemura, *Jump-type Hunt processes generated by lower bounded semi-Dirichlet forms*, Ann. Probab., **40** (2012), 858-889

[10] M. Hino, *Convergence of non-symmetric forms*, J. Math. Kyoto Univ., **38** (1998), 329-341

[11] Hornung U (ed.), Homogenization and Porous Media, Springer, 1997

[12] V. Jikov, S. Kozlov and O. Oleinik, Homogenization of Differential Operators and Integral Functionals, translated from the Russian by G. A. Yosifian, Springer-Verlag, 1994

[13] U. Mosco, *Composite media and asymptotic Dirichlet forms*, J. Funct. Anal., **123** (1994), 368-421

[14] Y. Ōshima, Semi-Dirichlet Forms and Markov Processes, De Gruyter, 2013

[15] A. Seesanea and T. Uemura, *Homogenization of diffusion processes with singular drifts and potentials via unfolding method*, J. Math. Anal. Appl., **556** (2026), Paper No. 130105, 29 pp. (arXiv:2306.14307, 2023)

[16] M. Tomisaki, *Homogenization of cádlág processes*, J. Math. Soc. Japan, **44** (1992), 281-305

[17] M. Tomisaki and T. Uemura, *Homogenization of symmetric Dirichlet forms*, J. Math. Soc. Japan, **74** (2022), 247-283

[18] T. Uemura, *On some path properties of symmetric stable-like processes for one dimension*, Potential Analysis, **16** (2002), 79-91