

RULED ZERO MEAN CURVATURE SURFACES IN THE THREE-DIMENSIONAL LIGHT CONE

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ABSTRACT. We obtain a complete classification of ruled zero mean curvature surfaces in the three-dimensional light cone. En route, we examine geodesics and screw motions in the space form, allowing us to discover helicoids. We also consider their relationship to catenoids using Weierstrass representations of zero mean curvature surfaces in the three-dimensional light cone.

1. INTRODUCTION

Rooted in the discoveries of non-Euclidean geometry, it is often interesting to examine which Euclidean geometric concepts and results can be applied to other geometries. Ever since Catalan showed that the standard helicoid is the only non-trivial ruled minimal surface in Euclidean space [4], classification problem of ruled surfaces satisfying certain curvature restrictions in various space forms has received plethora of interest across various three-dimensional spaces. These results include the study of:

- ruled Weingarten surfaces in Lorentzian 3-space $\mathbb{L}^3$ [7],
- ruled and helicoidal zero mean curvature (ZMC) surfaces in Lorentzian 3-space [16],
- ruled zero mean curvature surfaces in $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, Heisenberg group Nil_3 , and the Berger sphere [11, 19, 20],
- ruled Weingarten hypersurfaces in hyperbolic spaces $\mathbb{H}^{n+1}$, Lorentzian spaces and de Sitter spaces $\mathbb{S}^{n+1}$ [1, 2, 3], and
- ruled ZMC surfaces in isotropic 3-space [24].

The main purpose of this article is to present a classification of ruled ZMC surfaces in the three-dimensional light cone $\mathbb{Q}_+^3$, which is a space form equipped with a degenerate metric. Its curve theory and surface theory have been developed in [12, 13, 14, 15], which we review in our preparatory Section 2. We also review rotational ZMC surfaces, called *catenoids*, in $\mathbb{Q}_+^3$, found in [6], as they will play an important role.

We begin our main discussion in Section 3 by considering *geodesics* in $\mathbb{Q}_+^3$, as degenerate metric of the space form presents a meaningful obstacle. Then we find two important ruled ZMC surfaces in Section 4, to serve as examples unveiling the required ansatz to obtain a classification of ruled ZMC surfaces in $\mathbb{Q}_+^3$. In particular, noting that helicoids in Euclidean space can be characterized as ruled surfaces that are invariant under screw motions, we consider screw motions in Section 4.1, and show that the surface obtained via

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applying screw motions to a geodesic has zero mean curvature in Section 4.2, which we call *helicoids*. We also show that one of the catenoids reviewed, the parabolic catenoid, is in fact a ruled surface in Section 4.4. A noted feature of ZMC surfaces in $\mathbb{Q}_+^3$ is that they admit a *Weierstrass representation*¹ [13, 14, 18] as in the case of minimal surfaces in Euclidean space [25]. The Weierstrass representation allows for a ZMC surface to be represented in terms of a meromorphic function and a holomorphic 1-form called the *Weierstrass data*. Notably, the famous isometric deformation connecting catenoids to helicoids in Euclidean space admits a simple characterization in terms of its Weierstrass data, known as the associated family. Thus, in Section 5, we present a comprehensive examination of the relationship between catenoids and helicoids in $\mathbb{Q}_+^3$. In particular, we show that, as in the Euclidean case, every helicoid is in the associated family of some catenoid (Theorem 5.1), but unlike the Euclidean case, only certain catenoids admit a helicoid in its associated family (Theorem 5.2). Then, in Section 5.3, we introduce a Lawson-type correspondence between ZMC surfaces in the isotropic 3-space and $\mathbb{Q}_+^3$ and compare the associated families of catenoid in each space form under the correspondence.

Finally, in Section 6, we obtain a complete classification of ruled ZMC surfaces in $\mathbb{Q}_+^3$ (Theorem 6.2), showing that every ruled ZMC surface must be a helicoid or a parabolic catenoid up to isometries and homotheties of $\mathbb{Q}_+^3$.

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2. PRELIMINARIES

In this section, we briefly review the basic differential geometry of the three dimensional light cone, including curve and surface theory, mainly to set the notations to be used throughout. (We refer the readers to [12, 13] for a detailed introduction.)

2.1. $\mathbb{Q}_+^3$ as a quadric. We identify Lorentzian 4-space $\mathbb{L}^4$ with $\text{Herm}(2)$ as follows:

$$\mathbb{L}^4 \ni (x_0, x_1, x_2, x_3) \sim \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix} \in \text{Herm}(2).$$

Then for any $X, Y \in \text{Herm}(2) \cong \mathbb{L}^4$, the inner product can be expressed as

$$\langle X, Y \rangle := -\frac{1}{2} (\det(X + Y) - \det X - \det Y),$$

so that

$$\langle X, X \rangle = -\det X.$$

An arbitrary $F \in \text{SL}(2, \mathbb{C})$ acts on $\mathbb{L}^4$ as an orientation-preserving isometry via the action

$$(2.1) \quad \text{Herm}(2) \ni X \mapsto FXF^* \in \text{Herm}(2),$$

where F^* is the conjugate transpose of F . In fact, $\text{SL}(2, \mathbb{C})$ is a two-to-one covering of the group of orientation-preserving and origin-fixing isometries of $\mathbb{L}^4$.

¹ZMC surfaces in hyperbolic 3-space and in de Sitter 3-space have Weierstrass-type representations that is different in flavor, exploiting the method of infinite-dimensional Lie groups, known as loop groups.

The submanifolds of $\mathbb{L}^4$ we are interested in are

$$\begin{aligned}\mathbb{I}^3 &:= \{X \in \text{Herm}(2) : x_0 - x_3 = 0\}, \\ \mathbb{Q}_+^3 &:= \{X \in \text{Herm}(2) : \langle X, X \rangle = 0, \text{tr } X > 0\}, \\ \mathbb{Q}_-^3 &:= \{X \in \text{Herm}(2) : \langle X, X \rangle = 0, \text{tr } X < 0\}.\end{aligned}$$

In particular, the action of $\text{SL}(2, \mathbb{C})$ given by (2.1) acts as an orientation-preserving isometry of $\mathbb{Q}_+^3$. Conversely, any orientation preserving isometry of $\mathbb{Q}_+^3$ can be described by this action.

EXAMPLE 2.1. Rotations in $\mathbb{Q}_+^3$ can be described as isometries that fix a 2-dimensional subspace of $\mathbb{L}^4$ (see, for example, [6, 8]). Normalizing the 2-dimensional subspace based on the induced metric, they can be described by

$$D_1(\mu) := \begin{pmatrix} \mu & 0 \\ 0 & \frac{1}{\mu} \end{pmatrix}, \quad D_2(\mu) := \begin{pmatrix} 0 & \mu \\ -\frac{1}{\mu} & 0 \end{pmatrix}, \quad P(\mu) := \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}.$$

for some $\mu \in \mathbb{C} \setminus \{0\}$.

2.2. Curve theory. Given a unit-speed regular curve $\gamma : I \rightarrow \mathbb{Q}_+^3 \subset \mathbb{L}^4$ given on an interval I , define

$$\kappa := -\frac{1}{2}\langle \gamma'', \gamma'' \rangle, \quad \mathbf{T} := \gamma', \quad \text{and} \quad \mathbf{N} := \kappa\gamma - \gamma''.$$

Then there exist uniquely a function τ and a vector field $\mathbf{B}$ along γ which form a null basis of $\mathbb{L}^4$ with $\det(\gamma, \mathbf{T}, \mathbf{N}, \mathbf{B}) < 0$:

$\langle X, Y \rangle$					
		γ	$\mathbf{T}$	$\mathbf{N}$	$\mathbf{B}$
X	Y				
γ		0	0	1	0
$\mathbf{T}$		0	1	0	0
$\mathbf{N}$		1	0	0	0
$\mathbf{B}$		0	0	0	1

Using such $\{\gamma, \mathbf{T}, \mathbf{N}, \mathbf{B}\}$ as a moving frame along the curve γ , the Frenet equations are given by

$$(2.2) \quad \begin{pmatrix} \gamma' \\ \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \kappa & 0 & -1 & 0 \\ 0 & -\kappa & 0 & -\tau \\ \tau & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma \\ \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix},$$

The values κ and τ are referred to as the *cone curvature* and the *cone torsion*, respectively. See [12, 15] for details.

2.3. Surface theory. Let $\mathcal{U}$ be a simply-connected domain in $\mathbb{R}^2$. Given an immersion $X : \mathcal{U} \subset \mathbb{R}^2 \rightarrow \mathbb{Q}_+^3$ with coordinates $(u, v) = (u^1, u^2) \in \mathcal{U}$, there is a unique map $G : \mathcal{U} \rightarrow \mathbb{Q}_-^3$ which satisfies

$$\langle G, G \rangle = \langle G, X_u \rangle = \langle G, X_v \rangle = \langle G, X \rangle - 1 = 0,$$

which is called the *lightlike Gauss map* of X [10] (also referred to as the *associated surface* [14]). The first and second fundamental forms of X are then given by

$$\mathbf{g} := \langle X_i, X_j \rangle du^i du^j, \quad \mathbf{A} := \langle G, X_{ij} \rangle du^i du^j,$$

respectively, from which the definition of mean curvature and (extrinsic) Gauss curvature for X follow:

$$H := \frac{1}{2} \operatorname{tr}(\mathbf{g}^{-1} \mathbf{A}), \quad K := \det(\mathbf{g}^{-1} \mathbf{A}).$$

If a surface has $H \equiv c$ for some nonzero real constant c , then the surface is called a *constant mean curvature (CMC- c) surface*. If $H \equiv 0$, then the surface is referred to as a *zero mean curvature (ZMC) surface*.

When a surface is conformally parametrized, we will introduce the complex structure via $z = u + iv$.

An immersion $X : \mathcal{U} \rightarrow \mathbb{Q}_+^3$ admits three different representations, each of which we will make use in this manuscript: An immersion X can be viewed as a graph of any function $f : \mathcal{U} \rightarrow \mathbb{R}$ over the surface

$$(u, v) \mapsto \begin{pmatrix} u^2 + v^2 & u + iv \\ u - iv & 1 \end{pmatrix}$$

so that

$$X(u, v) = e^{f(u, v)} \begin{pmatrix} u^2 + v^2 & u + iv \\ u - iv & 1 \end{pmatrix}.$$

When we represent an immersion X as a graph using the function f , we will denote it by X_f .

On the other hand, using the fact that the map

$$\mathbb{C}^2 \setminus \{(0, 0)\} \ni \begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} z \\ w \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix}^* = \begin{pmatrix} z\bar{z} & z\bar{w} \\ \bar{z}w & w\bar{w} \end{pmatrix} \in \mathbb{Q}_+^3$$

is onto, we can represent an immersion via

$$X = \begin{pmatrix} A \\ C \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix}^* =: \varphi \varphi^*$$

for some complex-valued functions A and C defined on $\mathcal{U}$. We will refer to such $\varphi : \mathcal{U} \rightarrow \mathbb{C}^2$ as the *lift of X into $\mathbb{C}^2$* .

Finally, for any $F : \mathcal{U} \rightarrow \operatorname{SL}(2, \mathbb{C})$, we can also represent an immersion X via

$$X = F \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} F^*.$$

We will refer to such F as the *lift of X into $\operatorname{SL}(2, \mathbb{C})$* . Using this last characterization, we introduce the following notion:

DEFINITION 2.2. We say two immersions X and $\tilde{X}$ are *equal up to isometries and homothety of $\mathbb{Q}_+^3$* if they satisfy $\tilde{X}(s, t) = rFX(s, t)F^*$ for some $r \in \mathbb{R}^+$ and $F \in \operatorname{SL}(2, \mathbb{C})$, and denote this by

$$X \simeq \tilde{X}.$$

REMARK 2.3. For $c \in \mathbb{C} \setminus \{0\}$, we have

$$\begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}^* = c\bar{c} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

If $F_c := F \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}$, then

$$X_c := F_c \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} F_c^* = c\bar{c} F \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} F^* = c\bar{c} X, \quad dF_c F_c^{-1} = dF F^{-1}.$$

In other words, homothetic images of X can be obtained via F_c .

2.4. Representations of ZMC immersions. Suppose $X = X_f$ is an immersion regarded as a graph using some function $f : \mathcal{U} \rightarrow \mathbb{R}$. Then the mean curvature of X_f can be calculated as

$$(2.3) \quad H = \frac{1}{2} e^{-2f(u,v)} (f_{uu}(u,v) + f_{vv}(u,v)).$$

EXAMPLE 2.4 (A FAMILY OF CMC SURFACES IN $\mathbb{Q}_+^3$). Let us consider the surface $X_f : \mathcal{U} \rightarrow \mathbb{Q}_+^3$ given via the function $f : \mathcal{U} \rightarrow \mathbb{R}$

$$f(u, v) := d + \ln \operatorname{sech}(au + bv + c)$$

for some constants $a, b, c, d \in \mathbb{R}$ with $a^2 + b^2 \neq 0$. Then one can directly check that the mean curvature of X_f is

$$H = -e^{-2d}(a^2 + b^2)/2.$$

If $H \equiv 0$ so that X_f is a ZMC surface, then f must be harmonic, so $f = \varphi + \bar{\varphi}$ for some holomorphic function φ in a simply connected domain. Then, after a conformal change of parameters if necessary, we see the following:

Lemma 2.5 ([14]). $X : \mathcal{U} \rightarrow \mathbb{Q}_+^3$ has zero mean curvature if and only if

$$X(z) = \varphi(z)\varphi(z)^*, \quad \varphi(z) := \begin{pmatrix} A(z) \\ C(z) \end{pmatrix}$$

for some holomorphic functions A and C .

Now we would like to see how the lift of a ZMC immersion X into $\mathbb{C}^2$ induces the lift into $\operatorname{SL}(2, \mathbb{C})$: A holomorphic $F \in C^\omega(\mathcal{U}, \operatorname{SL}(2, \mathbb{C}))$ is called *null* if $\det F_z = 0$. Given an arbitrary holomorphic $\varphi(z) = \begin{pmatrix} A(z) \\ C(z) \end{pmatrix}$ that is the lift of a ZMC immersion to $\mathbb{C}^2$, one can find a null-holomorphic

$$F = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in C^\omega(\mathcal{U}, \operatorname{SL}(2, \mathbb{C}))$$

such that

$$F \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} F^* = \begin{pmatrix} A \\ C \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix}^*$$

via

$$(2.4) \quad F = \begin{pmatrix} A & 0 \\ C & A^{-1} \end{pmatrix} \begin{pmatrix} 1 & -E \\ 0 & 1 \end{pmatrix}, \quad E := \int \frac{((1/A)')^2}{(C/A)'} dz.$$

For a given null-holomorphic F we define meromorphic functions G, g and holomorphic 1-forms Ω, ω by

$$(2.5) \quad dF F^{-1} = \begin{pmatrix} G & -G^2 \\ 1 & -G \end{pmatrix} \Omega, \quad F^{-1} dF = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \omega.$$

This is the *Weierstrass representation* of ZMC surfaces in $\mathbb{Q}_+^3$ [13, 14, 17, 18], and (g, ω) is called *Weierstrass data*. We call G and g the hyperbolic Gauss map and the secondary Gauss map, respectively. The Hopf differential is

$$Q := \Omega dG = \omega dg.$$

For $\lambda \in \mathbb{C} \setminus \{0\}$, the change of Weierstrass data

$$(2.6) \quad (g, \omega) \mapsto (g, \lambda \omega)$$

induces a transformation, where the metric, Hopf differential, and the second fundamental form change as follows:

$$\mathbf{g} \mapsto |\lambda|^2 \mathbf{g}, \quad Q \mapsto \lambda Q, \quad \mathbf{A} = 2 \operatorname{Re} Q \mapsto \mathbf{A}_\lambda = 2 \operatorname{Re} \lambda Q.$$

When $|\lambda|^2 = 1$, this gives an isometric deformation of a ZMC surface, commonly referred to as the *associated family of ZMC surfaces*.

2.5. Catenoids in $\mathbb{Q}_+^3$. As examples of ZMC surfaces, we review catenoids in $\mathbb{Q}_+^3$, ZMC surfaces that are invariant under rotations [6]. For real constants a, b, c , consider

$$(2.7) \quad \varphi_a^E(z) := e^{iaz} \begin{pmatrix} e^{iz} \\ e^{-iz} \end{pmatrix}, \quad \varphi_b^H(z) := e^{ibz} \begin{pmatrix} e^z \\ e^{-z} \end{pmatrix}, \quad \varphi_c^P(z) := e^{icz} \begin{pmatrix} z \\ 1 \end{pmatrix}.$$

These are lifts to $\mathbb{C}^2$ of elliptic, hyperbolic, parabolic catenoids in $\mathbb{Q}_+^3$, respectively. Their lifts to $\operatorname{SL}(2, \mathbb{C})$ are

$$\begin{aligned} F_a^E(z) &:= \begin{pmatrix} e^{i(a+1)z} & -\frac{(a+1)^2}{4a} e^{-i(a-1)z} \\ e^{i(a-1)z} & -\frac{(a-1)^2}{4a} e^{-i(a+1)z} \end{pmatrix}, \\ F_b^H(z) &:= \begin{pmatrix} e^{i(b+1)z} & -\frac{i(b-i)^2}{4b} e^{-(ib-1)z} \\ e^{i(b-1)z} & -\frac{i(b+i)^2}{4b} e^{-(ib+1)z} \end{pmatrix}, \\ F_c^P(z) &:= \begin{pmatrix} ze^{icz} & -\frac{1}{2}(2+icz)e^{-icz} \\ e^{icz} & -\frac{1}{2}ice^{-icz} \end{pmatrix}. \end{aligned}$$

Their Weierstrass data can be written as

$$g^E(w) = g^H(w) = g^P(w) = w$$

and

$$(2.8) \quad \omega^E = \left(-\frac{1}{4} + \frac{1}{4a^2}\right) \frac{dw}{w^2}, \quad \omega^H = \left(-\frac{1}{4} - \frac{1}{4b^2}\right) \frac{dw}{w^2}, \quad \omega^P = -\frac{1}{4} \frac{dw}{w^2}.$$

REMARK 2.6. Note that there is only one parabolic catenoid up to isometries and homothety of $\mathbb{Q}_+^3$. For CMC-1 surfaces in de Sitter three-space, a similar fact has been observed in [9].

3. GEODESICS OF $\mathbb{Q}_+^3$

As the induced metric on $\mathbb{Q}_+^3$ is degenerate, we will consider geodesics in $\mathbb{Q}_+^3$ by using characterizations of geodesics in better known quadrics of $\mathbb{L}^4$: hyperbolic 3-space $\mathbb{H}^3$ and de Sitter 3-space $\mathbb{S}_1^3$. In both cases, geodesics are obtained as intersections of each corresponding space and a two dimensional plane in $\mathbb{L}^4$ which passes through the origin; thus, they are curves in a totally geodesic surface, i.e. a totally umbilic surface with vanishing mean and (extrinsic) Gaussian curvature, whose geodesic curvature vanishes.

3.1. Planes in $\mathbb{Q}_+^3$. We first define planes as follows:

DEFINITION 3.1. A *plane* is an immersion in $\mathbb{Q}_+^3$ that is totally geodesic, that is, whose second fundamental form $\mathbf{A}$ vanishes everywhere.

We will characterize planes within the class of totally umbilic surfaces in $\mathbb{Q}_+^3$: these are given by intersections with affine 3-planes, namely,

$$S[M, q] := \{X \in \mathbb{Q}_+^3 : \langle X, M \rangle = q\}$$

for some $M \in \mathbb{L}^4$ and $q \in \mathbb{R} \setminus \{0\}$. Then it can be directly checked that the lightlike Gauss map of $S[m, q]$ is given by

$$G = -\frac{1}{2q^2} \langle M, M \rangle X + \frac{1}{q} M$$

so that its mean curvature and Gaussian curvature satisfy

$$H = \frac{1}{2q^2} \langle M, M \rangle, \quad K = \frac{1}{4q^4} \langle M, M \rangle^2.$$

Thus:

Lemma 3.2. *An immersion is part of a plane if and only if it is part of $P[M, q]$ defined via*

$$P[M, q] := \{X \in \mathbb{Q}_+^3 : \langle X, M \rangle = q, \langle M, M \rangle = 0\}.$$

3.2. Geodesics. Now that we have the notion of a plane, we define geodesics as follows:

DEFINITION 3.3. A regular curve in a plane of $\mathbb{Q}_+^3$ is called a *geodesic* if its geodesic curvature vanishes in the plane.

To obtain an explicit formulation of geodesics, let us suppose that a plane is given via $P[M, q]$. We will now calculate the metric induced on $P[M, q]$ from the ambient space by constructing a coordinate chart. Choose any nonzero lightlike vector $\tilde{M}$ such that $\langle M, \tilde{M} \rangle = q \neq 0$, and note that $\text{span}\{M, \tilde{M}\}$ must have signature $(-+)$. Thus,

$$\mathfrak{R} := \text{span}\{M, \tilde{M}\}^\perp \cong \mathbb{E}^2,$$

where $\mathbb{E}^2$ denotes the usual Euclidean 2-plane.

We define a bijection $\psi : \mathfrak{R} \rightarrow P[M, q]$ by

$$\psi(Y) = Y + \tilde{M} - \frac{1}{2q} \langle Y, Y \rangle M$$

and its inverse by

$$\psi^{-1}(X) = X - \tilde{M} - \frac{1}{q}\langle X, \tilde{M} \rangle M.$$

Viewing ψ as a coordinate chart, let $\gamma : I \rightarrow P[m, q]$ be a unit-speed curve with $Y : I \rightarrow \mathfrak{R}$ such that

$$\gamma = \psi \circ Y.$$

Since

$$\gamma' = (Y + \tilde{M} - \frac{1}{2q}\langle Y, Y \rangle M)' = Y' - \frac{1}{q}\langle Y, Y' \rangle M,$$

it must follow that

$$\langle \gamma', \gamma' \rangle = \langle Y', Y' \rangle.$$

Therefore, any plane $P[m, q]$ is isometric to $\mathfrak{R} \cong \mathbb{E}^2$, and thus $\gamma : I \rightarrow P[m, q]$ is a curve in $P[m, q]$ with vanishing geodesic curvature if and only if $Y := \psi^{-1} \circ \gamma : I \rightarrow \mathfrak{R} \cong \mathbb{E}^2$ is a line.

To obtain explicit parametrizations of geodesics, let $Y : I \rightarrow \mathfrak{R}$ be a line parametrized by arc-length so that

$$Y(s) = Vs + W$$

for any constant $V, W \in \mathfrak{R} = \text{span}\{M, \tilde{M}\}^\perp$ with $\langle V, V \rangle = 1$. We then have

$$\begin{aligned} \gamma(s) &= \psi \circ Y(s) = Vs + W + \tilde{M} - \frac{1}{2q}(s^2 + 2\langle V, W \rangle s + \langle W, W \rangle)M \\ &= -\frac{1}{2q}Ms^2 + (V - \frac{1}{q}\langle V, W \rangle M)s + W + \tilde{M} - \frac{1}{2q}\langle W, W \rangle M \\ &:= \frac{1}{2}\vec{a}s^2 + \vec{b}s + \vec{c}. \end{aligned}$$

Then we can check directly that

$$(3.1) \quad \langle \vec{a}, \vec{a} \rangle = \langle \vec{a}, \vec{b} \rangle = \langle \vec{b}, \vec{c} \rangle = \langle \vec{c}, \vec{c} \rangle = 0, \quad \langle \vec{a}, \vec{c} \rangle = -\langle \vec{b}, \vec{b} \rangle = -1.$$

On the other hand, let $\gamma : I \rightarrow \mathbb{Q}_+^3$ be given via

$$\gamma(s) = \frac{1}{2}\vec{a}s^2 + \vec{b}s + \vec{c},$$

where $\vec{a}, \vec{b}, \vec{c} \in \mathbb{L}^4$ satisfy (3.1). Then we have

$$\langle \gamma(s), \vec{a} \rangle = -1,$$

so that γ is a curve in the plane $P[\vec{a}, -1]$. Therefore, using $\psi^{-1} : P[\vec{a}, -1] \rightarrow \mathfrak{R}$, we write $Y := \psi^{-1} \circ \gamma$ with $\vec{c} = \tilde{M}$ as

$$Y = \frac{1}{2}\vec{a}s^2 + \vec{b}s + \vec{c} - \vec{c} + \langle \frac{1}{2}\vec{a}s^2 + \vec{b}s + \vec{c}, \vec{c} \rangle \vec{a} = \vec{b}s$$

so that Y is a line.

Summarizing:

Theorem 3.4. *A curve $\gamma : I \rightarrow \mathbb{Q}_+^3$ is a geodesic in $\mathbb{Q}_+^3$ if and only if there exist some $\vec{a}, \vec{b}, \vec{c} \in \mathbb{L}^4$ satisfying*

$$(3.2) \quad \begin{array}{c|ccc} & \langle X, Y \rangle & & & \\ \hline & Y & \vec{a} & \vec{b} & \vec{c} \\ \hline X & \vec{a} & 0 & 0 & -1 \\ & \vec{b} & 0 & 1 & 0 \\ & \vec{c} & -1 & 0 & 0 \end{array}$$

such that

$$(3.3) \quad \gamma(s) = \frac{1}{2}\vec{a}s^2 + \vec{b}s + \vec{c}.$$

Using the parametrization, we can also deduce the following:

Lemma 3.5. *All geodesics are congruent to each other.*

Proof. Let $\vec{a}, \vec{b}, \vec{c}$ be arbitrary vectors of $\mathbb{L}^4$ which satisfy (3.2). By applying an isometry of $\mathbb{Q}_+^3$, we may assume that $\vec{c} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then (3.2) implies that

$$\vec{a} = \begin{pmatrix} 2 & w \\ \bar{w} & \frac{1}{2}w\bar{w} \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & \frac{1}{2}(we^{-i\theta} + \bar{w}e^{i\theta}) \end{pmatrix}$$

for some $w \in \mathbb{C}$ and $\theta \in \mathbb{R}$. Let $F := \begin{pmatrix} e^{-i\theta/2} & 0 \\ -\frac{1}{2}\bar{w}e^{i\theta/2} & e^{i\theta/2} \end{pmatrix} \in \text{SL}(2, \mathbb{C})$, then

$$F\vec{a}F^* = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad F\vec{b}F^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad F\vec{c}F^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and thus the claim follows. $\square$

3.3. Geodesics as space curves. To make connection with the curve theory reviewed in Section 2.2, let γ be a geodesic given by the unit-speed parametrization as in Theorem 3.4, so that

$$\gamma'(s) = \vec{a}s + \vec{b}, \quad \gamma''(s) = \vec{a}.$$

Thus the cone curvature of γ vanishes:

$$\kappa = -\frac{1}{2}\langle \gamma'', \gamma'' \rangle = 0.$$

On the other hand, since

$$\text{span}\{\gamma, \mathbf{T}, \mathbf{N}\} = \text{span}\{\gamma, \gamma', \gamma''\} = \text{span}\{\vec{a}, \vec{b}, \vec{c}\}$$

is constant in s , we have that $\mathbf{B} : I \rightarrow \text{span}\{\gamma, \mathbf{T}, \mathbf{N}\}^\perp$ is a constant vector, and hence the cone torsion τ also vanishes.

To consider the converse, let us assume that $\gamma : I \rightarrow \mathbb{Q}_+^3$ is a unit speed curve with $\kappa = \tau = 0$. Then the Frenet equation (2.2) implies $\gamma''' = T'' = -N' = \vec{0}$. Hence, γ'' is a constant vector, say $\vec{a} \in \mathbb{L}^4$. Since $\kappa = 0$ if and only if γ'' is lightlike, we have $\langle \vec{a}, \vec{a} \rangle = 0$.

Integrating once with respect to s , we obtain $\gamma'(s) = \vec{a}s + \vec{b}$ for some $\vec{b} \in \mathbb{L}^4$, and since γ is unit speed,

$$1 = \langle \gamma', \gamma' \rangle = \langle \vec{a}s + \vec{b}, \vec{a}s + \vec{b} \rangle = 2\langle \vec{a}, \vec{b} \rangle s + \langle \vec{b}, \vec{b} \rangle,$$

allowing us to conclude that $\langle \vec{a}, \vec{b} \rangle = 0$, $\langle \vec{b}, \vec{b} \rangle = 1$.

Integrating once more with respect to s , we may write

$$\gamma(s) = \frac{1}{2}\vec{a}s^2 + \vec{b}s + \vec{c}$$

for some $\vec{c} \in \mathbb{L}^4$. The fact that γ takes values in $\mathbb{Q}_+^3$ tells us

$$0 = \langle \gamma(s), \gamma(s) \rangle = \langle \vec{a}, \vec{c} \rangle s^2 + s^2 + 2\langle \vec{b}, \vec{c} \rangle s + \langle \vec{c}, \vec{c} \rangle,$$

so that $\langle \vec{a}, \vec{c} \rangle = -1$, $\langle \vec{b}, \vec{c} \rangle = 0$, and $\langle \vec{c}, \vec{c} \rangle = 0$.

Thus we conclude:

Theorem 3.6. *A unit-speed curve in $\mathbb{Q}_+^3$ is a geodesic if and only if its cone curvature and cone torsion vanish.*

REMARK 3.7. In Riemannian manifolds, a geodesic is determined by its initial position and initial velocity. However, that is not the case in $\mathbb{Q}_+^3$. For example, let γ be given as in (3.3) and take $\vec{a}$, $\vec{b}$, and $\vec{c}$ as in the proof of Lemma 3.5 with $\theta = 0$. Then any arbitrary choice of $w = i\mu \in i\mathbb{R}$ yields geodesics all with the same initial position and initial velocity. This is due to the freedom of choice of $\tilde{M}$.

4. TWO EXAMPLES OF RULED ZMC SURFACES $\mathbb{Q}_+^3$

In this section, we will examine two important ruled ZMC surfaces in $\mathbb{Q}_+^3$, with an eye on obtaining a complete classification of all ruled ZMC surfaces in $\mathbb{Q}_+^3$.

4.1. Screw motions in $\mathbb{Q}_+^3$. One of the most important examples of a ruled minimal surface in Euclidean space is a helicoid, obtained by applying a certain screw motion to a geodesic. Our first step in the classification of ruled ZMC surfaces in $\mathbb{Q}_+^3$ is to mimic the Euclidean case, and apply a certain screw motion to a geodesic, and see if the resulting surface has zero mean curvature. Thus, we first devote our attention to screw motions in $\mathbb{Q}_+^3$, by first examining isometries that form a one-parameter subgroup under composition:

FACT 4.1. Let $\varphi : (-\epsilon, \epsilon) \rightarrow \text{SL}(2, \mathbb{C})$ be a one-parameter subgroup of isometries of $\mathbb{Q}_+^3$, so that

$$\varphi(s+t) = \varphi(s)\varphi(t).$$

Hence $\varphi(s)^{-1}(\varphi(s+t) - \varphi(s)) = \varphi(t) - \varphi(0)$, which implies

$$\varphi(s)^{-1}\varphi(s)' = \varphi(0)' \in \mathfrak{sl}(2, \mathbb{C}).$$

Then $\varphi(s) = e^{s\varphi(0)'}$. For $a, b, c \in \mathbb{C}$ with $\Delta := \sqrt{-(a^2 + bc)} \neq 0$,

$$\exp \left(s \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right) = \begin{pmatrix} \cos \Delta s + \frac{a}{\Delta} \sin \Delta s & \frac{b}{\Delta} \sin \Delta s \\ \frac{c}{\Delta} \sin \Delta s & \cos \Delta s - \frac{a}{\Delta} \sin \Delta s \end{pmatrix}.$$

So let us consider all cases of $A \in \mathfrak{sl}(2, \mathbb{C})$, so that $A^2 = -(\det A)I_2$, generating such one-parameter subgroups, where the rotation matrices of Example 2.1 will be used.

Case 1. Suppose that $\det A \neq 0$. Then $\lambda := \sqrt{-\det A}$ is an eigenvalue of A and

$$A = M \operatorname{diag}(\lambda, -\lambda) M^{-1}$$

for some $M \in \operatorname{SL}(2, \mathbb{C})$. Then the one-parameter subgroup $s \mapsto e^{sA}$ is similar to $s \mapsto D_1(e^{\lambda s})$. If λ is purely imaginary, φ are elliptic rotations. If λ is real, φ are hyperbolic rotations. If λ is neither real nor purely imaginary, then φ are *screw motions*.

Case 2. Suppose that $\det A = 0$. Then $A = \begin{pmatrix} \alpha\beta & -\alpha^2 \\ \beta^2 & -\alpha\beta \end{pmatrix}$ for some $\alpha, \beta \in \mathbb{C}$ and $e^{sA} = I + sA$.

Let

$$\lambda := \sqrt{-i\alpha/\beta}, \quad h := 2i\alpha\beta \quad \text{for } \alpha, \beta \in \mathbb{C} \setminus \{0\}.$$

Then

$$D_2(\lambda)e^{sA}D_2(\lambda)^{-1} = \varphi(hs),$$

where

$$\varphi(k) := \begin{pmatrix} 1 + i\frac{k}{2} & \frac{k}{2} \\ \frac{k}{2} & 1 - i\frac{k}{2} \end{pmatrix} = BP(k)B^{-1}, \quad B := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}.$$

So the one-parameter subgroup $s \mapsto e^{sA}$ is similar to $s \mapsto P(sh)$ in $\operatorname{SL}(2, \mathbb{C})$ for some $h \in \mathbb{C}$. If α or β is 0, it's trivial. Note that, if $h = re^{i\theta}$, then

$$\begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \begin{pmatrix} 1 & sre^{i\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & rs \\ 0 & 1 \end{pmatrix},$$

so it is conjugate to a single parabolic rotation. In conclusion, we obtain the following:

Proposition 4.2. *Any one-parameter subgroup of isometries of $\mathbb{Q}_+^3$ is similar to*

$$D_1(e^{\lambda s}) = \begin{pmatrix} e^{\lambda s} & 0 \\ 0 & e^{-\lambda s} \end{pmatrix}, \quad \text{or} \quad P(rs) = \begin{pmatrix} 1 & rs \\ 0 & 1 \end{pmatrix},$$

for some $\lambda \in \mathbb{C}$ and $r \in \mathbb{R}$.

Now that we have the notions of geodesics and screw motions, we define the following special classes of surfaces:

DEFINITION 4.3. Let $X : \mathcal{U} \rightarrow \mathbb{Q}_+^3$ be an immersion.

- If X is invariant under screw motions, then X is called a *helicoidal surface*.
- If X is foliated by geodesics, then X is called a *ruled surface*.

4.2. Helicoids. Now let us consider the surface obtained by applying a screw motion to a geodesic:

$$\begin{aligned} X(u, v) &:= \begin{pmatrix} e^{(a+ib)v} & 0 \\ 0 & e^{-(a+ib)v} \end{pmatrix} \begin{pmatrix} u^2 & u \\ u & 1 \end{pmatrix} \begin{pmatrix} e^{(a+ib)v} & 0 \\ 0 & e^{-(a+ib)v} \end{pmatrix}^* \\ &= \begin{pmatrix} e^{2av}u^2 & e^{2ibv}u \\ e^{-2ibv}u & e^{-2av} \end{pmatrix}, \end{aligned}$$

where a and b are some real constants. Then one can calculate the lightlike Gauss map G and the corresponding fundamental forms to find that $H \equiv 0$. Thus, as in the case of minimal surfaces in Euclidean space, we define helicoids in $\mathbb{Q}_+^3$ as follows:

DEFINITION 4.4. For arbitrary nonzero real numbers a and b , we call the image of $\mathbb{R}^2$ by

$$H^{a,b}(u, v) := \begin{pmatrix} e^{(a+ib)v} & 0 \\ 0 & e^{-(a+ib)v} \end{pmatrix} \begin{pmatrix} u^2 & u \\ u & 1 \end{pmatrix} \begin{pmatrix} e^{(a+ib)v} & 0 \\ 0 & e^{-(a+ib)v} \end{pmatrix}^*$$

the *standard helicoid* of $\mathbb{Q}_+^3$. Any surface that is congruent to X up to homotheties and isometries of $\mathbb{Q}_+^3$ are referred to as *helicoids* (see Figure 1, left).

Now we note some important geometric facts about helicoids in $\mathbb{Q}_+^3$:

Lemma 4.5. *The following hold:*

- For any nonzero real numbers a and b , the curve $v \mapsto H^{a,b}(\frac{1}{2\sqrt{a^2+b^2}}, v)$ is a unit-speed helix and has constant cone curvature $\kappa = 2(a^2 - b^2)$ and constant cone torsion $\tau = 4ab$.
- The metric of the surface is given by $\mathbf{g} = du^2 + 4au du dv + 4(a^2 + b^2)u^2 dv^2$, so u -parameter curves and v -parameter curves of $H^{a,b}$ meet at constant angles, but not perpendicularly.
- The metric is singular if $u = 0$.

Proof. The proof follows from direct calculations. □

REMARK 4.6. For $a = 0$, we have

$$H^{0,b}(u, v) = \begin{pmatrix} u^2 & e^{2ibv}u \\ e^{-2ibv}u & 1 \end{pmatrix}.$$

Then the surface is the intersection of $\mathbb{Q}_+^3$ and the hyperplane $x_0 - x_3 = 1$ in $\mathbb{L}^4$, a *horosphere* (see Figure 1, center):

$$\{(x_0, x_1, x_2, x_3) \in \mathbb{Q}_+^3 : x_0 + x_3 = x_1^2 + x_2^2\}.$$

Also when $b = 0$, we have

$$H^{a,0}(u, v) = \begin{pmatrix} e^{2av}u^2 & u \\ u & e^{-2av} \end{pmatrix},$$

which is a lightlike surface in $\mathbb{Q}_+^3$.

In particular, applying screw motion to a geodesic does not result in orthogonal parametrization of the resulting helicoid, which makes the case of $\mathbb{Q}_+^3$ stand out from the cases of other space forms. However, we can find a conformal reparametrization of the standard helicoid as follows: If we let $\tilde{v} := 2bv$ and $\tilde{u} := 2av + \ln u$, then

$$(4.1) \quad \tilde{H}^{a,b}(\tilde{u}, \tilde{v}) := H^{a,b}\left(e^{\tilde{u} - \frac{a}{b}\tilde{v}}, \frac{1}{2b}\tilde{v}\right)$$

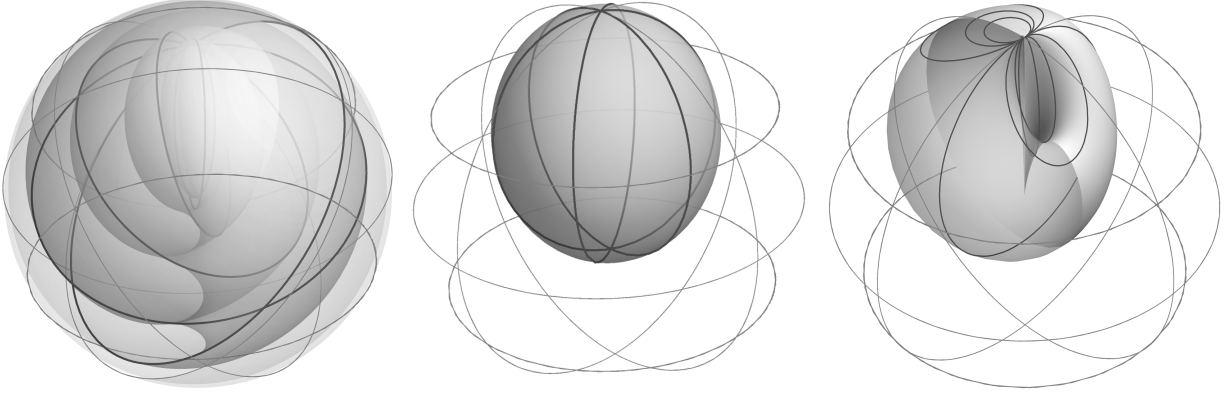


FIGURE 1. A helicoid (left), a horosphere (center) and a parabolic catenoid (right) in the ball model of $\mathbb{Q}_+^3$ (cf. [6]). Curves represent geodesics.

is conformally parametrized, and

$$\begin{aligned}
 \tilde{H}^{a,b}(u, v) &= e^{-\frac{a}{b}v} \begin{pmatrix} e^{i\frac{v}{2}} & 0 \\ 0 & e^{-i\frac{v}{2}} \end{pmatrix} \begin{pmatrix} e^{2u} & e^u \\ e^u & 1 \end{pmatrix} \begin{pmatrix} e^{i\frac{v}{2}} & 0 \\ 0 & e^{-i\frac{v}{2}} \end{pmatrix}^* \\
 (4.2) \quad &= \varphi_{i\frac{a}{2b}}^{\text{Hel}}(u + iv) \varphi_{i\frac{a}{2b}}^{\text{Hel}}(u + iv)^*, \quad \varphi_c^{\text{Hel}}(z) := e^{cz} \begin{pmatrix} e^z \\ 1 \end{pmatrix}.
 \end{aligned}$$

Direct calculations show that the v -parameter curves do not have constant cone curvature, hence are not helices.

REMARK 4.7. Note that $H^{a,b}[\mathbb{R}^+ \times \mathbb{R}] = \tilde{H}^{a,b}[\mathbb{R}^2]$ and $H^{a,b}(-u, v) = R_0 H^{a,b}(u, v) R_0^*$ for $R_0 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. That is,

$$H^{a,b}(-u, v) = R_1 \circ H^{a,b}(u, v), \quad R_1(x_0, x_1, x_2, x_3) := (x_0, -x_1, -x_2, x_3).$$

We may regard $H^{a,b}[\mathbb{R}^2]$, which has singularities at the v -axis, as the analytic extension of the spacelike surface $\tilde{H}^{a,b}[\mathbb{R}^2]$, which is intrinsically flat, as every ZMC surface in $\mathbb{Q}_+^3$ is intrinsically flat. (See also [6, § 4.5].)

4.3. Geodesics on helicoids and the singular set. Now we consider geodesics on helicoids given globally by $H^{a,b} : \mathbb{R}^2 \rightarrow \mathbb{Q}_+^3$. In particular, we wish to examine the behavior of geodesics on $H^{a,b}$ around the singular set $\{u = 0\}$.

We will first find the geodesics of helicoids by finding an (almost) orthonormal basis of the tangent bundle. Calculating the induced metric of $H^{a,b}$ as

$$ds^2 = (du + 2au dv)^2 + 4b^2 u^2 dv^2,$$

we reparametrize the surface as in (4.1) so that the surface is conformally parametrized by $z = \tilde{u} + i\tilde{v} \in \mathbb{C}_1 \cong \mathbb{C}$:

$$ds^2 = e^{2(\tilde{u} - \frac{a}{b}\tilde{v})} (d\tilde{u}^2 + d\tilde{v}^2) = |e^{(1 + \frac{a}{b}i)z} dz|^2.$$

Note that

$$(\tilde{u}, \tilde{v}) = (\ln u + 2av, 2bv).$$

Consider another change of coordinates

$$\mathbb{C}_1 \ni z \xrightarrow{\rho} \zeta = e^{(1+\frac{a}{b}i)z} \in \mathbb{C}_2 \cong \mathbb{C} \setminus \{0\}$$

which transforms the induced metric into

$$ds^2 = \frac{1}{|1 + \frac{a}{b}i|^2} |d\zeta|^2.$$

Thus, any geodesic in $\mathbb{C}_2$ satisfies

$$(\alpha - i\beta)\zeta + (\alpha + i\beta)\bar{\zeta} + 2\gamma = 0$$

for some real constants α, β, γ . We distinguish two cases.

Case 1. Suppose $\gamma \neq 0$. Substituting ζ by $e^{(1+\frac{a}{b}i)(\tilde{u}+i\tilde{v})}$ yields

$$\alpha \cos c_1 + \beta \sin c_1 = -\gamma e^{-(\tilde{u}-\frac{a}{b}\tilde{v})} = -\gamma u^{-1}, \quad \text{where } c_1 = c_1(\tilde{u}, \tilde{v}) := \frac{a}{b}\tilde{u} + \tilde{v}.$$

Then

$$|u| = \frac{|\gamma|}{|\alpha \cos c_1 + \beta \sin c_1|} \geq \frac{|\gamma|}{|\alpha| + |\beta|} > 0.$$

Thus, in this case, geodesics do not intersect the singular set (see Figure 2).

Case 2. Suppose $\gamma = 0$. In this case, $\text{Arg } \zeta = \text{constant}$. Noting that $\zeta = e^{\tilde{u}-\frac{a}{b}\tilde{v}} e^{i(\frac{a}{b}\tilde{u}+\tilde{v})}$, we conclude that any component of the lift of the geodesic by ρ can be written as

$$\frac{a}{b}\tilde{u} + \tilde{v} = c_1,$$

where c_1 is some real constant. Now, the corresponding points in the (u, v) -plane must lie on the curve parametrized by $\tilde{u}$ where

$$u = \exp\left(\left(1 + \frac{a^2}{b^2}\right)\tilde{u} - \frac{a}{b}c_1\right), \quad v = \frac{\tilde{v}}{2b} = \frac{c_1 - \frac{a}{b}\tilde{u}}{2b}.$$

In particular, we see that $u \rightarrow 0^+$ as $\tilde{u} \rightarrow -\infty$, but then $v \rightarrow \pm\infty$; thus, in this case, geodesics asymptotically approach the singular set $\{u = 0\}$, without ever intersecting it (see Figure 2).

We summarize as follows:

Theorem 4.8. *There is no geodesic intersecting the singular set in the helicoids.*

REMARK 4.9. The situation in the isotropic three-space $\mathbb{I}^3$ is different, where the standard helicoid can be parameterized by $H(u, v) := (u \cos av, u \sin av, bv)$, $(u, v) \in \mathbb{R}^2$, up to isometries and homothety. For each $v \in \mathbb{R}$, let $\gamma(u) := H(u, v)$. Then $\gamma[\mathbb{R}]$ is a geodesic of $\mathbb{I}^3$, and $\gamma[\mathbb{R}^+]$ and $\gamma[\mathbb{R}^-]$ are geodesics of $H[\mathbb{R}^+ \times \mathbb{R}]$ and $H[\mathbb{R}^- \times \mathbb{R}]$, respectively, which converge to the singular point $H(0, v)$.

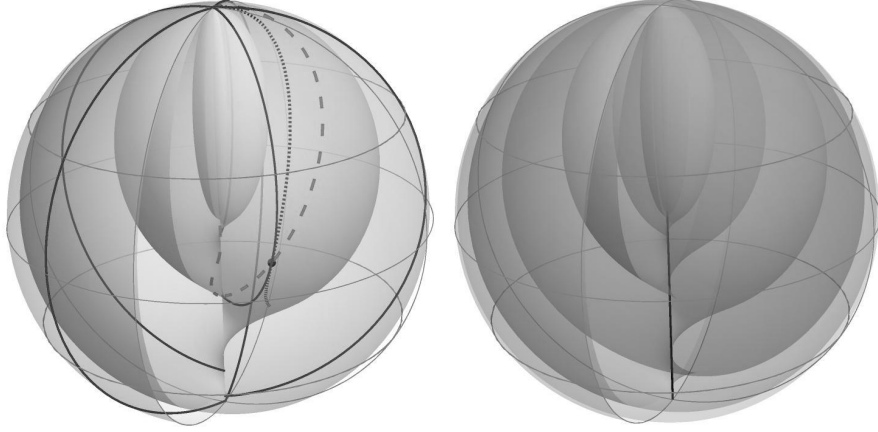


FIGURE 2. Left: $\tilde{H}^{a,b}[\mathbb{R}^2]$. The solid curves are (halves of) the rulings, the dotted curve is a geodesic of case 1 and the dashed curve is a geodesic of case 2 in §4.3. Right: $H^{a,b}[\mathbb{R}^2]$. The line is the singular set.

4.4. Parabolic catenoids. We now give another important example of a ruled ZMC surface in $\mathbb{Q}_+^3$: the parabolic catenoid reviewed in Section 2.5. To check that the parabolic catenoid is a ruled surface, we first observe that if we write $z = u + iv$ and φ_c^P is from (2.7), then

$$(4.3) \quad C_c^P(u, v) := \varphi_c^P(u + iv) \varphi_c^P(u + iv)^* = e^{-2cv} \begin{pmatrix} 1 & iv \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u^2 & u \\ u & 1 \end{pmatrix} \begin{pmatrix} 1 & iv \\ 0 & 1 \end{pmatrix}^*.$$

The fact that the parabolic catenoid is a ruled surface follows from the following reparametrization:

$$(4.4) \quad C_c^P(e^{2cv}u, v) = \begin{pmatrix} 1 & iv \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{cv} & 0 \\ 0 & e^{-cv} \end{pmatrix} \begin{pmatrix} u^2 & u \\ u & 1 \end{pmatrix} \begin{pmatrix} e^{cv} & 0 \\ 0 & e^{-cv} \end{pmatrix}^* \begin{pmatrix} 1 & iv \\ 0 & 1 \end{pmatrix}^*$$

so that the parabolic catenoid is a surface obtained by applying isometries to a geodesic (see Figure 1, right).

REMARK 4.10. The map

$$\mathbb{R} \ni c \mapsto p_c(v) := \begin{pmatrix} 1 & iv \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{cv} & 0 \\ 0 & e^{-cv} \end{pmatrix}$$

in (4.4) is not a one-parameter subgroup, that is, $p_c(v_1 + v_2) \neq p_c(v_1)p_c(v_2)$. Hence it is not a screw motion. On the other hand, the map

$$\mathbb{R} \ni c \mapsto q_c(v) := e^{-cv} \begin{pmatrix} 1 & iv \\ 0 & 1 \end{pmatrix} = \exp \begin{pmatrix} -cv & -iv \\ 0 & -cv \end{pmatrix}$$

in (4.3) induces a one-parameter subgroup, that is, $q_c(v_1 + v_2) = q_c(v_1)q_c(v_2)$. However, $q_c(v)$ does not represent an isometry.

5. HELICOIDS AND THE ASSOCIATED FAMILY OF CATENOIDS

In the Euclidean case, helicoids and catenoids are related by an isometric deformation, called the associated family. In this section, we examine whether helicoids in $\mathbb{Q}_+^3$ can also be found in the associated family of catenoids in $\mathbb{Q}_+^3$.

5.1. Associated family of catenoids in $\mathbb{Q}_+^3$. Recall from (2.6) that a ZMC surface with Weierstrass data (g, ω) admits an isometric deformation known as the *associated family* by a change in the Weierstrass data via

$$(g, \omega) \mapsto (g, \lambda \omega)$$

for any unit complex constant λ . Taking the Weierstrass data as

$$(g, \omega_\delta) = \left(w, \delta \frac{dw}{w^2} \right),$$

for some $\delta \in \mathbb{R} \setminus \{0\}$, we note from (2.8) that the resulting immersion X_δ is

- an elliptic catenoid if $\delta > -\frac{1}{4}$,
- a parabolic catenoid if $\delta = -\frac{1}{4}$, or
- a hyperbolic catenoid if $\delta < -\frac{1}{4}$.

Therefore, any surface with Weierstrass data

$$(g, \omega_\delta) = \left(w, \delta \frac{dw}{w^2} \right),$$

for any $\delta \in \mathbb{C} \setminus \{0\}$ must be in the associated family of a catenoid, given by the Weierstrass data

$$(g, \omega_{|\delta|}) = \left(w, |\delta| \frac{dw}{w^2} \right).$$

5.2. Helicoids and the associated family of catenoids. Turning our attention to the conformally parametrized (standard) helicoid $\tilde{H}^{a,b}$ in (4.2), we first note that we may assume without loss of generality that $b = \frac{1}{2}$. Then the lift of $\tilde{H}^a$ to $\mathbb{C}^2$ is given in (4.2) by

$$\varphi_a^{\text{Hel}}(z) := e^{iaz} \begin{pmatrix} e^z \\ 1 \end{pmatrix}.$$

Thus we can find the lift F^a to $\text{SL}(2, \mathbb{C})$ using (2.4):

$$F^a(z) = \begin{pmatrix} e^{\frac{z}{2}} & 0 \\ 0 & e^{-\frac{z}{2}} \end{pmatrix} \begin{pmatrix} 1 & -\frac{(1+ia)^2}{1+2ia} \\ 1 & \frac{a^2}{1+2ia} \end{pmatrix} \begin{pmatrix} e^{iaz} & 0 \\ 0 & e^{-iaz} \end{pmatrix} \begin{pmatrix} e^{\frac{z}{2}} & 0 \\ 0 & e^{-\frac{z}{2}} \end{pmatrix}.$$

This in turn allows us to find the Weierstrass data using (2.5), so that

$$\begin{aligned} (G, \Omega) &= \left(\frac{a-i}{a} e^z, a^2 e^{-z} dz \right) \\ (g, \omega) &= \left(-\frac{a(1+ia)}{2a-i} e^{(-1-2ia)z}, -e^{(1+2ia)z} dz \right). \end{aligned}$$

Making a change of coordinate via $w := g$, we may normalize the Weierstrass data as

$$(\tilde{g}, \tilde{\omega}) = \left(w, -\frac{a(a-i)}{(2a-i)^2} \frac{dw}{w^2} \right).$$

Therefore, we have:

Theorem 5.1. *Every helicoid in $\mathbb{Q}_+^3$ is in the associated family of some catenoid in $\mathbb{Q}_+^3$.*

However, the converse is not true, namely, there are catenoids that do not have helicoids in the associated family. To see this, we note that for

$$\delta(a) := -\frac{a(a-i)}{(2a-i)^2},$$

one can directly check that for real functions of x and y of a given by

$$x(a) + iy(a) := -\frac{1}{4} - \delta(a),$$

we have

$$(x(a)^2 + y(a)^2)^2 + \frac{1}{4}x(a)(x(a)^2 + y(a)^2) - \frac{1}{64}y(a)^2 = 0,$$

which is the formula for a cardioid. As a varies from $-\infty$ to ∞ , the image starts from $(-1/4, 0)$ and wraps around in the counterclockwise direction.

In particular, we have

$$|\delta(a)| \in (0, \frac{1}{2\sqrt{3}}],$$

so that when $|c| > \frac{1}{2\sqrt{3}}$, the catenoids with Weierstrass data $(g, \omega_c) = (z, c\frac{dw}{w^2})$ have no helicoids in its associated family. We summarize:

Theorem 5.2. *Let X_δ be the ZMC catenoid in $\mathbb{Q}_+^3$ given by the Weierstrass data*

$$g = \zeta, \quad \omega = \frac{\delta}{\zeta^2} d\zeta \quad \text{for } \delta \in \mathbb{R}^+.$$

Then the number of helicoids and catenoids in the associated family of X_δ varies depending upon δ , which can be described as follows (see also Figure 3):

- $0 < \delta < \frac{1}{4}$: two elliptic catenoids and two helicoids (Fig. 4),
- $\delta = \frac{1}{4}$: one elliptic catenoid, two helicoids, and one parabolic catenoid (Fig. 5),
- $\frac{1}{4} < \delta < \frac{1}{2\sqrt{3}}$: one elliptic catenoid, one hyperbolic catenoid, four helicoids (Fig. 6),
- $\delta = \frac{1}{2\sqrt{3}}$: one elliptic catenoid, one hyperbolic catenoid, two helicoids (Fig. 7),
- $\frac{1}{2\sqrt{3}} < \delta$: one elliptic catenoid, one hyperbolic catenoid.

5.3. Associated family of catenoids under Lawson-type correspondence between $\mathbb{I}^3$ and $\mathbb{Q}_+^3$. Lawson-type correspondence between ZMC surfaces in the isotropic 3-space $\mathbb{I}^3$ and the 3-dimensional light cone $\mathbb{Q}_+^3$ has been established in [17] as a generalization of the Umehara-Yamada perturbation [23].

In this section, we derive a similar type of correspondence between ZMC surfaces in $\mathbb{I}^3$ and $\mathbb{Q}_+^3$, and consider the correspondence between the associated family of catenoids in $\mathbb{I}^3$ and $\mathbb{Q}_+^3$. The derivation is obtained efficiently by viewing both classes of surfaces as graphs: Suppose that $X : \mathcal{U} \rightarrow \mathbb{Q}_+^3$ is an immersion represented as a graph of a function $f : \mathcal{U} \rightarrow \mathbb{R}$, that is,

$$X(u, v) = e^{f(u, v)} \begin{pmatrix} u^2 + v^2 & u + iv \\ u - iv & 1 \end{pmatrix}.$$

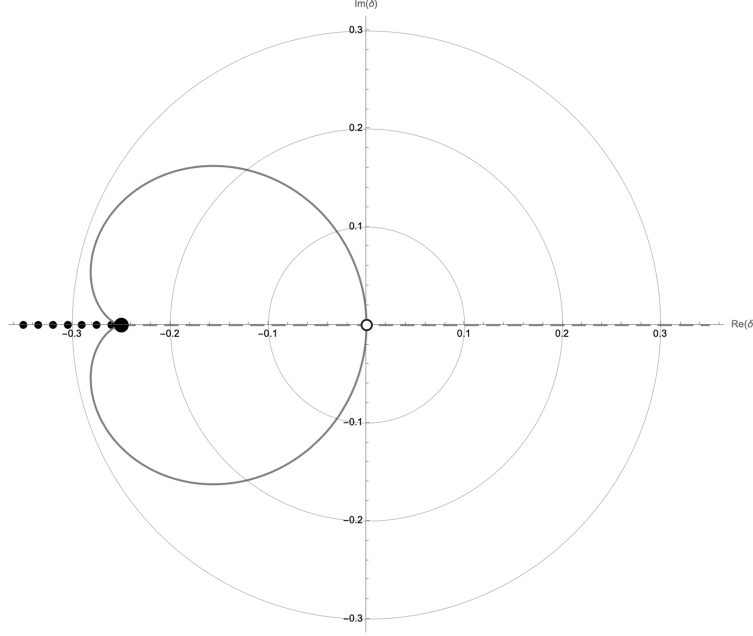


FIGURE 3. Types of surface given by the Weierstrass data $(g, \omega) = (w, \delta \frac{dw}{w^2})$, depending on the value of $\delta \in \mathbb{C} \setminus \{0\}$. The dashed line corresponds to elliptic catenoids, the black point parabolic catenoid, dotted line hyperbolic catenoids, and cardioid helicoids.

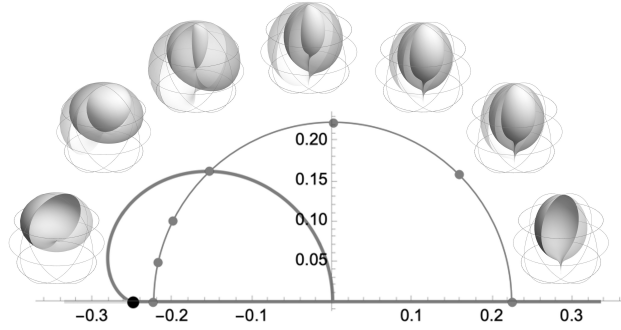


FIGURE 4. Associated families of catenoid given by the Weierstrass data $(g, \omega) = (w, \frac{1}{2\sqrt{5}} \frac{dw}{w^2})$.

We have seen in (2.3) that X is a ZMC surface if and only if f is a harmonic map. However, it is known [21] that any ZMC surface in $\mathbb{I}^3$ must be a graph of a harmonic map, giving us the following correspondence between ZMC surfaces in $\mathbb{I}^3$ and $\mathbb{Q}_+^3$:

Lemma 5.3. *The following map*

$$(5.1) \quad T : \mathbb{I}^3 \rightarrow \mathbb{Q}_+^3, \quad T(x, y, l) := e^l \begin{pmatrix} x^2 + y^2 & x + iy \\ x - iy & 1 \end{pmatrix}$$

sends ZMC surfaces in $\mathbb{I}^3$ to ZMC surfaces in $\mathbb{Q}_+^3$.

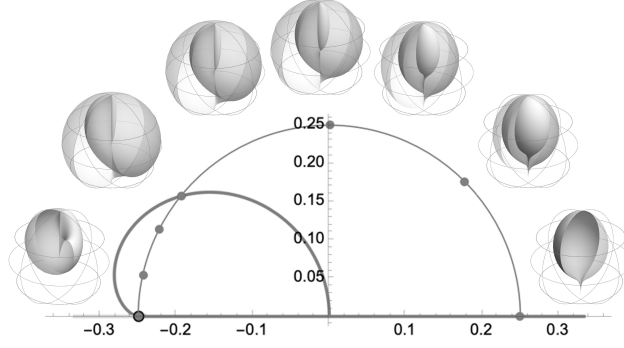


FIGURE 5. Associated families of catenoid given by the Weierstrass data $(g, \omega) = (w, \frac{1}{4} \frac{dw}{w^2})$.

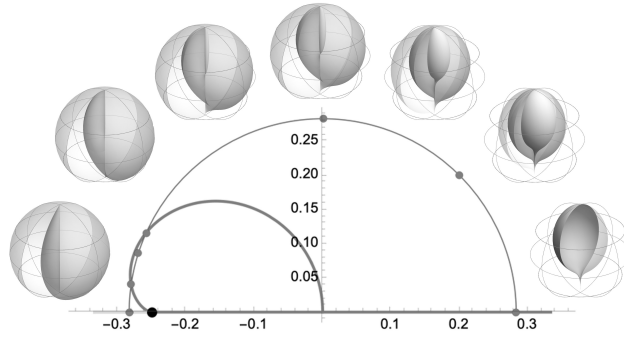


FIGURE 6. Associated families of catenoid given by the Weierstrass data $(g, \omega) = (w, \frac{1}{\sqrt{13}} \frac{dw}{w^2})$.

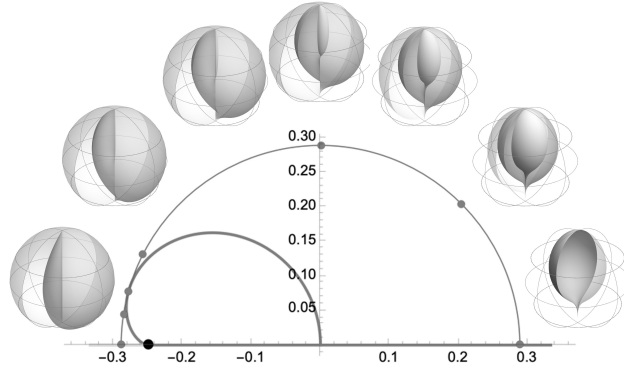


FIGURE 7. Associated families of catenoid given by the Weierstrass data $(g, \omega) = (w, \frac{1}{2\sqrt{3}} \frac{dw}{w^2})$.

Note that

$$T(x, y, l) = P(x + iy) D_1(e^{-l/2}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \left(P(x + iy) D_1(e^{-l/2}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)^{\star}.$$

EXAMPLE 5.4. The image of the plane $l = f(x, y) = ax + by + c$ in $\mathbb{I}^3$ under T in (5.1) is

$$X(u, v) = e^{2(au+bv+c)} \begin{pmatrix} u^2 + v^2 & u + iv \\ u - iv & 1 \end{pmatrix},$$

which is the parabolic catenoid.

We shall now examine the relationship between the associated families of catenoids in $\mathbb{I}^3$ and $\mathbb{Q}_+^3$. For arbitrary real constants α and β , consider

$$\Phi(\tilde{z}) := e^{\alpha+i\beta}(\tilde{z}, -i\tilde{z}, \ln \tilde{z}) \in \mathbb{C}^3, \quad \tilde{z} := e^{u+iv} \in \mathbb{C}.$$

With $r := e^\alpha$,

$$\begin{aligned} \tilde{X}_{\alpha,\beta}(u, v) &:= \operatorname{Re}(\Phi(z)) \\ &= r(e^u \cos(v + \beta), e^u \sin(v + \beta), u \cos \beta - v \sin \beta) \end{aligned}$$

are associate families of catenoids under homotheties in $\mathbb{I}^3$. By Lemma 5.3,

$$X_{\alpha,\beta}(u, v) := e^{r(u \cos \beta - v \sin \beta)} \begin{pmatrix} e^{2(u+\alpha)} & e^{u+\alpha} e^{i(v+\beta)} \\ e^{u+\alpha} e^{-i(v+\beta)} & 1 \end{pmatrix}$$

is a ZMC surface in $\mathbb{Q}_+^3$ for each $\alpha, \beta \in \mathbb{R}$. Then using $c := \frac{1}{2}e^{\alpha+i\beta} \in \mathbb{C} \setminus \{0\}$, we can rewrite $X_{\alpha,\beta} = X_c$ as

$$\begin{aligned} (5.2) \quad X_c(u, v) &= X_{\alpha,\beta}(u, v) \\ &= e^\alpha D_1(e^{\frac{1}{2}(\alpha+i\beta)}) \varphi_c(z) \varphi_c(z)^* D_1(e^{\frac{1}{2}(\alpha+i\beta)})^* \\ &\simeq \varphi_c(z) \varphi_c(z)^*, \end{aligned}$$

where $\varphi_c(z) := e^{cz} \begin{pmatrix} e^z \\ 1 \end{pmatrix}$.

To find the Weierstrass data for X_c , we use (5.2) to assume without loss of generality that

$$X_c(z) = \varphi_c(z) \varphi_c(z)^*,$$

that is, the lift of X_c to $\mathbb{C}^2$ is $\varphi_c(z)$. Then the lift F_c of X_c to $\operatorname{SL}(2, \mathbb{C})$ for $c \neq -\frac{1}{2}$ can be found using (2.4) as

$$F_c(z) := \begin{pmatrix} e^{\frac{z}{2}} & 0 \\ 0 & e^{-\frac{z}{2}} \end{pmatrix} \begin{pmatrix} 1 & -\frac{(c+1)^2}{2c+1} \\ 1 & -\frac{c^2}{2c+1} \end{pmatrix} \begin{pmatrix} e^{cz} & 0 \\ 0 & e^{-cz} \end{pmatrix} \begin{pmatrix} e^{\frac{z}{2}} & 0 \\ 0 & e^{-\frac{z}{2}} \end{pmatrix},$$

and we can see that F_c is indeed null-holomorphic. Direct calculations then show that

$$\begin{aligned} (G, \Omega) &= \left((1 + \frac{1}{c})e^z, -c^2 e^{-z} dz \right), \\ (g, \omega) &= \left(-\frac{c(c+1)}{2c+1} e^{-(2c+1)z}, -e^{(2c+1)z} dz \right). \end{aligned}$$

Now we make a coordinate change $w := g$ so that

$$(\tilde{g}_c, \tilde{\omega}_c) = \left(w, -\frac{c(c+1)}{(2c+1)^2} \frac{dw}{w^2} \right).$$

Since the map $\tilde{\delta}(c)$ given by

$$\tilde{\delta}(c) = -\frac{c(c+1)}{(2c+1)^2}$$

is a surjection onto $\mathbb{C} \setminus \{-\frac{1}{4}\}$, we conclude that every surface in the associated families of elliptic or hyperbolic catenoids in $\mathbb{Q}_+^3$ corresponds to a surface in the associated family of catenoids in $\mathbb{I}^3$, under the correspondence in Lemma 5.3.

In particular, if X_c is a ZMC surface constructed using Weierstrass data $(\tilde{g}_c, \tilde{\omega}_c)$, then it is

- an elliptic catenoid when $c \in \mathbb{R} \setminus \{-\frac{1}{2}\}$ so that $\tilde{\delta}(c) \in (-\frac{1}{4}, \infty)$,
- a hyperbolic catenoid when $c = -\frac{1}{2} + i\tilde{c}$ for $\tilde{c} \in \mathbb{R} \setminus \{0\}$ so that $\tilde{\delta}(c) \in (-\infty, -\frac{1}{4})$, or
- a helicoid when $c \in i\mathbb{R} \setminus \{0\}$ or $c \in -1 + i\mathbb{R} \setminus \{-1\}$.

REMARK 5.5. When $c = -\frac{1}{2}$, $X_{-\frac{1}{2}}(z)$ and its lift $F_{-\frac{1}{2}}(z)$ are

$$X_{-\frac{1}{2}}(z) = \begin{pmatrix} e^{\frac{z+\bar{z}}{2}} & e^{\frac{z-\bar{z}}{2}} \\ e^{-\frac{z+\bar{z}}{2}} & e^{-\frac{z-\bar{z}}{2}} \end{pmatrix} \quad \text{and} \quad F_{-\frac{1}{2}}(z) = \begin{pmatrix} e^{\frac{z}{2}} & \frac{z}{4}e^{\frac{z}{2}} \\ e^{-\frac{z}{2}} & (1 + \frac{z}{4})e^{-\frac{z}{2}} \end{pmatrix}.$$

Direct calculations show that

$$(G, \Omega) = \left(-e^z, -\frac{e^{-z}}{4} dz\right), \quad (g, \omega) = \left(-\frac{z+2}{4}, -dz\right).$$

Now we use the coordinate change $w := g$ to see that the Weierstrass data of this case is

$$(\tilde{g}_{-\frac{1}{2}}, \tilde{\omega}_{-\frac{1}{2}}) = (w, 4dw).$$

Thus, $X_{-\frac{1}{2}}$ is the trivial Enneper cousin (see [5] or [22]), and with $z = u + iv$,

$$\begin{aligned} (5.3) \quad X_{-\frac{1}{2}}(u, v) &= \begin{pmatrix} e^u & e^{iv} \\ e^{-iv} & e^{-u} \end{pmatrix} \\ &= \begin{pmatrix} e^{\frac{u}{2}} & 0 \\ 0 & e^{-\frac{u}{2}} \end{pmatrix} \begin{pmatrix} e^{i\frac{v}{2}} & 0 \\ 0 & e^{-i\frac{v}{2}} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{i\frac{v}{2}} & 0 \\ 0 & e^{-i\frac{v}{2}} \end{pmatrix}^* \begin{pmatrix} e^{\frac{u}{2}} & 0 \\ 0 & e^{-\frac{u}{2}} \end{pmatrix}^*. \end{aligned}$$

It is bi-rotationally invariant with constant $K \equiv -\frac{1}{4}$.

We take a change of parameters via

$$(u, v) = (\tilde{u} + \tilde{v}, \tilde{u} - \tilde{v}).$$

Then we have

$$X_{-\frac{1}{2}}(\tilde{u}, \tilde{v}) = \begin{pmatrix} e^{\lambda\tilde{u}} & 0 \\ 0 & e^{-\lambda\tilde{u}} \end{pmatrix} \begin{pmatrix} e^{\bar{\lambda}\tilde{v}} & 0 \\ 0 & e^{-\bar{\lambda}\tilde{v}} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{\bar{\lambda}\tilde{v}} & 0 \\ 0 & e^{-\bar{\lambda}\tilde{v}} \end{pmatrix}^* \begin{pmatrix} e^{\lambda\tilde{u}} & 0 \\ 0 & e^{-\lambda\tilde{u}} \end{pmatrix}^*,$$

where $\lambda = \frac{1}{2}(1 + i)$. Thus the surface is also bi-helicoidal, obtained by applying two screw motions to a single point.

REMARK 5.6. The example of trivial Enneper cousin (5.3) is an example of a helicoidal ZMC surface in $\mathbb{Q}_+^3$ that is not in the associated family of catenoids. Thus it would be interesting to classify all helicoidal ZMC surface in $\mathbb{Q}_+^3$. Also it remains to see if the example of trivial Enneper cousin is the only bi-helicoidal ZMC surface in $\mathbb{Q}_+^3$.

5.4. Lightlike Gauss maps of catenoids and helicoids. Let $X : \mathcal{U} \rightarrow \mathbb{Q}_+^3$ be a conformally parametrized ZMC immersion with conformal coordinates (u, v) . Then the lightlike Gauss map $G : \mathcal{U} \rightarrow \mathbb{Q}_-^3$ can be viewed as a surface into $\mathbb{Q}_-^3$, called the *associated surface* of X in [14], and the following facts are known:

REMARK 5.7. Let $X : \mathcal{U} \rightarrow \mathbb{Q}_+^3$ be a ZMC immersion with conformal coordinates $(u, v) \in \mathcal{U}$. If the lightlike Gauss map G is immersed, then G is also a conformally parametrized ZMC immersion. Furthermore, the first fundamental form $\mathbf{g}_G$ of G is given in terms of the first fundamental form of X via

$$\mathbf{g}_G = -K\mathbf{g}_X.$$

Normalizing helicoids $\tilde{H}^a(u, v)$ in (4.2) with $b = \frac{1}{2}$ so that

$$\tilde{H}^a(u, v) = e^{-2av} \begin{pmatrix} e^{2u} & e^{u+iv} \\ e^{u-iv} & 1 \end{pmatrix},$$

the lightlike Gauss map $G^a(u, v)$ of $\tilde{H}^a(u, v)$ is given by

$$G^a(u, v) = -2e^{2av} \begin{pmatrix} a^2 + 1 & a(a-i)e^{-u+iv} \\ a(a+i)e^{-u-iv} & a^2e^{-2u} \end{pmatrix}.$$

Setting $e^{\alpha+i\beta} := a - i$ for $\alpha, \beta \in \mathbb{R}$, we then notice

$$\begin{aligned} G^a(u, v) &= -2e^{2av} \begin{pmatrix} e^{2\alpha} & ae^{\alpha+i\beta}e^{-u+iv} \\ ae^{\alpha-i\beta}e^{-u-iv} & a^2e^{-2u} \end{pmatrix} \\ &= -2ae^\alpha D_2\left(\frac{i}{\sqrt{a}}\right) D_1(e^{-\frac{1}{2}(\alpha+i\beta)}) \tilde{H}^a(-u, -v) D_1(e^{-\frac{1}{2}(\alpha+i\beta)})^* D_2\left(\frac{i}{\sqrt{a}}\right)^* \\ &\simeq \tilde{H}^a(-u, -v), \end{aligned}$$

so that $G^a(u, v)$ is again a helicoid. We can similarly check that if a ZMC surface in $\mathbb{Q}_+^3$ is a catenoid, then its lightlike Gauss map is also a catenoid:

Proposition 5.8. *Let $X : \mathcal{U} \rightarrow \mathbb{Q}_+^3$ be a helicoid or (an elliptic, a hyperbolic, a parabolic) catenoid. Then the lightlike Gauss map of X is also a helicoid or (an elliptic, a hyperbolic, a parabolic) catenoid in $\mathbb{Q}_-^3$, respectively.*

6. CLASSIFICATION OF RULED ZMC SURFACES IN $\mathbb{Q}_+^3$

In this section, we show that any ruled ZMC surface in $\mathbb{Q}_+^3$ must be either a helicoid (4.2) or a parabolic catenoid (4.4) up to isometry and homothety. As noted in Lemma 4.5, the surface obtained by applying screw motions to a geodesic results in a ZMC surface, which we call a helicoid; however, helices and geodesics do not meet orthogonally (even though they meet at a constant angle). Therefore, the standard techniques of classifying ruled ZMC surfaces do not work in $\mathbb{Q}_+^3$.

To overcome this obstacle, we will view a ruled surface as an application of isometries to a geodesic, and find the condition on the isometries for the resulting surface to have zero mean curvature.

Thus, without loss of generality, we assume that any spacelike ruled surface in $\mathbb{Q}_+^3$ is parameterized as

$$X(s, t) = F(s)\delta(t)F(s)^\star$$

for the geodesic $\delta(t) := \begin{pmatrix} t^2 & t \\ t & 1 \end{pmatrix}$ and some curve F in $\text{SL}(2, \mathbb{C})$ with $F(0) = I_2$. Let

$$X_*^F(s, t) := F(s)^{-1}X_*(s, t)(F(s)^{-1})^\star, \quad \Omega(s) := F(s)^{-1}F(s)' = \begin{pmatrix} \alpha(s) & \beta(s) \\ \gamma(s) & -\alpha(s) \end{pmatrix}.$$

Then, direct calculations show that

$$\begin{aligned} X^F(s, t) &= \delta(t), & X_s^F(s, t) &= \Omega(s)\delta(t) + \delta(t)\Omega(s)^\star, & X_t^F(s, t) &= \delta(t)', \\ X_{ss}^F(s, t) &= F(s)^{-1}F(s)''\delta(t) + 2\Omega(s)\delta(t)\Omega(s)^\star + \delta(t)(F(s)^{-1}F(s)'')^\star, \\ X_{st}^F(s, t) &= \Omega(s) \begin{pmatrix} 2t & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 2t & 1 \\ 1 & 0 \end{pmatrix} \Omega(s)^\star, & X_{tt}^F(s, t) &= \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Here we note that

$$F(s)^{-1}F(s)'' = (F^{-1}F')' + (F^{-1}F')^2 = \Omega(s)' - \det \Omega(s)I_2,$$

where I_2 is the 2×2 identity matrix.

Now let $G(s, t)$ be the lightlike Gauss map of $X(s, t)$. Then

$$G^F(s, t) := F(s)^{-1}G(s, t)(F(s)^{-1})^\star$$

satisfies

$$\langle G^F, G^F \rangle = \langle G^F, X_s^F \rangle = \langle G^F, X_t^F \rangle = \langle G^F, X^F \rangle - 1 = 0.$$

To proceed further, let

$$a_0 := \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}, \quad a_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad a_2 := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad a_3 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

We regard a_3 as a point in $\mathbb{Q}_+^3$, and a_0, a_1, a_2 as a basis of $T_{a_3}\mathbb{Q}_+^3$.

We realize that $\delta(t)$ is obtained by rotating a_3 by the parabolic rotation $P(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$.

That is, $\delta(t) = P(t)a_3P(t)^\star$. We let

$$\mathbf{f}_\iota(t) := P(t)a_\iota P(t)^\star, \quad \iota = 0, 1, 2, 3.$$

Then for any t , $\{\mathbf{f}_0(t), \mathbf{f}_1(t), \mathbf{f}_2(t)\}$ is a basis of $T_{\delta(t)}\mathbb{Q}_+^3$, and we see that

$$\begin{aligned} X^F(s, t) &= \mathbf{f}_3(t), & X_t^F(s, t) &= \mathbf{f}_1(t), \\ \Omega(s) &= \frac{1}{2}(\alpha(t^2 - 1) + (\beta + \gamma)t)\mathbf{f}_0 + \frac{1}{2}(2\alpha t + \beta + \gamma)\mathbf{f}_1 - \frac{i}{2}(\beta - \gamma)\mathbf{f}_2 - \alpha\mathbf{f}_3, \\ X_s^F(s, t) &= (2t\alpha_1(s) + \beta_1(s) - t^2\gamma_1(s))\mathbf{f}_1(t) \\ &\quad + (2t\alpha_2(s) + \beta_2(s) - t^2\gamma_2(s))\mathbf{f}_2(t) - 2(\alpha_1(s) - t\gamma_1(s))\mathbf{f}_3(t), \end{aligned}$$

where $*_1 := \text{Re}(*)$, $*_2 := \text{Im}(*)$ for $* = \alpha, \beta, \gamma$.

Then

$$G^F(s, t) = \mathbf{f}_0(t) + \frac{2(\alpha_1(s) - t\gamma_1(s))}{2t\alpha_2(s) + \beta_2(s) - t^2\gamma_2(s)} \mathbf{f}_2(t) - \frac{2(\alpha_1(s) - t\gamma_1(s))^2}{(2t\alpha_2(s) + \beta_2(s) - t^2\gamma_2(s))^2} \mathbf{f}_3(t),$$

and

$$\begin{aligned} E &:= \langle X_s, X_s \rangle = \langle X_s^F, X_s^F \rangle, & L &:= \langle G, X_{ss} \rangle = \langle G^F, X_{ss}^F \rangle, \\ F &:= \langle X_s, X_t \rangle = \langle X_s^F, X_t^F \rangle, & M &:= \langle G, X_{st} \rangle = \langle G^F, X_{st}^F \rangle, \\ G &:= \langle X_t, X_t \rangle = \langle X_t^F, X_t^F \rangle, & N &:= \langle G, X_{tt} \rangle = \langle G^F, X_{tt}^F \rangle. \end{aligned}$$

The mean curvature H of X is

$$H = \frac{\mathcal{C}(s, t)}{2(t^2\gamma_2(s) - 2t\alpha_2(s) - \beta_2(s))^3},$$

where $\mathcal{C}(s, t)$ is a fourth-order polynomial

$$\mathcal{C}(s, t) := c_0(s) + c_1(s)t + c_2(s)t^2 + c_3(s)t^3 + c_4(s)t^4$$

of t , such that

$$\begin{aligned} c_4(s) &:= 2\gamma_2(s)(\gamma_1(s)^2 + \gamma_2(s)^2), \\ c_3(s) &:= -8\alpha_2(s)(\gamma_1(s)^2 + \gamma_2(s)^2) + 2\gamma_2(s)\gamma_1'(s) - 2\gamma_1(s)\gamma_2'(s). \end{aligned} \tag{6.1}$$

Thus X has zero mean curvature if and only if

$$c_0(s) = c_1(s) = c_2(s) = c_3(s) = c_4(s) = 0 \quad \text{for all } s.$$

From $c_4(s) = 0$, we see that $\gamma_2(s) = 0$ for all s . Now to use (6.1), we distinguish two cases: $\alpha_2(s_0) \neq 0$ for some s_0 or $\alpha_2(s) = 0$ for all s .

Case 1. $\alpha_2(s_0) \neq 0$ for some s_0 . We may assume that $\alpha_2(s) \neq 0$ for all s in some interval containing s_0 . From the vanishing of $c_3(s)$ in (6.1), we conclude that $\gamma_1(s) = 0$ for all s . Then $\Omega(s)$ is an upper triangular matrix, and an interesting analysis can be carried out. In this case, the coefficients of the numerator of the ZMC surface equation become the following:

$$\begin{aligned} c_0(s) &= 2\beta_2(s)\alpha_1(s)' + \alpha_1(s)(4\alpha_2(s)\beta_1(s) - 2\beta_2(s)') - 4\alpha_1(s)^2\beta_2(s), \\ c_1(s) &= 4\alpha_2(s)\alpha_1(s)' - 4\alpha_1(s)\alpha_2(s)', \quad c_2(s) = c_3(s) = c_4(s) = 0. \end{aligned}$$

From the vanishing of $c_1(s)$, we conclude that

$$\alpha_1(s) = d\alpha_2(s) \tag{6.2}$$

for some real constant d . We distinguish two cases.

Case 1-1. Suppose that $d \neq 0$ so that

$$\alpha_1(s) \neq 0 \quad \text{for all } s. \tag{6.3}$$

Let $c = 1/d$. Then,

$$\frac{c_0(s)}{\alpha_1(s)^2} = -4\beta_2(s) - 2\left(\frac{\beta_2(s)}{\alpha_1(s)}\right)' + 4c\beta_1(s),$$

and $c_0(s) = 0$ is equivalent to

$$(6.4) \quad \beta_1(s) = \frac{1}{c}\beta_2(s) + \frac{1}{2c} \left(\frac{\beta_2(s)}{\alpha_1(s)} \right)'.$$

Thus, X has zero mean curvature if and only if (6.2) and (6.4) hold while α_1 and β_2 are arbitrary smooth functions.

By defining f and g by

$$(6.5) \quad \alpha_1(s) := g(s), \quad \beta_2(s) := f(s)g(s),$$

we have

$$\alpha_2(s) := cg(s), \quad \beta_1(s) := \frac{1}{c}f(s)g(s) + \frac{1}{2c}f(s)',$$

so that

$$\Omega(s) = \Omega_1(s) + \Omega_2(s),$$

where

$$\Omega_1(s) = \begin{pmatrix} (1+ic)g(s) & (\frac{1}{c} + i)f(s)g(s) \\ 0 & -(1+ic)g(s) \end{pmatrix}, \quad \Omega_2(s) = \begin{pmatrix} 0 & \left(\frac{f(s)}{2c}\right)' \\ 0 & 0 \end{pmatrix}.$$

With $\mathcal{G}(s) := \int g(s)ds$,

$$(6.6) \quad F_1(s) := \begin{pmatrix} e^{(1+ic)\mathcal{G}(s)} & 0 \\ 0 & e^{-(1+ic)\mathcal{G}(s)} \end{pmatrix}, \quad F_2(s) := \begin{pmatrix} 1 & \frac{f(s)}{2c} \\ 0 & 1 \end{pmatrix},$$

and the matrix-valued function

$$F(s) := F_1(s) F_2(s)$$

is the (unique) solution to $F(s)^{-1}F(s)' = \Omega(s)$ with $F(0) = I_2$. Therefore

$$X(s, t) = F_1(s)F_2(s)\delta(t)F_2(s)^*F_1(s)^*$$

with F_1, F_2 as in (6.6) for some smooth real-valued functions $f, \mathcal{G}$. Note that for fixed s , the image of $t \mapsto F_2(s)\delta(t)F_2(s)^*$ is the same as the image of $t \mapsto \delta(t)$. Hence, $X(s, t) = F_1(s)\delta(t)F_1(s)^*$.

Finally we notice from (6.3) and (6.5) that $g(s) \neq 0$, hence we conclude that $\mathcal{G}$ is a strictly monotone function, hence it has an inverse function $s = h(\mathcal{G})$ and we can take $\mathcal{G}$ as a new variable. We abuse notation by calling it s again, and conclude that

$$X(s, t) = \begin{pmatrix} e^{(1+ic)s} & 0 \\ 0 & e^{-(1+ic)s} \end{pmatrix} \begin{pmatrix} t^2 & t \\ t & 1 \end{pmatrix} \begin{pmatrix} e^{(1+ic)s} & 0 \\ 0 & e^{-(1+ic)s} \end{pmatrix}^*.$$

Then for any nonzero real number a , we may set $s = as'$ and $b = ca$, and by calling s' as s again, and conclude that

$$(6.7) \quad X(s, t) = \begin{pmatrix} e^{(a+ib)s} & 0 \\ 0 & e^{-(a+ib)s} \end{pmatrix} \begin{pmatrix} t^2 & t \\ t & 1 \end{pmatrix} \begin{pmatrix} e^{(a+ib)s} & 0 \\ 0 & e^{-(a+ib)s} \end{pmatrix}^*.$$

That is, any generic ruled ZMC surface in $\mathbb{Q}_+^3$ is congruent to one of the above.

It is already shown that the surface in (6.7) is a ruled ZMC surface in $\mathbb{Q}_+^3$.

Case 1-2. Suppose that $d = 0$. Then $\alpha_1(s) = 0$ for all s , from which it follows that $c_0(s) = c_1(s) = 0$ for all s . In this case, we have

$$\Omega(s) = \begin{pmatrix} i\alpha_2(s) & \beta_1(s) + i\beta_2(s) \\ 0 & -i\alpha_2(s) \end{pmatrix}.$$

We can easily solve $F(s)^{-1}F(s)' = \Omega(s)$ with $F(0) = I_2$ to obtain

$$(6.8) \quad F(s) = \begin{pmatrix} 1 & B_1(s) + iB_2(s) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{iA(s)} & 0 \\ 0 & e^{-iA(s)} \end{pmatrix},$$

where $A(s) := \int_0^s \alpha_2(\tilde{s})d\tilde{s}$, $B_1(s) + iB_2(s) := \int_0^s (\beta_1(\tilde{s}) + i\beta_2(\tilde{s}))e^{2iA(\tilde{s})}d\tilde{s}$. So we conclude that $F(s)$ is of the form in (6.8) for some real-valued functions A, B_1, B_2 .

Case 2. $\alpha_2(s) = 0$ for all s . Then $c_2(s) = -6\beta_2(s)\gamma_1(s)^2$. We again have two cases:

Case 2-1. Suppose that $\beta_2(s_1) \neq 0$ for some s_1 . By continuity, we may assume without loss of generality that $\beta_2(s) \neq 0$ for all s , so that $\gamma_1(s) = 0$ for all s . Then

$$\begin{aligned} c_0(s) &= 2\alpha_1(s)'\beta_2(s) - 2\alpha_1(s)\beta_2(s)' - 4\alpha_1(s)^2\beta_2(s), \\ c_1(s) &= c_2(s) = c_3(s) = c_4(s) = 0. \end{aligned}$$

Hence the mean curvature is identically zero if and only if

$$\alpha_1(s)'\beta_2(s) - \alpha_1(s)\beta_2(s)' - 2\alpha_1(s)^2\beta_2(s) = 0,$$

or equivalently

$$\beta_2(s) = c_1\alpha_1(s)e^{-2\int_0^s \alpha_1(\tilde{s})d\tilde{s}} = c_2A(s)'e^{-A(s)}.$$

In this case,

$$(6.9) \quad \Omega(s) = \begin{pmatrix} \alpha_1(s) & \beta_1(s) + i\beta_2(s) \\ 0 & -\alpha_1(s) \end{pmatrix},$$

and $F^{-1}F' = \Omega$ with $F(0) = I_2$ yield

$$(6.10) \quad F(s) = \begin{pmatrix} 1 & B_1(s) + iB_2(s) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{A(s)/2} & 0 \\ 0 & e^{-A(s)/2} \end{pmatrix},$$

where $A(s) := 2\int_0^s \alpha_1(\tilde{s})d\tilde{s}$, $B_1(s) := \int_0^s \beta_1(\tilde{s})e^{A(\tilde{s})}d\tilde{s}$, and $B_2(s) := c_2A(s)$.

Conversely, if F is defined by (6.10) with arbitrary $A(s)$ and $B_1(s)$ with $B_2(s) := c_2A(s)$, where c_2 is an arbitrary real number, then $\Omega(s) := F(s)^{-1}F(s)'$ satisfies (6.9) with $\alpha_1(s) = \frac{1}{2}A(s)'$, $\beta_1(s) = B_1(s)'e^{-A(s)}$, $\beta_2(s) = B_2(s)'e^{-A(s)}$.

Case 2-2. Suppose that $\beta_2(s) = 0$ for all s . Then coefficients c_0, c_1, c_2, c_3, c_4 are all zero. In this case, $\Omega(s)$ is real and trace-free, i.e. $\Omega(s) \in \mathfrak{sl}(2, \mathbb{R})$. Then F such that $F^{-1}F' = \Omega$ is real-valued, i.e. $F(s) \in \text{SL}(2, \mathbb{R})$. This means that $X = F(s)\delta(t)F(s)^*$ is real-valued, which in turn means that the x_2 -component of $X(s, t)$ is zero, hence the image of X lies in

$$\mathbb{Q}_+^3 \cap \{x_1 = 0\} = \{(x_0, x_1, x_2, x_3) : x_0^2 + x_1^2 + x_3^2 = 0, x_2 = 0\}.$$

Since it is not spacelike, we exclude this case.

In summary, we have the following:

Proposition 6.1. *A surface given by $X(s, t) := F(s)\delta(t)F(s)^\star$ is a ruled ZMC surface in $\mathbb{Q}_+^3$, if $\delta(t) := \begin{pmatrix} t^2 & t \\ t & 1 \end{pmatrix}$ and*

$$(6.11) \quad F(s) := \begin{pmatrix} e^{as} & 0 \\ 0 & e^{-as} \end{pmatrix} \begin{pmatrix} e^{ibs} & 0 \\ 0 & e^{-ibs} \end{pmatrix}, \quad \text{or}$$

$$F(s) := \begin{pmatrix} 1 & B_1(s) + iB_2(s) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{iA(s)} & 0 \\ 0 & e^{-iA(s)} \end{pmatrix}, \quad \text{or}$$

$$(6.12) \quad F(s) := \begin{pmatrix} 1 & B_1(s) + ic_2A(s) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{A(s)/2} & 0 \\ 0 & e^{-A(s)/2} \end{pmatrix}$$

for arbitrary real numbers a, b, c and arbitrary real-valued functions A, B_1, B_2 , is a ruled ZMC surface.

Conversely, any ruled ZMC surface in $\mathbb{Q}_+^3$ is congruent to one of the above.

Now we analyse the shapes of the ruled ZMC surfaces given by (6.11) and (6.12). First of all, we see that for F in (6.11),

$$X(s, t) = \begin{pmatrix} * & ** \\ * & 1 \end{pmatrix},$$

that is, the surface is the intersection of $\mathbb{Q}_+^3$ and the hyperplane $x_0 - x_3 = 1$ in $\mathbb{L}^4$. So it is simply the horosphere.

Now let X be given by F in (6.12). Direct calculations show that

$$x_2(s) = c_2 s e^{-2s}, \quad x_0(s) - x_3(s) = e^{-2s}.$$

Then

$$(6.13) \quad x_2 = c_3(x_0 - x_3) \ln(x_0 - x_3), \quad c_3 = -c_2/2 \in \mathbb{R} \setminus \{0\},$$

that is, the ruled ZMC surface in (6.12) is the intersection of $\mathbb{Q}_+^3$ and the surface given by (6.13).

We show that this is a ruled surface. If we let

$$x_0 - x_3 = e^s,$$

then

$$X = \vec{a}(s) \frac{x_1^2}{2} + \vec{b}(s)x_1 + \vec{c}(s),$$

where

$$\vec{a}(s) := \begin{pmatrix} 2e^{-s} & 0 \\ 0 & 0 \end{pmatrix}, \quad \vec{b}(s) := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \vec{c}(s) := e^s \begin{pmatrix} (c_3 s)^2 & ic_3 s \\ -ic_3 s & 1 \end{pmatrix}.$$

If we let

$$x_1 = u, \quad c_3 s = v,$$

then

$$(6.14) \quad X(u, v) = \begin{pmatrix} 2e^{-cv} & 0 \\ 0 & 0 \end{pmatrix} \frac{u^2}{2} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} u + e^{cv} \begin{pmatrix} v^2 & iv \\ -iv & 1 \end{pmatrix}.$$

Note that if $c = 0$ then

$$X(u, v) = \begin{pmatrix} u^2 + v^2 & u + iv \\ u - iv & 1 \end{pmatrix},$$

which is the surface given in (6.11).

To see how (6.14) is obtained from the geodesic $X(u, 0)$, we see that

$$\begin{aligned} X(u, v) &= \left(\Phi_1(v) \Phi_2(v) \begin{pmatrix} u \\ 1 \end{pmatrix} \right) \left(\Phi_1(v) \Phi_2(v) \begin{pmatrix} u \\ 1 \end{pmatrix} \right)^* \\ &= (\Phi_1(v) \Phi_2(v)) \begin{pmatrix} u^2 & u \\ u & 1 \end{pmatrix} (\Phi_1(v) \Phi_2(v))^*, \end{aligned}$$

where

$$\Phi_1(v) = \begin{pmatrix} 1 & iv \\ 0 & 1 \end{pmatrix}, \quad \Phi_2(v) = \begin{pmatrix} e^{-cv/2} & 0 \\ 0 & e^{cv/2} \end{pmatrix},$$

that is, we apply to the geodesic $\Phi(u, 0)$ the screw motion which is a composition of the hyperbolic rotation $\Phi_2(v)$ and the parabolic rotation $\Phi_1(v)$.

Theorem 6.2. *The following maps*

$$H^{a,b}(u, v) := \begin{pmatrix} e^{(a+ib)v} & 0 \\ 0 & e^{-(a+ib)v} \end{pmatrix} \begin{pmatrix} u^2 & u \\ u & 1 \end{pmatrix} \begin{pmatrix} e^{(a+ib)v} & 0 \\ 0 & e^{-(a+ib)v} \end{pmatrix}^*$$

and

$$\tilde{C}_c^P(u, v) := \begin{pmatrix} 1 & iv \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-\frac{cv}{2}} & 0 \\ 0 & e^{\frac{cv}{2}} \end{pmatrix} \begin{pmatrix} u^2 & u \\ u & 1 \end{pmatrix} \begin{pmatrix} e^{-\frac{cv}{2}} & 0 \\ 0 & e^{\frac{cv}{2}} \end{pmatrix}^* \begin{pmatrix} 1 & iv \\ 0 & 1 \end{pmatrix}^*$$

are ruled ZMC surfaces in $\mathbb{Q}_+^3$, where a, b, c are real constants with $b \neq 0$. $H^{a,b}$ is the helicoid and $\tilde{C}_c^P$ is the parabolic catenoid. When $a = 0$, $c = 0$, they are standard horosphere.

Conversely, any spacelike ruled ZMC surface in $\mathbb{Q}_+^3$ must be one of the above up to the homothety and isometries of $\mathbb{Q}_+^3$.

Proof. The first claim follows from Subsections 4.2 and 4.4. The second claim follows from the contents of this section. $\square$

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