NONEXISTENCE OF PHANTOM CATEGORIES ON VERY GENERAL NONCOMMUTATIVE PROJECTIVE PLANES

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Abstract. We show that very general noncommutative projective planes do not admit phantom categories.

Contents

1. Introduction

Let X be a smooth projective variety. A full triangulated subcategory $\mathcal{A} \subset \mathbf{D}^{\mathrm{b}} \operatorname{coh}(X)$ is called *admissible* if the inclusion admits both left and right adjoints. Such categories are also called *geometric noncommutative schemes* [22]. A nontrivial geometric noncommutative scheme is called a *quasi-phantom category* if it has a finite Grothendieck group and trivial Hochschild homology, and it is called a *phantom category* if, in addition, it has trivial Grothendieck group.

In [19], Kuznetsov conjectured that quasi-phantom categories do not exist, and pointed out that this nonexistence could be used to prove the termination of semi-orthogonal decompositions. However, this conjecture was disproved by the explicit construction of quasi-phantom categories on classical Godeaux surfaces [8] and Burniat surfaces [3]. Subsequently, phantom categories were constructed on the products of surfaces with a quasi-phantom category [12] and on determinantal Barlow surfaces [9]. These surfaces on which the phantom category was constructed are all of the non-negative Kodaira dimension. In particular, they are non-rational. Therefore, it is interesting to consider whether phantom categories exist on rational surfaces.

Among others, there was a folklore conjecture that phantom categories do not exist on del Pezzo surfaces, especially on $\mathbb{P}^2$. A del Pezzo surface is either a blow-up of $\mathbb{P}^2$ at most eight points in general position or $\mathbb{P}^1 \times \mathbb{P}^1$. This conjecture was solved in the affirmative

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by Pirozhkov [23]. Borisov and Kemboi extend this result on del Pezzo surfaces to a bigger class of rational surfaces:

Theorem 1.1 (=[10, Theorem 3.1]). *Let X be the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ at a finite set of points on a smooth cubic curve E such that the restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(E)$ is injective. Then $\mathbf{D}^{\mathrm{b}} \operatorname{coh}(X)$ admits no phantom categories.*

The assumption of Theorem 1.1 is satisfied if the points on E are in very general position. However, the assumption that all points lie on a cubic curve is a nontrivial closed condition as soon as the number of points is more than nine. In this sense, the points as in Theorem 1.1 are in special position.

In contrast, Krah [18] constructed phantom categories on blow-ups of $\mathbb{P}^2$ at ten points in general position:

Theorem 1.2 (=[18, Theorem 1.1]). *Let X be the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ at 10 closed points $p_1, \dots, p_{10} \in \mathbb{P}^2_{\mathbb{C}}$ in general position. Then there exists an exceptional collection $(\mathcal{L}_1, \dots, \mathcal{L}_{13})$ consisting of line bundles such that its right orthogonal complement $\langle \mathcal{L}_1, \dots, \mathcal{L}_{13} \rangle^{\perp}$ is a phantom category.*

Thus the existence of phantom categories is a subtle problem even for rational surfaces. Moreover, Theorem 1.1 and Theorem 1.2 mean that a surface without phantom categories may deform to surfaces with phantom categories.

In this paper, we initiate the study of phantom categories in the broader context of noncommutative algebraic geometry. More specifically, we study flat deformations of $\operatorname{coh}(\mathbb{P}^2)$ as abelian categories [20]. According to [27, Theorem 7.2], they can be realized as *noncommutative projective planes*, defined as the quotient category

$$\operatorname{qgr}(A) := \operatorname{grmod}(A) / \operatorname{tors}(A),$$

where A is a three-dimensional Artin-Schelter (AS-)regular quadratic (graded) algebra, $\operatorname{grmod}(A)$ denotes the category of finitely generated graded A-modules, and $\operatorname{tors}(A)$ denotes the Serre subcategory of finite-dimensional modules. It is known that $\mathbf{D}^{\mathrm{b}} \operatorname{qgr}(A)$ is a geometric noncommutative scheme [22, Theorem 5.8].

Three-dimensional quadratic AS-regular algebras are classified by triples $(E, \sigma, \mathcal{L})$, where $E \subset \mathbb{P}^2$ is a cubic divisor, $\sigma \in \mathrm{Aut}(E)$ is an automorphism, and $\mathcal{L} \in \mathrm{Pic}(E)$ is a very ample invertible sheaf. The cubic divisor E is considered to be embedded in the noncommutative projective plane via a pair of adjoint functors (see (2.9)).

$$\mathbb{L}j^* \colon \mathbf{D}^{\mathrm{b}} \operatorname{qgr}(A) \rightleftarrows \mathbf{D}^{\mathrm{b}} \operatorname{coh}(E) \colon j_*$$

While $\mathbf{D}^{\mathrm{b}} \operatorname{coh}(\mathbb{P}^2)$ has no phantom categories, this result does not immediately imply the nonexistence of phantoms on noncommutative projective planes as we pointed out above. Nevertheless, using a noncommutative analogue of the methods of Pirozhkov [23] and Borisov-Kemboi [10], we obtain the following result.

Theorem 1.3 (=Theorem 3.11, MAIN RESULT). *Let A be a three-dimensional AS-regular quadratic algebra associated to a geometric triple $(E, \sigma, \mathcal{L})$. Assume that E is nonsingular and σ is a translation of infinite order. Then $\mathbf{D}^{\mathrm{b}} \operatorname{qgr}(A)$ admits no phantom categories.*

This theorem shows that very general noncommutative deformations of the projective plane do not admit phantom categories. The assumptions of Theorem 1.3 are rather technical, and we expect:

Conjecture 1.4. *Every noncommutative projective plane admits no phantom categories.*

Our methods do not directly extend to arbitrary noncommutative projective planes. One reason for this limitation is that some results from [4] cannot be applied to arbitrary noncommutative projective planes. Consequently, Proposition 1.6 and Proposition 1.7 (see below) cannot be extended to arbitrary noncommutative projective planes.

Borisov and Kemboi conjectured that any smooth projective surface which admits an effective (anti-)canonical divisor does not admit phantom categories [10, Conjecture 1.3]. This conjecture naturally extends to the noncommutative setting, especially to blow-ups at points on the anti-canonical divisor E of noncommutative projective planes [25]. Namely, we expect:

Conjecture 1.5. *Any blow-up at points on the anti-canonical divisor E of a noncommutative projective plane admits no phantom categories.*

1.1. **Outline.** In Section 2.1 we recall fundamental definitions and results on admissible subcategories and phantom categories. In Section 2.2 we recall the definition and results about AS-regular algebras. In Section 2.3 we recall the notion of spherical functor and results. The main result of this section is Corollary 2.33, whose proof is deferred to Section A. Section 3.1 and Section 3.2 develop key technical tools required for our main theorem. The following proposition proved in Section 3.1 roughly corresponds to Step 3 of the proof of Theorem 1.1.

Proposition 1.6 (=Proposition 3.2)**.** *Assume that the order of the automorphism σ associated with the three-dimensional AS-regular algebra A is infinite. Then the set $\{j_*\mathcal{O}_p \mid p \in E\}$ is a spanning class of $\mathbf{D}^{\mathrm{b}}\operatorname{qgr}(A)$.*

The proof relies on Artin-Tate-Van den Bergh's results on finitely generated graded modules over AS-regular algebras [4, Propositions 7.5, 7.9]. The infinite order condition on σ is necessary for applying these results.

The following proposition proved in Section 3.2 roughly corresponds to Step 1 of the proof of Theorem 1.1.

Proposition 1.7 (=Proposition 3.10)**.** *If the automorphism σ has infinite order, then for a very general point $p \in E$, there are no objects $M \in \mathbf{D}^{\mathrm{b}}\operatorname{qgr}(A)$ with $\operatorname{Supp}(\mathbb{L}j^*M) = \{p\}$.*

Based on these propositions, we prove the main theorem in Section 3.3.

Notation and Conventions. Throughout the paper, $\mathbf{k}$ is a base field, assumed to be algebraically closed unless specified otherwise. All modules are right modules. The notation M_A is used to specify that M is a right module over a ring A.

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2. Preliminaries

2.1. **Admissible subcategories and phantom categories.** In this section we review the basic definition (for details, see [16]). Let $\mathcal{T}$ be a $\mathbf{k}$-linear triangulated category, where $\mathbf{k}$ is a base field. For $X, Y \in \mathcal{T}$, we define a graded $\mathbf{k}$-vector space

$$\mathbb{R}\mathrm{Hom}_{\mathcal{T}}(X, Y) \coloneqq \bigoplus_{\ell \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(X, Y[\ell]).$$

A triangulated category $\mathcal{T}$ is *of finite type over* $\mathbf{k}$ if $\dim_{\mathbf{k}} \mathbb{R}\mathrm{Hom}_{\mathcal{T}}(X, Y) < \infty$ for all $X, Y \in \mathcal{T}$. We assume that any triangulated category is of finite type over a base field $\mathbf{k}$ throughout this paper.

Definition 2.1. A full triangulated subcategory $\mathcal{A} \subset \mathcal{T}$ is called *admissible* if the inclusion functor admits a left and right adjoint functor.

Definition 2.2. A *semi-orthogonal decomposition* of $\mathcal{T}$ is an ordered pair of admissible subcategories $(\mathcal{A}, \mathcal{B})$ of $\mathcal{T}$ which satisfies the following conditions:

(1) $\mathbb{R}\mathrm{Hom}(b, a) = 0$ for any $a \in \mathcal{A}$ and $b \in \mathcal{B}$.
(2) $\mathcal{T}$ is the smallest triangulated category which contains $\mathcal{A}$ and $\mathcal{B}$.

If $(\mathcal{A}, \mathcal{B})$ is a semi-orthogonal decomposition of $\mathcal{T}$, we write $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$.

The conditions (1) and (2) imply that for any $x \in \mathcal{T}$, there exists a unique exact triangle up to isomorphism

$$b \to x \to a \to b[1]$$

where $a \in \mathcal{A}$ and $b \in \mathcal{B}$. The correspondences $x \mapsto a$ and $x \mapsto b$ induce functors $\mathcal{T} \to \mathcal{A}$ and $\mathcal{T} \to \mathcal{B}$, respectively. These are called the *left projection onto* $\mathcal{A}$ and the *right projection onto* $\mathcal{B}$.

Definition 2.3. An object $E \in \mathcal{T}$ is called *exceptional* if $\mathbb{R}\mathrm{Hom}_{\mathcal{T}}(E, E) = \mathbf{k}[0]$, i.e., $\mathrm{Hom}_{\mathcal{T}}(E, E[\ell]) = 0$ when $\ell \neq 0$ and $\mathrm{Hom}_{\mathcal{T}}(E, E) = \mathbf{k}$. An *exceptional collection* in $\mathcal{T}$ is a sequence of exceptional objects $(E_1, \dots, E_n)$ which satisfies $\mathbb{R}\mathrm{Hom}_{\mathcal{T}}(E_i, E_j) = 0$ for all $i > j$.

Definition 2.4. An exceptional collection $(E_1, \dots, E_n)$ is called *strong* if in addition, $\mathrm{Hom}_{\mathcal{T}}(E_i, E_j[k]) = 0$ for all $i < j$ and $k \neq 0$.

Definition 2.5. The *Grothendieck group* of $\mathcal{T}$, denoted $\mathsf{K}_0(\mathcal{T})$, is the abelian group generated by the isomorphism classes of $\mathcal{T}$, with the relations of the form $[X] - [Y] + [Z]$ for all exact triangle $X \to Y \to Z \to X[1]$.

By definition of the Grothendieck group, any exact functor $F\colon \mathcal{T}_1 \to \mathcal{T}_2$ induces a group homomorphism $\mathsf{K}_0(\mathcal{T}_1) \to \mathsf{K}_0(\mathcal{T}_2)$.

If we have a semi-orthogonal decomposition $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$, then the inclusion functors induce an isomorphism

$$\mathsf{K}_0(\mathcal{T}) \simeq \mathsf{K}_0(\mathcal{A}) \oplus \mathsf{K}_0(\mathcal{B}).$$

Definition 2.6. Let X and Y be smooth projective varieties and let $E \in \mathbf{D}^{\mathrm{b}} \operatorname{coh}(X \times Y)$. The *integral functor* $\Phi_E\colon \mathbf{D}^{\mathrm{b}} \operatorname{coh}(X) \to \mathbf{D}^{\mathrm{b}} \operatorname{coh}(Y)$ is defined by

$$\Phi_E(F) = p_*(q^*(F) \overset{\mathbb{L}}{\otimes} E)$$

where $q\colon X \times Y \to X$ and $p\colon X \times Y \to Y$ are projections. The object E is called the *kernel of the integral functor* Φ_E.

Definition 2.7 ([22])**.** A *geometric noncommutative scheme* is an admissible subcategory of $\mathbf{D}^{\mathrm{b}} \operatorname{coh}(X)$ where X is a smooth projective variety.

By definition of geometric noncommutative schemes, any admissible subcategory of a geometric noncommutative scheme is also a geometric noncommutative scheme.

Example 2.8. A triangulated category with a full exceptional collection is a geometric noncommutative scheme by [22, Theorem 5.8]. By Theorem 2.21, every noncommutative projective plane is a geometric noncommutative scheme.

Definition 2.9. A nontrivial geometric noncommutative scheme $\mathcal{A}$ is called a *phantom category* if $\mathsf{K}_0(\mathcal{A}) = 0$.

Remark 2.10. In this paper, we adopt a slightly different definition of phantom category. It is known that if the base field $\mathbf{k}$ is algebraically closed and has infinite transcendence degree over its prime field, this definition coincides with the common one, as shown in [12, Theorem 5.5].

2.2. **Noncommutative projective planes.**

2.2.1. *Three dimensional Artin-Schelter regular quadratic algebras.* Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a $\mathbb{Z}$-graded algebra. We say that A is *positively graded* if $A_{<0} = 0$. All $\mathbb{Z}$-graded algebras are assumed to be over $\mathbf{k}$ and positively graded unless otherwise stated. Let $\mathrm{Grmod}(A)$ denote the category of graded right modules over A. The *truncation* is an endofunctor of $\mathrm{Grmod}(A)$ defined by

$$\mathrm{Grmod}(A) \to \mathrm{Grmod}(A),\ M \mapsto M_{\geq m} := \bigoplus_{k\geq m} M_k \tag{2.1}$$

for $m \in \mathbb{Z}$, similarly, the graded module $M_{>m}$ is also defined. The n-th *shift* of gradings for $n \in \mathbb{Z}$ is an autoequivalence of $\mathrm{Grmod}(A)$ defined by

$$\mathrm{Grmod}(A) \to \mathrm{Grmod}(A),\ M \mapsto M(n). \tag{2.2}$$

The graded module $M(n)$ is defined by $M(n)_k = M_{n+k}$. For a positively graded algebra A, the ideal $A_{>0}$ is two-sided, and there exists a canonical isomorphism of graded bimodules $A_0 \simeq A/A_{>0}$.

Definition 2.11. Let A be a positively $\mathbb{Z}$-graded algebra. We say A is *d-dimensional Artin-Schelter (AS-)regular* if the following conditions are satisfied.

1. A is connected, i.e., $A_0 = \mathbf{k}$,
2. $\mathrm{gl.dim}\, A = d$,
3. $\dim A_n$ is bounded by a polynomial in n,
4. (Gorenstein condition) $\mathrm{Ext}^i_A(\mathbf{k}, A) \simeq \begin{cases} \mathbf{k} & \text{if } i = d, \\ 0 & \text{if } i \neq d. \end{cases}$

Any three-dimensional AS-regular algebra generated in degree one is either a quadratic or a cubic algebra ([7, Theorem 1.5]). In this paper, we focus on three-dimensional quadratic AS-regular algebras defined as follows.

Definition 2.12. Let A be an AS-regular algebra. We say that A is a *three-dimensional quadratic (AS-)regular algebra* if the minimal resolution of $\mathbf{k}_A$ in $\mathrm{Grmod}(A)$ has the form

$$0 \to A(-3) \to A(-2)^{\oplus 3} \to A(-1)^{\oplus 3} \to A \to \mathbf{k}_A \to 0. \tag{2.3}$$

Let $\mathrm{grmod}(A)$ be an abelian category of finitely generated graded A-modules, and let $\mathrm{tors}(A)$ be a Serre subcategory of $\mathrm{grmod}(A)$ whose objects are finite dimensional modules. The Serre quotient of $\mathrm{grmod}(A)$ by $\mathrm{tors}(A)$ will be denoted by

$$\pi\colon\ \mathrm{grmod}(A) \to \mathrm{qgr}(A) := \mathrm{grmod}(A)/\mathrm{tors}(A). \tag{2.4}$$

The functor π is exact. The truncation (2.1) and the shift of gradings (2.2) send any object in $\mathrm{tors}(A)$ to itself. Hence these endofunctors of $\mathrm{grmod}(A)$ induce the endofunctors of $\mathrm{qgr}(A)$. We use the same notation for these endofunctors of $\mathrm{qgr}(A)$.

2.2.2. *Geometric algebras.* Let $(X, \sigma, \mathcal{L})$ be a triple of a projective variety X, an automorphism $\sigma \in \operatorname{Aut}(X)$, and a very ample invertible sheaf $\mathcal{L} \in \operatorname{Pic}(X)$. Set $V = H^0(X, \mathcal{L})$ and consider X is embedded into $\operatorname{Proj}(\operatorname{Sym} V)$ by $\mathcal{L}$.

Definition 2.13. The *geometric algebra* $A(X, \sigma, \mathcal{L})$ associated to the triple $(X, \sigma, \mathcal{L})$ as above is the quotient algebra of the tensor algebra $T(V)$ by the two-sided ideal generated by $R = \{f \in V \otimes V \mid f(p, \sigma(p)) = 0 \text{ for any point } p \in X\}$.

Theorem 2.14 ([5])**.** *For a three-dimensional AS-regular quadratic algebra A, there exists a geometric triple $(E, \sigma, \mathcal{L})$ of the following types whose geometric algebra $A(E, \sigma, \mathcal{L})$ is isomorphic to A.*

(1) $E = \mathbb{P}^2$, $\sigma \in \operatorname{Aut}(E)$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(1)$, or
(2) $E \subset \mathbb{P}^2$ is a divisor of degree 3, $\sigma \in \operatorname{Aut}(E)$ and $\mathcal{L} = \mathcal{O}_E(1)$, such that $\sigma^*\mathcal{L} \not\cong \mathcal{L}$, $(\sigma^2)^*\mathcal{L} \otimes_E \mathcal{L} \simeq \sigma^*\mathcal{L} \otimes_E \sigma^*\mathcal{L}$.

Conversely, a geometric algebra associated to a triple $(E, \sigma, \mathcal{L})$ as above is a three-dimensional quadratic AS-regular algebra.

Let $(E, \sigma, \mathcal{L})$ be a geometric triple of type (2) as in the above theorem. Then the *twisted homogeneous coordinate ring* associated to the triple $(E, \sigma, \mathcal{L})$ is a $\mathbb{Z}$-graded algebra $B(E, \sigma, \mathcal{L})$ which is constructed in the following way.

$$B(E, \sigma, \mathcal{L}) = \bigoplus_n B_n$$

where $B_n = H^0(E, \mathcal{L}_n)$, $\mathcal{L}_n = \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}}$ and $\mathcal{L}^{\sigma^\ell} = (\sigma^\ell)^*\mathcal{L}$. The multiplication is defined by

$$\begin{aligned} B_n \otimes B_m &= H^0(E, \mathcal{L}_n) \otimes H^0(E, \mathcal{L}_m) \\ &\simeq H^0(E, \mathcal{L}_n) \otimes H^0(E, \mathcal{L}_m^{\sigma^n}) \xrightarrow{\otimes} H^0(E, \mathcal{L}_{n+m}) = B_{n+m}. \end{aligned}$$

The isomorphism in the second row is obtained by $\sigma^{n*} \colon H^0(E, \mathcal{L}_m) \simeq H^0(E, \mathcal{L}_m^{\sigma^n})$. By the definition of the algebras $A(E, \sigma, \mathcal{L})$ and $B(E, \sigma, \mathcal{L})$, there is a canonical map $A(E, \sigma, \mathcal{L}) \to B(E, \sigma, \mathcal{L})$.

Proposition 2.15. *(1) The canonical map $A(E, \sigma, \mathcal{L}) \to B(E, \sigma, \mathcal{L})$ is surjective.*
(2) The kernel of the canonical map is generated by an element g of degree three in the center of $A(E, \sigma, \mathcal{L})$ and g is unique up to scalar multiplication.
(3) There exists an equivalence

$$\Gamma_* \colon \operatorname{coh}(E) \to \operatorname{qgr}(B(E, \sigma, \mathcal{L})) \tag{2.5}$$

$$\mathcal{E} \mapsto \bigoplus_{n \in \mathbb{Z}} H^0\left(E, \mathcal{E} \otimes \bigotimes_{i=0}^{n-1} \sigma^{i*}\mathcal{L}\right). \tag{2.6}$$

Proof. See [5, Theorem 2] and [6, Theorem 1.3]. □

For simplicity, we write $A(E, \sigma, \mathcal{L})$ and $B(E, \sigma, \mathcal{L})$ as A and B respectively.

Definition 2.16. Let g be a central element of A as in Proposition 2.15 and let M be a graded A-module. The morphism of graded A-module

$$\cdot g \colon M(-3) \to M \tag{2.7}$$

is defined by $m \mapsto mg$.

Definition 2.17. Let M be a graded A-module. The graded A-module $K_{M,n}$ is defined by

$$K_{M,n} := \ker\left(g^n\colon M \to M(3n)\right) \tag{2.8}$$

for $n \in \mathbb{Z}_{>0}$. We also define $K_{M,\infty} := \bigcup_{n\in\mathbb{Z}_{>0}} K_{M,n}$. If M is clear, we simply write K_n and K_∞ instead of $K_{M,n}$ and $K_{M,\infty}$ respectively.

By Proposition 2.15, we have $A/g \simeq B$. There exists a pair of adjoint functors

$$j^*\colon \operatorname{qgr}(A) \rightleftarrows \operatorname{qgr}(B) \overset{(2.5)}{\simeq} \operatorname{coh}(E)\colon j_*$$

where j^* corresponds to the extension of scalars $M \mapsto M \otimes_A {}_AB_B$ and j_* corresponds to the restriction of scalars. The functor j_* is fully faithful and exact. Moreover, we have a pair of adjoint functors

$$\mathbb{L}j^*\colon \mathbf{D}^{\mathrm{b}}\operatorname{qgr}(A) \rightleftarrows \mathbf{D}^{\mathrm{b}}\operatorname{qgr}(B) \simeq \mathbf{D}^{\mathrm{b}}\operatorname{coh}(E)\colon \mathbb{R}j_* = j_*. \tag{2.9}$$

By (2.9), we have an exact triangle

$$M(-3) \xrightarrow{\cdot g} M \xrightarrow{\eta_M} j_*\mathbb{L}j^*M \to M(-3)[1] \tag{2.10}$$

where $M \in \mathbf{D}^{\mathrm{b}}\operatorname{qgr}(A)$ and η is the unit of the adjunction (2.9). By the long exact sequence of this triangle, we obtain the following lemma.

Lemma 2.18. *Let $M \in \operatorname{grmod}(A)$. Then we have*
(1) $\mathbb{L}^1j^*M \simeq j^*\left(K_{M,1}(-3)\right)$,
(2) $\mathbb{L}^kj^*M = 0$ if $k \neq 0, 1$.
In particular, if M has no g-torsion element, $\mathbb{L}^1j^*M = 0$.

The direct computation shows that:

Lemma 2.19. *The shift of gradings* (2.2) *is compatible with the autoequivalence of* $\operatorname{coh}(E)$

$$\sigma_*\left(- \otimes \mathcal{L}\right) \tag{2.11}$$

via the equivalence $\operatorname{coh}(E) \simeq \operatorname{qgr}(B)$ *as in* (2.5). *i.e., there exists an isomorphism*

$$\Gamma_*(\mathcal{E})(1) \simeq \Gamma_*(\sigma_*(\mathcal{E} \otimes \mathcal{L})) \tag{2.12}$$

where $\mathcal{E} \in \operatorname{coh}(E)$. Hence, $j_*(\mathcal{E})(1) \simeq j_*(\sigma_*(\mathcal{E} \otimes \mathcal{L}))$. In particular, we have an isomorphism

$$\left(j_*\mathcal{O}_p\right)(n) \simeq j_*\left(\mathcal{O}_{\sigma^n(p)}\right) \tag{2.13}$$

for any $p \in E$ and $n \in \mathbb{Z}$.

Let us introduce the explicit description of the Serre functor of $\mathbf{D}^{\mathrm{b}}\operatorname{qgr}(A)$ for a very general three-dimensional AS-regular quadratic algebra A.

Theorem 2.20. *If the automorphism σ is a translation and E is a smooth elliptic curve then the Serre functor S_A of $\mathbf{D}^{\mathrm{b}}\operatorname{qgr}(A)$ is of the form*

$$S_A(M) = M(-3)[2].$$

Proof. See [26, Corollaries 9.3, 9.4] and [21, Theorem A.4]. □

Moreover, it is known that there exists a full exceptional collection which is a noncommutative analogue of Beilinson's theorem, which addresses the case of $\mathbb{P}^2$.

Theorem 2.21. *Let $\mathcal{O}(n) := \pi(A(n))$. The sequence $(\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2))$ is a full strong exceptional collection in $\mathbf{D}^{\mathrm{b}}\operatorname{qgr}(A)$. In particular, $\mathbf{D}^{\mathrm{b}}\operatorname{qgr}(A)$ has a tilting object.*

Proof. See [1, Theorem 7.1]. □

From this theorem, there exists an isomorphism

$$\mathsf{K}_0(A) := \mathsf{K}_0(\mathbf{D}^{\mathrm{b}}\,\mathrm{qgr}(A)) \simeq \mathbb{Z}^3. \tag{2.14}$$

2.2.3. *GK dimension.*

Definition 2.22. Let M be a finitely generated graded A-module. A *Hilbert series* $h_M(t)$ is defined by

$$h_M(t) = \sum_{n\in\mathbb{Z}} \dim(M_n)t^n \in \mathbb{Z}[[t]][t^{-1}].$$

Example 2.23. The Hilbert series of A is given by $h_A(t) = 1/(1-t)^3$ because of the minimal resolution (2.3).

The assumption $\mathrm{gl.dim}\, A = 3$ implies that there exists a projective resolution

$$0 \to P_3 \to P_2 \to P_1 \to P_0 \to M \to 0,$$

so that

$$h_M(t) = \sum_{i=0}^{3} (-1)^i h_{P_i}(t).$$

Since an indecomposable projective object of $\mathrm{grmod}\, A$ is isomorphic to $A(\ell)$ for some $\ell \in \mathbb{Z}$ (see [5, p. 40], [4, p. 339]), combined with Example 2.23 this implies that

$$h_M(t) = \frac{r}{(1-t)^3} + \frac{a}{(1-t)^2} + \frac{b}{1-t} + f(t) \tag{2.15}$$

for uniquely determined $a, b, r \in \mathbb{Z}$ and $f(t) \in \mathbb{Z}[t^{\pm}]$. This immediately implies:

Lemma 2.24. *For any $M \in \mathrm{grmod}\, A$ there exists a unique polynomial $P_M(x) \in \mathbb{Q}[x]$ such that $P_M(d) = \dim M_d$ for sufficient large $d \in \mathbb{Z}$. Moreover, if $M, N \in \mathrm{grmod}(A)$ are isomorphic in $\mathrm{qgr}(A)$, then $P_M(x) = P_N(x)$.*

Definition 2.25. The *Gelfand-Kirillov dimension* (or the *GK-dimension*) of a nontrivial graded A-module M is a pole order of Hilbert polynomial $h_M(t)$ at $t = 1$. Let $\mathrm{GKdim}\, M$ denote the GK-dimension of a graded module M.

For $0 \neq M \in \mathrm{grmod}\, A$ we have the following quadchotomy by (2.15).

$$\mathrm{GKdim}\, M = \begin{cases} 3 & \text{if } r > 0 \\ 2 & \text{if } r = 0 \text{ and } a > 0 \\ 1 & \text{if } r = a = 0 \text{ and } b > 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.16}$$

We also have

$$\mathrm{GKdim}\, M = \deg P_M(x) + 1$$

for the polynomial $P_M(x)$ as in Lemma 2.24. The degree of the zero polynomial is defined to be -1.

Definition 2.26. Let us denote the localization of graded algebra A with respect to the homogeneous element g by $\Lambda := A[g^{-1}]$, and by $\Lambda_0 \subseteq \Lambda$ its degree 0 part.

Proposition 2.27 (=[4, Proposition 7.5])**.** *The following categories are equivalent:*

(1) finite dimensional Λ_0-modules V,

(2) finitely generated graded A-modules N such that $\dim N_n$ is bounded, modulo g-torsion modules.

Proposition 2.28 (=[4, Corollary 7.9]). *If the order of the automorphism σ associated to A is infinite, then Λ_0 has no nontrivial finite dimensional representation. In particular, any finitely generated graded A-module N whose dimension $\dim N_n$ is bounded is a g-torsion module.*

According to the two results above, any graded A-module M with $\operatorname{GKdim} M \leq 1$ is a g-torsion module if the order of σ is infinite.

2.3. **Spherical functor.** In this section we assume that any triangulated category and any exact functor have dg enhancements. Hence, there exist functorial cones. Let $F\colon \mathcal{T}_1 \to \mathcal{T}_2$ be an exact functor admitting right and left adjoint functor $R, L\colon \mathcal{T}_2 \to \mathcal{T}_1$. Consider the canonical triangles

$$FR \xrightarrow{\epsilon} \mathrm{id} \to T \to FR[1]$$

and

$$\mathrm{id} \xrightarrow{\eta} RF \to C \to \mathrm{id}[1] \tag{2.17}$$

where $\eta\colon \mathrm{id}_{\mathcal{T}_2} \to RF$ is the unit, and $\epsilon\colon FR \to \mathrm{id}_{\mathcal{T}_1}$ is the counit of adjunction. The functor T is called a *twist* and C is a *cotwist* of F.

Definition 2.29. An exact functor $F\colon \mathcal{T}_1 \to \mathcal{T}_2$ which admits a left adjoint L and right adjoint R is called *spherical* if the cotwist C is an equivalence and $R \simeq CL$.

Theorem 2.30 (=[2, Theorem 2.3]). *For a spherical functor F, the twist T is an equivalence.*

Example 2.31. Let A be a three-dimensional quadratic AS-regular algebra and let $(E, \sigma, \mathcal{L})$ be the triple associated to A. Assume that E is a smooth elliptic curve and σ is a translation. The restriction functor $\mathbb{L}j^*\colon \mathbf{D}^{\mathrm{b}} \operatorname{qgr}(A) \to \mathbf{D}^{\mathrm{b}} \operatorname{coh}(E)$ is spherical. Indeed, the right adjoint functor to $\mathbb{L}j^*$ is the functor j_*, and the left adjoint functor $j_!$ is defined by

$$j_!(-) = S_A^{-1} j_* S_E \overset{\text{Theorem 2.20}}{=} \left(j_*((-) \otimes_E \omega_E)\right)(3)[-1] = \left(j_*(-)\right)(3)[-1]$$

where S_A, S_E are the Serre functors of $\mathbf{D}^{\mathrm{b}} \operatorname{qgr}(A)$, $\mathbf{D}^{\mathrm{b}} \operatorname{coh}(E)$ respectively. Considering the canonical exact triangle

$$M(-3) \xrightarrow{\cdot g} M \to j_* \mathbb{L}j^* M \to M(-3)[1],$$

the spherical cotwist functor $C\colon M \mapsto M(-3)[1]$ is an autoequivalence. It is easy to see that $Cj_!(M) \simeq j_*(M)$. Therefore, $\mathbb{L}j^*$ is a spherical functor.

Theorem 2.32 (=[2, Proposition 2.1]). *Let $F\colon \mathcal{T}_1 \to \mathcal{T}_2$ be a spherical functor such that the spherical cotwist is isomorphic to the Serre functor of $\mathcal{T}_1$ up to a shift. If $\mathcal{A} \subset \mathcal{T}_1$ is an admissible subcategory, then the composition $\mathcal{A} \hookrightarrow \mathcal{T}_1 \to \mathcal{T}_2$ is also a spherical functor.*

By Example 2.31, we can apply this theorem to the restriction functor $\mathbb{L}j^*$.

Corollary 2.33. *Let $\mathcal{B} \subset \mathbf{D}^{\mathrm{b}} \operatorname{qgr}(A)$ be an admissible subcategory and $\mathrm{pr}_{\mathcal{B}}^R$ be the right projection onto $\mathcal{B}$. There exists an exact triangle in $\mathbf{D}^{\mathrm{b}} \operatorname{coh}(E \times E)$*

$$K_{LR} \to \mathcal{O}_\Delta \to K_T \to K_{LR}[1] \tag{2.18}$$

such that $\Phi_{K_{LR}} \simeq \mathbb{L}j^* \mathrm{pr}_{\mathcal{B}}^R j_*$ and $\Phi_{K_T} \simeq T$ where T is a spherical twist functor associated to the spherical functor $\mathcal{B} \hookrightarrow \mathbf{D}^{\mathrm{b}} \operatorname{qgr}(A) \xrightarrow{\mathbb{L}j^*} \mathbf{D}^{\mathrm{b}} \operatorname{coh}(E)$. Thus for any $F \in \mathbf{D}^{\mathrm{b}} \operatorname{coh}(E)$, there exists an exact triangle

$$\mathbb{L}j^* \mathrm{pr}_{\mathcal{B}}^R (j_* F) \to F \to T(F) \to \mathbb{L}j^* \mathrm{pr}_{\mathcal{B}}^R (j_* F)[1] \tag{2.19}$$

in $\mathbf{D}^{\mathrm{b}} \operatorname{coh}(E)$.

We defer the proof of Corollary 2.33 to Section A. The main issue is to check that the triangle (2.19) of functors is realized by a distinguished triangle of kernels for integral functors.

3. Proof of Main theorem

3.1. **A spanning class of noncommutative projective planes.** In this section, let A be a three-dimensional AS-regular algebra and let $(E, \sigma, \mathcal{L})$ be a geometric triple associated to A. Assume that E is a smooth elliptic curve.

Lemma 3.1. *Suppose that the order of the automorphism σ is infinite. Let M be a bounded complex in $\operatorname{qgr}(A)$. If $\mathbb{L}j^*M = 0$, then $M = 0$.*

Proof. Consider the canonical triangle in $\mathbf{D}^{\mathrm{b}} \operatorname{qgr}(A)$

$$M(-3) \xrightarrow{\cdot g} M \to j_*\mathbb{L}j^*M = 0 \to M(-3)[1].$$

Thus for any $i \in \mathbb{Z}$, the morphism induced on the i-th cohomology

$$H^i(M)(-3) \xrightarrow{\cdot g} H^i(M) \tag{3.1}$$

is an isomorphism in $\operatorname{qgr}(A)$. By Lemma 2.24, we have $P_{H^i(M)}(x+3) = P_{H^i(M)}(x)$, and this implies that $P_{H^i(M)}(x)$ is constant. Hence, we have

$$1 \geq \operatorname{GKdim} H^i(M)$$

for any $i \in \mathbb{Z}$ and hence, $\dim H^i(M)_n$ is bounded. By Proposition 2.28, $H^i(M)$ is a g-torsion module. Since the multiplication map (3.1) is an isomorphism in $\operatorname{qgr}(A)$, we have $H^i(M) = 0$ in $\operatorname{qgr}(A)$ for any $i \in \mathbb{Z}$. So, $M = 0$ in $\mathbf{D}^{\mathrm{b}} \operatorname{qgr}(A)$. □

This immediately implies the following proposition:

Proposition 3.2. *Suppose that the order of automorphism of σ is infinite. Then the set $\{j_*\mathcal{O}_p \mid p \in E\}$ is a spanning class of $\mathbf{D}^{\mathrm{b}} \operatorname{qgr}(A)$.*

Proof. Let $M \in \mathbf{D}^{\mathrm{b}} \operatorname{qgr}(A)$, and assume that $\mathbb{R}\mathrm{Hom}(M, j_*\mathcal{O}_p) = 0$ for any $p \in E$. Then, by the adjunction $\mathbb{L}j^* \dashv j_*$, we have $\mathbb{R}\mathrm{Hom}(\mathbb{L}j^*M, \mathcal{O}_p) = 0$ for any $p \in E$. Since the set $\{\mathcal{O}_p\}_{p \in E}$ is a spanning class of $\mathbf{D}^{\mathrm{b}} \operatorname{coh}(E)$, we have $\mathbb{L}j^*M = 0$, and hence $M = 0$ by Lemma 3.1. Therefore, the set $\{j_*\mathcal{O}_p\}_{p \in E}$ is a spanning class of $\mathbf{D}^{\mathrm{b}} \operatorname{qgr}(A)$. □

3.2. **Support of graded modules restricted to the anti-canonical divisor.** In this section, let A be a three-dimensional quadratic AS-regular algebra and let $(E, \sigma, \mathcal{L})$ be a geometric triple associated to A. Assume that E is a smooth elliptic curve.

Definition 3.3. Let us define a subset $E^{\mathrm{sp}} \subset E$ of special points as

$$E^{\mathrm{sp}} := \{p \in E \mid \operatorname{Supp}(\mathbb{L}j^*M) = \{p\} \text{ for some } M \in \mathbf{D}^{\mathrm{b}} \operatorname{qgr}(A)\}. \tag{3.2}$$

For any $M \in \mathbf{D}^{\mathrm{b}} \operatorname{qgr}(A)$, there exists a Leray spectral sequence

$$E_2^{p,q} = \mathbb{L}^p j^* H^q(M) \Rightarrow \mathbb{L}^{p+q} j^*(M). \tag{3.3}$$

By Lemma 2.18, we obtain that $E_2^{p,q} = 0$ if $p \neq 0, 1$ and hence, this spectral sequence degenerates at E_2-page. Thus there exists an exact sequence

$$0 \to E_2^{1,n-1} = \mathbb{L}^1 j^* H^{n-1}(M) \to \mathbb{L}^n j^* M \to E_2^{0,n} = \mathbb{L}^0 j^* H^n(M) \to 0$$

so that we have

$$\operatorname{Supp}(\mathbb{L}j^*M) = \bigcup_{n \in \mathbb{Z}} \operatorname{Supp}(\mathbb{L}j^* H^n(M)).$$

This implies that

$$E^{\mathrm{sp}} = \{p \in E \mid \operatorname{Supp}(\mathbb{L}j^*M) = \{p\} \text{ for some } M \in \operatorname{qgr}(A)\}. \tag{3.4}$$

Lemma 3.4. *There exists a decomposition of E^{sp} as the following form:*

$$E^{\mathrm{sp}} = E_1 \cup E_2$$

where

$$E_1 = \{p \in E \mid \operatorname{Supp}(\mathbb{L}j^*M) = \{p\} \textit{ for some } g\textit{-torsion module } M\}, \tag{3.5}$$

$$E_2 = \{p \in E \mid \operatorname{Supp}(\mathbb{L}j^*M) = \{p\} \textit{ for some } M \textit{ such that } \mathbb{L}^1j^*M = 0\}. \tag{3.6}$$

Proof. For any $p \in E^{\mathrm{sp}}$, there exists $M \in \operatorname{qgr}(A)$ such that $\operatorname{Supp}(\mathbb{L}j^*M) = \{p\}$ by (3.4). Set $Q \coloneqq M/K_\infty$ where K_∞ is defined in Definition 2.17. Note that K_∞ is a g-torsion module by definition and $\mathbb{L}^1j^*Q = 0$ by Lemma 2.18. Then by the long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{L}^1j^*K_\infty \arrow[r, "\sim"] & \mathbb{L}^1j^*M \arrow[r] & \mathbb{L}^1j^*Q = 0 \arrow[dll, rounded corners] & \\
& j^*K_\infty \arrow[r] & j^*M \arrow[r] & j^*Q \arrow[r] & 0,
\end{tikzcd} \tag{3.7}$$

we have $\operatorname{Supp}(\mathbb{L}j^*K_\infty) \subset \operatorname{Supp}(\mathbb{L}j^*M) = \{p\}$. If $\operatorname{Supp}(\mathbb{L}j^*K_\infty) = \{p\}$, then $p \in E_1$ since K_∞ is a g-torsion module. If $\operatorname{Supp}(\mathbb{L}j^*K_\infty) = \emptyset$, that is $\mathbb{L}j^*K_\infty = 0$, then $K_\infty = 0$ by Lemma 3.1. Hence $M \simeq Q$ and $p \in E_2$ since $\mathbb{L}^1j^*Q = 0$. □

Lemma 3.5. *Let $M \in \operatorname{grmod}(A)$. Assume that M is annihilated by g, i.e., $j_*\mathcal{F} \simeq M$ for some $\mathcal{F} \in \operatorname{coh}(E)$. Then*

$$\operatorname{Supp}(\mathbb{L}^1j^*M) = \sigma^{-3}\operatorname{Supp}(\mathcal{F}).$$

Proof. If $\mathcal{F}$ is zero, then $M \simeq j_*\mathcal{F}$ is also a zero object. Then both j^*M and $\mathbb{L}^1j^*M$ are obviously zero, and hence these supports are empty sets. Therefore, we may assume that $\mathcal{F} \neq 0$. Since M is annihilated by g, we have $M = K_1$ where K_1 is defined in Definition 2.17. By Lemma 2.18, we have

$$\mathbb{L}^1j^*M \simeq j^*\left(M(-3)\right) \simeq j^*((j_*\mathcal{F})(-3)).$$

By Lemma 2.19, we have

$$\left(j_*\mathcal{F}\right)(-1) \simeq j_*(\sigma_*^{-1}(\mathcal{F}) \otimes \mathcal{L}^{-1}).$$

This inductively implies that

$$\left(j_*\mathcal{F}\right)(-3) \simeq j_*\left(\left(\sigma_*^{-3}\mathcal{F}\right) \otimes \sigma^{2*}\mathcal{L}^{-1} \otimes \sigma^*\mathcal{L}^{-1} \otimes \mathcal{L}^{-1}\right)$$

and hence, $\mathbb{L}^1j^*M \simeq (\sigma_*^{-3}\mathcal{F}) \otimes \sigma^{2*}\mathcal{L}^{-1} \otimes \sigma^*\mathcal{L}^{-1} \otimes \mathcal{L}^{-1}$. Therefore, we have

$$\operatorname{Supp}(\mathbb{L}^1j^*M) = \operatorname{Supp}(\sigma_*^{-3}\mathcal{F}) = \sigma^{-3}\operatorname{Supp}(\mathcal{F}).$$

□

Let us define the map

$$\deg\colon \mathsf{K}_0(E) \coloneqq \mathsf{K}_0(\mathbf{D}^{\mathrm{b}}\operatorname{coh}(E)) \to \mathbb{Z} \tag{3.8}$$

which is given by $\mathcal{F} \mapsto \chi(\mathcal{F})$ where $\mathcal{F} \in \operatorname{coh}(E)$ and $\chi(F) = \sum(-1)^i \dim H^i(E, \mathcal{F})$. It is known that there exists an isomorphism

$$\operatorname{Pic}(E) \oplus \mathbb{Z} \simeq \mathsf{K}_0(E) \tag{3.9}$$

which is defined by the direct sum of $\sum n_p p \mapsto \sum n_p[\mathcal{O}_p]$ for a Weil divisor $\sum n_p p$ on E and $n \mapsto n[\mathcal{O}_E]$ for $n \in \mathbb{Z}$ (see [13, Exercise I.6.11]). Note that the map defined in (3.8) coincides with the map of the degree of a Weil divisor on $\operatorname{Pic}(E)$, via the map (3.9).

Lemma 3.6. *If M is a g-torsion module such that* $\mathrm{Supp}(j^*M)$ *is a finite set, then it is an extension of skyscraper sheaves of the form $j_*\mathcal{O}_p$ for $p \in E$; in particular,* $\mathrm{Supp}(\mathbb{L}j^*M)$ *is also a finite set.*

Proof. Let $n = \min\{\ell \mid Mg^\ell = 0\}$. We prove the assertion by using induction on n. If $n = 1$, we have $M \simeq j_*j^*M$. Since $\mathrm{Supp}(j^*M)$ is a finite set, j^*M is an iterated extension of skyscraper sheaves. Hence, M is also an iterated extension of the skyscraper sheaves of the form $j_*\mathcal{O}_p$ for $p \in E$. By Lemma 3.5, the support of $\mathbb{L}j^*M$ is a finite set. If $n > 1$, there exists an exact sequence

$$0 \to K = Mg \to M \to Q = M/Mg \to 0.$$

We obtain a long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{L}^1 j^*K \arrow[r] & \mathbb{L}^1 j^*M \arrow[r] & \mathbb{L}^1 j^*Q \arrow[dll, rounded corners] & \\ & j^*K \arrow[r] & j^*M \arrow[r] & j^*Q \arrow[r] & 0. \end{tikzcd} \tag{3.10}$$

Since $\mathrm{Supp}(j^*Q)$ is a subset of $\mathrm{Supp}(j^*M)$, it is finite. As Q is annihilated by g, it is an iterated extension of skyscraper sheaves and $\mathrm{Supp}(\mathbb{L}j^*Q)$ is a finite set, similar to the case when $n = 1$. This implies that $\mathrm{Supp}(j^*K)$ is a finite set by (3.10). By definition, $Kg^{n-1} = 0$. By induction hypothesis, K is an iterated extension of skyscraper sheaves. Therefore, M is an iterated extension of skyscraper sheaves. □

Lemma 3.7. *If M is a g-torsion module such that* $\mathrm{Supp}(j^*M)$ *is a non-empty finite set, then the class of $\mathbb{L}j^*M$ in $\mathsf{K}_0(E)$ is a non-trivial sum of elements of the form $[\mathcal{O}_p] - [\mathcal{O}_{\sigma^{-3}(p)}]$, where $p \in E$.*

Proof. By Lemma 3.6, M is an iterated extension of skyscraper sheaves of the form $j_*\mathcal{O}_p$ for $p \in E$. Hence, in the Grothendieck group $\mathsf{K}_0(A)$, the class of M is the sum of classes $[j_*\mathcal{O}_p]$ for $p \in E$. By Lemma 2.18 and (2.13), we have

$$\mathbb{L}j^*[j_*\mathcal{O}_p] = [\mathbb{L}^0 j^*j_*\mathcal{O}_p] - [\mathbb{L}^1 j^*j_*\mathcal{O}_p] = [\mathcal{O}_p] - [\mathcal{O}_{\sigma^{-3}(p)}].$$

Hence the assertion holds. □

Lemma 3.8. *If σ is an infinite order translation, then the set E_1 as defined in* (3.5) *is empty.*

Proof. Assume that there exists a g-torsion module M such that $\mathrm{Supp}(\mathbb{L}j^*M) = \{p\}$ for some $p \in E$, i.e., $p \in E_1$. Since $\mathbb{L}j^*M$ is supported at p, the class of $\mathbb{L}j^*M$ is proportional to $[p]$. However, by Lemma 3.7, it can be also represented as a sum of the classes $[\mathcal{O}_q] - [\mathcal{O}_{\sigma^{-3}(q)}]$ for various points $q \in E$. Note that since σ is a translation, the difference $[\mathcal{O}_q] - [\mathcal{O}_{\sigma^{-3}(q)}]$ does not depend on the point q. Indeed, for $q_1, q_2 \in E$, we obtain $\sigma^{-3}(q_1) - \sigma^{-3}(q_2) = q_1 - q_2$ in $\mathrm{Pic}(E)$ since σ is a translation. This implies that $[\mathcal{O}_{q_1}] - [\mathcal{O}_{\sigma^{-3}(q_1)}] = [\mathcal{O}_{q_2}] - [\mathcal{O}_{\sigma^{-3}(q_2)}]$. Hence, we obtain a relation

$$k[\mathcal{O}_p] = r([\mathcal{O}_p] - [\mathcal{O}_{\sigma^{-3}(p)}])$$

for some integers k, r, where by Lemma 3.7 $r > 0$. Since the degree of the right hand side is zero, we have $k = 0$ and hence, the class $[\mathcal{O}_p] - [\mathcal{O}_{\sigma^{-3}(p)}]$ is annihilated by some positive integer. Since the class $[\mathcal{O}_q] - [\mathcal{O}_{\sigma^{-3}(p)}]$ is constant for any point q, we obtain

$$r([\mathcal{O}_p] - [\mathcal{O}_{\sigma^{-3}(p)}]) = \sum_{i=1}^{r}([\mathcal{O}_{\sigma^{-3i+3}(p)}] - [\mathcal{O}_{\sigma^{-3i}(p)}]) = [\mathcal{O}_p] - [\mathcal{O}_{\sigma^{-3r}(p)}].$$

The class vanishes in $\mathsf{K}_0(E)$ if and only if $p = \sigma^{-3r}(p)$, which cannot happen since σ is an infinite order translation. $\square$

For $\eta \in \mathsf{K}_0(E)$, let us define $E_{2,\eta} \subset E_2$ as

$$E_{2,\eta} := \{p \mid \mathrm{Supp}(\mathbb{L}j^*M) = \{p\} \text{ for some } M \in \mathrm{qgr}(A) \text{ such that } \mathbb{L}^1 j^*M = 0, [M] = \eta\}.$$

If $\deg \mathbb{L}j^*\eta \leq 0$, then $E_{2,\eta} = \emptyset$. Indeed, assume $E_{2,\eta} \neq \emptyset$ and take $p \in E_{2,\eta}$. Then there exists $M \in \mathrm{grmod}(A)$ such that $\mathrm{Supp}(\mathbb{L}j^*M) = \{p\}$ and $\mathbb{L}^1 j^*M = 0$. Since j^*M is supported at p, we have $\deg \mathbb{L}j^*\eta = \deg[j^*M] = \dim H^0(E, j^*M) > 0$.

Lemma 3.9. *For any $\eta \in \mathsf{K}_0(A)$ which satisfies $n := \deg \mathbb{L}j^*\eta > 0$, $E_{2,\eta}$ is finite. In particular, $E_2 = \bigcup_{\eta \in \mathsf{K}_0(A)} E_{2,\eta}$ is at most countable.*

Proof. Let $p, q \in E_{2,\eta}$. Then there exist $M, N \in \mathrm{grmod}(A)$ which satisfy the following conditions:

$$\mathbb{L}^1 j^*M = \mathbb{L}^1 j^*N = 0 \text{ and } \mathrm{Supp}(\mathbb{L}j^*M) = \{p\},\ \mathrm{Supp}(\mathbb{L}j^*N) = \{q\}.$$

Since j^*M and j^*N are supported at p and q respectively, we have the following equation of K-classes on E

$$[\mathbb{L}j^*M] = [j^*M] = n[\mathcal{O}_p],\ [\mathbb{L}j^*N] = [j^*N] = n[\mathcal{O}_q]$$

where $n = \deg \mathbb{L}j^*\eta > 0$. Since $\eta = [M] = [N]$, then $\mathbb{L}j^*[M] = \mathbb{L}j^*[N]$. i.e.,

$$n\left([\mathcal{O}_p] - [\mathcal{O}_q]\right) = 0.$$

By (3.9), the divisor $p - q \in \mathrm{Pic}(E)$ is annihilated by n. Therefore, we can define the map

$$E_{2,\eta} \to E[n],\ p \mapsto p - q$$

for a fixed point $q \in E_{2,\eta}$ and this map is injective where $E[n]$ is a set of divisors annihilated by n. Hence, $E_{2,\eta}$ is a finite set since $E[n]$ is a finite set. Finally, since $\mathsf{K}_0(A) \simeq \mathbb{Z}^3$ by (2.14), the set $E_2 = \bigcup_{\eta \in \mathsf{K}_0(A)} E_{2,\eta}$ is at most countable. $\square$

Proposition 3.10. *If the automorphism σ is an infinite order translation, then the set E^{sp} is at most countable. As a consequence, we have $E \setminus E^{\mathrm{sp}} \neq \emptyset$.*

Proof. It follows from Lemma 3.4, Lemma 3.8, and Lemma 3.9. $\square$

3.3. **Proof of Theorem 1.3.** In this section, we prove the main theorem using a noncommutative analogue of [10, Theorem 3.1], which addresses the commutative case.

Let A be a three-dimensional quadratic AS-regular algebra and let $(E, \sigma, \mathcal{L})$ be a geometric triple associated to A described in Theorem 2.14. We assume that E is an elliptic curve, and the automorphism σ is a translation with infinite order.

Theorem 3.11. *Let $\mathbf{D}^{\mathrm{b}}\,\mathrm{qgr}(A) = \langle \mathcal{A}, \mathcal{B} \rangle$ be a semi-orthogonal decomposition with $K_0(\mathcal{B}) = 0$. Then $\mathcal{B}$ is trivial. In particular, there are no phantom categories on noncommutative projective planes with infinite order translation.*

Proof. By Proposition 3.2, it is enough to show that $j_*\mathcal{O}_p \in \mathcal{A}$ for any $p \in E$. Consider the decomposition of $j_*\mathcal{O}_p$

$$B_p \to j_*\mathcal{O}_p \to A_p \to B_p[1] \tag{3.11}$$

where $A_p \in \mathcal{A}$ and $B_p \in \mathcal{B}$. Then we only have to prove that $\mathbb{L}j^*B_p = 0$ for any $p \in E$ since $B_p = 0$ if and only if $\mathbb{L}j^*B_p = 0$ by Lemma 3.1.

By Corollary 2.33, there exists an exact triangle on E

$$\mathbb{L}j^*B_p \to \mathcal{O}_p \to C_p := T(\mathcal{O}_p) \to \mathbb{L}j^*B_p[1] \tag{3.12}$$

where T is a spherical twist associated to $\mathcal{B} \hookrightarrow \mathbf{D}^b \operatorname{qgr}(A) \overset{\mathbb{L}j^*}{\to} \mathbf{D}^{\mathrm{b}} \operatorname{coh}(E)$. Since $\mathcal{O}_p$ is a spherical object, then so is C_p. Moreover, since the K-class of $\mathbb{L}j^* B_p$ vanishes by our assumption for $\mathcal{B}$, we have

$$[\mathcal{O}_p] = [C_p] \tag{3.13}$$

in $\mathsf{K}_0(E)$. It is known that any spherical object on E is isomorphic to either a simple vector bundle or a skyscraper sheaf up to shift by [11, Proposition 4.13]. Hence the equation (3.13) implies that for any $p \in E$, there exists an isomorphism

$$C_p \simeq \mathcal{O}_p[2a_p] \tag{3.14}$$

for some $a_p \in \mathbb{Z}$. Since T sends skyscraper sheaves to skyscraper sheaves up to shift, the autoequivalence T is of the form

$$T \simeq \rho_*(- \otimes \mathcal{L})[n] \tag{3.15}$$

for some $\rho \in \operatorname{Aut}(E)$, $\mathcal{L} \in \operatorname{Pic}(E)$, and $n \in \mathbb{Z}$ by [15, Corollary 4.3]. Then we claim that $\rho = \mathrm{id}_E$, $n = 0$, and $\mathcal{L} \simeq \mathcal{O}_E$. Indeed, by (3.14), we have $\rho = \mathrm{id}_E$. By (3.12), we have $\operatorname{Supp}(\mathbb{L}j^* B_p) \subset \{p\}$. By Proposition 3.10, there exists a point $p \notin E^{\mathrm{sp}}$. If $p \notin E^{\mathrm{sp}}$, then $\mathbb{L}j^* B_p = 0$. Moreover, by (3.12) we have

$$\mathcal{O}_p \simeq C_p \tag{3.16}$$

for $p \notin E^{\mathrm{sp}}$. This implies $n = 0$. Finally, consider the exact triangle obtained by (2.19)

$$\mathbb{L}j^* B_{\mathcal{O}} \to \mathcal{O}_E \to \mathcal{L} \to \mathbb{L}j^* B_{\mathcal{O}}[1]$$

where $B_{\mathcal{O}} = \operatorname{pr}^R_{\mathcal{B}}(j_* \mathcal{O}_E)$. The equation of the K-classes

$$[\mathcal{O}_E] = [\mathcal{L}]$$

implies that $\mathcal{O}_E \simeq \mathcal{L}$.

Therefore, the morphism of kernels on $E \times E$ that corresponds to $\mathrm{id} \to T$ is given by some element ψ of $\operatorname{Hom}_{E \times E}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$. Since the morphism space is one-dimensional, ψ is either an isomorphism or a trivial morphism. By (3.16), ψ is not trivial and hence it is an isomorphism. As a consequence, we have $\mathbb{L}j^* B_p = 0$ for any $p \in E$ and the assertion holds. $\square$

Appendix A. Proof of Corollary 2.33

In this section we prove Corollary 2.33. Let Q be a finite acyclic quiver and $\Lambda = \mathbf{k}Q/I$, where I is a two-sided ideal of the path algebra $\mathbf{k}Q$.

Since Q is finite and acyclic, the algebra Λ has finite global dimension. Moreover, the enveloping algebra $\Lambda^e = \Lambda^{\mathrm{op}} \otimes_{\mathbf{k}} \Lambda$ has finite global dimension. Indeed, Λ^e is a quotient algebra of the path algebra $\mathbf{k}(Q^{\mathrm{op}} \times Q)$, where Q^{op} is the opposite quiver of Q and $Q^{\mathrm{op}} \times Q$ is the product of the quivers (see [14, Proposition 3]). As $Q^{\mathrm{op}} \times Q$ is finite and acyclic, the enveloping algebra Λ^e has finite global dimension as well.

Lemma A.1. *Let Λ be a finite dimensional algebra and let $\Lambda^e = \Lambda^{\mathrm{op}} \otimes_{\mathbf{k}} \Lambda$ be an enveloping algebra. For a right Λ-module W, there exists the following adjunction*

$${}_{\Lambda}(-) \otimes_{\mathbf{k}} W_{\Lambda} \colon \operatorname{mod} \Lambda^{\mathrm{op}} \rightleftarrows \operatorname{mod} \Lambda^e \colon {}_{\Lambda} \operatorname{Hom}_{\operatorname{mod} \Lambda}(W_{\Lambda}, (-)),$$

where a left module structure of ${}_{\Lambda} \operatorname{Hom}_{\operatorname{mod} \Lambda}(W_{\Lambda}, M)$ for a bimodule M is inherited from the left module structure of M. In particular, we have an adjunction

$${}_{\Lambda}(-) \overset{\mathbb{L}}{\otimes}_{\mathbf{k}} W_{\Lambda} = {}_{\Lambda}(-) \otimes_{\mathbf{k}} W_{\Lambda} \colon \mathbf{D}^{\mathrm{b}} \operatorname{mod} \Lambda^{\mathrm{op}} \rightleftarrows \mathbf{D}^{\mathrm{b}} \operatorname{mod} \Lambda^e \colon {}_{\Lambda} \mathbb{R}\operatorname{Hom}_{\operatorname{mod} \Lambda}(W_{\Lambda}, (-)).$$

Lemma A.2. *Let $\mathbf{D}^{\mathrm{b}} \operatorname{mod} \Lambda = \langle \mathcal{A}, \mathcal{B} \rangle$ be a semi-orthogonal decomposition. Then there exists a semi-orthogonal decomposition*

$$\mathbf{D}^{\mathrm{b}} \operatorname{mod} \Lambda^e = \langle \mathbf{D}^{\mathrm{b}} \operatorname{mod} \Lambda^{\mathrm{op}} \otimes_{\mathbf{k}} \mathcal{A}, \mathbf{D}^{\mathrm{b}} \operatorname{mod} \Lambda^{\mathrm{op}} \otimes_{\mathbf{k}} \mathcal{B} \rangle$$

where $\Lambda^e = \Lambda^{\mathrm{op}} \otimes_{\mathbf{k}} \Lambda$ is the enveloping algebra.

Proof. Since Λ^e has finite global dimension, the derived category $\mathbf{D}^{\mathrm{b}} \operatorname{mod} \Lambda^e$ is the smallest triangulated category containing projective modules over Λ^e. Since Λ^e is the quotient of the path algebra $\mathbf{k}(Q^{\mathrm{op}} \times Q)$, any indecomposable projective module is of the form

$$P_{(i,j)} = \Lambda e_j \otimes_{\mathbf{k}} e_i \Lambda \in \operatorname{mod} \Lambda^{\mathrm{op}} \otimes_{\mathbf{k}} \operatorname{mod} \Lambda$$

where i, j are vertices of Q. Therefore, we have

$$\mathbf{D}^{\mathrm{b}} \operatorname{mod} \Lambda^e = \langle \operatorname{mod} \Lambda^{\mathrm{op}} \otimes_{\mathbf{k}} \operatorname{mod} \Lambda \rangle$$

and this shows that $\mathbf{D}^{\mathrm{b}} \operatorname{mod} \Lambda^{\mathrm{op}} \otimes_{\mathbf{k}} \mathcal{A}$ and $\mathbf{D}^{\mathrm{b}} \operatorname{mod} \Lambda^{\mathrm{op}} \otimes_{\mathbf{k}} \mathcal{B}$ generate $\mathbf{D}^{\mathrm{b}} \operatorname{mod} \Lambda^e$ as a triangulated category. By the adjunction Lemma A.1, for $M, N \in \operatorname{mod} \Lambda^{\mathrm{op}}$, $A \in \mathcal{A}$ and $B \in \mathcal{B}$, we have

$$\begin{aligned}
\mathbb{R}\mathrm{Hom}_{\Lambda^e}(M \otimes_{\mathbf{k}} B, N \otimes_{\mathbf{k}} A) &= \mathbb{R}\mathrm{Hom}_{\operatorname{mod} \Lambda^{\mathrm{op}}}(M, \mathbb{R}\mathrm{Hom}_{\operatorname{mod} \Lambda}(B, N \otimes_{\mathbf{k}} A)) \\
&= \mathbb{R}\mathrm{Hom}_{\operatorname{mod} \Lambda^{\mathrm{op}}}(M, N \otimes_{\mathbf{k}} \mathbb{R}\mathrm{Hom}_{\operatorname{mod} \Lambda}(B, A)) \\
&= \mathbb{R}\mathrm{Hom}_{\operatorname{mod} \Lambda^{\mathrm{op}}}(M, N) \otimes_{\mathbf{k}} \mathbb{R}\mathrm{Hom}_{\operatorname{mod} \Lambda}(B, A) \\
&= 0.
\end{aligned}$$

This implies the semi-orthogonality. □

Lemma A.3. *Let $\mathcal{B}$ be an admissible subcategory of the $\mathbf{D}^{\mathrm{b}} \operatorname{mod} \Lambda$. Then the right and the left projections onto $\mathcal{B}$ have dg enhancements.*

Proof. For the diagonal Λ-bimodule Λ, there exists an exact triangle

$$P_{\mathcal{B}} \to \Lambda \to P_{\mathcal{A}} \to P_{\mathcal{B}}[1],$$

where $P_{\mathcal{A}}$ and $P_{\mathcal{B}}$ are objects belonging to the subcategories $\mathbf{D}^{\mathrm{b}} \operatorname{mod} \Lambda^{\mathrm{op}} \otimes_{\mathbf{k}} \mathcal{A}$ and $\mathbf{D}^{\mathrm{b}} \operatorname{mod} \Lambda^{\mathrm{op}} \otimes_{\mathbf{k}} \mathcal{B}$, respectively. The functors $\Phi_{P_{\mathcal{A}}} \colon M \mapsto M \overset{\mathbb{L}}{\otimes}_{\Lambda} P_{\mathcal{A}}$ and $\Phi_{P_{\mathcal{B}}} \colon M \mapsto M \overset{\mathbb{L}}{\otimes}_{\Lambda} P_{\mathcal{B}}$ correspond to the projection functors onto the admissible subcategories $\mathcal{A}$ and $\mathcal{B}$, respectively. Hence the projection functors onto $\mathcal{A}$ and $\mathcal{B}$ have dg enhancements. In particular, the right projection onto $\mathcal{B}$ has a dg enhancement. By applying the previous discussion to the semi-orthogonal decomposition $\langle \mathcal{B}, \mathcal{B}^{\perp} \rangle$, the left projection onto $\mathcal{B}$ has a dg enhancement. □

Remark A.4. A result similar to Lemma A.3 is also obtained as a combination of general results such as [22, Proposition 3.8 and arguments in p. 75], but only up to quasi-equivalences. We decided to include Lemma A.3 and its proof in this paper, as it is unclear how to handle such quasi-equivalences in the proof of Proposition A.5.

Proposition A.5 (= Corollary 2.33)**.** *Let $\mathcal{B} \subset \mathbf{D}^{\mathrm{b}} \operatorname{qgr}(A)$ be an admissible subcategory and $\mathrm{pr}^R_{\mathcal{B}}$ be the right projection onto $\mathcal{B}$. There exists an exact triangle in $\mathbf{D}^{\mathrm{b}} \operatorname{coh}(E \times E)$*

$$K_{LR} \to \mathcal{O}_{\Delta} \to K_T \to K_{LR}[1]$$

such that $\Phi_{K_{LR}} \simeq \mathbb{L}j^* \mathrm{pr}^R_{\mathcal{B}} j_*$ and $\Phi_{K_T} \simeq T$, where T is a spherical twist functor associated to the spherical functor $\mathcal{B} \hookrightarrow \mathbf{D}^{\mathrm{b}} \operatorname{qgr}(A) \to \mathbf{D}^{\mathrm{b}} \operatorname{coh}(E)$. Thus for any $F \in \mathbf{D}^{\mathrm{b}} \operatorname{coh}(E)$, there exists an exact triangle

$$\mathbb{L}j^* \mathrm{pr}^R_{\mathcal{B}}(j_* F) \to F \to T(F) \to \mathbb{L}j^* \mathrm{pr}^R_{\mathcal{B}}(j_* F)[1]$$

in $\mathbf{D}^{\mathrm{b}} \operatorname{coh}(E)$.

Proof. Since $\mathbf{D}^{\mathrm{b}}\operatorname{qgr}(A)$ admits a full strong exceptional collection, there is a tilting object G by Theorem 2.21. We apply the lemmas above to $\Lambda = \operatorname{End}(G)$. There is a dg adjunction

$$- \otimes_\Lambda G \colon \mathscr{P}\!\mathit{erf}(\Lambda) \rightleftarrows \mathscr{P}\!\mathit{erf}(A) \colon \operatorname{Hom}(G, -) \tag{A.1}$$

where $\mathscr{P}\!\mathit{erf}(\Lambda)$ is a dg enhancement of the triangulated category of perfect dg modules $\operatorname{Perf}(\Lambda)$ and $\mathscr{P}\!\mathit{erf}(A)$ is a dg enhancement of $\mathbf{D}^{\mathrm{b}}\operatorname{qgr}(A)$ (see [17]). Since G is a tilting object, the adjunction induces an equivalence of triangulated categories between its homotopy categories.

$$- \overset{\mathbb{L}}{\otimes} G \colon \mathbf{D}^{\mathrm{b}} \operatorname{mod}\Lambda \simeq \mathbf{D}^{\mathrm{b}}\operatorname{qgr}(A) \colon \mathbb{R}\operatorname{Hom}(G, -) \tag{A.2}$$

Let $\mathcal{B}_\Lambda$ be a full dg subcategory of $\mathscr{P}\!\mathit{erf}(\Lambda)$ which corresponds to $\mathcal{B} \subset \mathbf{D}^{\mathrm{b}}\operatorname{qgr}(A)$ via the equivalence (A.2). By Lemma A.3, there exists a dg adjunction.

$$\mathcal{B}_\Lambda \rightleftarrows \mathscr{P}\!\mathit{erf}(\Lambda)$$

We have a dg adjunction $L \dashv R$

$$L \colon \mathcal{B}_\Lambda \rightleftarrows \mathscr{P}\!\mathit{erf}(\Lambda) \underset{}{\overset{(\mathrm{A.1})}{\rightleftarrows}} \mathscr{P}\!\mathit{erf}(A) \underset{j_*}{\overset{\mathbb{L}j^*}{\rightleftarrows}} \mathscr{P}\!\mathit{erf}(E) \colon R,$$

where $\mathscr{P}\!\mathit{erf}(E)$ is a dg enhancement of the triangulated category of perfect complexes on E. From the adjunction $L \dashv R$, we have an exact triangle of dg endofunctors of $\mathscr{P}\!\mathit{erf}(E)$

$$LR \xrightarrow{\epsilon} \mathrm{id} \to T \coloneqq \operatorname{Cone}(\epsilon) \to LR[1] \tag{A.3}$$

where $\epsilon \colon LR \to \mathrm{id}_{\mathscr{P}\!\mathit{erf}(E)}$ is a counit. By [24, Theorem 8.9], there exists an exact triangle

$$K_{LR} \to \mathcal{O}_\Delta \to K_T \to K_{LR}[1] \tag{A.4}$$

in $\mathbf{D}^{\mathrm{b}}\operatorname{coh}(E \times E)$ such that the exact triangle of integral functors associated to (A.4) corresponds to (A.3). By construction, $H^0(\Phi_{K_{LR}}) \simeq \mathbb{L}j^* \operatorname{pr}^R_{\mathcal{B}} j_*$ as the exact endofunctors of $\mathbf{D}^{\mathrm{b}}\operatorname{coh}(E)$. We have an exact triangle in $\mathbf{D}^{\mathrm{b}}\operatorname{coh}(E)$

$$\mathbb{L}j^* \operatorname{pr}^R_{\mathcal{B}} j_*(F) \to F \to T(F) \to \mathbb{L}j^* \operatorname{pr}^R_{\mathcal{B}} j_*(F)[1]$$

for any object $F \in \mathbf{D}^{\mathrm{b}}\operatorname{coh}(E)$. □

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