HYPERBOLIC L–SPACE KNOTS THAT SHARE THE SAME UPSILON TORSION FUNCTION

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Abstract. The Upsilon torsion function is a knot invariant introduced by Allen and Livingston as a unification of torsion orders in certain versions of knot Floer homology. Although it can be calculated from the full knot Floer complex, the calculation is often complicated. In this paper, we give the first infinite family of hyperbolic L–space knots which shares a common Upsilon torsion function. As a byproduct, we can prove that arbitrarily large values of two torsion orders can be realized by hyperbolic L–space knots, most of which are twisted torus knots.

1. Introduction

In [1], Allen and Livingston defined the Upsilon torsion function $\Upsilon_K^{\mathrm{Tor}(t)}$ for a knot K, which is a piecewise linear continuous function defined on the interval $[0,2]$. In contrast to the classical Upsilon function, which generalizes Ozsváth and Szabó's τ–invariant [16], the Upsilon torsion function $\Upsilon_K^{\mathrm{Tor}(t)}$ can be viewed as a generalization of two types of knot Floer torsion orders, namely $\mathrm{Ord}(K)$ and $\mathrm{Ord}'(K)$, as will be explained later. We remark that the Upsilon torsion function and two torsion orders are not concordance invariants.

The function $\Upsilon_K^{\mathrm{Tor}(t)}$ can be calculated from the full knot Floer complex $\mathrm{CFK}^\infty(K)$. However, such calculations tend to be complicated, even in the case of torus knots, and no prior studies have explored the Upsilon torsion function for a family of knots.

The main purpose of this paper is to calculate the Upsilon torsion function for a family of twisted torus knots of the form $T(p, pk+1; 2, 1)$ with $p \geq 2$ and $k \geq 1$ (see Section 2 for twisted torus knots), and to establish the following theorem.

Theorem 1.1. *There exist infinitely many hyperbolic L–space knots that share the same Upsilon torsion function.*

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As mentioned above, there are two types of torsion orders in knot Floer homology. The first one is introduced by Juhász, Miller and Zemke [5]. Recall that the minus version of knot Floer homology $\mathrm{HKF}^-(K)$ is a finitely generated module over the polynomial ring $\mathbb{F}_2[U]$. Let us denote $\mathrm{Tor}(\mathrm{HFK}^-(K))$ its $\mathbb{F}_2[U]$-torsion submodule. Then the torsion order of a knot K is defined as

$$\mathrm{Ord}(K) = \min\{k \geq 0 \mid U^k \cdot \mathrm{Tor}(\mathrm{HFK}^-(K)) = 0\} \in \mathbb{N} \cup \{0\}.$$

Of course, for the unknot O, $\mathrm{Ord}(O) = 0$. Since knot Floer homology detects the unknot [19], $\mathrm{Ord}(K) \geq 1$ when K is non-trivial.

The second is defined similarly in [3] by using the torsion submodule of Ozsváth, Stipsicz and Szabó's unoriented knot Floer homology $\mathrm{HFK}'(K)$, which is also a module over $\mathbb{F}_2[U]$ ([17]), instead of $\mathrm{HFK}^-(K)$. Hence

$$\mathrm{Ord}'(K) = \min\{k \geq 0 \mid U^k \cdot \mathrm{Tor}(\mathrm{HFK}'(K)) = 0\} \in \mathbb{N} \cup \{0\}.$$

Again, $\mathrm{Ord}'(K) = 0$ if and only if K is trivial. (For, $\mathrm{HFK}'(O) = \mathbb{F}_2[U]$, which is torsion-free [17, Corollary 2.15]. Conversely, if $\mathrm{HFK}'(K)$ is torsion-free, then $\mathrm{HFK}'(K) = \mathbb{F}_2[U] = \mathrm{HFK}'(O)$ [17, Proposition 3.5]. So, the unoriented knot Floer complexes $\mathrm{CFK}'(K)$ and $\mathrm{CFK}'(O)$ share the same homology, which implies chain homotopy equivalence between them [18, Proposition A.8.1]. Since setting $U = 0$ reduces the complex into the hat version of knot Floer complex [17, Proposition 2.4], we have $\widehat{\mathrm{HFK}}(K) \cong \widehat{\mathrm{HFK}}(O)$ by [18, Proposition A.3.5]. This implies $K = O$.)

As shown in [1], two types of torsion order can be unified in terms of the Upsilon torsion function $\Upsilon_K^{\mathrm{Tor}}(t)$. Specifically, the derivative of $\Upsilon_K^{\mathrm{Tor}}(t)$ near 0 equals to $\mathrm{Ord}(K)$, and $\Upsilon_K^{\mathrm{Tor}}(1) = \mathrm{Ord}'(K)$.

As a byproduct, we obtain the following corollary.

Corollary 1.2. *Let $N \geq 1$ be a positive integer. Then there exist infinitely many hyperbolic knots K_1 and K_2 with $\mathrm{Ord}(K_1) = N$ and $\mathrm{Ord}'(K_2) = N$. Furthermore, K_1 and K_2 are taken so that they are L–space knots, except $\mathrm{Ord}(K_1) = 1$ case.*

Remark 1.3. *We initially prepared the manuscript under the title "Hyperbolic knots with arbitrarily large torsion order in knot Floer homology". However, based on referee's comment pointing out that hyperbolic knots with arbitrarily large torsion order can be constructed easily by the following argument, we have revised the title accordingly.*

Let $T(p,q)$ be the (p,q)-torus knot. It is known that $\mathrm{Ord}(T(p,q)) = p-1$ for $1 < p < q$ ([5, Corollary 5.3]), and that $\mathrm{Ord}'(T(p,p+1)) = \lfloor \frac{p}{2} \rfloor$ ([3, Lemma 7.1]). On the other hand, every knot can be converted to *infinitely many hyperbolic knots by a single crossing change* [13]*. Since a single crossing change induces a genus one cobordism with two saddle points, the torsion orders can vary by at most two under such a change (see* [3, Corollary 1.6] *and* [5, Corollary 1.5]*). Therefore, by performing a single crossing change on a torus knot with sufficiently large torsion order, one obtains infinitely many hyperbolic knots with arbitrarily large torsion order.*

However, this method does not guarantee the existence of a hyperbolic knot realizing any given positive integer torsion order, due to the indeterminacy caused by the fact that the torsion orders can change by at most two. Corollary 1.2 addresses this issue: it ensures the existence of hyperbolic knots realizing every positive integer torsion order. Moreover, these knots can be L–space knots.

We pose a simple question.

Question 1.4. *Let M and N be positive integers. Does there exist a knot K with* $(\mathrm{Ord}(K), \mathrm{Ord}'(K)) = (M, N)$*?*

Remark 1.5. *For two types of torsion order, the original symbols are* $\mathrm{Ord}_v(K)$ *and* $\mathrm{Ord}_U(K)$ *(see* [5, 3]*).*

2. Twisted torus knots

A twisted torus knot is obtained from a torus knot of type (p, q) by twisting r adjacent strands by s full twists. The resulting knot is denoted by $T(p, q; r, s)$ as in literatures [8, 9, 10, 11].

Throughout this section, let K be the twisted torus knot $T(p, kp + 1; 2, 1)$ with $p \geq 2, k \geq 1$. Clearly, if $p = 2$, then $T(2, 2k + 1; 2, 1) = T(2, 2k+3)$. Also, Lee [10, 11] shows that $T(3, 3k+1; 2, 1) = T(3, 3k+2)$, and $T(4, 4k+1; 2, 1)$ is the $(2, 8k+3)$-cable of $T(2, 2k+1)$. We will show later that $T(p, kp+1; 2, 1)$ is hyperbolic if $p \geq 5$ (Proposition 2.7). Since these knots are the closure of a positive braid, it is fibered by [21]. In particular, the Seifert algorithm on a positive braided diagram gives a fiber, which is a minimal genus Seifert surface. Thus we know that it has genus $(kp^2 - kp + 2)/2$. Hence K is non-trivial.

Lemma 2.1. *K is an L–space knot.*

Proof. This follows from [22]. □

Lemma 2.2. *The Alexander polynomial* $\Delta_K(t)$ *of* K *is given by*

$$\Delta_K(t) = \begin{cases} 1 + \displaystyle\sum_{i=1}^{k}(-t+t^p)t^{(i-1)p} & \\ \quad + \displaystyle\sum_{i=1}^{p-3}\sum_{j=1}^{k}(-t^{ikp+1}+t^{ikp+2}-t^{ikp+2+i}+t^{(ik+1)p})t^{(j-1)p} & \text{if } p \geq 3, \\ \quad + \displaystyle\sum_{i=1}^{k+1}(-t^{kp(p-2)+1}+t^{kp(p-2)+2})t^{(i-1)p}, & \\ 1-t+t^2-\cdots+t^{2k+2}, & \text{if } p = 2. \end{cases}$$

Proof. When $p=2$, it is well known that

$$\Delta_K(t) = \frac{(1-t)(1-t^{2(2k+3)})}{(1-t^2)(1-t^{2k+3})} = 1-t+t^2-\cdots+t^{2k+2},$$

since $K=T(2,2k+3)$ as mentioned before.

Assume $p \geq 3$. The conclusion essentially follows from [15]. In his notation, our knot K is $\Delta(p,kp+1,2)$ with $r=p-1$. Hence

$$\Delta_K(t) = \frac{1-t}{(1-t^p)(1-t^{kp+1})}\cdot (1-(1-t)(t^{(p-1)(kp+1)+1}+t^{kp+1})-t^{p(kp+1)+2}).$$

The second factor is changed as

$$\begin{aligned} 1-(1-t)(t^{(p-1)(kp+1)+1}+t^{kp+1})-t^{p(kp+1)+2} &= 1-t^{(p-1)(kp+1)+1}-t^{kp+1} \\ +t^{(p-1)(kp+1)+2}+t^{kp+2}-t^{p(kp+1)+2} &= (1-t^{kp+1})+ \\ +t^{kp+2}(1-t^{(kp+1)(p-2)})+t^{(p-1)(kp+1)+2}(1-t^{kp+1}). \end{aligned}$$

Thus

$$\Delta_K(t) = \frac{1-t}{1-t^p}\cdot(1+t^{kp+2}\sum_{i=0}^{p-3}t^{i(kp+1)}+t^{(p-1)(kp+1)+2}).$$

We set

$$A = \sum_{i=1}^{k}(-t+t^p)t^{(i-1)p},$$

$$B = \sum_{i=1}^{p-3}\sum_{j=1}^{k}(-t^{ikp+1}+t^{ikp+2}-t^{ikp+2+i}+t^{(ik+1)p})t^{(j-1)p},$$

$$C = \sum_{i=1}^{k+1}(-t^{kp(p-2)+1}+t^{kp(p-2)+2})t^{(i-1)p}.$$

Then it is straightforward to calculate

$$(1-t^p)A = -t + t^p + t^{kp+1} - t^{(k+1)p},$$

$$(1-t^p)B = -t^{kp+1} + t^{(k+1)p} + (1-t)\sum_{i=0}^{p-3} t^{(i+1)kp+i+2} + t^{kp(p-2)+1} - t^{kp(p-2)+2},$$

$$(1-t^p)C = -t^{kp(p-2)+1} + t^{kp(p-2)+2} + t^{kp(p-2)+1+(k+1)p} - t^{kp(p-2)+2+(k+1)p}.$$

Hence

$$\begin{aligned}(1-t^p)(1+A+B+C) &= 1-t+(1-t)\sum_{i=0}^{p-3} t^{(i+1)kp+i+2} \\ &\quad + (1-t)t^{kp(p-2)+1+(k+1)p} \\ &= (1-t)(1+t^{kp+2}\sum_{i=0}^{p-3} t^{i(kp+1)} + t^{(p-1)(kp+1)+2}).\end{aligned}$$

This shows that $\Delta_K(t) = 1+A+B+C$ as desired. □

For a polynomial, the term "gaps of exponents" refers to the sequence of differences between the exponents of consecutive non-zero terms.

Corollary 2.3. *The gaps of the exponents of the Alexander polynomial of K are*

$$(1, p-1)^k, (1,1,1,p-3)^k, (1,1,2,p-4)^k, \dots, (1,1,p-3,1)^k, 1, 1, (p-1,1)^k$$

if $p \geq 3$, and 1^{2k+2} if $p = 2$. Here, the power indicates the repetition. (We remark that the above sequence is $(1,2)^k, 1, 1, (2,1)^k$ when $p = 3$.)

To prove that our twisted torus knot $K = T(p, kp+1; 2, 1)$ is hyperbolic when $p \geq 5$, we give a more general result by using [4]. A knot k is called a *fully positive braid knot* if it is the closure of a positive braid which contains at least one full twist.

Proposition 2.4. *Let k be a fully positive braid knot. If k is a tunnel number one, satellite knot, then k is a cable knot.*

Proof. By [14], k has a torus knot $T(r,s)$ as a companion. We may assume that $1 < r < s$. Then Theorem 1.2 of [4] claims that the pattern P is represented by a positive braid in a solid torus.

Let us recall the construction of [14]. Starting from a 2-bridge link $K_1 \cup K_2$, consider the solid torus $E(K_2)$ containing K_1. Remark that K_1 and K_2 are unknotted. For the companion knot $T(r,s)$, consider a homeomorphism from $E(K_2)$ to the tubular neighborhood of $T(r,s)$, which sends the preferred longitude of $E(K_2)$ to the regular fiber of the Seifert fibration in the exterior of $T(r,s)$. Hence our pattern knot P, which is defined under preserving preferred longitudes, is obtained from K_1 by adding rs positive full twists to $E(K_2)$. Since K_1 is unknotted, we can set the pattern P as the closure of a positive braid

$$\sigma_{i(1)}\sigma_{i(2)}\dots\sigma_{i(n-1)}(\sigma_1\sigma_2\dots\sigma_{n-1})^{nrs}$$

for some $n \ge 2$, where $\{i(1), i(2), \dots, i(n-1)\} = \{1, 2, \dots, n-1\}$. (If the initial part before rs full twists contains more than $n-1$ generators, then the Seifert algorithm gives a fiber surface of the closure K_1, which has positive genus.)

For two braids β_1 and β_2, we write $\beta_1 \sim \beta_2$ if they are conjugate or equivalent.

Claim 2.5. $\sigma_{i(1)}\sigma_{i(2)}\dots\sigma_{i(n-1)}(\sigma_1\sigma_2\dots\sigma_{n-1})^{nrs} \sim (\sigma_1\sigma_2\dots\sigma_{n-1})^{nrs+1}$.

Proof of Claim 2.5. Put $F = (\sigma_1\sigma_2\dots\sigma_{n-1})^{nrs}$, which is central in the braid group. First, write $\sigma_{i(1)}\sigma_{i(2)}\dots\sigma_{i(n-1)}F = U_1\sigma_1U_2F$, where U_i is a word without σ_1, which is possibly empty. Then $U_1\sigma_1U_2F \sim \sigma_1U_2FU_1 \sim \sigma_1U_2U_1F$. Next, set $U_2U_1 = V_1\sigma_2V_2$, where V_i is a (possibly, empty) word without σ_1, σ_2. Note that σ_1 and V_1 commute. Then

$$\sigma_1U_2U_1F = \sigma_1V_1\sigma_2V_2F \sim V_1\sigma_1\sigma_2V_2F \sim \sigma_1\sigma_2V_2FV_1 \sim \sigma_1\sigma_2V_2V_1F.$$

Repeating this procedure, we have the conclusion. □

Thus the pattern P is the closure of a braid $(\sigma_1\sigma_2\dots\sigma_{n-1})^{nrs+1}$. This means that k is a cable knot. □

Remark 2.6. *Lee* [11, Question 1.2] *asks whether* $T(p, q; r, s)$ *is a cable knot, if it is a satellite knot under a condition that* $1 < p < q$, $r \ne q$, r *is not a multiple of* p, $1 < r \le p + q$ *and* $s > 0$. *Proposition 2.4 gives a positive answer if the knot has tunnel number one, which is known to be true when* $r \in \{2, 3\}$ *(see* [7]*).*

Proposition 2.7. *If* $p \ge 5$*, then* K *is hyperbolic.*

Proof. First, $K = T(p, kp+1; 2, 1)$ is a torus knot if and only if $p = 2, 3$ by [10, Theorem 1.1]. Hence we know that our knot is not a torus knot.

Assume that K is a satellite knot for a contradiction. We remark that K has tunnel number one. (A short arc at the extra full twist gives an unknotting tunnel.) Proposition 2.4 shows that K is the $(n, nrs+1)$-cable of $T(r, s)$. Then $K = T(4, 4m + 1; 2, 1)$ for some $m \ge 1$ by [11]. This is a contradiction, because of Lemma 2.2 and $p \ge 5$. Thus we have shown that K is neither a torus knot nor a satellite knot, so K is hyperbolic. □

3. Upsilon torsion function

In this section, we determine the Upsilon torsion function $\Upsilon^{\mathrm{Tor}}(K)$ of $K = T(p, kp+1; 2, 1)$. Since K is an L–space knot (Lemma 2.1), the full knot Floer complex $\mathrm{CFK}^\infty(K)$ is determined by the Alexander polynomial ([20]). It has the form of staircase diagram described by the gaps of Alexander polynomial. If the gaps are given as a sequence $a_1, a_2, \dots, a_n$, then the terms give the length of horizontal and vertical steps. More precisely, let g be the genus of K. Start at the vertex $(0, g)$ on the coordinate plane. Go right a_1 steps, and down a_2 steps, and so on. Finally, we reach $(g, 0)$. By the symmetry of the Alexander polynomial, the staircase inherits the symmetry along the line $y = x$.

We follow the process in [1, Appendix]. However, we assign a modified filtration level FL to each generator of the complex. If a generator x has the coordinate (a, b), then $\mathrm{FL}(x) = tb + (2-t)a$. In fact, for any $t \in [0, 2]$, FL defines a real-valued function on $\mathrm{CFK}^\infty(K)$. Then, for all $s \in \mathbb{R}$, $\mathcal{F}_s$ is spanned by all vectors $x \in \mathrm{CFK}^\infty(K)$ such that $\mathrm{FL}(x) \leq s$. The collection $\{\mathcal{F}_s\}$ gives a filtration on $\mathrm{CFK}^\infty(K)$. See [12]. (Remark that this filtration level is just the twice of that used in [1].) Since $\mathcal{F}_s \subset \mathcal{F}_u$ if $s \leq u$, a generator $x_i \in \mathcal{F}_u$ can be added by $x_j \in \mathcal{F}_s$, without any change of the filtration level. That, $\mathrm{FL}(x_i) = \mathrm{FL}(x_i + x_j)$.

For the staircase complex, repeating a change of basis gradually splits the complex into a single isolated generator and separated arrows. Then the value of the Upsilon torsion function is given as the maximum difference between filtration levels among the arrows.

Since the Upsilon torsion function, defined on $[0, 2]$, is symmetric along $t = 1$, it suffices to consider the domain $[0, 1]$.

As the simplest case, we demonstrate the process when $p = 2$.

Example 3.1. *Let $p = 2$. Then $K = T(2, 2k+3)$ as mentioned before, and we show that its Upsilon torsion function $\Upsilon^{\mathrm{Tor}}_K(t) = t$ $(0 \leq t \leq 1)$, independent of k.*

By Corollary 2.3, the gaps of the exponents of the Alexander polynomial is $1, 1, \dots, 1$ (repeated $2k+2$ times). Hence the staircase diagram has the form as shown in Figure 1, where A_i has Maslov grading 0, but B_i has grading 1, and each arrow has length one.

Each generator is assigned the filtration level FL*. The difference between filtration levels among the generators is important. We have* $\mathrm{FL}(B_{i+1}) - \mathrm{FL}(A_i) = 2 - t$ *and* $\mathrm{FL}(B_i) - \mathrm{FL}(A_i) = t$*, because each arrow has length one. Thus we have*

$$\mathrm{FL}(A_0) \leq \mathrm{FL}(A_1) \leq \cdots \leq \mathrm{FL}(A_{k+1}),$$

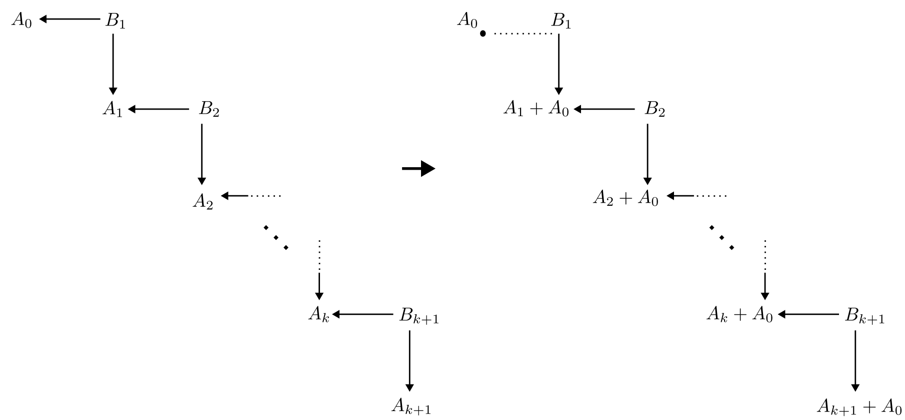

FIGURE 1. Left: The staircase diagram when $p = 2$. The generator A_i has grading 0, but B_i has grading 1. Each arrow has length one. Right: By adding A_0 to $A_1, \dots, A_{k+1}$, the generator A_0 is isolated from the complex.

where each equality occurs only when $t = 1$. Hence A_0 has the lowest filtration level among the generators with grading 0. Add A_0 to $A_1, \dots, A_{k+1}$. Then the generator A_0 is isolated from the complex as shown in Figure 1. (Recall that we use $\mathbb{F}_2$ coefficients.) In the remaining part of the complex, $A_1 + A_0$ is the lowest, since $\mathrm{FL}(A_i + A_0) = \mathrm{FL}(A_i)$ for $i = 1, 2, \dots, k + 1$. To simplify the notation, we keep the same symbol A_i, instead of $A_i + A_0$, after this, if no confusion can arise.

Add A_1 to the other generators with grading 0, except A_0. Then the arrow $B_1 \to A_1$ is split off from the complex. Repeating this process leads to the decomposition of the original staircase into one isolated generator A_0 and $k+1$ vertical arrows. For each arrow, the difference of filtration levels is equal to t, so the maximum difference is t among the arrows. This shows $\Upsilon_K^{\mathrm{Tor}}(t) = t$.

Theorem 3.2. *Let $K = T(p, pk + 1; 2, 1)$ with $p \geq 4$ and $k \geq 1$. The Upsilon torsion function $\Upsilon_K^{\mathrm{Tor}}(t)$ is given as*

$$\Upsilon_K^{\mathrm{Tor}}(t) = \begin{cases} (p-1)t & (0 \leq t \leq \frac{2}{p}) \\ 2 - t & (\frac{2}{p} \leq t \leq \frac{2}{p-2}) \\ (p-3)t & (\frac{2}{p-2} \leq t \leq \frac{4}{p}) \\ 2m + (-m-1)t & (\frac{2m}{p} \leq t \leq \frac{2m}{p-1},\ m = 2, \dots, \lfloor \frac{p-1}{2} \rfloor) \\ (p-2-m)t & (\frac{2m}{p-1} \leq t \leq \frac{2(m+1)}{p},\ m = 2, \dots, \lfloor \frac{p}{2} \rfloor - 1). \end{cases}$$

In particular, $\Upsilon_K^{\mathrm{Tor}}(1) = \lfloor \frac{p-2}{2} \rfloor$.

Proof. Recall that the gaps are

$$(1, p{-}1)^k, (1, 1, 1, p{-}3)^k, (1, 1, 2, p{-}4)^k, \ldots, (1, 1, p{-}3, 1)^k, 1, 1, (p{-}1, 1)^k$$

by Corollary 2.3. We name the generators of the staircase as in Figures 2, 3 and 4.

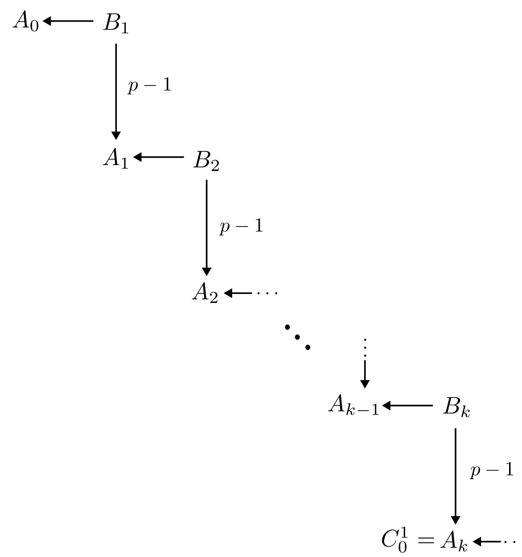

FIGURE 2. The first part corresponds to $(1, p{-}1)^k$. The generators A_i have Maslov grading 0, but B_i have 1. The number $p-1$ next to each vertical arrow indicates the length. Each horizontal arrow has length one. Here, $C_0^1 = A_k$.

In particular, we have the difference between filtration levels of certain generators with Maslov grading 0 as in Table 1. The argument is divided into 4 cases.

Case 1. $0 \leq t \leq \frac{2}{p}$**.** Then any difference in Table 1 is at least 0. Hence A_0 has the lowest filtration level among the generators with grading 0, whose filtration levels increase when we go to the right.

Exactly as in Example 3.1, the staircase complex is decomposed into a single isolated generator A_0 and separated vertical arrows $B_i \to A_i$, $D_i^j \to E_i^j$, $F_i^j \to C_i^j$, $H \to A_0'$ and $B_i' \to A_i'$ where $i = 1, 2, \ldots, k,\ j = 1, \ldots, p-3$. Hence the maximum difference of filtration levels on the arrows is $(p-1)t$. This gives $\Upsilon_K^{\mathrm{Tor}}(t) = (p-1)t$ for $0 \leq t \leq 2/p$.

Case 2. $\frac{2}{p} \leq t \leq \frac{4}{p}$**.** Then $\mathrm{FL}(A_0) \geq \mathrm{FL}(A_1) \geq \cdots \geq \mathrm{FL}(A_k)$. After A_k, the filtration levels increase among the generators with grading 0,

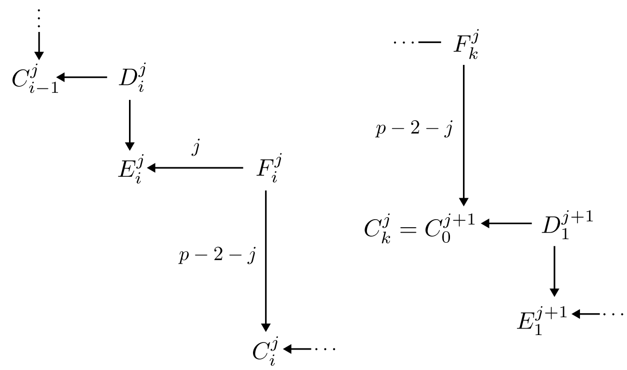

FIGURE 3. Left: The second part corresponds to $(1, 1, j, p-2-j)^k$ $(j = 1, \dots, p-3)$. The generators C_*^j and E_*^j have Maslov grading 0, but the others have 1. Right: This is a connecting part between $(1, 1, j, p-2-j)$ and $(1, 1, j+1, p-2-(j+1))$.

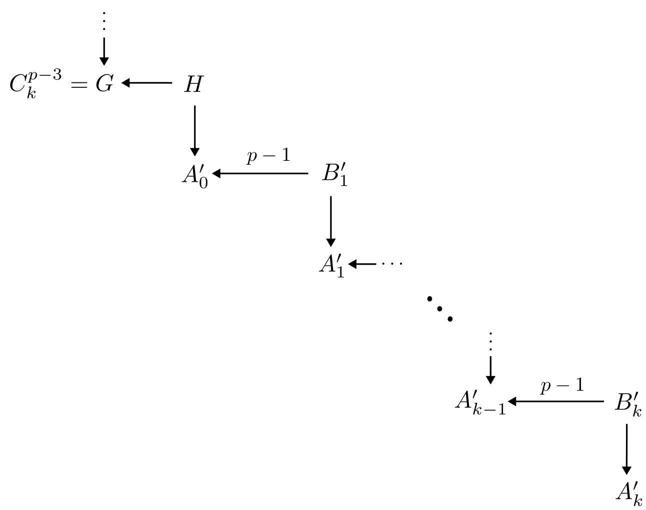

FIGURE 4. The last part corresponds to $1, 1, (p-1, 1)^k$. The generators G and A'_* have Maslov grading 0, but the others have 1.

so A_k is the lowest. Add A_k to the other generators with grading 0. Then A_k will be isolated, and the complex splits into two parts. We say that the first part, which starts at A_0 and ends at B_k, is *N-shaped*, but the second, which starts at D_1^1 and ends at A'_k, is *mirror N-shaped*. In general, if a "zigzag" complex starts and ends at horizontal arrows,

Difference	Indices
$\mathrm{FL}(A_i) - \mathrm{FL}(A_{i-1}) = 2 - pt$	$i = 0, \ldots, k$
$\mathrm{FL}(E_i^j) - \mathrm{FL}(C_{i-1}^j) = 2 - 2t \geq 0$	$i = 1, \ldots, k;\ j = 1, \ldots, p-3$
$\mathrm{FL}(C_i^j) - \mathrm{FL}(E_i^j) = (2-p)t + 2j$	$i = 1, \ldots, k;\ j = 1, \ldots, p-3$
$\mathrm{FL}(C_i^j) - \mathrm{FL}(C_{i-1}^j) = -pt + 2(j+1)$	$i = 1, \ldots, k;\ j = 1, \ldots, p-3$
$\mathrm{FL}(C_0^j) - \mathrm{FL}(C_0^{j-1}) = (-pt + 2j)k$	$j = 2, \ldots, p-3$
$\mathrm{FL}(A_0') - \mathrm{FL}(G) = 2 - 2t \geq 0$	
$\mathrm{FL}(A_i') - \mathrm{FL}(A_{i-1}') = -pt + 2p - 2 > 0$	$i = 1, \ldots, k$

TABLE 1. Difference between filtration levels of the generators with Maslov grading 0.

then it is N-shaped. If it starts and ends at vertical arrows, then it is mirror N-shaped.

For the first part, add A_{k-1} to the others with grading 0, which splits the arrow $A_{k-1} \leftarrow B_k$ off. Repeat this as in Case 1. Then the N-shaped complex is decomposed into separated horizontal arrows $A_{i-1} \leftarrow B_i$ ($i = 1, 2, \ldots, k$), each of which has difference $2-t$. The mirror N-shaped complex is also decomposed into vertical arrows similarly. Thus the maximum difference among them is $(p-3)t$.

Compare $2-t$ and $(p-3)t$. If $\frac{2}{p} \leq t \leq \frac{2}{p-2}$, then $2 - t \geq (p-3)t$. If $\frac{2}{p-2} \leq t \leq \frac{4}{p}$, then $2 - t \leq (p-3)t$. Hence $\Upsilon_K^{\mathrm{Tor}}(t) = 2 - t$ for $\frac{2}{p} \leq t \leq \frac{2}{p-2}$, and $(p-3)t$ for $\frac{2}{p-2} \leq t \leq \frac{4}{p}$.

Case 3. $\frac{2m}{p} \leq t \leq \frac{2m}{p-1}$ $(m = 2, \ldots, \lfloor (p-1)/2 \rfloor)$**.**

From Table 1, we see that $C_k^{m-1} = C_0^m$ is the lowest among the generators with grading 0. Adding this to the others with grading 0 decomposes the complex into one isolated generator C_0^m, the N-shaped one between A_0 and F_k^{m-1} and the mirror N-shaped one between D_1^m and A_k'.

As before, the mirror N-shaped complex can be decomposed into vertical arrows. The longest arrows has length $(p-2-m)t$.

The N-shaped complex is described in Figure 5. We have

$$\mathrm{FL}(A_0) \geq \mathrm{FL}(A_1) \geq \cdots \geq \mathrm{FL}(A_k = C_0^1) \geq \mathrm{FL}(C_1^1) \geq \cdots \geq \mathrm{FL}(C_k^1 = C_0^2)$$
$$\geq \mathrm{FL}(C_1^2) \geq \cdots \geq \mathrm{FL}(C_{k-1}^{m-1})$$

and $\mathrm{FL}(E_i^j) \geq \mathrm{FL}(C_{i-1}^j)$ $(i = 1, 2, \ldots, k; j = 1, 2 \ldots, m-1)$.

Hence C_{k-1}^{m-1} is the lowest. Adding this to the others with grading 0 on the left splits an N-shaped complex $C_{k-1}^{m-1} \leftarrow D_k^{m-1} \rightarrow E_k^{m-1} \leftarrow$

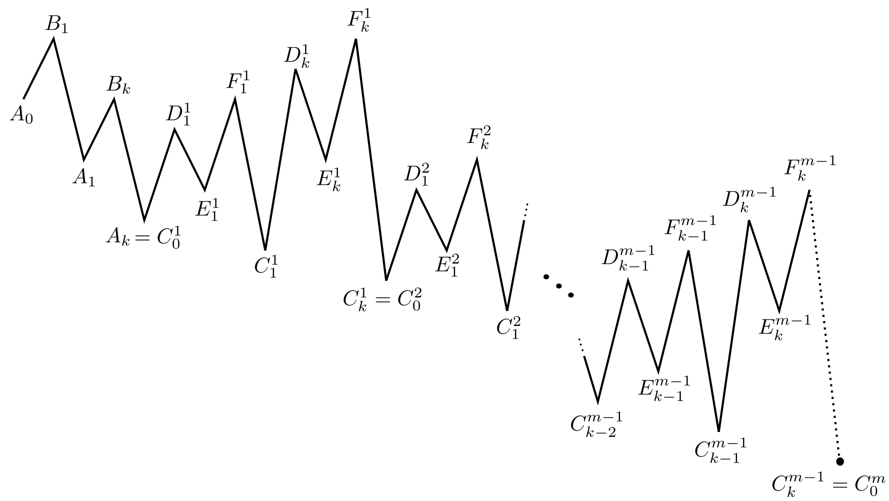

Figure 5. The N-shaped complex between A_0 and F_k^{m-1} after isolating the lowest vertex C_0^m, where $k = 2$. The height indicates the filtration level of each generator. As before, we keep the same notation for generators after a change of basis.

F_k^{m-1} off. For the remaining part, the lowest is C_{k-2}^{m-1}. Again, adding this to the others with grading 0 on the left splits an N-shaped complex $C_{k-2}^{m-1} \leftarrow D_{k-1}^{m-1} \to E_{k-1}^{m-1} \leftarrow F_{k-1}^{m-1}$ off. Repeat this, then we obtain an N-shaped complex between A_0 and B_k, and N-shaped complexes $C_{i-1}^j \leftarrow D_i^j \to E_i^j \leftarrow F_i^j$ $(i = 1, \dots, k; j = 1, \dots, m-1)$.

For the former, the process as in Case 2 yields separated southwest arrows (originally horizontal), each of which has difference $2 - t$. Let us consider the latter N-shaped ones. Since $\mathrm{FL}(F_i^j) - \mathrm{FL}(D_i^j) = -t + (2-t)j \geq 0$, add D_i^j to F_i^j. After that, add C_{i-1}^j to E_i^j. As shown in Figure 6, this change of basis decomposes the complex into a pair of arrows. One has difference $2-t$, and the other has difference $\mathrm{FL}(F_i^j) - \mathrm{FL}(C_{i-1}^j) = 2(j+1) + (-j-2)t$. Note $2 - t \leq 2(j+1) + (-j-2)t$. Furthermore, for $1 \leq j \leq m-1$, $j = m-1$ attains the maximum value, $2m + (-m-1)t$.

Hence we need to compare the values $(p-2-m)t$ and $2m+(-m-1)t$. Since $2m+(-m-1)t \geq (p-2-m)t$, we have $\Upsilon_K^{\mathrm{Tor}}(t) = 2m+(-m-1)t$ for this case.

Case 4. $\frac{2m}{p-1} \leq t \leq \frac{2(m+1)}{p}$ $(m = 2, \dots, \lfloor p/2 \rfloor - 1)$**.**

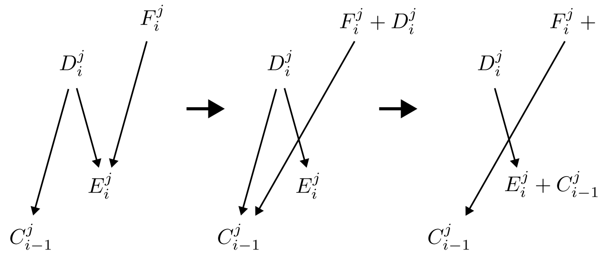

FIGURE 6. A change of basis for an N-shaped complex. Add D_i^j to F_i^j, and C_{i-1}^j to E_i^j.

As in Case 3, $C_k^{m-1} = C_0^m$ is the lowest. So, adding this to the others with grading 0 decomposes the complex into one isolated generator C_0^m, the N-shaped one and the mirror N-shaped one, again.

For the N-shaped complex, the situation is the same as in Case 3. Thus we have an arrow with maximum difference $2m+(-m-1)t$ from this N-shaped complex.

However, we need to handle the mirror N-shaped complex differently now.

First, consider the case where $\frac{2m}{p-1} \le t \le \frac{2m}{p-2}$. Then the filtration levels of the generators with grading 0 increase as going to the right. So, as in Case 3, this part can be decomposed into vertical arrows, and the longest has length $(p-2-m)t$.

Second, consider the case where $\frac{2m}{p-2} \le t \le \frac{2(m+1)}{p}$. Then $\mathrm{FL}(E_i^m) \ge \mathrm{FL}(C_i^m) \ge \mathrm{FL}(C_{i-1}^m)$ $(i = 1, 2, \dots, k)$, but the filtration levels of the remaining generators with grading 0, $C_0^{m+1}, E_1^{m+1}, C_1^{m+1}, \dots, G, A'_0, \dots A'_k$, increase as going to the right. See Figure 7.

Here, C_1^m is the lowest. Adding this to the others with grading 0 on the right splits a mirror N-shaped complex $D_1^m \to E_1^m \leftarrow F_1^m \to C_1^m$ off. Then C_2^m is the lowest in the remaining part. Repeating this yields mirror N-shaped complexes $D_i^m \to E_i^m \leftarrow F_i^m \to C_i^m$ $(i = 1, 2, \dots, k)$, and one more mirror N-shaped one between D_1^{m+1} and A'_k. For the last one, the previous process gives southeast arrows (originally vertical). For each mirror N-complex $D_i^m \to E_i^m \leftarrow F_i^m \to C_i^m$, we remark $\mathrm{FL}(F_i^m) - \mathrm{FL}(D_i^m) = (-m-1)t + 2m \ge 0$. Hence adding D_i^m to F_i^m yields a pair of southeast arrows as shown in Figure 8. Thus, only the arrows that are originally vertical remain, and the longest among them has length $(p-2-m)t$.

Finally, compare $2m + (-m-1)t$ and $(p-2-m)t$. Since $\frac{2m}{p-2} \le t \le \frac{2(m+1)}{p}$, the latter is bigger. Then $\Upsilon_K^{\mathrm{Tor}}(t) = (p-2-m)t$ for this case. $\square$

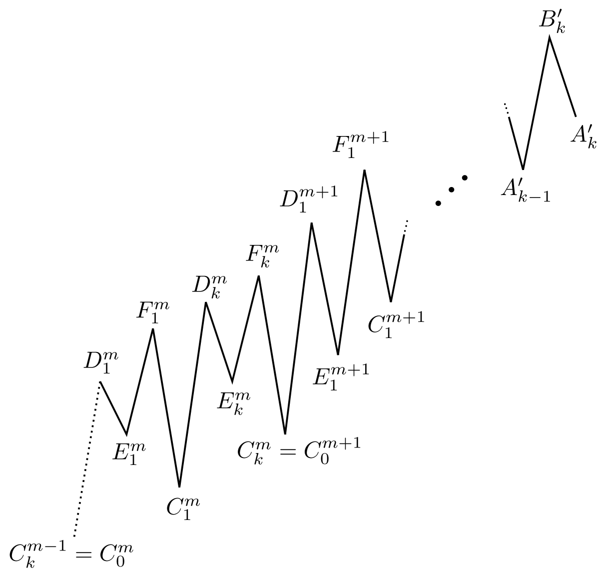

FIGURE 7. The mirror N-shaped complex when $\frac{2m}{p-2} \leq t \leq \frac{2(m+1)}{p}$, where $k = 2$. The generator C_1^m is the lowest.

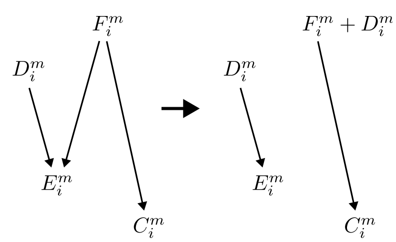

FIGURE 8. A change of basis for a mirror N-shaped complex. Adding D_i^m to F_i^m yields a pair of southeast arrows.

Example 3.3. *When $p = 6$,*

$$\Upsilon_K^{\mathrm{Tor}}(t) = \begin{cases} 5t & (0 \leq t \leq \frac{1}{3}) \\ 2 - t & (\frac{1}{3} \leq t \leq \frac{1}{2}) \\ 3t & (\frac{1}{2} \leq t \leq \frac{2}{3}) \\ 4 - 3t & (\frac{2}{3} \leq t \leq \frac{4}{5}) \\ 2t & (\frac{4}{5} \leq t \leq 1). \end{cases}$$

See Figure 9.

Example 3.4. *When $p = 3$, $K = T(3, 3k + 1; 2, 1) = T(3, 3k + 2)$ as stated before. By Corollary 2.3, the gaps of the exponents of the*

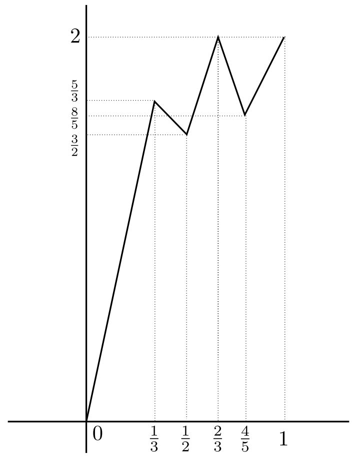

FIGURE 9. The Upsilon torsion function $\Upsilon_K^{\mathrm{Tor}}(t)$ of $K = T(6, 6k+1; 2; 1)$. Then $\Upsilon_K^{\mathrm{Tor}}(1) = 2$.

Alexander polynomial is $(1,2)^k, 1, 1, (2,1)^k$. *Then we have*

$$\Upsilon_K^{\mathrm{Tor}}(t) = \begin{cases} 2t & (0 \leq t \leq \frac{2}{3}) \\ 2 - t & (\frac{2}{3} \leq t \leq 1). \end{cases}$$

We omit the detail.

Proof of Theorem 1.1. Let $K = T(p, kp+1; 2, 1)$ with $p \geq 5$. Then K is a hyperbolic L–space knot by Lemma 2.1 and Proposition 2.7. After fixing p, Theorem 3.2 shows that the Upsilon torsion function does not depend on k. □

Finally, We prove Corollary 1.2.

Proof of Corollary 1.2. By [1], we have $\frac{d}{dt}\Upsilon_K^{\mathrm{Tor}}(0) = \mathrm{Ord}(K)$ and $\Upsilon_K^{\mathrm{Tor}}(1) = \mathrm{Ord}'(K)$. Thus, Theorem 3.2 immediately gives $\mathrm{Ord}(K) = p - 1$ and $\mathrm{Ord}'(K) = \lfloor (p-2)/2 \rfloor$ when $p \geq 4$.

By Lemma 2.1 and Proposition 2.7, the twisted torus knot $K = T(p, kp+1; 2, 1)$ is a hyperbolic L–space knot if $p \geq 5$. Since K has genus $(kp^2 - kp + 2)/2$, distinct choices of k, with a fixed p, give distinct knots.

Set $K_2 = K$ with $p = 2N + 3 \geq 5$. Then K_2 is hyperbolic and $\mathrm{Ord}'(K_2) = \lfloor (p-2)/2 \rfloor = N$.

If $N \geq 4$, then set $K_1 = K$ with $p = N + 1$. Then K_1 is hyperbolic and $\mathrm{Ord}(K_1) = p - 1 = N$.

To complete the proof, we need to give infinitely many hyperbolic knots K_1 whose $\mathrm{Ord}(K_1)$ takes each of the values $1, 2, 3$. Note that for an L–space knot K, $\mathrm{Ord}(K)$ is equal to the longest gap in the exponents of its Alexander polynomial [5, Lemma 5.1].

(1) By [5, Corollary 1.8], $\mathrm{Ord}(L) \leq \mathrm{br}(L)-1$ for any knot L, where $\mathrm{br}(L)$ is the bridge number of L. Hence if K_1 is a hyperbolic 2-bridge knot, then $\mathrm{Ord}(K_1)=1$.
(2) Let $K_1 = T(3,4;2,s)$ with $s \geq 2$. Then K_1 is a hyperbolic L–space knot ([9, 10, 22]). From the Alexander polynomial [15], we have $\mathrm{Ord}(K_1)=2$. Since K_1 has genus $s+3$, distinct choices of s give distinct knots.
(3) Finally, there are infinitely many hyperbolic L–space knots $\{k_n\}$, defined in [2, Section 2], each of which satisfies $\mathrm{Ord}(k_n)=3$ by [2, Proposition 3.1 (3)]. (See also [6, Proposition 5.1].)

□

Remark 3.5. *The twisted torus knot $K = T(p, pk+1; 2, 1)$ is an L–space knot (Lemma 2.1) and twist positive in the sense of* [6]*. Thus the proof of* [6, Theorem 1.3] *concludes that* $\mathrm{Ord}(K) = p-1$ *by showing* $p-1 \leq \mathrm{Ord}(K) \leq \mathrm{br}(K)-1 \leq i(K)-1 = p-1$*, where $i(K)$ is the braid index of K.*

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References

1. S. Allen and C. Livingston: *An Upsilon torsion function for knot Floer homology*, to appear, Math. Research Letters, arXiv:2208.04768.
2. K. Baker and M. Kegel: *Census L–space knots are braid positive, except for one that is not*, Algebr. Geom. Topol. **24** (2024), no. 1, 569–586.
3. S. Gong and M. Marengon: *Nonorientable link cobordisms and torsion order in Floer homologies*, Algebr. Geom. Topol. **23** (2023), no. 6, 2627–2672.
4. T. Ito: *Satellite fully positive braid links are braided satellite of fully positive braid links*, preprint. `arXiv:2402.01129`.
5. A. Juhász, M. Miller and I. Zemke: *Knot cobordisms, bridge index, and torsion in Floer homology*, J. Topology **13** (2020), no.4, 1701–1724.
6. S. Krishna and H. Morton: *Twist positivity, L–space knots, and concordance*, Selecta Math. (N.S.) **31** (2025), no. 1, Paper No. 11, 27 pp.
7. J. H. Lee: *Twisted torus knots $T(p,q;3,s)$ are tunnel number one*, J. Knot Theory Ramifications **20** (2011), no. 6, 807–811.
8. S. Lee: *Knot types of twisted torus knots*, J. Knot Theory Ramifications **26** (2017), no. 12, 1750074 (7 pages).

9. S. Lee: *Satellite knots obtained by twisting torus knots: hyperbolicity of twisted torus knots*, Int. Math. Res. Not. IMRN (2018), no. 3, 785–815.
10. S. Lee: *Positively twisted torus knots which are torus knots*, J. Knot Theory Ramifications **28** (2019), no. 3, 1950023, 13 pp.
11. S. Lee: *Cable knots obtained by positively twisting torus knots*, J. Knot Theory Ramifications **32** (2023), no. 3, Paper No. 2350018, 15 pp.
12. C. Livingston: *Notes on the knot concordance invariant upsilon*, Algebr. Geom. Topol. **17** (2017), no. 1, 111–130.
13. K. Miyazaki and K. Motegi: *Crossing change and exceptional Dehn surgery*, Osaka J. Math. **39** (2002), no. 4, 773–777.
14. K. Morimoto and M. Sakuma: *On unknotting tunnels for knots*, Math. Ann. **289** (1991), no. 1, 143–167.
15. H. Morton: *The Alexander polynomial of a torus knot with twists*, J. Knot Theory Ramifications **15** (2006), no.8, 1037–1047.
16. P. Ozsváth, A. Stipsicz and Z. Szabó: *Concordance homomorphisms from knot Floer homology*, Adv. Math. **315** (2017), 366–426.
17. P. Ozsváth, A. Stipsicz and Z. Szabó: *Unoriented knot Floer homology and the unoriented four-ball genus*, Int. Math. Res. Not. IMRN (2017), no.17, 5137–5181.
18. P. Ozsváth, A. Stipsicz and Z. Szabó: *Grid homology for knots and links*, Mathematical Surveys and Monographs, vol. 208, American Mathematical Society, Providence, RI, 2015.
19. P. Ozsváth and Z. Szabó: *Holomorphic disks and genus bounds*, Geom. Topol. **8** (2004), 311–334.
20. P. Ozsváth and Z. Szabó: *On knot Floer homology and lens space surgeries*, Topology **44** (2005), 1281–1300.
21. J. R. Stallings: *Constructions of fibred knots and links*, Proc. Sympos. Pure Math., XXXII American Mathematical Society, Providence, RI, 1978, pp. 55–60.
22. F. Vafaee: *On the knot Floer homology of twisted torus knots*, Int. Math. Res. Not. IMRN (2015), no. 15, 6516–6537.

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