Logarithmic Tate conjectures over finite fields

Kazuya Kato, Chikara Nakayama, Sampei Usui

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Abstract

We formulate an analogue of Tate conjecture on algebraic cycles, for the log geometry over a finite field. We show that the weight-monodromy conjecture follows from this conjecture and from the semi-simplicity of the Frobenius action. This conjecture suggests the existence of the monodromy cycle which gives the monodromy operator and an action of $\mathfrak{sl}(2)$ on the cohomology, and which lives in the world of log motives.

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Introduction

Let V be a projective smooth algebraic variety over a p-adic local field K. The monodromy operator acts on the ℓ-adic étale cohomology group $H^m_{\text{ét}}(V_{\overline{K}}, \mathbb{Q}_\ell)$ for each prime number $\ell \neq p$.

There are problems on the monodromy operator:

1. Weight-monodromy conjecture. In the case where the residue field is finite, this conjecture gives a strong relation of the weight of the action of the Frobenius and the action of the monodromy operator on the ℓ-adic étale cohomology group.

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2. The monodromy operator often has properties which are independent of ℓ. This independence suggests that the monodromy operator is in fact an algebraic cycle in the algebraic geometry over the residue field k of K. This subject was studied in C. Consani ([2]) (with Appendix by S. Bloch), C. Consani and M. Kim [3], and also in [10] Appendix.

These problems can be regarded as problems in logarithmic geometry over the residue field k. Assume that V has a strictly semistable reduction X. Then X is regarded as a log scheme over the standard log point s over k. By [13] Proposition 4.2, we have the identification

$$H^m_{\text{ét}}(V_{\overline{K}}, \mathbb{Q}_\ell) = H^m_{\text{logét}}(X_{\overline{s}(\log)}, \mathbb{Q}_\ell)$$

with the log étale cohomology group ([12], [15]), where $\overline{s}(\log)$ is the log separable closure of s. Furthermore, the action of $\mathrm{Gal}(\overline{K}/K)$ on the left-hand-side is identified with the action of the log fundamental group $\pi_1^{\log}(s) = \mathrm{Gal}(K^{\text{tame}}/K)$ on the right-hand-side, where $K^{\text{tame}} \subset \overline{K}$ denotes the maximal tame extension of K.

In this paper, assuming that k is a finite field, we formulate logarithmic analogues of the Tate conjecture on algebraic cycles. Such a conjecture was formulated and discussed in [9] Section 6 and in [10] Appendix without assuming the finiteness of k, but here in the case where k is a finite field, we formulate a stronger log Tate conjecture 1.12 using a log Tate curve over s. A log Tate curve is the reduction of a Tate elliptic curve over K, which is regarded as a log scheme over s. Our method is based on the fact that all log Tate curves over s are isogenous in the case where k is finite. Note that the above problems 1 and 2 may be reduced to the finite residue field case by specialization arguments.

The log Tate conjecture 1.12 says that if $m \geq 2r$, the map

$$\mathbb{Q}_\ell \otimes \mathrm{gr}^{m-r} K_{0,\lim}(X \times E^{m-2r}) \to H^m_{\text{logét}}(X_{\overline{s}(\log)}, \mathbb{Q}_\ell)(r)^G$$

is surjective. Here E is the log Tate curve over s, $X \times E^{m-2r}$ means the fiber product over s, $K_{0,\lim}$ is the inductive limit of the K-groups K_0 for the log modifications of $X \times E^{m-2r}$, gr^{m-r} is for the γ-filtration, $G = \pi_1^{\log}(s)$, and $(\cdot)^G$ is the part fixed by G.

We prove that if this log Tate conjecture is true and a certain semi-simplicity of the Frobenius action is true, then the weight-monodromy conjecture is true.

In Conjecture 3.6, we formulate a finer version

$$\mathbb{Q}_\ell \otimes K(X, m, r) \overset{\cong}{\to} H^m_{\text{logét}}(X_{\overline{s}(\log)}, \mathbb{Q}_\ell)(r)^G$$

of the log Tate conjecture 1.12, by using a subquotient $K(X, m, r)$ of $\mathbb{Q} \otimes \mathrm{gr}^{m-r} K_{0,\lim}(X \times E^{m-2r})$.

The log Tate conjecture 3.6 suggests that there is a unique element of $K(X \times X, 2d, d-1)$ with $d = \dim X$ which induces the monodromy operator on $H^m_{\text{ét}}(V_{\overline{K}}, \mathbb{Q}_\ell)$ for every m and $\ell \neq p$ and gives an action of $\mathfrak{sl}(2)$ on it. We call this conjectural element the *monodromy cycle*. In 4.13, we give an interpretation of the monodromy cycle by using log motives.

The above all seem to tell that the log Tate curve over a finite field has a special importance in arithmetic geometry.

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1 Log Tate conjecture

1.1. Let k be a finitely generated field over the prime field.

The usual Tate conjecture is that for a projective smooth algebraic variety X over k, the étale cohomology $H^{2r}_{\text{ét}}(X \otimes_k \overline{k}, \mathbb{Q}_\ell)(r)^G$, where $G = \text{Gal}(\overline{k}/k)$, is generated by the classes of algebraic cycles on X of codimension r. It is known that $H^m_{\text{ét}}(X \otimes_k \overline{k}, \mathbb{Q}_\ell)(r)^G = 0$ unless $m = 2r$.

1.2. Consider a log scheme $s = \text{Spec}(k)$ which is endowed with a log structure M_s such that $M_s/\mathcal{O}_s^\times = \mathbb{N}$.

By Hilbert 90, we have an exact sequence $0 \to k^\times \to M_s^{\text{gp}}(s) \to M_s^{\text{gp}}/\mathcal{O}_s^\times(s) \to 0$, and there is a section π of M_s such that $\mathbb{N} \overset{\cong}{\to} M_s/\mathcal{O}_s^\times$; $1 \mapsto \pi$. Such a π is called a *generator* of the log structure of s.

Let $\mathfrak{S} = \mathfrak{S}(s)$ be the category of projective vertical log smooth fs log schemes over s which have charts of the log structure Zariski locally. The reason why we put the last condition on the log structure is explained in 1.5 below.

By the above remark on the existence of a section π, s itself belongs to $\mathfrak{S}$.

For example, if K, k, and V with strictly semistable reduction X are as in Introduction, that is, if there is a projective smooth regular flat scheme $\mathfrak{X}$ over the valuation ring O_K of K of strictly semistable reduction and $X = \mathfrak{X} \otimes_{O_K} k$, then we can take $\text{Spec}(k)$ with the inverse image M_s of the natural log structure of $\text{Spec}(O_K)$ as our s, a prime element of K gives a generator of the log structure of s, and X with the inverse image of the natural log structure of $\mathfrak{X}$ belongs to $\mathfrak{S}$.

Throughout this paper, X denotes an object of $\mathfrak{S}$ unless otherwise stated.

For a prime number ℓ which is different from the characteristic of k, we denote

$$H^m(X)_\ell := H^m_{\text{logét}}(X_{\overline{s}(\log)}, \mathbb{Q}_\ell).$$

1.3. In this situation, $H^m(X)_\ell(r)^G$, where $G = \pi_1^{\log}(s)$, need not be zero for $m \neq 2r$.

The following conjecture is well-known at least if X is the reduction of V as in Introduction.

Conjecture 1.4. $H^m(X)_\ell(r)^G = 0$ *unless* $m \geq 2r$.

Now we consider the case $m \geq 2r$.

1.5. Let

$$K_{0,\lim}(X) = \varinjlim_{X'} K_0(X'),$$

where X' ranges over all log modifications ([9] 2.3.6) of X.

The condition that X has charts of the log structure Zariski locally ensures that X has sufficiently many log modifications to get a nice $K_{0,\lim}(X)$ as is explained in [9] Sections 2.2–2.4.

First, the case $m = 2r$ is formulated in the similar style to the classical case.

Conjecture 1.6. *The Chern class map*

$$\mathbb{Q}_\ell \otimes \mathrm{gr}^r K_{0,\lim}(X) \to H^{2r}(X)_\ell(r)^G$$

is surjective.

This conjecture was discussed in [9] Section 6.

For general $m \geq 2r$, we have

Conjecture 1.7. *Let $m \geq 2r$. Then the following holds.*

(1) *The map*

$$\mathbb{Q}_\ell \otimes \mathrm{gr}^{m-r} K_{0,\lim}(X \times \mathbb{G}_m^{m-2r}) \to H^m(X)_\ell(r)^G$$

is surjective.

(2) *The Chern class map*

$$\mathbb{Q}_\ell \otimes \mathrm{gr}^r KH_{2r-m,\lim}(X) \to H^m(X)_\ell(r)^G$$

is surjective. Here KH denotes the homotopy K-theory ([21]) and $KH_{i,\lim}(X)$ is defined as $\varinjlim_{X'} KH_i(X')$ in the same way as in 1.5.

Note that the index $2r - m$ in the K-group in (2) is ≤ 0.

(1) implies (2) as is explained in 1.13.

1.8. The conjectures (1) and (2) in Conjecture 1.7 were considered in the resp. part of [10] A.14. Actually, there in [10], we assumed that X was saturated over s. We put this assumption just because we were following the analogy with the theory of log Hodge structures with unipotent local monodromy and because the action of the inertia subgroup $\pi_1^{\log}(\bar{s}) = \mathrm{Gal}(\bar{s}(\log)/\bar{s})$ of $\pi_1^{\log}(s) = \mathrm{Gal}(\bar{s}(\log)/s)$ on $H^m(X)_\ell$ is unipotent for such an X (see Proposition 4.9).

1.9. Here we review the log Tate curve over s from the viewpoint of the theory of log abelian varieties ([7]). This viewpoint makes some stories (for example, the story of the multiplication by $a \in \mathbb{Z}$ in 3.5) transparent.

Let q be a section of the log structure of our log point s which does not belong to $k^\times$. Then we have a log elliptic curve $E^{(q)} = \mathbb{G}_{m,\log}^{(q)}/q^{\mathbb{Z}}$ which is an abelian group sheaf functor on the category of fs log schemes over s for the (classical) étale topology. Here $\mathbb{G}_{m,\log}$ is the sheaf $T \mapsto \Gamma(T, M_T^{\mathrm{gp}})$ and $(\cdot)^{(q)}$ means the part consisting of $t \in M_T^{\mathrm{gp}}$ such that locally, we have $q^m|t|q^n$ for some $m, n \in \mathbb{Z}$ such that $m \leq n$, that is, tq^{-m} and $q^n t^{-1}$ belong to M_T. This $E^{(q)}$ is not a representable functor, but it has big representable subfunctors $E^{(q,n)} = \mathbb{G}_{m,\log}^{(q,n)}/q^{\mathbb{Z}}$ for integers $n \geq 1$, called the models of $E^{(q)}$, where $\mathbb{G}_{m,\log}^{(q,n)}$ $(= \mathbb{G}_{m,\log}^{(q,(1/n)\mathbb{Z})}$ in the notation in [7] Section 1) is the part of $\mathbb{G}_{m,\log}$ consisting of t such that locally we have $q^r|t^n|q^{r+1}$ for some $r \in \mathbb{Z}$. Here we say that $E^{(q,n)}$ is big for the reasons explained in 1.10 below.

For a local field K as in Introduction, for a lifting $\tilde{q}$ of q to $K^\times$, $E^{(q,n)}$ is a reduction of the Tate elliptic curve over K of q-invariant $\tilde{q}$. As a scheme, $E^{(q,n)}$ is an n-gon (see [7] 1.4).

For $n \geq 2$, $E^{(q,n)}$ belongs to $\mathfrak{S}$ ($E^{(q,1)}$ does not have charts Zariski locally due to the self-intersection of the 1-gon). In this paper, a log Tate curve E means $E = E^{(q,n)}$ for some q and for some $n \geq 2$.

$E^{(q,n)}$ is semistable if the image of q in $\mathbb{N}$ under $M_s/\mathcal{O}_s^\times \cong \mathbb{N}$ is n and is saturated over s if this image is a multiple of n.

1.10. In the above, we called $E^{(q,n)}$ a big subfunctor for the following two reasons.

(1) $E^{(q,n)}$ covers $E^{(q)}$ modulo isogenies and translations in the sense that the canonical map from the disjoint union of $E^{(q^k, \frac{1}{n}\mathbb{Z}+\frac{1}{2n})}$ for $k \geq 1$ to $E^{(q)}$ is surjective.

(2) $E^{(q,n)}$ coincides with $E^{(q)}$ modulo log modifications in the sense that for every fs log scheme T over s and every $a \in E^{(q)}(T)$, the fiber product of $E^{(q,n)} \to E^{(q)} \overset{a}{\leftarrow} T$ is represented by a log modification of T.

These (1) and (2) can be deduced from [7] 1.4.

1.11. Assume that k is a finite field. Then all the log elliptic curves $E^{(q)}$ are isogenous. In fact, let π be a generator of the log structure of s. Then $q = \pi^n u$ for some integer $n \geq 1$ and $u \in k^\times$. Take an integer $m \geq 1$ such that $u^m = 1$. Then we have the homomorphism $E^{(q)} \to E^{(\pi^{mn})}$ induced by $\mathbb{G}_{m,\log} \to \mathbb{G}_{m,\log}$; $t \mapsto t^m$ and the projection $E^{(\pi^{mn})} \to E^{(\pi)}$, which are isogenies.

Conjecture 1.12. *Let $m \geq 2r$. Assume that k is a finite field. Let E be a log Tate curve over the standard log point over k. Then the map*

$$\mathbb{Q}_\ell \otimes \mathrm{gr}^{m-r} K_{0,\lim}(X \times E^{m-2r}) \to H^m(X)_\ell(r)^G$$

is surjective.

The definition of this map is explained in 1.13 below. As in explained in 1.14, this conjecture is independent of the choice of the log Tate curve E.

1.13. Assume $m \geq 2r$.

We have a diagram

$$\begin{array}{ccccc} \mathrm{gr}^{m-r} K_{0,\lim}(X \times E^{m-2r}) & \to & \mathrm{gr}^{m-r} K_{0,\lim}(X \times \mathbb{G}_m^{m-2r}) & \to & \mathrm{gr}^r KH_{2r-m,\lim}(X) \\ \downarrow & & & & \downarrow \\ \mathrm{Hom}_G(\mathrm{Sym}^{m-2r} H^1(E)_\ell, H^m(X)_\ell(r)) & & \to & & H^m(X)_\ell(r)^G. \end{array}$$

The upper left arrow is the pullback by the inclusion $\mathbb{G}_m \subset E$, and the lower arrow is the restriction to e_1^{m-2r}, where e_1 is the canonical base of $H^1(E)_\ell^G = \mathbb{Q}_\ell$.

The left vertical arrow is defined as

$$\mathrm{gr}^{m-r} K_{0,\lim}(X \times E^{m-2r}) \to H^{2(m-r)}(X \times E^{m-2r})(m-r)^G \to (H^m(X)_\ell \otimes H^1(E)_\ell^{\otimes(m-2r)}(m-r))^G$$

$$\to \mathrm{Hom}_G(H^1(E)_\ell^{\otimes(m-2r)}, H^m(X)_\ell(r)) \to \mathrm{Hom}_G(\mathrm{Sym}^{m-2r} H^1(E)_\ell, H^m(X)_\ell(r)).$$

Here the first arrow is the Chern class map, the second arrow is the projection to a Künneth component, the third arrow is by Poincaré duality, and the fourth arrow is the restriction to the direct summand $\mathrm{Sym}^{m-2r} H^1(E)_\ell$ of $H^1(E)_\ell^{\otimes(m-2r)}$.

The upper right arrow is defined by using the Mayer–Vietoris sequence for $\mathbb{P}^1 = (\mathbb{A}^1) \cup (\mathbb{A}^1)^-$, $\mathbb{A}^1 \cap (\mathbb{A}^1)^- = \mathbb{G}_m$, repeatedly.

Since the composite map $\mathrm{gr}^{m-r} K_{0,\lim}(X \times \mathbb{G}_m^{m-2r}) \to H^m(X)_\ell(r)^G$ is defined also as

$$\mathrm{gr}^{m-r} K_{0,\lim}(X \times \mathbb{G}_m^{m-2r}) \to H^{2m-2r}(X \times \mathbb{G}_m^{m-2r})_\ell(m-r)^G$$

$$\to (H^m(X)_\ell \otimes H^1(\mathbb{G}_m)_\ell^{\otimes(m-2r)})(m-r)^G = H^m(X)_\ell(r)^G$$

by using the Künneth decomposition, the above diagram is commutative.

The right vertical arrow was used in [10] in the studies of motives in log geometry.

1.14. Conjecture 1.12 is independent of the choice of (q, n) of the log Tate curve $E = E^{(q,n)}$. In fact, since $E^{(q,mn)} \to E^{(q,n)}$ is a log modification, $K_{0,\lim}$ in Conjecture 1.12 is independent of the choice of n if q is fixed. The isogenies in 1.11 have the isogenies in the converse directions, and the isogenies induce isomorphisms between $H^1(E^{(q,n)})_\ell$ for different (q, n).

1.15. By the diagram in 1.13, Conjecture 1.12 implies Conjecture 1.7 (1) and Conjecture 1.7 (1) implies Conjecture 1.7 (2).

If we admit Conjecture 1.6, the left vertical arrow in this diagram induces a surjection from $\mathbb{Q}_\ell \otimes$ the upper K-group as is seen from the definition of the left vertical arrow, and hence Conjecture 1.12 becomes equivalent to the statement that the lower arrow in this diagram is surjective.

2 Weight-monodromy conjecture

2.1. We review the monodromy operator and the weight-monodromy conjecture. Let q be a section of the log structure of s which does not belong to $k^\times$. Let $I = \pi_1^{\log}(\bar{s})$. We have a canonical homomorphism

$$a_q : I \to \mathbb{Z}_\ell(1)\ ;\ \sigma \mapsto (\sigma(q^{1/\ell^n})/q^{1/\ell^n})_{n \geq 1}$$

with finite cokernel.

Let X be an object of $\mathfrak{S}$ and let $m \geq 0$. Then for some open subgroup I' of I, the action of I' on $H^m(X)_\ell$ is unipotent and factors through $a_q : I' \to \mathbb{Z}_\ell(1)$. Taking an element σ of I' such that $a_q(\sigma) \neq 0$, we define the monodromy operator as

$$N_q := a_q(\sigma)^{-1} \log(\sigma)\ :\ H^m(X)_\ell \to H^m(X)_\ell(-1),$$

which is independent of such a σ. Here $\log(\sigma)$ is the logarithm of the unipotent action of σ on $H^m(X)_\ell$.

If q' is another section of the log structure of s which does not belong to $k^\times$ and if c is the rational number such that the classes of q' and q^c in $\mathbb{Q} \otimes M_s^{\mathrm{gp}}/\mathcal{O}_s^\times \cong \mathbb{Q}$ coincide, we have $N_{q'} = c^{-1} N_q$.

In the situation where the choice of q is not important (as in the situation in the rest of this Section 2), we denote N_q simply as N.

2.2. In the case where k is a finite field, the weight-monodromy conjecture says the following. For the monodromy filtration W of the monodromy operator N on $H^m(X)_\ell$, the action of the geometric Frobenius on W_r/W_{r-1} is of weight $m+r$ for every r.

Strong results on the weight-monodromy conjecture were obtained by P. Scholze in [18], but it is still a conjecture.

Theorem 2.3. *Assume the log Tate conjecture* 1.12 *in general (not only for X and m below). Assume the following* (i) *and* (ii)*.*

(i) *There is a non-degenerate pairing*

$$H^m(X)_\ell \times H^m(X)_\ell \to \mathbb{Q}_\ell(-m)$$

which is compatible with the action of G.

(ii) *The Frobenius action on $H^m(X)_\ell^{N=0}$ is semisimple.*

Then the weight-monodromy conjecture for X is true.

The condition of the existence of the pairing in (i) is satisfied if X is the reduction of V as in Introduction, by the hard Lefschetz on V. This assumption (i) is used in 2.7 below.

Lemma 2.4. *Let $v \in H^m(X)_\ell(r)^G$ with $m \geq 2r$. Then v is in the image of N^{m-2r}.*

Proof. By the log Tate conjecture 1.12, we have a G-homomorphism $\mathrm{Sym}^{m-2r} H^1(E)_\ell \to H^m(X)_\ell(r)$ which sends e_1^{m-2r} to v (see the last remark in 1.13). Since e_1^{m-2r} belongs to the image of $N^{m-2r} : \mathrm{Sym}^{m-2r} H^1(E)_\ell(m-2r) \to \mathrm{Sym}^{m-2r} H^1(E)_\ell$, v belongs to the image of $N^{m-2r} : H^m(X)_\ell(m-r) \to H^m(X)_\ell(r)$. □

2.5. To prove Theorem 2.3, by [9] Proposition 2.1.13 (cf. [17], [20] Proposition 2.4.2.1, [22]), we are reduced to the case where X is strictly semistable.

Then we can use the Rapoport–Zink spectral sequence in [14].

Lemma 2.6. *Let $v \in H^m(X)_\ell^{N=0}$ and assume that v is of weight $w \leq m$. Then v is in the image of N^{m-w}.*

Proof. We may assume $v \neq 0$ and v is an eigenvector of Frobenius with eigenvalue α. Since $N(v) = 0$, in the Rapoport–Zink spectral sequence ([14] Proposition 1.8.3), v appears in $H^{2t+w}(Z)_\ell(t)$ for some projective smooth scheme Z over k (in the non-log sense) and for some integer t. By Poincaré duality, the Frobenius eigenvalue α^{-1} appears in $H^{2t-w}(Z)_\ell(t)$ for some Z and t with the eigenvector v'. Then $v \otimes v' \in H^{m+2t-w}(X \times Z)_\ell(t)^G$. Note that $m + 2t - w \geq 2t$. By Lemma 2.4, $v \otimes v'$ belongs to the image of N^{m-w}, that is, to the image of $N^{m-w} \otimes 1$ on $H^m(X)_\ell \otimes H^{2t-w}(Z)_\ell(t)$. Hence v belongs to the image of N^{m-w}. □

2.7. We complete the proof of Theorem 2.3.

By the existence of the pairing in (i), the determinantial weight (the average weight) of $H^m(X)_\ell$ is m. We prove that this together with Lemma 2.6 proves Theorem 2.3. Fix a lifting of the Frobenius to $\pi_1^{\log}(s)$. Fix a base t of $\mathbb{Q}_\ell(1)$ and consider $tN : H^m(X)_\ell \to H^m(X)_\ell$. By the condition (ii), we have a decomposition $H^m(X)_\ell^{tN=0} = \bigoplus_w V_w$, where V_w

is the part of weight w. Take a base $(v_{w,i})_i$ of each V_w for $w \le m$. By Lemma 2.6, for each $w \le m$ and i, there is a $v'_{w,i} \in H^m(X)_\ell$ such that $v_{w,i} = (tN)^{m-w}(v'_{w,i})$. We may assume that this $v'_{w,i}$ is of weight $w + 2(m-w) = 2m - w$. For $w \le m$ and $0 \le j \le m - w$, let $V_{w,+2j}$ be the subspace of $H^m(X)_\ell$ generated by $(N^{m-w-j}(v'_{w,i}))_i$ of weight $w + 2j$. Thus $V_{w,+0} = V_w$. Let $V = \bigoplus_{w \le m, 0 \le j \le m-w} V_{w,+2j}$. It is sufficient to prove $H^m(X)_\ell = V$. The average weight of V is m by construction. On the other hand, we see that the weights of $H^m(X)_\ell/V$ are $> m$. If $H^m(X)_\ell \ne V$, this contradicts that the average weight of $H^m(X)_\ell$ is m.

3 Log K_0 and the finer log Tate conjecture

3.1. In the classical theory, for a projective smooth scheme X over a finite field, it is conjectured that $\mathbb{Q}_\ell \otimes \mathrm{gr}^r K_0(X) \overset{\cong}{\to} H^{2r}(X)_\ell(r)^G$. That is, not only the surjectivity, the isomorphism is conjectured. For the log case, $K_{0,\mathrm{lim}}$ is too big to have the isomorphism even in the case where k is finite. In fact, if X is a curve, for each log modification along a singular point of X, $\dim_{\mathbb{Q}} \mathbb{Q} \otimes \mathrm{Pic} = \mathbb{Q} \otimes \mathrm{gr}^1 K_0$ increases by one, and hence $\mathbb{Q} \otimes \mathrm{gr}^1 K_{0,\mathrm{lim}}$ usually becomes of infinite dimension. To improve this situation, the following log K_0 may be nice.

3.2. For an element a of the inductive limit of $H^0(X', M^{\mathrm{gp}}_{X'}/\mathcal{O}^\times_{X'})$ for log modifications X' of X, let $l_a \in K_{0,\mathrm{lim}}(X)$ be the class of the line bundle defined by the image of a under the connecting map $H^0(X', M^{\mathrm{gp}}_{X'}/\mathcal{O}^\times_{X'}) \to H^1(X', \mathcal{O}^\times_{X'})$. Let

$$K_0^{\log}(X)$$

be the quotient of the ring $K_{0,\mathrm{lim}}(X)$ by the ideal generated by $l_a - 1$ for all a. Then the Chern class maps to the ℓ-adic cohomology factor through $K_0^{\log}(X)$. This is because the Chern character of l_a in $\bigoplus_r H^{2r}(X)_\ell(r)$ is 1 by the following lemma, and since the Chern character is a ring homomorphism, it kills the ideal generated by all $l_a - 1$. Hence the Chern class maps kill this ideal.

Lemma 3.3. *The Chern class of l_a in $H^2(X)_\ell(1)$ vanishes.*

Proof. This is because the Chern class map on $\mathrm{Pic}(X) = H^1(X, \mathcal{O}^\times_X)$ factors through $H^1(X, M^{\mathrm{gp}}_X)$ as is seen by the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z}/\ell^n\mathbb{Z}(1) & \to & \mathbb{G}_m & \overset{\ell^n}{\to} & \mathbb{G}_m & \to & 0 \\ & & \| & & \cap & & \cap & & \\ 0 & \to & \mathbb{Z}/\ell^n\mathbb{Z}(1) & \to & \mathbb{G}_{m,\log} & \overset{\ell^n}{\to} & \mathbb{G}_{m,\log} & \to & 0. \end{array}$$

□

Conjecture 3.4. *In the case where k is a finite field, we have*

$$\mathbb{Q}_\ell \otimes \mathrm{gr}^r K_0^{\log}(X) \overset{\cong}{\to} H^{2r}(X)_\ell(r)^G.$$

3.5. To consider the general $m \geq 2r$, we introduce a subgroup $K(X,m,r)$ of $\mathbb{Q} \otimes \mathrm{gr}^{m-r} K_0^{\log}(X \times E^{m-2r})$.

For an integer $a \geq 1$, the multiplication by $a : E^{(q)} \to E^{(q)}$ induces a morphism (also denoted by a) $E' = E^{(q,an)} \to E = E^{(q,n)}$. For $1 \leq i \leq t$, by the pullback via the morphism $X \times (E')^t \to X \times E^t$ induced by the evident morphisms on all components except a on the i-th E', we have a homomorphism $a_i^* : K_0^{\log}(X \times E^t) \to K_0^{\log}(X \times E^t)$.

Define $K(X,m,r)$ to be the part of $\mathbb{Q} \otimes \mathrm{gr}^{m-r} K_0^{\log}(X \times E^{m-2r})$ consisting of all elements x satisfying the following conditions (i) and (ii).

(i) $a_i^* x = ax$ for $1 \leq i \leq m-2r$ and for any $a \geq 1$.
(ii) x is invariant under the action of the Symmetric group S_{m-2r}.

We define $K(X,m,r) = 0$ in the case $m < 2r$.
We have $K(X,2r,r) = \mathbb{Q} \otimes \mathrm{gr}^r K_0^{\log}(X)$.

Conjecture 3.6. *In the case where k is a finite field, we have*

$$\mathbb{Q}_\ell \otimes K(X,m,r) \overset{\cong}{\to} H^m(X)_\ell(r)^G.$$

3.7. We have the following relations of the conjectures. Conjecture 3.4 is a finer version of Conjecture 1.6, and Conjecture 3.6 is a finer version of Conjecture 1.12.

The idea of Conjecture 3.6 comes from Conjecture 3.4 as follows. By Conjecture 3.4, we expect

$$\mathbb{Q}_\ell \otimes \mathrm{gr}^{m-r} K_0^{\log}(X \times E^{m-2r}) \overset{\cong}{\to} H^{2m-2r}(X \times E^{m-2r})_\ell(m-r)^G.$$

Consider elements of the right-hand-side satisfying the conditions (i) and (ii). By the condition (i) and by the fact that the pullbacks $a^* : H^t(E)_\ell \to H^t(E)_\ell$ for integers $a \geq 1$ and $t \geq 0$ coincide with the multiplication by a^t, we get the Künneth component

$$H^m(X)_\ell \otimes H^1(E)_\ell^{\otimes(m-2r)}(m-r)^G.$$

By the condition (ii), we get

$$\mathrm{Hom}_G(\mathrm{Sym}^{m-2r} H^1(E)_\ell, H^m(X)_\ell(r)).$$

If we admit the weight-monodromy conjecture, the last thing should be equal to $H^m(X)_\ell(r)^G$.

3.8. Let V be as in Introduction with strictly semistable reduction X, assume that k is a finite field, and let

$$L(H^m(V),s) = \det(1 - \varphi \cdot \sharp(k)^{-s}\ ;\ H^m(V_{\overline{K}}, \mathbb{Q}_\ell)^{N=0})^{-1}$$

be the Euler factor of L, where φ is the geometric Frobenius. In [1], a conjecture of S. Bloch on the order of the pole of $L(H^m(V),s)$ at $s = r \in \mathbb{Z}$ is discussed. He used his higher Chow group in the conjecture. We can give a modified version of his conjecture:

Conjecture 3.9. *The order of the pole of $L(H^m(V),s)$ at $s = r \in \mathbb{Z}$ is equal to $\dim_{\mathbb{Q}}(K(X,m,r))$.*

In fact, this order of the pole is equal to the multiplicity of the eigenvalue 1 in the action of φ on $H^m(V_{\overline{K}}, \mathbb{Q}_\ell(r))^{N=0} = H^m(X)_\ell(r)^{N=0}$. Hence if we assume the semisimplicity of the action of φ as is widely believed, the log Tate conjecture 3.6 implies this conjecture.

3.10. The authors expect that an advantage of our method is that by working with K_0, we can apply the usual method of algebraic cycles, for example, the intersection theory. The rationality and the positivity of the intersection number may be meaningful.

We consider the following new intersection theory.

Assume $m, m', r, r' \in \mathbb{Z}$, $m, m' \geq 0$, $m+m' = 2d$ ($d = \dim(X)$), $m-2r = m'-2r' \geq 0$. Then we have the pairing

$$K(X, m, r) \times K(X, m', r') \to \mathbb{Q}$$

obtained as (letting $t = m - 2r = m' - 2r'$)

$$\mathrm{gr}^{m-r} K_{0,\lim}(X \times E^t) \times \mathrm{gr}^{m'-r'} K_{0,\lim}(X \times E^t) \to \mathrm{gr}^{(m+m')-(r+r')} K_{0,\lim}(X \times E^t)$$
$$= \mathrm{gr}^{d+t} K_{0,\lim}(X \times E^t) \to \mathbb{Q},$$

where the last arrow is the left vertical arrow of the diagram in [9] Proposition 2.4.9 which we use by taking $X \times E^t$ and s here as X and Y there, respectively.

Conjecture 3.11. *This pairing $K(X, m, r) \times K(X, m', r') \to \mathbb{Q}$ is a perfect pairing of finite dimensional $\mathbb{Q}$-vector spaces.*

3.12. This pairing has the following property. If $a \in K(X, m, r)$, $b \in K(X, m', r')$, and if $\alpha \in H^m(X)_\ell(r)^G$ and $\beta \in H^{m'}(X)_\ell(r')^G$ are the images of a and b, respectively, the pairing $(a, b) \in \mathbb{Q}$ coincides with $\tilde{\alpha} \cup \beta$, where $\tilde{\alpha}$ is an element of $H^m(X)_\ell(m-r)$ such that $N_q^{m-2r}(\tilde{\alpha}) = \alpha$ and $\cup$ is the cup product $H^m(X)_\ell(m-r) \times H^{m'}(X)_\ell(r') \to H^{2d}(X)(m-r+r') = H^{2d}(X)_\ell(d) \to \mathbb{Q}_\ell$. Here q in N_q is the q which we used for $E = E^{(q,n)}$.

In 5.6, we consider an example of this.

4 The monodromy cycle

4.1. Assume that k is a finite field. If we use $X \times X$ as X in Conjecture 3.6 and use the Künneth decomposition and Poincaré duality on the log étale cohomology, we would obtain an isomorphism

$$\mathbb{Q}_\ell \otimes K(X \times X, 2d, d-1) \xrightarrow{\cong} \bigoplus_m \mathrm{Hom}_G(H^m(X)_\ell, H^m(X)_\ell(-1)),$$

where $d = \dim X$.

Based on this, we expect that for a section q of the log structure of s which does not belong to $k^\times$, we have a unique element of $K(X \times X, 2d, d-1)$ which induces the monodromy operators $N_q : H^m(X)_\ell \to H^m(X)_\ell(-1)$ for all m and all $\ell \neq p$ and also the analogous monodromy operator $N_q : H^m_{\mathrm{logcrys}}(X) \to H^m_{\mathrm{logcrys}}(X)(-1)$ of the log crystalline cohomology ([5]).

We call this expected element the *monodromy cycle*.

Conjecture 4.2. *A monodromy cycle exists.*

4.3. By the relation of the monodromy filtration and the theorem of Jacobson–Morozov discussed in [4] 1.6.8, the weight-monodromy conjecture 2.2 is equivalent to the following statement. Fix a lifting of the geometric Frobenius from $\mathrm{Gal}(\bar{k}/k)$ to $\pi_1^{\log}(s)$ and fix a base of $\mathbb{Q}_\ell(1)$. Then we have a unique action of the Lie algebra $\mathfrak{sl}(2)$ on $H^m(X)_\ell$ such that the monodromy operator N on $H^m(X)_\ell$ is the action of the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}(2)$, and the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathfrak{sl}(2)$ acts on the part of $H^m(X)_\ell$ of Frobenius weight w as the multiplication by $w - m$.

In the following 4.4, we give a geometric understanding of this action of $\mathfrak{sl}(2)$ by using the monodromy cycle.

In 4.13, we will try to understand the monodromy cycle and this action of $\mathfrak{sl}(2)$ by using the theory of log motives [9].

4.4. The map

$$\mathrm{gr}^{d+1} K_0^{\log}(X \times X \times E \times E) \to H^{2(d+1)}(X \times X \times E \times E)_\ell(d+1)$$

induces, by the condition (i) in 3.5, a homomorphism

$$K(X \times X, 2d, d-1) \to (H^{2d}(X \times X)_\ell(d) \otimes H^1(E)_\ell \otimes H^1(E)_\ell(1))^G$$

$$= \mathrm{Hom}_G(\mathrm{End}(H^1(E)_\ell), \textstyle\bigoplus_m \mathrm{End}(H^m(X)_\ell)),$$

and the condition (ii) in 3.5 tells that the image of this map is contained in the space of homomorphisms $\mathrm{End}(H^1(E)_\ell) \to \bigoplus_m \mathrm{End}(H^m(X)_\ell)$ which kill the scaler matrices $\mathbb{Q}_\ell$ in $\mathrm{End}(H^1(E)_\ell)$. Note that $\mathrm{End}(H^1(E)_\ell)/\mathbb{Q}_\ell$ is identified with the part $\mathfrak{sl}(H^1(E)_\ell)$ of $\mathrm{End}(H^1(E)_\ell)$ of trace 0. Thus we have

$$K(X \times X, 2d, d-1) \to \mathrm{Hom}_G(\mathfrak{sl}(H^1(E)_\ell), \textstyle\bigoplus_m \mathrm{End}(H^m(X)_\ell)).$$

If $[N] \in K(X \times X, 2d, d-1)$ is a monodromy cycle, the induced homomorphism

$$\mathfrak{sl}(H^1(E)_\ell) \to \textstyle\bigoplus_m \mathrm{End}(H^m(X)_\ell)$$

is the action of $\mathfrak{sl}(2)$ mentioned in 4.3. This homomorphism sends the monodromy operator $N_q : H^1(E)_\ell \to H^1(E)_\ell(-1)$ of E to the monodromy operators $N_q : H^m(X)_\ell \to H^m(X)_\ell(-1)$ of X. (Here this q need not be equal to the q which we used to define the log Tate curve E.)

Remark 4.5. In [10] A.17, we conjectured the existence of a monodromy cycle as an element of $\mathrm{gr}^{d-1} KH_{-2,\lim}(X \times X)$ not assuming the finiteness of k. But this element does not give an action of $\mathfrak{sl}(2)$. The action of $\mathfrak{sl}(2)$ appears only by the power of the log Tate curve.

4.6. In [9] Section 5.1, the category of log motives $LM_R(S)$ (the logarithmic version of the category of Grothendieck motives) over an fs log scheme S was defined fixing a non-empty set R of prime numbers which are invertible on S, by using the homological equivalence for ℓ-adic realizations for $\ell \in R$. We consider the case where S is our log point s and k is a finite field, and R is the set of all prime numbers which are not the characteristic of k. We denote $LM_R(s)$ by $\mathfrak{M} = \mathfrak{M}(s)$.

Objects of $\mathfrak{M}$ are direct summands of the objects $h(X)(r)$ which are associated to objects X of $\mathfrak{S}$ and integers r in $\mathbb{Z}$. For objects X and X' of $\mathfrak{S}$ and for $r, r' \in \mathbb{Z}$, we have a surjective homomorphism

$$\mathbb{Q} \otimes \mathrm{gr}^{\dim(X)+r'-r} K_0^{\log}(X \times X') \to \mathrm{Hom}_{\mathfrak{M}}(h(X)(r), h(X')(r')),$$

and this is an isomorphism if the finer ℓ-adic log Tate conjecture 3.4 is true for some $\ell \in R$.

4.7. Let

$$\mathcal{H}om : \mathfrak{M} \times \mathfrak{M} \to \mathfrak{M}, \quad \mathcal{E}nd : \mathfrak{M} \to \mathfrak{M}$$

be internal hom and the internal end, respectively. Using the dual log motive M^* of M, they are written as $(M, M') \mapsto M^* \otimes M'$ and $M \mapsto M^* \otimes M$, respectively. Let $\mathfrak{A} = \mathcal{E}nd(H^1(E)) = H^1(E)^* \otimes H^1(E)$. Then $\mathfrak{A}$ is a ring object of $\mathfrak{M}$. It is hence a Lie algebra object with the commutator Lie bracket. Let $\mathfrak{sl}(H^1(E))$ be the kernel of the canonical morphism $H^1(E)^* \otimes H^1(E) \to 1$, where 1 is the unit object of $\mathfrak{M}$. Then $\mathfrak{sl}(H^1(E))$ is a Lie algebra object of $\mathfrak{M}$.

We try to understand the monodromy cycle of a log motive $M \in \mathfrak{M}$ as a canonical homomorphism of Lie algebra objects $\mathfrak{sl}(H^1(E)) \to \mathcal{E}nd(M)$, that is, a canonical action of $\mathfrak{sl}(H^1(E))$ on M in $\mathfrak{M}$.

4.8. Consider the full subcategory $\mathfrak{M}_{\mathrm{sat}} = \mathfrak{M}_{\mathrm{sat}}(s)$ of $\mathfrak{M}$ consisting of all objects which are isomorphic to a direct summand of $h(X)(r)$ for some $X \in \mathfrak{S}$ which is saturated over s and for some $r \in \mathbb{Z}$.

Note that $X \in \mathfrak{S}$ is saturated over s if and only if X is reduced as a scheme ([19] Theorem II.4.2).

If X and Y are saturated over s, the fiber product $X \times Y$ over s is saturated. Hence in $\mathfrak{M}$, $\mathfrak{M}_{\mathrm{sat}}$ is stable under tensor products, duals, and direct summands.

If $X \in \mathfrak{S}$ is semistable, then X is saturated. Though we feel that the properties semistable and saturated are similar good properties, we use the latter to define a subcategory in the above because the property semistable is not stable under the fiber products over s.

By the construction of the category $\mathfrak{M}$ in [9], the category of classical Grothendieck motives over k with morphisms considered modulo homological equivalence with respect to ℓ-adic realizations for all $\ell \neq \mathrm{char}(k)$ is regarded as a full subcategory of $\mathfrak{M}_{\mathrm{sat}}$. If Conjecture 3.4 is true for some $\ell \in R$, this full subcategory coincides with the category of classical Grothendieck motives over k with morphisms considered modulo rational equivalence.

Proposition 4.9. *Let M be an object of $\mathfrak{M}_{\mathrm{sat}}$. Then the action of the inertia subgroup $\pi_1^{\log}(\bar{s})$ of $\pi_1^{\log}(s)$ on the ℓ-adic realization M_ℓ of M is unipotent.*

This follows from the next proposition.

Proposition 4.10. *Let $f : Y \to s$ be a morphism from an fs log scheme to the standard log point and ℓ a prime invertible on s.*

(1) *Assume that the cokernel of*

$$\mathbb{Z}_Y = f^{-1}(M_s^{\mathrm{gp}}/\mathcal{O}_s^\times) \to M_Y^{\mathrm{gp}}/\mathcal{O}_Y^\times$$

is torsion-free. Then for every $m \geq 0$ and $n \geq 0$, the action of every element g of $\pi_1^{\log}(\overline{s})$ on $H^m(Y_{\overline{s}(\log)}, \mathbb{Z}/\ell^n\mathbb{Z}) = R^m f_{\text{logét},*}(\mathbb{Z}/\ell^n\mathbb{Z})_{\overline{s}(\log)}$ satisfies $(g-1)^{m+1} = 0$.

(2) *The assumption on f in (1) is satisfied if f is saturated.*

Proof. (2) is [8] Proposition (A.4.1).

We prove (1). Let $\overset{\circ}{Y}$ be the underlying scheme of Y endowed with the étale topology. By the Leray spectral sequence

$$E_2^{i,j} = H^i_{\text{ét}}(\overset{\circ}{Y}, \Psi^j) \Rightarrow H^{i+j}_{\text{logét}}(Y_{\overline{s}(\log)}, \mathbb{Z}/\ell^n\mathbb{Z}),$$

it is enough to show that the action of $\pi_1^{\log}(\overline{s})$ on the nearby cycle

$$\Psi^j := R^j(Y_{\overline{s}(\log)} \xrightarrow{\pi} Y \xrightarrow{\varepsilon} \overset{\circ}{Y})_*(\mathbb{Z}/\ell^n\mathbb{Z}) = R^j\varepsilon_*\pi_*\mathbb{Z}/\ell^n\mathbb{Z}$$

is trivial. This is a local problem and can be checked stalkwise as follows. Let $y \in Y$. Then the argument in the proof of [14] Lemma 1.8.2, which treats the semistable case, works if the homomorphism $f\colon \pi_1^{\log}(\overline{y}) \to \pi_1^{\log}(\overline{s})$ of profinite groups has a section. For an fs log point x, $\pi_1^{\log}(\overline{x}) = \mathrm{Hom}\,(M_{\overline{x}}^{\mathrm{gp}}/\mathcal{O}_{\overline{x}}^\times, \prod_{\ell'} \mathbb{Z}_{\ell'}(1))$, where ℓ' ranges over all prime numbers which are invertible on x. Hence the existence of the section follows from the fact that the cokernel of the injection $M_{\overline{s}}^{\mathrm{gp}}/\mathcal{O}_{\overline{s}}^\times \to M_{\overline{y}}^{\mathrm{gp}}/\mathcal{O}_{\overline{y}}^\times$ is torsion-free. □

4.11. For a log Tate curve E, the log motive $H^1(E)$ belongs to $\mathfrak{M}_{\mathrm{sat}}$ because it is $H^1(E^{(q,n)})$ for q and $n \geq 2$ such that the image of q in $\mathbb{N} = M_s/\mathcal{O}_s^\times$ is a multiple of n and $E^{(q,n)}$ is saturated for such (q,n). Hence for any $r \geq 0$ and any classical Grothendieck motive C over k, $\mathrm{Sym}^r H^1(E) \otimes C$ belongs to $\mathfrak{M}_{\mathrm{sat}}$.

Now Proposition 4.9, [11] Proposition 1.10 and the log Tate conjecture 3.6 suggest the following.

Conjecture 4.12. *Let M be an object of $\mathfrak{M}_{\mathrm{sat}}$. Then M is isomorphic to*

$$\bigoplus_{r\geq 0} \mathrm{Sym}^r(H^1(E)) \otimes C_r$$

for some classical Grothendieck motives C_r over k which are 0 for almost all r.

4.13. For M as in Conjecture 4.12, the desired action of $\mathfrak{sl}(H^1(E))$ on M is given as the tensor product of the natural action on $\mathrm{Sym}^r(H^1(E))$ and the trivial action on C_r. This action is well-defined (we assume Conjecture 3.4 here). In fact, C_r is canonical because it represents the functor $D \mapsto \mathrm{Hom}\,(\mathrm{Sym}^r(H^1(E)) \otimes D, M)$ on the category of classical Grothendieck motives over k, and the morphism $\mathrm{Sym}^r(H^1(E)) \otimes C_r \to M$ comes from this property of C_r, and hence the object C_r and the isomorphism in Conjecture 4.12 are canonical.

5 Examples

Assume that k is a finite field. The next theorem proves a part of a special case of Conjecture 3.6.

Theorem 5.1. *Assume that X is a strictly semistable curve. Then the map $\mathbb{Q}_\ell \otimes K(X,1,0) \to H^1(X)_\ell^G$ is surjective.*

5.2. We prove Theorem 5.1 till 5.5. In 5.5, we use the additivity of the Albanese map, which will be seen in the next section by using theta functions.

Let I be the set of all irreducible components of X and let J be the set of all singular points of X. By using the base change to a finite extension of k, we may assume that all points in J are k-rational. We may assume that X is connected.

Define the free abelian group H^1 by the exact sequence

$$0 \to \mathbb{Z} \to \bigoplus_{i\in I} \mathbb{Z} \to \bigoplus_{j\in J} \mathbb{Z} \to H^1 \to 0.$$

This is identified with the exact sequence of cohomology groups for $* =$ zar or ét

$$0 \to H^0_*(X,\mathbb{Z}) \to H^0_*(U,\mathbb{Z}) \to \bigoplus_{j\in J} H^1_{*,j}(X,\mathbb{Z}) \to H^1_*(X,\mathbb{Z}) \to H^1_*(U,\mathbb{Z}),$$

where U is the non-singular part of X and $H^1_{*,j}(X,\cdot)$ is the cohomology with support. That is,

$$H^1 = H^1_{\text{zar}}(X,\mathbb{Z}) = H^1_{\text{ét}}(X,\mathbb{Z}).$$

Take the $\mathbb{Z}$-dual of the above exact sequence

$$0 \to H_1 \to \bigoplus_{j\in J} \mathbb{Z} \to \bigoplus_{i\in I} \mathbb{Z} \to \mathbb{Z} \to 0,$$

where H_1 is the $\mathbb{Z}$-dual of H^1.

Then the theory of the Rapoport–Zink spectral sequence ([14]) gives the following. $H^1(X)_\ell$ has an increasing filtration W such that $W_{-1} = 0$, $W_2 = H^1(X)_\ell$, $W_0 = \mathbb{Q}_\ell \otimes H^1$, $W_1/W_0 = \bigoplus_{i\in I} H^1(X_i)_\ell$, where X_i denotes the irreducible component $i \in I$, $W_2/W_1 = \mathbb{Q}_\ell(-1) \otimes H_1$, and the monodromy operator N_q for a generator q of the log structure of s coincides with the composition

$$H^1(X)_\ell \to \mathbb{Q}_\ell(-1) \otimes H_1 \to \mathbb{Q}_\ell(-1) \otimes H^1 \to H^1(X)_\ell(-1),$$

in which the middle arrow is induced by the composition of the canonical maps

$$H_1 \to \bigoplus_{j\in J} \mathbb{Z} \to H^1.$$

The map $\mathbb{Q} \otimes H_1 \to \mathbb{Q} \otimes H^1$ is an isomorphism because it is induced by the pairing

$$\mathbb{Q} \otimes H_1 \times \mathbb{Q} \otimes H_1 \to \mathbb{Q}$$

which is the restriction of the positive definite pairing

$$\mathbb{Q}^J \times \mathbb{Q}^J \to \mathbb{Q}; ((x_j)_j, (y_j)_j) \mapsto \sum_j x_j y_j$$

and hence is positive definite. That is, the weight-monodromy conjecture is true for this $H^1(X)_\ell$.

Lemma 5.3. *Let q be a section of the log structure of s which does not belong to $k^\times$. Then the image of $H^1_{\mathrm{zar}}(X,\mathbb{Z}) \to H^1_{\mathrm{zar}}(X, M_X^{\mathrm{gp}})$ induced by the homomorphism $\mathbb{Z} \to M_X^{\mathrm{gp}}$; $1 \mapsto q$ is finite.*

Proof. We use the result in Kajiwara [6]. Working on the Zariski site of X, consider the quotient sheaf $\mathcal{F} = M_X^{\mathrm{gp}}/(\mathcal{O}_X^\times \oplus q^{\mathbb{Z}})$. Then $\mathcal{F} \cong \bigoplus_{j\in J} (a_j)_*\mathbb{Z}$, where a_j is the inclusion map $j \to X$. From the exact sequence $0 \to \mathcal{O}_X^\times \oplus q^{\mathbb{Z}} \to M_X^{\mathrm{gp}} \to \mathcal{F} \to 0$, we obtain an exact sequence of cohomology

$$\textstyle\bigoplus_{j\in J} \mathbb{Z} \to \mathrm{Pic}(X) \oplus H^1 \to H^1_{\mathrm{zar}}(X, M_X^{\mathrm{gp}}) \to 0.$$

By [6] Theorem 2.19, this exact sequence induces an exact sequence of subgroups

$$0 \to H_1 \to \mathrm{Pic}(X)_0 \oplus H^1 \to H^1_{\mathrm{zar}}(X, M_X^{\mathrm{gp}})_0 \to 0.$$

Here $\mathrm{Pic}(X)_0$ is the kernel of $\mathrm{Pic}(X) \to \bigoplus_{i\in I} \mathrm{Pic}(X_i) \to \bigoplus_{i\in I} \mathbb{Z}$, where the second arrow is given by the degree maps, and $H^1_{\mathrm{zar}}(X, M_X^{\mathrm{gp}})_0$ is the kernel of a homomorphism $H^1_{\mathrm{zar}}(X, M_X^{\mathrm{gp}}) \to \mathbb{Z}$ which is called the degree map in [6]. Consider the exact sequence of sheaves $0 \to \mathcal{O}_X^\times \to \bigoplus_{i\in I} (p_i)_*(\mathcal{O}_{X_i}^\times) \to \mathcal{G} \to 0$, where p_i is the inclusion morphism $X_i \to X$, and the stalk of $\mathcal{G}$ at $x \in X$ is $k^\times$ if x is a singular point and is 0 otherwise. By the finiteness of k, we have that the kernel of $\mathrm{Pic}(X) \to \bigoplus_{i\in I} \mathrm{Pic}(X_i)$ is finite. Since the kernel of the degree map $\mathrm{Pic}(X_i) \to \mathbb{Z}$ is finite by the finiteness of k, we have that $\mathrm{Pic}(X)_0$ is finite. Hence by the above [6] Theorem 2.19, we have an exact sequence $0 \to \mathbb{Q}\otimes H_1 \to \mathbb{Q}\otimes H^1 \to \mathbb{Q}\otimes H^1_{\mathrm{zar}}(X, M_X^{\mathrm{gp}})_0 \to 0$. Since $\mathbb{Q}\otimes H_1 \to \mathbb{Q}\otimes H^1$ is an isomorphism, the map $\mathbb{Q}\otimes H^1 \to \mathbb{Q}\otimes H^1_{\mathrm{zar}}(X, M_X^{\mathrm{gp}})$ is the zero map. □

Lemma 5.4. *Consider the exact sequence $0 \to \mathbb{Z} \xrightarrow{q} \mathbb{G}_{m,\log}^{(q)} \to E^{(q)} \to 0$ on X. The connecting map*

$$\Phi : \mathbb{Q}\otimes_{\mathbb{Z}} E^{(q)}(X) \to \mathbb{Q}\otimes H^1$$

is surjective.

Proof. By Lemma 5.3, we have the surjection $\mathbb{Q}\otimes H^0(X, \mathbb{G}_{m,\log}/q^{\mathbb{Z}}) \to \mathbb{Q}\otimes H^1$. Since X is vertical, for every local section t of M_X, locally there is an $n\in\mathbb{Z}$ such that $t|q^n$ and $t^{-1}|q^n$. Hence $H^0(X, \mathbb{G}_{m,\log}/q^{\mathbb{Z}}) = H^0(X, \mathbb{G}_{m,\log}^{(q)}/q^{\mathbb{Z}}) = E^{(q)}(X)$. □

5.5. By [9] Section 3.1, for $a \in E^{(q)}(X)$, we have an element $C(a)$ of $\mathrm{gr}^1 K_{0,\lim}(X\times E)$ whose image under $\mathrm{gr}^1 K_{0,\lim}(X\times E) \to H^1(X)_\ell^G$ coincides with $\Phi(a)$. This element is obtained as follows (see the proof of Proposition 3.1.4 in [9]).

Let $E = E^{(q,n)}$ and let Y be the log modification of $E\times E$ defined as $Y = \tilde{Y}/(q^{\mathbb{Z}}\times q^{\mathbb{Z}})$, where $\tilde{Y}$ is the log modification of $\mathbb{G}_{m,\log}^{(q,n)} \times \mathbb{G}_{m,\log}^{(q,n)}$ which represents the following functor. For an fs log scheme T over s, $\tilde{Y}(T)$ is the set of all $(t_1,t_2) \in (\mathbb{G}_{m,\log}^{(q,n)} \times \mathbb{G}_{m,\log}^{(q,n)})(T)$ such that for each $r\in\mathbb{Z}$, we have either $q^r t_1 | t_2$ or $t_2 | q^r t_1$ locally on T. Then the diagonal $\Delta = E \to E\times E$ factors through a strict closed immersion $\Delta \to Y$. The ideal of $\mathcal{O}_Y$ which defines Δ is an invertible and hence is a line bundle on Y. Let Z be a log modification of $X\times E$ such that the composition $Z \to X\times E \xrightarrow{(a,\mathrm{id.})} E^{(q)}\times E$ factors as

$Z \xrightarrow{a'} Y \to E \times E \to E^{(q)} \times E$. Let $C(a) \in \mathrm{gr}^1 K_{0,\lim}(X \times E)$ be the pullback of the class of this line bundle in $\mathrm{gr}^1 K_0(Y)$ under a'.

Let $\Psi(a)$ be the image of $C(a) - C(0)$ in $\mathrm{gr}^1 K_0^{\log}(X \times E)$. Since the image of $C(a)$ in $H^1(X)_\ell^G$ is $\Phi(a)$ and the image of $C(0)$ in $H^1(X)_\ell^G$ is $\Phi(0) = 0$, we have:

(1) The following diagram is commutative.

$$\begin{array}{ccc} E^{(q)}(X) & \xrightarrow{\Psi} & \mathrm{gr}^1 K_0^{\log}(X \times E) \\ {\scriptstyle\Phi}\downarrow & & \downarrow \\ H^1 & \to & H^1(X)_\ell^G \end{array}$$

We will prove the following (2) in 6.10 below.

(2) The map $\Psi : E^{(q)}(X) \to \mathrm{gr}^1 K_0^{\log}(X \times E)$ is a homomorphism.

(Though this (2) can be deduced from the theory of the log Jacobian variety of X and its self-duality described in [9] Section 6.2, we give a proof of (2) in Section 6 below using the theta function of a log Tate curve over our log point s because we think the log Tate curve is important in this paper and its theta function may have a value and because the full details of [9] Section 6.2 are not yet published.) By (2), we have

(3) The image of Ψ in $\mathbb{Q} \otimes \mathrm{gr}^1 K_0^{\log}(X \times E)$ is contained in $K(X, 1, 0)$.

By Lemma 5.4, the above (1) and (3), and the surjectivity of $\mathbb{Q}_\ell \otimes H^1 \to H^1(X)_\ell^G$, we have Theorem 5.1.

5.6. Let X be as in Theorem 5.1 and assume that all singular points are k-rational. Take the Tate elliptic curve $E^{(q,n)}$, where q is a generator of the log structure of s. Then we have a commutative diagram

$$\begin{array}{ccc} K(X,1,0) \times K(X,1,0) & \to & \mathbb{Q} \\ \downarrow & & \| \\ \mathbb{Q} \otimes H^1 \times \mathbb{Q} \otimes H^1 & \to & \mathbb{Q}. \end{array}$$

Here the upper arrow is as in 3.10 and the lower arrow is $(\mathbb{Q} \otimes H^1) \otimes (\mathbb{Q} \otimes H^1) \cong (\mathbb{Q} \otimes H_1) \otimes (\mathbb{Q} \otimes H^1) \to \mathbb{Q}$ in which the first arrow is by the isomorphism $\mathbb{Q} \otimes H_1 \xrightarrow{\cong} \mathbb{Q} \otimes H^1$ in 5.2.

The next theorem is about the existence of a monodromy cycle in a special case.

Theorem 5.7. *Assume that X is a strictly semistable curve.*

(1) *The map $\mathbb{Q}_\ell \otimes K(X \times X, 2, 0) \to H^2(X \times X)_\ell^G = \bigoplus_m \mathrm{Hom}_G(H^m(X)_\ell, H^m(X)_\ell(-1))$ is surjective.*

(2) *There is an element of $K(X \times X, 2, 0)$ which induces a homomorphism*

$$\mathfrak{sl}(H^1(E)) \to \mathcal{E}nd(h(X))$$

of Lie algebra objects in the category $\mathfrak{M}$, which sends the monodromy operator $N_q : H^1(E)_\ell \to H^1(E)_\ell(-1)$ to the monodromy operator $N_q : H^m(X)_\ell \to H^m(X)_\ell(-1)$ for every m and every ℓ.

Proof. We may assume that all singular points of X are k-rational.

(1) is proved as follows. By 5.2, $H^2(X \times X)^G_\ell = H^1(X)^G_\ell \otimes H^1(X)^G_\ell$ and it is generated by the image of $H^1 \otimes H^1$. By Theorem 5.1, it is generated by the image of $K(X, 1, 0) \times K(X, 1, 0) \to K(X \times X, 2, 0) \to H^2(X \times X)^G_\ell$.

(2) is proved as follows. The surjective homomorphism $\mathbb{Q} \otimes E^{(q)}(X) \to \mathbb{Q} \otimes H^1$ and the homomorphism $\Psi : E^{(q)}(X) \to \mathrm{gr}^1 K_0^{\log}(X \times E)$ give a homomorphism $H^1(E) \otimes H^1 \to H^1(X)$ of log motives in $\mathfrak{M}$, where $H^1(E) \otimes H^1$ means the log motive $H^1(E)^{\oplus r}$ if we fix a $\mathbb{Z}$-base $(e_i)_{1 \le i \le r}$ of H^1. By taking the dual log motives ([9] 3.2.8), we obtain a homomorphism $H^1(X) \to H^1(E) \otimes H_1$ of log motives, and the composition $H^1(E) \otimes H^1 \to H^1(X) \to H^1(E) \otimes H_1 \to H^1(E) \otimes H^1$ is the identity morphism. Hence $H^1(E) \otimes H^1$ is regarded as a direct summand of the log motive $h(X)$. We define the Lie action of $\mathfrak{sl}(H^1(E))$ on $h(X)$ by using the evident action on $H^1(E)$ on this direct summand, and using the trivial action on the other direct summand. This gives the monodromy operator as stated in (2).

This Lie action is induced by an element of $K(X \times X, 2, 0)$ obtained as follows. The pairing $(\mathbb{Q} \otimes H^1) \times (\mathbb{Q} \otimes H^1) \to \mathbb{Q}$ in 5.6 is perfect. Take elements $a_1, \dots, a_n$ of $K(X, 1, 0)$ such that if α_i denotes the image of a_i in $\mathbb{Q} \otimes H^1$, then $(\alpha_i)_i$ is a base of $\mathbb{Q} \otimes H^1$. Let $b_1, \dots, b_n$ be elements of $K(X, 1, 0)$ such that if β_i denotes the image of b_i in $\mathbb{Q} \otimes H^1$, then $(\beta_i)_i$ is the dual base of $(\alpha_i)_i$ for the pairing $(\mathbb{Q} \otimes H^1) \times (\mathbb{Q} \otimes H^1) \to \mathbb{Q}$. The desired element of $K(X \times X, 2, 0)$ is obtained as $\sum_{i=1}^n a_i \cdot b_i$, where $\cdot$ is the pairing $K(X, 1, 0) \times K(X, 1, 0) \to K(X \times X, 2, 0)$. □

5.8. In this paper, if we drop the assumption that X is projective, many points stop working.

For example, consider a reduction X of the rigid analytic Hopf surface $S := (\mathbb{A}^2 \smallsetminus \{(0, 0)\})^{\mathrm{an}} / q^{\mathbb{Z}}$ over a p-adic local field K, where q is a non-zero element of the maximal ideal m_K. Though S is not an algebraic variety, we have a reduction X of S which is a proper vertical log smooth scheme over s, and which is not projective. This X represents the functor $T \mapsto \Gamma(T, (F \times E^{(q,n)})/\mathbb{G}_m)$ for the diagonal action of $\mathbb{G}_m$, where F is the subsheaf of $\mathcal{O} \times \mathcal{O}$ consisting of those (a, b) such that either a or b belongs to $\mathcal{O}^\times$, and q is regarded as a section of the log structure of s. Note that $S = ((\mathbb{A}^2 \smallsetminus \{(0, 0)\})^{\mathrm{an}} \times \mathbb{G}_m^{\mathrm{an}}/q^{\mathbb{Z}})/\mathbb{G}_m^{\mathrm{an}}$. We have the projection $X \to \mathbb{P}^1_k = F/\mathbb{G}_m$ whose all fibers are $E^{(q,n)}$.

We have that $H^3(X)_\ell(2)^G = H^3(X)_\ell(2) = \mathbb{Q}_\ell$ is non-zero, so Conjecture 1.4 is not true for this X.

We have $H^1(X)^G_\ell = H^1(X)_\ell = \mathbb{Q}_\ell$ but it does not come from a G-homomorphism $H^1(E)_\ell \to H^1(X)_\ell$. So Conjecture 1.12 is not true for this X.

6 Theta functions

Here we describe the theory of theta functions on a log Tate curve over a standard log point. This is the reduction to k of the theory of theta functions on the Tate curve over a p-adic local field K described, for example, in [16]. The latter theory is the p-adic analogue of the classical theory over $\mathbb{C}$ reviewed in 6.1 below.

This is used in 5.5 in the previous section.

6.1. Let $q \in \mathbb{C}^\times$, $|q| < 1$, and consider the elliptic curve $E = \mathbb{C}^\times/q^{\mathbb{Z}}$ over $\mathbb{C}$. We have the theta functions θ and θ_a for $a \in \mathbb{C}^\times$, which are meromorphic functions on $\mathbb{C}^\times$, defined as follows:

$$\theta(t) = \prod_{n=0}^{\infty} (1 - q^n t) \cdot \prod_{n=1}^{\infty} (1 - q^n t^{-1}), \qquad \theta_a(t) = \theta(t/a).$$

Hence $\theta = \theta_1$. We have

(1) $\theta(qt) = -t^{-1}\theta(t), \quad \theta_a(qt) = -at^{-1}\theta_a(t)$.
(2) The divisor of θ_a on $\mathbb{C}^\times$ is the pullback of the divisor (a) on E.

From (1), we have:

(3) For $a, b \in \mathbb{C}^\times$, the function $f(t) = \theta_{ab}(t)\theta_1(t)\theta_a(t)^{-1}\theta_b(t)^{-1}$ satisfies $f(qt) = f(t)$ and hence is a meromorphic function on $E = \mathbb{C}^\times/q^{\mathbb{Z}}$.

By (2), the divisor of f is the divisor $(ab) + (1) - (a) - (b)$ on E. This gives a proof of the fact that the divisor $(ab) + (1) - (a) - (b)$ on E is a principal divisor.

6.2. We now go to the log geometry.

In general, for an fs log scheme S, let $\mathcal{R}_S$ be the sheaf of rings over $\mathcal{O}_S$ obtained from $\mathcal{O}_S$ by inverting all non-zero-divisors in $\mathcal{O}_S$. Let $\mathcal{Q}_S$ be the log structure on $\mathcal{R}_S$ associated to the pre-log structure $M_S \to \mathcal{R}_S$.

The group sheaf $\mathcal{Q}_S^{\mathrm{gp}}$ is generated by the subgroup sheaves $\mathcal{R}_S^\times$ and M_S^{gp}. We regard $\mathcal{Q}_S^{\mathrm{gp}}/M_S^{\mathrm{gp}}$ as the log version of the sheaf $\mathcal{R}_S^\times/\mathcal{O}_S^\times$ of Cartier divisors. An element D of $H^0(S, \mathcal{Q}_S^{\mathrm{gp}}/M_S^{\mathrm{gp}})$ gives an M_S^{gp}-torsor on S defined to be the inverse image of D in $\mathcal{Q}_S^{\mathrm{gp}}$. It is like the $\mathcal{O}_S^\times$-torsor associated to an element of $H^0(S, \mathcal{R}_S^\times/\mathcal{O}_S^\times)$. We have the commutative diagram

$$\begin{array}{ccc} H^0(S, \mathcal{R}_S^\times/\mathcal{O}_S^\times) & \to & H^1(S, \mathcal{O}_S^\times) \\ \downarrow & & \downarrow \\ H^0(S, \mathcal{Q}_S^{\mathrm{gp}}/M_S^{\mathrm{gp}}) & \to & H^1(S, M_S^{\mathrm{gp}}). \end{array}$$

6.3. Now we work over a standard log point s.

Let

$$E = E^{(q,n)}, \quad \tilde{E} = \mathbb{G}_{m,\log}^{(q,n)} \quad (1.9) \quad \text{with } n \geq 2,$$

so $E = \tilde{E}/q^{\mathbb{Z}}$.

Then there is a unique section $\theta = \theta(t)$ ($t \in \tilde{E} \subset \mathbb{G}_{m,\log}^{(q)}$) of $\mathcal{Q}_{\tilde{E}}^{\mathrm{gp}}$ satisfying the following conditions (i) and (ii).

(i) $\theta(qt) = -t^{-1}\theta(t)$.

(ii) On the open part $\{t \in \mathbb{G}_{m,\log} \mid 1|t|q\}$ of $\tilde{E}$, $\theta = (1-\alpha(t))(1-\alpha(qt^{-1})) \in \mathcal{R}_{\tilde{E}}^\times$, where α is the structural map $M \to \mathcal{O}$ of the log structure.

The restriction of $\theta(t)$ to $\mathbb{G}_m \subset \tilde{E}$ is $1 - t$.

6.4. Let X be an object of $\mathfrak{S}$ and let $a \in E^{(q)}(X)$. We will define the theta function θ_a as a section of $\mathcal{Q}^{\mathrm{gp}}$ of a certain covering $\tilde{E}_{X,a}$ of a log modification $E_{X,a}$ of $X \times E$.

First we consider the case $X = E$ and a is the identity map $\delta : E \to E$. In this case, $E_{E,\delta} = Y$ and $\tilde{E}_{E,\delta} = \tilde{Y}$ in 5.5. The restriction of θ_δ to $\mathbb{G}_m \times \mathbb{G}_m \subset \tilde{Y}$ is to be $1 - t_1^{-1}t_2$.

We define θ_δ as the unique section of $\mathcal{Q}^{\mathrm{gp}}$ of $\tilde{Y}$ characterized by the following properties (i) and (ii).

(i) $\theta_\delta(qt_1, qt_2) = \theta_\delta(t_1, t_2)$ and $\theta_\delta(t_1, qt_2) = -t_1t_2^{-1}\theta_\delta(t_1, t_2)$.

(ii) On the part $\{(t_1, t_2) \mid 1|t_1|q,\ 1|t_2|q\}$ of $\tilde{Y}$, $\theta_\delta(t_1, t_2) = (1-\alpha(t_1^{-1}t_2))(1-\alpha(qt_2^{-1}t_1)) \in \mathcal{R}^\times$ if $t_1|t_2$, and $\theta_\delta(t_1, t_2) = -t_1^{-1}t_2(1-\alpha(t_2^{-1}t_1))(1-\alpha(qt_2^{-1}t_1)) \in \mathcal{Q}^{\mathrm{gp}}$ if $t_2|t_1$.

The restriction of θ_δ to $\{1\} \times \tilde{E}$ is θ.

Lemma 6.5. *By the canonical map $\mathcal{R}^\times/\mathcal{O}^\times \to \mathcal{Q}^{\mathrm{gp}}/M^{\mathrm{gp}}$ on $\tilde{Y}$, the Cartier divisor of the diagonal E in Y is sent to the class of θ_δ.*

Proof. This is clear by the explicit form of θ_δ. □

6.6. Let X be an object of $\mathfrak{S}$ and let $a \in E^{(q)}(X)$. We define a log modification $E_{X,a}$ of $X \times E$ to be the fiber product of $X \times E \overset{(a,1)}{\to} E^{(q)} \times E \leftarrow Y$. We define $\tilde{E}_{X,a}$ to be the fiber product of $E_{X,a} \to Y \leftarrow \tilde{Y}$. Let θ_a be the pullback of θ_δ under $\tilde{E}_{X,a} \to \tilde{Y}$.

Note that $q^{\mathbb{Z}} \times q^{\mathbb{Z}}$ acts on $\tilde{E}_{X,a}$. The action of $(q, q) \in q^{\mathbb{Z}} \times q^{\mathbb{Z}}$ does not change θ_a and the pullback by the action of $(1, q)$ changes θ_a to $-\tilde{a}t^{-1}\theta_a$, where $\tilde{a}$ (resp. t) is the composition of $\tilde{E}_{X,a} \to \tilde{Y}$ and the first (resp. second) projection $\tilde{Y} \to \mathbb{G}_{m,\log}$.

6.7. Since the class of θ_a in $\mathcal{Q}^{\mathrm{gp}}/M^{\mathrm{gp}}$ is invariant under the action of $q^{\mathbb{Z}} \times q^{\mathbb{Z}}$, the M^{gp}-torsor associated to this class descends to an M^{gp}-torsor on $E_{X,a}$. We denote it by $\mathcal{T}(a)$.

Proposition 6.8. *Let $a_i \in E^{(q)}(X)$, $n(i) \in \mathbb{Z}$ for $1 \le i \le r$, and assume that $\sum_{i=1}^r n(i) = 0$ and $\prod_{i=1}^r a_i^{n(i)} = 1$ in the group $E^{(q)}(X)$. Let $E_{X,a_1,\dots,a_r}$ be the fiber product of E_{X,a_i} ($1 \le i \le r$) over $X \times E$ which is a log modification of $X \times E$.*

Then on $E_{X,a_1,\dots,a_r}$, $\prod_{i=1}^r \mathcal{T}(a_i)^{n(i)}$ is a trivial torsor.

Proof. For $r \ge 1$, let $Y^{(r)}$ (resp. $\tilde{Y}^{(r)}$) be the log modification of E^{r+1} (resp. $\tilde{E}^{r+1}$) defined by the condition on $(t_1, \dots, t_{r+1})$ that (t_i, t_{r+1}) belongs to Y (resp. $\tilde{Y}$) for $1 \le i \le r$. Hence $Y^{(1)} = Y$ (resp. $\tilde{Y}^{(1)} = \tilde{Y}$). Then $E_{X,a_1,\dots,a_r}$ is the fiber product of $X \times E \to (E^{(q)})^r \times E \leftarrow Y^{(r)}$, where the first arrow is $(x, t) \mapsto (a_1(x), \dots, a_r(x), t)$. Let $\tilde{E}_{X,a_1,\dots,a_r}$ be the fiber product of $E_{X,a_1,\dots,a_r} \to Y^{(r)} \leftarrow \tilde{Y}^{(r)}$. We pullback θ_{a_i} ($1 \le i \le r$) to $\tilde{E}_{X,a_1,\dots,a_r}$. Consider $f = \prod_{i=1}^r \theta_{a_i}^{n(i)}$ there. For the action of $(q^{\mathbb{Z}})^{r+1}$, f does not change under the action of the element γ_i ($1 \le i \le r$) whose i-th component and $r+1$-component are q and whose other components are 1. For the pullback by the action of γ whose $r+1$-th component is q and whose other components are 1, $\gamma^*(f)f^{-1} \in \mathbb{G}_{m,\log}^{(q)}$ and its image in $E^{(q)}$ is trivial by the assumption on a_i. Hence $\gamma^*(f)f^{-1}$ is a locally constant function on $\tilde{E}_{X,a_1,\dots,a_r}$ with values in $q^{\mathbb{Z}}$, which we denote by c. Let g be the section $t^{-c}\prod_{i=1}^r \theta_{a(i)}^{n(i)}$ of $\mathcal{Q}^{\mathrm{gp}}$ of $\tilde{E}_{X,a_1,\dots,a_r}$. Then g is invariant under the action of $(q^{\mathbb{Z}})^{r+1}$ and hence is a section of $\mathcal{Q}^{\mathrm{gp}}$ of $E_{X,a_1,\dots,a_r}$. This gives a section of the torsor $\prod_{i=1}^r \mathcal{T}(a_i)^{n(i)}$. □

Lemma 6.9. *Let Y be an object of $\mathfrak{S}$ and let $a, b \in \mathrm{Pic}(Y)$. If the images of a and b in $H^1(Y, M_Y^{\mathrm{gp}})$ coincide, their images in $\mathrm{gr}^1 K_0^{\log}(Y)$ coincide.*

Proof. This follows from the exact sequence $H^0(Y, M_Y^{\mathrm{gp}}/\mathcal{O}_Y^\times) \to H^1(Y, \mathcal{O}_Y^\times) \to H^1(Y, M_Y^{\mathrm{gp}})$. □

6.10. Now (2) in 5.5 follows from Lemma 6.5, Proposition 6.8, and Lemma 6.9.

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Kazuya KATO
Department of Mathematics
University of Chicago
Chicago, Illinois, 60637, USA
kkato@uchicago.edu

Chikara NAKAYAMA
Department of Economics
Hitotsubashi University
2-1 Naka, Kunitachi, Tokyo 186-8601, Japan
c.nakayama@r.hit-u.ac.jp

Sampei USUI$^\sharp$
Graduate School of Science
Osaka University

$\sharp$ the deceased