HOMOLOGY GROUPS OF SEVERAL ORIENTED GRASSMANNIANS

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Abstract. This is a report of a computation of the homology groups of Grassmannians $\widetilde{Gr}(m,n)$ of oriented m-subspaces in $\mathbb{R}^{m+n}$ for $(m,n) = (3,3)$, $(3,4)$, $(3,5)$, $(4,4)$, $(4,5)$, $(5,5)$, which shows that the integral homology of $\widetilde{Gr}(5,5)$ has 4-torsion in dimensions 10 and 14. The computation method in this report can be applied to all oriented Grassmannians.

1. Introduction

Although the cohomology of the Grassmannian $Gr(m,n)$ consisting of m dimensional subspaces in the real vector space $\mathbb{R}^{m+n}$ is fully understood, even the homology of the oriented Grassmannian $\widetilde{Gr}(m,n)$ consisting of oriented subspaces is still not clearly understood. The reason is due to the existence of torsions other than 2.

A Morse-Smale function on a manifold induces a chain complex whose homology is isomorphic to the singular homology of the manifold. This chain complex is called a *Morse complex*. In [5], the author investigated integral Morse complexes on the Grassmannians $\widetilde{Gr}(m,n)$, and obtained a matrix representation of the boundary maps (see Theroem 2.1). This allows us to compute the elementary divisors of the boundary maps. The computation results, which are check by Sage, for the cases $(m,n) = (3,3)$, $(3,4)$, $(3,5)$, $(4,4)$, $(4,5)$, $(5,5)$ are shown in Tables in Section 4. Through a general method (see Proposition 3.1) of homology computation of finitely generated chain complexes, we can use the results in Section 4 to get explicitly the singular homologies of those oriented Grassmannians as summarized in Table in Section 5.1. The method of computation based on Therorem 2.1 is applicable to all oriented Grassmannians.

In [2], the homology groups were computed for oriented Grassmannians $\widetilde{Gr}(m,n)$, including the cases $(m,n) = (3,3)$, $(3,4)$, $(3,5)$, $(4,4)$. The results agree with ours except for $(m,n) = (4,4)$ in dimensions 6 and 9. Other computation of $H_*(\widetilde{Gr}(3,3);\mathbb{Z})$ is described in [1, Table 6].

Both the existence and non-existence of 4-torsion in various oriented Grassmannians are investigated in [3, Theorem 8.8, Remark 8.10]. As shown

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in Table 7, the oriented Grassmannian $\widetilde{Gr}(5,5)$ has 4-torsion in the homology groups of dimensions 10 and 14. In light of the results in [3], $\widetilde{Gr}(5,5)$ is considered to be the lowest dimensional oriented Grassmannian having torsion other than 2.

2. Boundary maps of the Morse complex

The integral curves $t \mapsto \gamma(t)$ of the gradient flow, called *gradient lines*, of a function on a closed Riemannian manifold converge to critical points of the function as $t \to \pm\infty$. If the function satisfies the Morse-Smale transversality condition, the space of gradient lines between two critical points with Morse indices differing by k is a $k-1$ dimensional manifold. The *Morse complex* is a chain complex over $\mathbb{Z}$ generated by critical points of the Morse-Smale function and with boundary maps defined by counting the gradient lines with sign between critical points with Morse indices differing by 1. *The homology of a Morse complex is isomorphic to the singular homology of the manifold* (see [5] and references cited therein).

Given a subspace V of dimension m in the real vector space $\mathbb{R}^{m+n}$, there exists a unique symmetric matrix X_V whose $+1$ eigenspace is V and -1 eigenspace is the orthogonal complement of V. Let H be a diagonal matrix with diagonal elements $h_1, \cdots, h_{m+n}$. If $h_1 < \cdots < h_{m+n}$, then the function defined by $V \mapsto X_V \mapsto \mathrm{trace}(HX_V)$ on the Grassmannian $Gr(m,n)$ of m-subspaces is a Morse-Smale function, which we denote by $h : Gr(m,n) \to \mathbb{R}$.

Let $[k]$ be the set of integers $\{1, \cdots, k\}$, and $M(m,n)$ the set of increasing functions $\mu : [m] \to [m+n]$. For each $\mu \in M(m,n)$ and $\alpha \in [m]$, define a function μ_α by

$$\mu_\alpha(a) = \begin{cases} \mu(a) & (\ a \neq \alpha\ ) \\ \mu(a) - 1 & (\ a = \alpha\ ) \end{cases}$$

and set $c(\mu) = \{\alpha \in [m] \mid \mu_\alpha \in M(m,n)\}$.

Let $\{e_1, \cdots, e_{m+n}\}$ be the standard basis of $\mathbb{R}^{m+n}$. For each $\mu \in M(m,n)$, we denote by $V_\mu \in Gr(m,n)$ the subspace spanned by vectors $e_{\mu(1)}, \cdots, e_{\mu(m)}$. Then the set of critical points of the function h coincides with the set $\{V_\mu \mid \mu \in M(m,n)\}$, and the Morse index $\mathrm{Ind}(V_\mu)$ of a critical point V_μ equals $\sum_{a=1}^{m}(\mu(a) - a)$. The Grassmannian $Gr(m,n)$ has dimension mn, and the Morse indices are $0 \leq \mathrm{Ind}(V_\mu) \leq mn$. For two critical points V_μ and $V_{\mu'}$, there exist gradient lines from V_μ to $V_{\mu'}$ if and only if $\mu' = \mu_\alpha$ for some $\alpha \in c(\mu)$.

Given an integer k, let s_k be the number of elements $\mu \in M(m,n)$ such that the Morse index $\mathrm{Ind}(V_\mu)$ of the critical point V_μ is equal to k. The generating function of $\{s_k\}$ equals the q-binomial coefficient;

$$\sum_{k=0}^{mn} s_k q^k = \binom{m+n}{m}_q . \tag{2.1}$$

We denote by $\widetilde{Gr}(m,n)$ the *Grassmannian of oriented m-subspaces of* $\mathbb{R}^{m+n}$. Let $V_\mu^{\pm 1} \in \widetilde{Gr}(m,n)$ be the subspace $V_\mu \in Gr(m,n)$ endowed with orientation defined by an ordered basis $\{\pm e_{\mu(1)}, e_{\mu(2)}, \cdots, e_{\mu(m)}\}$. A covering map $\widetilde{Gr}(m,n) \to Gr(m,n)$ is defined by ignoring orientations. The composition $\widetilde{h}$ of this covering and the function h is a Morse-Smale function of $\widetilde{Gr}(m,n)$, whose Morse complex will be denote by $\mathcal{C}_*(m,n)$. The critical point set of $\widetilde{h}$ is $\{V_\mu^\epsilon \mid \mu \in M(m,n), \varepsilon = \pm 1\}$. Denote by $\langle V_\mu^\varepsilon \rangle$ the generators of $\mathcal{C}_*(m,n)$. In [5] we proved the following :

Theorem 2.1. *The k-th boundary map* ∂_* *of* $\mathcal{C}_k(m,n)$ *is given by*

$$\partial_k \langle V_\mu^\varepsilon \rangle = -\sum_{\alpha \in c(\mu)} (-1)^{\sum_{a=1}^{\alpha-1}(\mu(a)-a)} \left((-1)^{m+\mu(\alpha)} \langle V_{\mu_\alpha}^\varepsilon \rangle + \langle V_{\mu_\alpha}^{-\varepsilon} \rangle \right)$$

3. A general method of computing homology

Given a chain complex $\mathcal{C} = (C_*, \partial_*)$ with coefficients in $\mathbb{Z}$, let r_k denote the rank of C_k, σ_k the number of elementary divisors of $\partial_k : C_k \to C_{k-1}$ equal to 1, and ρ_k the number of those greater than 1. The rank β_k of free part of k dimensional homology group is called the *k-th Betti number*. The homology of the chain complex $\mathcal{C}$ is computed as follows (see, for example, [4]) :

Proposition 3.1. *The k-th Betti number* β_k *of* $\mathcal{C}$ *equals* $r_k - (\sigma_k + \sigma_{k+1} + \rho_k + \rho_{k+1})$. *Let* $\{\epsilon_1, \cdots, \epsilon_{\rho_{k+1}}\}$ *denote the elementary divisors of the boundary map* ∂_{k+1} *greater than* 1. *Then we have*

$$H_k(\mathcal{C}) \cong \mathbb{Z}^{\beta_k} \oplus \mathbb{Z}/\epsilon_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/\epsilon_{\rho_{k+1}}\mathbb{Z}.$$

4. Elementary divisors of the boundary maps

We denote the boundary map $\mathcal{C}_k(m,n) \to \mathcal{C}_{k-1}(m,n)$ of the Morse complex of $\widetilde{h} : \widetilde{Gr}(m,n) \to \mathbb{R}$ by ∂_k, the rank of $\mathcal{C}_k(m,n)$ by r_k, the number of elementary divisors of ∂_k equal to 1 by σ_k, those greater than 1 by ρ_k, respectively. The rank r_k equals $2s_k$, where s_k is obtained in formula (2.1).

Since Theorem 2.1 yields all data to represent the boundary map ∂_k as a matrix, if a linear order is specified on the sets $\{\langle V_\mu^\varepsilon \rangle\}$ of generators of both $\mathcal{C}_k(m,n)$ and $\mathcal{C}_{k-1}(m,n)$, we obtain the matrix representing ∂_k by a straightforward computation. Then it is possible to use, for example, SageMath to get the elementary divisors of ∂_k.

As an example of the above computation, we show that the boundary map $\partial_{11} : \mathcal{C}_{11}(5,5) \to \mathcal{C}_{10}(5,5)$ is represented by the matrix $\begin{pmatrix} A & B \\ B & A \end{pmatrix}$, where the blocks A and B are given by

$$A = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

$$B = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}$$

Then by using Sage function ”`elementary_divisor`”, we find that the non-zero elementary divisors of ∂_{11} consist of eighteen 1's and one 4, namely $\sigma_{11} = 18$ and $\rho_{11} = 1$ in this case.

The resulting values of r_k, σ_k, and ρ_k are shown in Tables 1 through 6 in the cases $(m, n) = (3, 3)$, $(3, 4)$, $(3, 5)$, $(4, 4)$, $(4, 5)$, $(5, 5)$. The elementary divisors greater than 1 of those boundary maps equal to 2, except ∂_{11} *and* ∂_{15} *of the case* $(m, n) = (5, 5)$*, which have an elementary divisor equal to* 4.

k	0	1	2	3	4	5	6	7	8	9
r_k	2	2	4	6	6	6	6	4	2	2
σ_k	0	1	1	2	3	2	3	2	1	1
ρ_k	0	0	0	1	0	0	0	1	0	0

TABLE 1. The case $(3, 3)$

k	0	1	2	3	4	5	6	7	8	9	10	11	12
r_k	2	2	4	6	8	8	10	8	8	6	4	2	2
σ_k	0	1	1	2	3	3	4	4	3	3	2	1	1
ρ_k	0	0	0	1	0	0	1	1	0	0	1	0	0

TABLE 2. The case $(3, 4)$

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
r_k	2	2	4	6	8	10	12	12	12	12	10	8	6	4	2	2
σ_k	0	1	1	2	3	4	5	6	5	6	5	4	3	2	1	1
ρ_k	0	0	0	1	0	0	1	0	0	0	1	0	0	1	0	0

TABLE 3. The case $(3, 5)$

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
r_k	2	2	4	6	10	10	14	14	16	14	14	10	10	6	4	2	2
σ_k	0	1	1	2	3	4	5	6	6	6	6	5	4	3	2	1	1
ρ_k	0	0	0	1	0	0	1	2	0	0	2	1	0	0	1	0	0

TABLE 4. The case $(4, 4)$

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
r_k	2	2	4	6	10	12	16	18	22	22	24	22	22	18	16	12	10	6	4	2	2
σ_k	0	1	1	2	3	5	6	8	9	10	11	11	10	9	8	6	5	3	2	1	1
ρ_k	0	0	0	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	0	0

TABLE 5. The case $(4, 5)$

k	0	1	2	3	4	5	6	7	8	9	10	11	12
r_k	2	2	4	6	10	14	18	22	28	32	36	38	40
σ_k	0	1	1	2	3	5	7	9	11	14	16	18	18
ρ_k	0	0	0	1	0	1	1	1	1	0	1	1	1

k	13	14	15	16	17	18	19	20	21	22	23	24	25
r_k	40	38	36	32	28	22	18	14	10	6	4	2	2
σ_k	20	18	18	16	14	11	9	7	5	3	2	1	1
ρ_k	0	1	1	1	0	1	1	1	1	0	1	0	0

TABLE 6. The case $(5, 5)$

5. MAIN RESULT

5.1. $H_k(\widetilde{Gr}(m,n);\mathbb{Z})$. Applying Proposition 3.1 to the results in Section 4, we obtain Table 7 of the homology groups $H_k = H_k(\widetilde{Gr}(m,n);\mathbb{Z})$ for

$(m,n) = (3,3)$, $(3,4)$, $(3,5)$, $(4,4)$, $(4,5)$, $(5,5)$. We see that $H_{10}(\widetilde{Gr}(5,5);\mathbb{Z})$ *and* $H_{14}(\widetilde{Gr}(5,5);\mathbb{Z})$ *are isomorphic to* $\mathbb{Z}/4\mathbb{Z}$.

(m,n)	(3,3)	(3,4)	(3,5)	(4,4)	(4,5)	(5,5)
H_0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
H_1	0	0	0	0	0	0
H_2	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
H_3	0	0	0	0	0	0
H_4	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^2$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$
H_5	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
H_6	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
H_7	0	0	$\mathbb{Z}$	0	0	$\mathbb{Z}/2\mathbb{Z}$
H_8	0	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}^4$	$\mathbb{Z}^3$	$\mathbb{Z}^2$
H_9	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$
H_{10}		0	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$
H_{11}		0	$\mathbb{Z}$	0	0	$\mathbb{Z}/2\mathbb{Z}$
H_{12}		$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}$
H_{13}			0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$
H_{14}			0	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$
H_{15}			$\mathbb{Z}$	0	0	$\mathbb{Z}/2\mathbb{Z}$
H_{16}				$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$
H_{17}					$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}^2\oplus\mathbb{Z}/2\mathbb{Z}$
H_{18}					0	$\mathbb{Z}/2\mathbb{Z}$
H_{19}					0	$\mathbb{Z}/2\mathbb{Z}$
H_{20}					$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
H_{21}						$\mathbb{Z}$
H_{22}						$\mathbb{Z}/2\mathbb{Z}$
H_{23}						0
H_{24}						0
H_{25}						$\mathbb{Z}$

TABLE 7. $H_*(\widetilde{Gr}(m,n);\mathbb{Z})$ with $(m,n) = (3,3)$, $(3,4)$, $(3,5)$, $(4,4)$, $(4,5)$, $(5,5)$.

5.2. $H_k(\widetilde{Gr}(m,n);\mathbb{Z}/2\mathbb{Z})$. We remark that the universal coefficient theorem describes $\mathbb{Z}/2\mathbb{Z}$ homology in terms of $\mathbb{Z}$ homology;

$$H_k(\widetilde{Gr}(m,n);\mathbb{Z}/2\mathbb{Z}) \cong H_k(\widetilde{Gr}(m,n);\mathbb{Z})\otimes\mathbb{Z}/2\mathbb{Z}\oplus\mathrm{Tor}(H_{k-1}(\widetilde{Gr}(m,n);\mathbb{Z}),\mathbb{Z}/2\mathbb{Z}).$$

On the other hand, if we use $\mathbb{Z}/2\mathbb{Z}$ as coefficients of the Morse complex of $\widetilde{h}:\widetilde{Gr}(m,n)\to\mathbb{R}$, the resulting Morse complex $\mathcal{C}_*(m,n)\otimes\mathbb{Z}/2\mathbb{Z}$ calculates the singular homology $H_k(\widetilde{Gr}(m,n);\mathbb{Z}/2\mathbb{Z})$. Since the rank of the boundary map of $\mathcal{C}_*(m,n)\otimes\mathbb{Z}/2\mathbb{Z}$ equals the number σ_k of the elementary divisors equal to 1, we also get the $\mathbb{Z}/2\mathbb{Z}$ Betti numbers of $\widetilde{Gr}(m,n)$ directly from

Tables in Section 4;

$$\dim_{\mathbb{Z}/2\mathbb{Z}} H_k(\widetilde{Gr}(m,n);\mathbb{Z}/2\mathbb{Z}) = r_k - (\sigma_k + \sigma_{k+1}).$$

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