Writhe invariants
of 3-regular spatial graphs

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Abstract

We give a necessary condition for two diagrams of 3-regular spatial graphs with the same underlying abstract graph G to represent isotopic spatial graphs. The test works by reading off the writhes of the knot diagrams coming from a collection of cycles in G in each diagram, and checking whether the writhe tuples differ by an element in the image of a certain map of $\mathbb{Z}$-modules determined by G. We exemplify by using our result to distinguish, for each $n \geq 3$, all elements in a certain infinite family of embeddings of the Möbius ladder M_n into $\mathbb{R}^3$. We also connect these writhe tuples to a classical invariant of spatial graphs due to Wu and Taniyama.

1   Introduction

The classical question in knot theory of determining whether two knots or links are isotopic naturally leads to the generalization where we consider instead embeddings of (finite) graphs into $\mathbb{R}^3$, that is, spatial graphs. An algorithm for determining whether two spatial graphs are isotopic has recently been published [2], but its computational complexity sets it beyond the realm of practical applicability. In the case of 3-regular spatial graphs (meaning, where all vertices have degree 3), the classification problem has been reduced to a comparison of elements in a semigroup defined by a certain presentation [4, Theorem 2].

One can sometimes obstruct the existence of an isotopy between two given spatial graphs by distinguishing the multisets of links contained in them as unions of cycles; in other words, reducing the problem to one of classical knot theory. However, many examples resist this technique – consider, for instance, the embeddings of the complete bipartite graph on $3 + 3$ vertices $K_{3,3}$ depicted in Figure 1. Here, all links that can be found as subgraphs of each embedding are unknots. Moreover, these spatial graphs are mirror images of one another, which typically makes the task of distinguishing them more difficult.

In this note, we give a rather elementary test for comparing 3-regular spatial graphs, which works in particular for distinguishing the pair in Figure 1. The starting point is the notion of the writhe $w(D)$ of a diagram D for a knot K, that is, the signed count of crossings in D. It is well-known that $w(D)$ carries no information about K – one can easily produce a diagram of K with arbitrary writhe. However, given a spatial graph Γ, we can fix a diagram D for Γ and simultaneously consider the writhes of all knot diagrams corresponding to cycles in the abstract
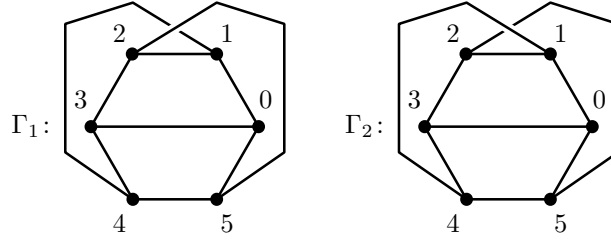

Figure 1: Two spatial graphs of type K$_{3,3}$.

underlying graph G of Γ. Our main result constrains the possible values of such a "writhe tuple". More precisely, denoting by $\mathrm{E}(G)$ the edge set of G, we associate to each set C of cycles in G a $\mathbb{Z}$-linear map

$$C^* : \mathbb{Z}^{\mathrm{E}(G)} \to \mathbb{Z}^C,$$

and relate the possible writhe-tuples $w_C(D)$ of diagrams D of Γ as follows:

Theorem 1 (Invariance of $[w_C(D)]$)**.** *Let Γ_1, Γ_2 be 3-regular spatial graphs with the same underlying abstract graph G, and let C be a set of cycles of G. Let D_1, D_2 be diagrams of Γ_1, Γ_2, respectively, with matching cyclic orders. If Γ_1 is isotopic to Γ_2, then*

$$w_C(D_2) - w_C(D_1) \in C^*(\mathbb{Z}^{\mathrm{E}(G)}).$$

This theorem is re-stated and proved in the text as Theorem 3.1. The "matching cyclic orders" assumption is a minor condition, easily circumvented when one wishes to apply the theorem for obstructing an isotopy between Γ_1 and Γ_2.

We also show that for each $n \geq 3$, Theorem 1 can be applied to prove that the elements in a particular infinite family of embeddings of the Möbius ladder M$_n$ are pairwise non-isotopic (Example 3.6), vastly generalizing the example in Figure 1. At the end, we explain how a slightly coarser equivalence class of writhe tuples than the one in Theorem 1 can be recovered from an invariant due to Wu and Tanayama [8, 7] that classifies spatial graphs "up to homology".

Structure of this article

Section 2 lays out the terminology necessary for stating and proving our main result. We group these preliminaries into three subsections, pertaining to knots, graphs, and spatial graphs. Most of the material is standard and can be skipped and consulted occasionally for clarifying our conventions, with two exceptions: Definitions 2.1 and 2.4 are not standard, and play a key role in this article.

In Section 3 we prove Theorem 1 and use it to show that the spatial graphs in Figure 1 are not isotopic. This example is then generalized to certain families of embeddings of the Möbius ladder M$_n$, for each $n \geq 3$.

Section 4 collects a few additional observations, namely on limitations of Theorem 1, and in Section 5 we briefly explain the Wu invariant and its connection to Theorem 1.

Acknowledgements

2 Preliminaries

We shall work in the piecewise-linear ("PL") category, following the conventions in Rourke-Sanderson's textbook [5].

2.1 Knots

In this article, we will define a **knot** to be a PL subspace $K \subset \mathbb{R}^3$ that is PL-homeomorphic to $\mathbb{S}^1$. We do not take the homeomorphism as part of the structure; in particular, knots will not carry orientations. We say K **has a diagram** if its image under the standard projection $\pi \colon \mathbb{R}^3 \to \mathbb{R}^2$ has only double-point self-intersections, and they are all transverse ("**crossings**"). Then, the **diagram** of K is $\pi(K)$ together with the data, at each crossing, of which strand crosses over which. After choosing an orientation for K, one associates to each crossing in a diagram D a sign and recording whether the over-strand moves right-to-left or left-to-right of the under-strand; see Figure 2. Although one reads off crossing signs by choosing an orientation of K, reversing this orientation does not change the sign, so the sign of a crossing is well-defined for diagrams of unoriented knots. If n_+, n_- are, respectively, the number of positive and negative crossings, then the **writhe** of D is the integer
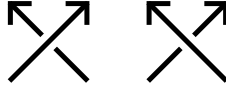
$$w(D) := n_+ - n_-.$$

Figure 2: A positive crossing (left) and a negative crossing (right).

Two knots K_1, K_2 are **isotopic** if there is a PL isotopy from $\mathrm{id}_{\mathbb{R}^3}$ to a PL self-homeomorphism Φ of $\mathbb{R}^3$ with $\Phi(K_1) = K_2$. It is well-known that the diagram of a knot K determines K up to isotopy, and every knot is isotopic to one that has a diagram. As we are interested in knots only up to isotopy, we loosely speak of "a diagram of K", even when K does not have a diagram, to mean "the diagram of a knot isotopic to K".

As Figure 3 illustrates, one can easily find isotopic knots with diagrams of different writhes. In fact, this trick of "adding a kink" to a diagram allows one to find isotopic knots whose diagrams realize any desired integer. The writhe of a diagram is thus very far from being a knot invariant.
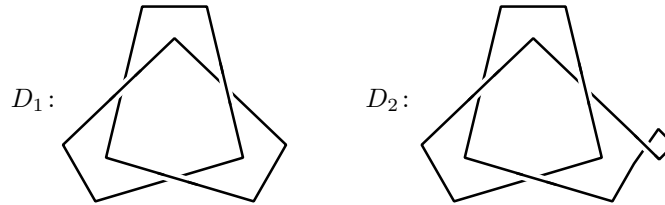

Figure 3: Diagrams of isotopic knots with $w(D_1) = 3$ and $w(D_2) = 2$.

2.2 Graphs

A **graph** will be understood as the data of a set $\mathrm{V}(G)$ of **vertices**, a set $\mathrm{E}(G)$ of **edges**, and a pair of functions $s_G, t_G \colon \mathrm{E}(G) \to \mathrm{V}(G)$. Thus we allow **loops** (that is, edges e with

$s_G(e) = t_G(e))$ and multiple edges between the same two vertices, and edges that are not loops carry an orientation. The set of **half-edges** of G is the product $\mathrm{E}(G) \times \{0, 1\}$. A half-edge $(e, 0)$ (respectively $(e, 1)$) is **incident** to a vertex v if $s_G(e) = v$ (respectively $t_G(e) = v$). The **degree** of v is the number of half-edges incident to v; in other words:

$$\deg(v) := \# s_G^{-1}(v) + \# t_G^{-1}(v) \in \mathbb{N} \cup \{\infty\}.$$

Given $n \in \mathbb{N}$, we say G is n-**regular** if all vertices of G have degree n.

The **topological realization** of G is the space

$$|G| := \left(\mathrm{V}(G) \sqcup \coprod_{e \in \mathrm{E}(G)} [0, 1]_e \right) / \sim,$$

where $s_G(e) \sim 0_e$ and $t_G(e) \sim 1_e$ for every $e \in \mathrm{E}(G)$. We are interested exclusively in the situation where G is finite, so $|G|$ embeds as a PL subspace in $\mathbb{R}^3$. Any two PL structures on $|G|$ are PL-homeomorphic restricting to the identity on $\mathrm{V}(G)$, and respecting the orientations of all topological edges $[0, 1]_e$, so from now on we always understand $|G|$ to carry a canonical polyhedral structure.

A **cycle** of length $k \in \mathbb{N}_{\geq 1}$, or k-**cycle**, in G is a sub-graph comprised of k distinct vertices $v_1, \dots, v_k$ and k distinct edges $e_1, \dots, e_k$, such that for each $i \in \mathbb{Z}/k$, we have $\{s_G(e_i), t_G(e_i)\} = \{v_i, v_{i+1}\}$. This definition is independent of the orientation of the edges.

Definition 2.1. Given a finite set C of cycles in a graph G, we define the map of free $\mathbb{Z}$-modules

$$C^* \colon \mathbb{Z}^{\mathrm{E}(G)} \to \mathbb{Z}^C$$
$$e \mapsto \sum_{\gamma \ni e} \gamma.$$

Example 2.2. Figure 4 depicts the complete bipartite graph $G = \mathrm{K}_{3,3}$ on $3 + 3$ vertices. We refrain from specifying edge orientations, as they will not play a role in what follows. $\mathrm{K}_{3,3}$ has 9 edges and 15 cycles: 9 of length 4, and 6 of length 6. We present the matrix of C^* in the standard bases, for C the set of 6-cycles.

2.3 Spatial graphs

Let a G be a finite graph. A **spatial graph** of type G is a PL embedding $\Gamma \colon |G| \hookrightarrow \mathbb{R}^3$. The **support** of Γ is the set $|\Gamma| := \Gamma(|G|)$. Denoting by $\pi \colon \mathbb{R}^3 \twoheadrightarrow \mathbb{R}^2$ the standard projection, we say Γ **has a diagram** if for the standard projection $\pi \colon \mathbb{R}^3 \to \mathbb{R}^2$, the composition $\pi \circ \Gamma$ is injective, except for finitely-many transverse double point intersections ("**crossings**") between interior points of edges.

Then, the **diagram** of Γ is the following data:

1. for each $v \in \mathrm{V}(G)$, the point $\pi(\Gamma(v))$,

2. for each $e \in \mathrm{E}(G)$, the PL-immersed curve $\pi(\Gamma([0, 1]_e))$, alongside the orientation inherited from $[0, 1]_e$,

3. at each **crossing** $p \in \mathbb{R}^2$, the datum of which of the two strands meeting at p is over-crossing.

We often use the same terminology to refer to the vertices/edges of G, to the Γ-images of their topological realizations, and to their π-projections.
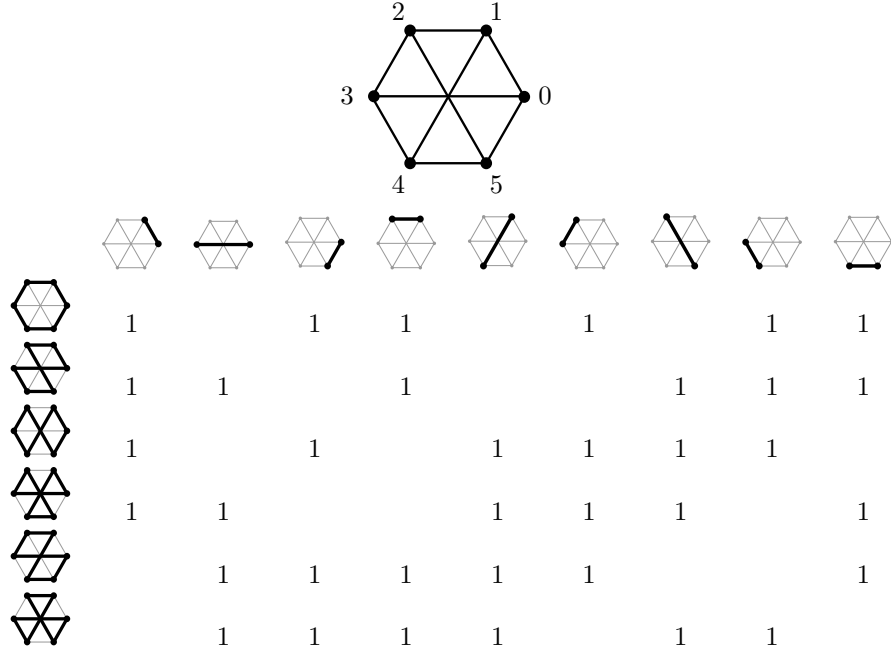
Figure 4: The graph $K_{3,3}$ and the matrix of C^*, with C the set of 6-cycles.

The spatial graph Γ associates to each half-edge $(e,0)$ (resp. $(e,1)$) of G the arc $\Gamma([0,\tfrac{1}{2}]_e)$ (resp. $\Gamma([\tfrac{1}{2},1]_e)$) in $\mathbb{R}^3$. For each $v \in V(G)$, the diagram D induces a cyclic ordering of these arcs, obtained by reading them off counterclockwise as their projections emanate from v, which in turn cyclically orders the half-edges. Two diagrams D_1, D_2 of spatial graphs of the same type G have **matching cyclic orders** if for every vertex $v \in V(G)$ they induce the same cyclic orders of the half-edges at v.

Two spatial graphs Γ_1, Γ_2 of type G, are **isotopic**, written $\Gamma_1 \cong \Gamma_2$, if there is a PL isotopy of $\mathbb{R}^3$ from $\mathrm{id}_{\mathbb{R}^3}$ to a map Φ with $\Phi \circ \Gamma_1 = \Gamma_2$. A diagram determines a spatial graph Γ up to isotopy, and one can always isotope a spatial graph Γ to one with a diagram, so we will often say "a diagram of Γ" to mean "the diagram of a spatial graph isotopic to Γ".

Kauffman has given the following characterization of when two diagrams represent isotopic spatial graphs:

Theorem 2.3 (Reidemeister moves for spatial graphs [3, Theorem 2.1]). *Any two diagrams representing isotopic spatial graphs differ by plane isotopies and a finite sequence of moves in five explicit types, illustrated in Figure 5 for the 3-regular case.*[1]

Evidently, the converse statement also holds: diagrams related by a sequence of these moves represent isotopic spatial graphs.

Moves R1 – R3 are the incarnations of the classical Reidemeister moves for knots. Moves R4 and R5 involve a vertex of Γ, which in general might have any degree. We illustrate the degree 3 case because our main result concerns 3-regular spatial graphs. Note that the induced cyclic ordering on the half-edges at a vertex is only affected by moves of type R5 at that vertex.

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[1] Kauffman's list contains one additional variant of R3, where the moving strand crosses *over* the other two, but that move can be deduced from the one we present, together with R2. Similarly, the two R4 moves we show, where the moving strand transitions between crossing one and two strands, have variants where it shifts between crossing zero and three strands. But again, using R2 it is easy to see that those are redundant.
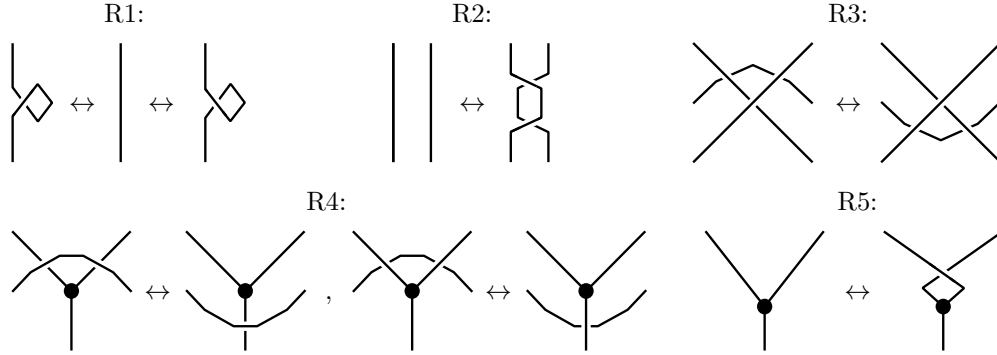
Figure 5: Reidemeister moves for 3-regular spatial graphs.

Writhe tuples

A spatial graph Γ of type G specifies, for each cycle γ of G, a knot $\Gamma(|\gamma|)$, and a diagram D of Γ yields a diagram D_γ of that knot.

Definition 2.4. Let Γ be a spatial graph of type G, and D a diagram of Γ. Given a set C of cycles in G, the **writhe tuple** of D associated to C is

$$w_C(D) := (w(D_\gamma))_{\gamma \in C} \in \mathbb{Z}^C.$$

We emphasize that although the edges of Γ are oriented, these orientations play no role in computing $w_C(D)$. The signs of the crossings to be tallied into each entry $w(D_\gamma)$ are given by choosing an orientation for the knot $\Gamma(|\gamma|)$ and using it for attaching signs to the crossings in D_γ. Example 2.5 shows a crossing in a spatial graph diagram D that is counted with different signs depending on which D_γ is being considered.

Example 2.5. Figure 6 depicts diagrams D_1, D_2 of two spatial graphs Γ_1, Γ_2 of type $K_{3,3}$, and their writhe tuples associated to the set C of 6-cycles.

$$w_C(D_1) = (\quad 0, \quad 0, \quad 1, \quad -1, \quad 0, \quad -1 \quad)$$
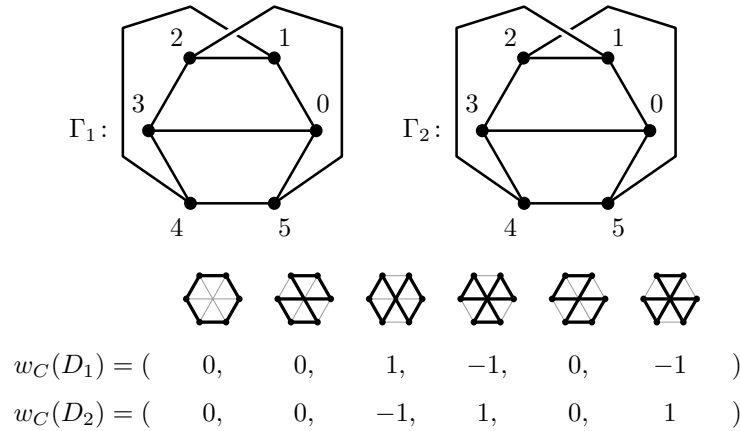$$w_C(D_2) = (\quad 0, \quad 0, \quad -1, \quad 1, \quad 0, \quad 1 \quad)$$

Figure 6: Diagrams for two spatial graphs of type $K_{3,3}$, and their writhe tuples associated to the set C of 6-cycles.

Although, as we observed in Section 2.1, the writhe of a knot diagram contains no information about the knot, we will show that the writhe tuple of a spatial graph diagram may contain information about the spatial graph.

3   Main result

In this section we shall prove Theorem 1, which we re-state here for the reader's convenience:

Theorem 3.1 (Invariance of $[w_C(D)]$)**.** *Let Γ_1, Γ_2 be 3-regular spatial graphs of the same type G, and let C be a set of cycles of G. Let D_1, D_2 be diagrams of Γ_1, Γ_2, respectively, with matching cyclic orders. If $\Gamma_1 \cong \Gamma_2$, then*

$$w_C(D_2) - w_C(D_1) \in C^*(\mathbb{Z}^{\mathrm{E}(G)}).$$

Note that the hypothesis on the cyclic orders is no significant loss of generality. In applying Theorem 3.1 to show that D_1 and D_2 do not represent isotopic spatial graphs, one can first modify them with R5 moves to ensure that this condition is met.

We will prove Theorem 3.1 by studying how the writhe tuple is affected by each Reidemeister move. The 3-regularity assumption plays a role only in describing the effect of R5-type moves.

Lemma 3.2 (Moves R1–R4)**.** *Let D, D' be diagrams of spatial graphs of type G, and let C be a set of cycles of G.*

1. *If D' differs from D by an R1 move at the edge $e \in \mathrm{E}(G)$, then $w_C(D') = w_C(D) + \varepsilon C^*(e)$, where $\varepsilon = 1$ if the move adds a positive crossing or removes a negative crossing; otherwise $\varepsilon = -1$.*

2. *If D' differs from D by a move of type $\mathrm{R}2, \mathrm{R}3$ or $\mathrm{R}4$, then $w_C(D') = w_C(D)$.*

Note that crossings involved in an R1 move have a well-defined sign, so the condition in item 1 makes sense independently of any choice of cycle.

Proof. (1) Applying an R1 move at e adds the same value ε to the writhe of D_γ for each $\gamma \in C$ containing e. The writhes of diagrams associated to other cycles remain unchanged. Hence,

$$w_C(D') - w_C(D) = \varepsilon \sum_{\gamma \ni e} \gamma = \varepsilon C^*(e).$$

(2) An R2 move affects only the writhes of knot diagrams D_γ for cycles γ containing both edges involved. In that case, the added crossings come with opposite signs, so either way $w(D_\gamma)$ remains unaltered. Similarly, R3 merely displaces crossings between the same edges, so it has no effect on $w(D_\gamma)$ for any cycle γ, and again $w_C(D') = w_C(D)$.

As for R4, clearly $w(D'_\gamma) = w(D_\gamma)$ if γ does not contain the edge e crossing over/under the relevant vertex v, so assume that $\gamma \ni e$. Next, note that γ contains either none, or exactly two among the half-edges incident to v. In the first case, there is again clearly no effect on $w(D_\gamma)$. If γ contains the two half-edges at v that gain or lose a crossing, then on D_γ this is indistinguishable from an R2 move. Otherwise, D_γ merely has one crossing displaced. Either way, $w(D'_\gamma) = w(D_\gamma)$ and we conclude $w_C(D') = w_C(D)$. □

Corollary 3.3 (Realizing tuples)**.** *Let D be a diagram of a spatial graph Γ of type G, let C be a set of cycles in G, and let $w \in w_C(D) + C^*(\mathbb{Z}^{\mathrm{E}(G)})$. Then Γ has a diagram D' with $w_C(D') = w$.*

Proof. If $w - w_C(D) = C^*((n_e)_{e \in \mathrm{E}(G)})$, we modify D by performing $|n_e|$ moves of type R1 at each edge e, adding crossings of the same sign as n_e. Lemma 3.2 (1) guarantees the resulting diagram D' has w as its writhe tuple. $\qquad \square$

In describing the effect of the move R5, we will use the same notation C^* for the extended map

$$C^* \colon \left(\tfrac{1}{2}\mathbb{Z}\right)^{\mathrm{E}(G)} \to \left(\tfrac{1}{2}\mathbb{Z}\right)^C,$$

where $\tfrac{1}{2}\mathbb{Z} := \{\ldots, -1, -\tfrac{1}{2}, 0, \tfrac{1}{2}, 1, \ldots\}$ is the free $\mathbb{Z}$-module of half-integers.

Lemma 3.4 (Move R5). *Let D, D' be diagrams of 3-regular spatial graphs of type G, and let C be a set of cycles of G. Suppose D' differs from D by an R5 move at the vertex v. Let $e_1, e_2 \in \mathrm{E}(G)$ be the edges at v that gain or lose a crossing, and e_3 the remaining edge.*
    Then

$$w(D') = w(D) + C^* \left(\tfrac{\varepsilon}{2}(e_1 + e_2 - e_3)\right),$$

where $\varepsilon = 1$ if the move adds a crossing, and $\varepsilon = -1$ if it removes a crossing.

Note that e_1, e_2, e_3 are not necessarily distinct, as G might have a loop at v.

Proof. Each $\gamma \in C$ contains either none, or exactly two of the half-edges at v, and

$$w(D'_\gamma) - w(D_\gamma) = \begin{cases} \varepsilon & \text{if } e_1, e_2 \in \gamma, \\ 0 & \text{otherwise.} \end{cases} \qquad (\star)$$

For the element $u := \tfrac{\varepsilon}{2}(e_1 + e_2 - e_3)$, we have

$$C^*(u) = \frac{\varepsilon}{2} \left(\sum_{\gamma \ni e_1} \gamma + \sum_{\gamma \ni e_2} \gamma - \sum_{\gamma \ni e_3} \gamma \right).$$

In this sum, the cycles γ containing both e_1 and e_2 (hence not e_3) have coefficient $\tfrac{\varepsilon}{2}(1+1) = \varepsilon$, those containing e_1 and e_3 have coefficient $\tfrac{\varepsilon}{2}(1 - 1) = 0$, and similarly for those containing e_2 and e_3. Obviously if γ contains none among e_1, e_2, e_3, its coefficient is also 0. In all cases, this is precisely the value of $(\star)$, and so

$$w_C(D') - w_C(D) = C^*(u). \qquad \square$$

Proof of Theorem 3.1. By Theorem 2.3, D_2 differs from D_1 by a sequence of Reidemeister moves. Thus, Lemmas 3.2 and 3.4 combined show that $w_C(D_2) - w_C(D_1)$ is a sum of elements of the form $\varepsilon C^*(e) \in C^*(\mathbb{Z}^{\mathrm{E}(G)})$, and of the form $C^*(\tfrac{\varepsilon}{2}(e_1 + e_2 - e_3))$ with e_1, e_2, e_3 the edges at a vertex. The condition on induced cyclic orders guarantees that, moreover, an even number of R5 moves is applied at each vertex, so the summands of the second type can be grouped into pairs associated to the same vertex.
    Now, for every two elements

$$u = \tfrac{\varepsilon}{2}(e_1 + e_2 - e_3), \qquad u' = \tfrac{\varepsilon'}{2}(e'_1 + e'_2 - e'_3)$$

with $\varepsilon, \varepsilon' \in \{\pm 1\}$ and $\{e_1, e_2, e_3\} = \{e'_1, e'_2, e'_3\}$, the sum $u + u'$ has coefficients in $\{-1, 0, 1\}$, so $u + u' \in \mathbb{Z}^{\mathrm{E}(G)}$. Hence, the joint contribution of each pair has the form

$$C^*(u + u') \in C^*(\mathbb{Z}^{\mathrm{E}(G)}). \qquad \square$$

Example 3.5. The diagrams for the spatial graphs Γ_1, Γ_2 from Example 2.5 have matching cyclic orders of the half-edges at every vertex. Their writhe tuples associated to the set C of 6-cycles differ by

$$w_C(D_2) - w_C(D_1) = (0, 0, -2, 2, 0, 2).$$

One now checks that the map C^*, whose matrix is shown in Example 2.2, does not have this tuple in its image. A quick way to see this is by noting that all columns of that matrix have entries adding up to 4 (in other words, each edge of $K_{3,3}$ is in exactly 4 length-6 cycles). Yet the entry sum of $w_C(D_2) - w_C(D_1)$ is not a multiple of 4. Thus, by Theorem 3.1, we conclude $\Gamma_1 \ncong \Gamma_2$.

The interested reader can check that the same argument could have been carried out using as C the set of 4-cycles instead.

Finally, it is worth mentioning the following connection, kindly pointed out to us by Ryo Nikkuni, to a $\mathbb{Z}$-valued invariant for spatial graphs Γ of type $K_{3,3}$, called the Simon invariant $S(\Gamma)$. For a definition, see Taniyama's account [6, Section 4]; the crucial point for us is that, by performing a signed count of how many times each pair of edges appears in a cycle, one can observe that for the writhe tuple $w_C(D) = (w_1, \ldots, w_6)$ of any diagram D of Γ, we have $w_1 - w_2 - w_3 + w_4 - w_5 + w_6 = 3S(\Gamma)$.

On the other hand, a standard computation shows that there is an isomorphism

$$\varphi\colon \mathbb{Z}^C/C^*(\mathbb{Z}^{E(G)}) \to \mathbb{Z} \oplus \mathbb{Z}/2$$
$$[\gamma_1] = [\gamma_4] = [\gamma_6] \mapsto (1, 0)$$
$$[\gamma_2] = [\gamma_3] = [\gamma_5] \mapsto (-1, 1),$$

where $\gamma_1, \ldots, \gamma_6$ are the 6-cycles of $K_{3,3}$ (ordered as above). Thus, the first coordinate of $\varphi([w_C(D)])$ equals $3S(\Gamma)$, and therefore Theorem 3.1 can tell apart any two spatial graphs of type $K_{3,3}$ that are distinguished by the Simon invariant.

Example 3.5 can be vastly generalized:

Example 3.6. Given $n \in \mathbb{N}_{\geq 1}$, the **Möbius ladder** with n rungs M_n is the 3-regular graph with vertex set $\mathbb{Z}/2n$, and

- for each $k \in \mathbb{Z}/2n$, an edge e_k connecting k to $k + 1$,

- for each $l \in \mathbb{Z}/n$, an edge r_l connecting the two vertices that have l as their mod-n residue

(with some choice of edge orientations, which will be irrelevant). See Figure 7.

For each $n \geq 1$ and odd integer m, we consider the spatial graph Γ_n^m of type M_n with diagram D_n^m as in Figure 8: there are exactly $|m|$ crossings, all between e_0 and e_n, and their sign matches the sign of m (if e_0, e_n are oriented, respectively, from 0 to 1 and from n to $n+1$).

We will use Theorem 3.1 to prove that if $n \geq 3$, then $\Gamma_n^m \ncong \Gamma_n^{m'}$ whenever $m \neq m'$. (For $n = 1$ or $n = 2$ we recover, respectively, the theta-graph and the complete graph on 4 vertices K_4. In either case, one easily checks that for every odd m, m', we have $\Gamma_n^m \cong \Gamma_n^{m'}$.)

Let us then fix $n \geq 3$ once and for all. We consider:

- the set C_4 of 4-cycles obtained by choosing $l \in \mathbb{Z}/n$, and taking the edges r_l, r_{l+1} alongside the two e_k with $k \equiv l \mod n$,

- the set C_+ of $(n+1)$-cycles obtained by choosing $k \in \mathbb{Z}/2n$ and taking the edge r_k, along with $e_k, e_{k+1}, \ldots, e_{k+n-1}$,

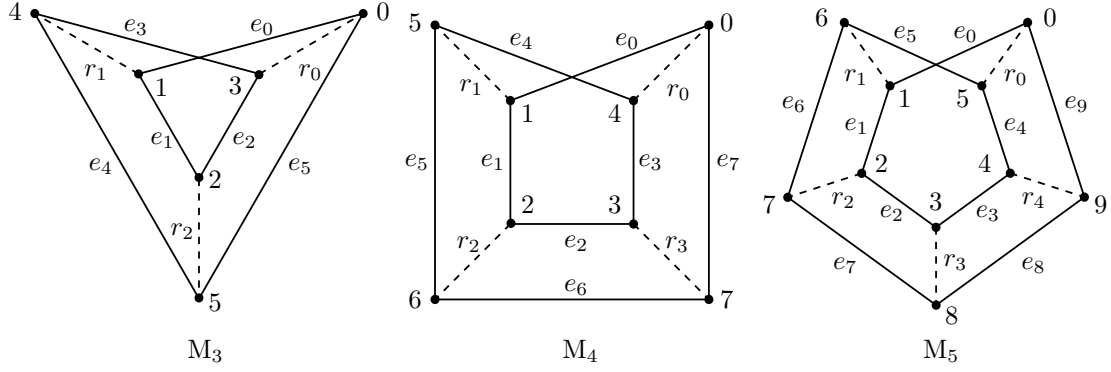- the $2n$-cycle γ comprised of the edges e_k over all $k \in \mathbb{Z}/2n$.

Figure 7: The Möbius ladders M_3, M_4 and M_5.
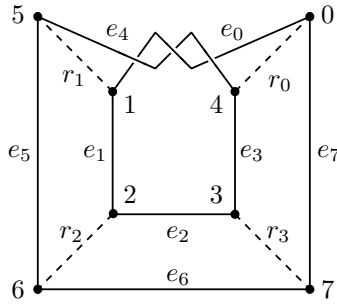

Figure 8: The diagram D_4^3 of Γ_4^3.

We now put $C := C_4 \cup C_+ \cup \{\gamma\}$. For each odd m, there are precisely two non-zero entries in $w_C(D_n^m)$: the 4-cycle containing e_0 and e_n has coordinate $-m$, and γ has coordinate m. We illustrate the system of equations relevant for comparing $\Gamma_n^m, \Gamma_n^{m'}$ in the $n = 4$ case:

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	r_0	r_1	r_2	r_3	$w_C(D_n^{m'}) - w_C(D_n^m)$
	1				1				1	1			$-(m' - m)$
C_4		1				1				1	1		0
			1				1				1	1	0
				1				1	1			1	0
	1	1	1	1					1				0
		1	1	1	1					1			0
			1	1	1	1					1		0
				1	1	1	1					1	0
C_+					1	1	1	1	1				0
	1					1	1	1		1			0
	1	1					1	1			1		0
	1	1	1					1				1	0
γ	1	1	1	1	1	1	1	1					$m' - m$

We will show that this system has no solution if $m \neq m'$, and therefore, by Theorem 3.1, we conclude $\Gamma_n^m \ncong \Gamma_n^{m'}$. For each $l \in \mathbb{Z}/n$, the sum of the two "C_+" equations involving r_l, minus the "γ" equation, shows that the r_l-coordinate of any solution is $-\frac{m'-m}{2}$. Using this information, the sum of all "C_4" equations (of which there are n) tells us that the e_k-coordinates add up

to $(n-1)(m'-m)$. From the "γ" equation, we then deduce $(n-2)(m'-m) = 0$, which, as $n \neq 2$, implies $m' = m$.

4   Additional remarks

3-regularity

The 3-regularity assumption in Theorem 3.1 is essential, as the next example shows.

Example 4.1. The diagrams D_1, D_2 in Figure 9 represent isotopic spatial graphs and induce the same cyclic orders on half-edges at each vertex. They are however not 3-regular.
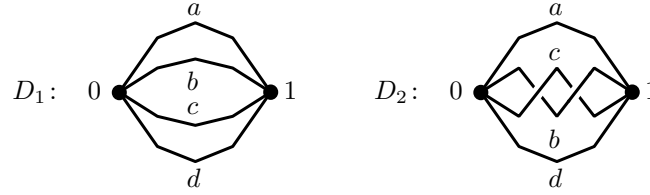

Figure 9: Diagrams for two isotopic spatial graphs.

The reader can easily check that for C the set of all 2-cycles (of which there are 6), we have $w_C(D_2) - w_C(D_1) \notin C^*(\mathbb{Z}^{\mathrm{E}(G)})$.

Choosing C

In applying Theorem 3.1 to distinguish spatial graphs of type G given as diagrams D_1, D_2, one must settle on a set C of cycles. Note that the map C^* satisfies the following naturality property: given a subset $C' \subseteq C$, the corresponding map $(C')^*$ is the composition

$$\mathbb{Z}^{\mathrm{E}(G)} \xrightarrow{C^*} \mathbb{Z}^C \twoheadrightarrow \mathbb{Z}^{C'},$$

where the map to the right is the canonical projection. Each writhe tuple $w_{C'}(D_i)$ is also the projection of $w_C(D_i)$. Thus, if $w_{C'}(D_2) - w_{C'}(D_1) \notin (C')^*(\mathbb{Z}^{\mathrm{E}(G)})$, then also $w_C(D_2) - w_C(D_1) \notin C^*(\mathbb{Z}^{\mathrm{E}(G)})$. In other words, the method is, unsurprisingly, more powerful if the larger set C is used.

Invisible crossing changes

A **crossing change** on a spatial graph diagram is the operation of flipping the data at a crossing of which strand crosses over which. The following proposition indicates a situation where Theorem 3.1 has no chance of obstructing the existence of an isotopy between spatial graphs.

Proposition 4.2 (Invisible crossing changes)**.** *Let D, D' be diagrams of 3-regular spatial graphs of type G, and let C be a set of cycles of G. If D and D' differ only by a crossing change between strands in the same edge, or in adjacent edges, then*

$$w_C(D') - w_C(D) \in C^*(\mathbb{Z}^{\mathrm{E}(G)}).$$

Proof. Suppose first that both strands at the relevant crossing p of D are in the same edge e. The sign of p is the same in the diagram D_γ of every $\gamma \in C$ containing e, and the crossing change alters that sign in every γ, adding the same value $2\varepsilon = \pm 2$ to $w(D_\gamma)$ for every γ containing e. All other cycles are unaffected, so $w_C(D') - w_C(D) = C^*(2\varepsilon e)$. (In other words, from the perspective of the writhe tuple, this crossing change is indistinguishable from performing two R1 moves at e.)

If the crossing change happens between two different edges e_1, e_2 meeting at v, then the cycles $\gamma \in C$ for which $w(D_\gamma)$ gets affected are precisely the ones containing both e_1 and e_2. Since e_1, e_2 are adjacent, the sign of p in D_γ is the same for every $\gamma \in C$, and the crossing change has the effect of adding the same value $2\varepsilon = \pm 2$ to $w(D_\gamma)$ for γ containing e_1 and e_2. As far as the writhe tuple is concerned, this is the same as applying two R5 moves at v, or its mirrored version, between e_1 and e_2. Hence, if D° is the diagram obtained from D by performing these two moves (instead of the crossing change), we have $w_C(D^\circ) = w_C(D')$. Since D° and D represent isotopic spatial graphs, and they have matching cyclic orders, we conclude by Theorem 3.1 that

$$w_C(D') - w_C(D) = w_C(D^\circ) - w_C(D) \in C^*(\mathbb{Z}^{\mathrm{E}(G)}). \qquad \square$$

5 Connection to the Wu invariant

In 1995, Taniyama gave a classification of spatial graphs without degree restrictions "up to homology" (a coarser equivalence than isotopy) [7], by specializing an invariant due to Wu for embedded CW-complexes in Euclidean spaces [8]. For another exposition, see for example the paper of Flapan-Fletcher-Nikkuni [1, Section 2]. In this section, we briefly present Taniyama's very explicit description of the Wu invariant for spatial graphs, and connect it to Theorem 3.1.

Given a graph G, consider the free $\mathbb{Z}$-module $Z(G)$ spanned by the unordered pairs $E^{e,f} = E^{f,e} := \{e, f\}$ of edges that do not share a vertex. Then define, for each vertex v and edge e not incident to v, the following element of $Z(G)$:

$$V^{v,e} := \sum_{\substack{f \text{ incident to } v \\ f \text{ disjoint from } e}} \delta^{v,f} E^{e,f}, \quad \text{where } \delta^{v,f} := \begin{cases} 1 & \text{if } s_G(f) = v, \\ -1 & \text{if } t_G(f) = v. \end{cases}$$

Note that in this sum, a loop f at v is counted once with each sign, so the coefficient of $E^{e,f}$ vanishes. We let $B(G) \subseteq Z(G)$ be the submodule generated by all $V^{v,e}$, and define the **linking module** of G to be

$$L(G) := Z(G)/B(G).$$

The Wu invariant of a spatial graph Γ is then an element of $L(G)$, which can be read off of a diagram D as follows: for each pair of edges e, f of G, let $\ell_D(e, f) = \ell_D(f, e)$ be the signed number of crossings between e, f (read off using the orientations of e, f). The **Wu invariant** of Γ is

$$\mathcal{L}(\Gamma) = \left[\sum_{e, f \text{ disjoint}} \ell_D(e, f) E^{e,f} \right] \in L(G).$$

Taniyama shows that this equality (which we will take as a definition) recovers Wu's invariant, and is therefore independent of the diagram D [7, Proposition 2.1]. Note that crossings in D between edges that share a vertex do not contribute to the sum, which should remind us of Proposition 4.2!

For our purposes, it will be convenient to express the linking module as a modified quotient

$$L(G) = \tilde{Z}(G)/\tilde{B}(G),$$

where $\tilde{Z}(G)$ differs from $Z(G)$ in that it is has as $\mathbb{Z}$-basis the variables $E^{e,f}$ for *all* unordered pairs of edges e, f (including when e, f share a vertex, possibly even coinciding). Accordingly, we also define $\tilde{B}(G) \subseteq \tilde{Z}(G)$ as the submodule generated by

1. the variables $E^{e,f}$ where e and f share a vertex, and

2. the elements
$$\tilde{V}^{v,e} := \sum_{f \text{ incident to } v} \delta^{v,f} E^{e,f},$$

   taken over all vertices v and edges e that are not incident.

This results in a quotient that is canonically isomorphic to $L(G)$ as defined above, and the Wu invariant of a spatial graph Γ with diagram D can be expressed as a sum over *all* edges:

$$\mathcal{L}(\Gamma) = \left[\sum_{e,f \in \mathrm{E}(G)} l_D(e,f) E^{e,f} \right].$$

From now on, assume G is 3-regular. Before presenting the main result of this section, we need to introduce the following notation:

- For each $v \in \mathrm{V}(G)$, with incident edges e, f, g (loops counted twice), define
$$u_v := \tfrac{1}{2}(e + f + g) \in (\tfrac{1}{2}\mathbb{Z})^{\mathrm{E}(G)}.$$

- Denote by $\mathcal{C}_G \subseteq (\tfrac{1}{2}\mathbb{Z})^{\mathrm{E}(G)}$ the submodule spanned by $\mathrm{E}(G) \cup \{u_v \mid v \in \mathrm{V}(G)\}$. Recall the observation at the heart of Lemma 3.4, that when C^* is regarded as the extended map $(\tfrac{1}{2}\mathbb{Z})^{\mathrm{E}(G)} \to (\tfrac{1}{2}\mathbb{Z})^C$, we have

$$C^*(u_v - g) = \sum_{\gamma \ni e,f} \gamma \in \mathbb{Z}^C. \tag{†}$$

   In particular, $C^*(\mathcal{C}_G) \subseteq \mathbb{Z}^C$.

- Given edges e, f in a cycle γ of G, put

$$\varepsilon_\gamma^{e,f} := \begin{cases} 1 & \text{if } e, f \text{ have matching orientation along } \gamma, \\ -1 & \text{otherwise.} \end{cases}$$

With this notation in place, we can now clarify how the invariance of Writhe tuple cosets given by Theorem 3.1 can be seen as a consequence of the Wu invariant.

Proposition 5.1 (Writhe tuple from the Wu invariant). *If G is a 3-regular graph, then the $\mathbb{Z}$-module homomorphism*

$$\varphi \colon \tilde{Z}(G) \to \mathbb{Z}^C$$
$$E^{e,f} \mapsto \sum_{\gamma \ni e,f} \varepsilon_\gamma^{e,f} \gamma$$

descends to a map $\phi \colon L(G) \to \mathbb{Z}^C/C^*(\mathcal{C}_G)$.

Moreover, if D is a diagram of a spatial graph Γ of type G, then $\phi(\mathcal{L}(\Gamma)) = [w_C(D)]$.

Proof. For the first part, we need to check that $\varphi(E^{e,f}) \in C^*(\mathcal{C}_G)$ whenever e and f share a vertex, and that $\varphi(\tilde{V}^{v,e}) \in C^*(\mathcal{C}_G)$ for every vertex v and disjoint edge e.

Let us first deal with the former. For $e = f$, we see

$$\varphi(E^{e,e}) = \sum_{\gamma \ni e} \gamma = C^*(e),$$

and if e and f are distinct edges sharing a vertex v, then $\varepsilon_\gamma^{e,f} = -\delta_1 \delta_2$ (independently of γ), so

$$\varphi(E^{e,f}) = -\delta_1 \delta_2 \sum_{\gamma \ni e,f} \gamma \stackrel{(\dagger)}{=} -\delta_1 \delta_2 C^*(u_v - g),$$

where g is the third edge at v.

We next compute $\varphi(\tilde{V}^{v,e})$ for every vertex v and disjoint edge e. Denoting by e_1, e_2, e_3 the edges at v appearing in the definition of $\tilde{V}^{v,e}$, we abbreviate $\delta^{v,e_i} =: \delta_i$. The key is the following observation: for every cycle γ containing e, e_1 and e_2, we have

$$\delta_1 \varepsilon_\gamma^{e_1,e} = -\delta_2 \varepsilon_\gamma^{e_2,e} \qquad (\star)$$

(and similarly for the other pairs of indices). To see this, one should first note that, since e_1 and e_2 are in a cycle with e, they cannot be loops, so they do not appear twice in the sum defining $\tilde{V}^{v,e}$, and hence $e_1 \neq e_2$. One can then easily see that

$$\varepsilon_\gamma^{e_1,e} \varepsilon_\gamma^{e_2,e} = \varepsilon_\gamma^{e_1,e_2} = -\delta_1 \delta_2,$$

from which $(\star)$ follows.

We now see

$$\varphi(\tilde{V}^{v,e}) = \delta_1 \varphi(E^{e_1,e}) + \delta_2 \varphi(E^{e_2,e}) + \delta_3 \varphi(E^{e_3,e}).$$

Expanding the summand coming from e_1, we obtain

$$\delta_1 \varphi(E^{e_1,e}) = \delta_1 \sum_{\gamma \ni e_1,e} \varepsilon_\gamma^{e_1,e} \gamma$$

$$= \delta_1 \sum_{\gamma \ni e_1,e_2,e} \varepsilon_\gamma^{e_1,e} \gamma + \delta_1 \sum_{\gamma \ni e_1,e_3,e} \varepsilon_\gamma^{e_1,e} \gamma.$$

Using $(\star)$, this can be rewritten as

$$-\delta_2 \sum_{\gamma \ni e_1,e_2,e} \varepsilon_\gamma^{e_2,e} \gamma - \delta_3 \sum_{\gamma \ni e_1,e_3,e} \varepsilon_\gamma^{e_3,e} \gamma.$$

The first summand thus cancels one of the summands coming from e_2, and the second cancels one summand coming from e_3. By the same reasoning, the remaining summands coming from e_2, e_3 cancel each other, and we conclude $\varphi(\tilde{V}^{v,e}) = 0$.

For the second part, we see that $\phi(\mathcal{L}(\Gamma))$ is represented by

$$\sum_{e,f \in \mathrm{E}(G)} \ell_D(e,f) \varphi(E^{e,f}) = \sum_{e,f \in \mathrm{E}(G)} \ell_D(e,f) \sum_{\gamma \ni e,f} \varepsilon_\gamma^{e,f} \gamma,$$

where it is clear that for each $\gamma \in C$, the γ-coordinate is

$$\sum_{\gamma \ni e,f} \varepsilon_\gamma^{e,f} \ell_D(e,f) = w(D_\gamma). \qquad \square$$

It is worth emphasizing that this proof strongly relies on the description of the linking module $L(G)$ in terms of the "tilde" modules $\tilde{Z}(G)$ and $\tilde{B}(G)$. We use the extra variables $E^{e,f}$ (for e, f intersecting), and the modified vertex elements $\tilde{V}^{v,e}$ so that when computing $\varphi(\tilde{V}^{v,e})$, we get *precisely* three summands $\delta_i \varphi(E^{e_i,e})$. Using the standard definition, the summands where e_i intersects e would not appear, and the ensuing computation, where everything neatly cancels out, would not be possible.

We finish by drawing the reader's attention to the fact that the residue class of $w_C(D)$ in Proposition 5.1 is with respect to $C^*(\mathcal{C}_G)$, which is in general larger than the submodule $C^*(\mathbb{Z}^{\mathrm{E}(G)})$ featured in our main result, Theorem 3.1. This might suggest that Theorem 3.1 refines the criterion given by the Wu invariant. However, this is merely an artifact of the linking module being defined without a fixed choice of cyclic orders of half-edges at vertices, whereas our main result requires one such choice. We leave it as an exercise for the interested reader to explain how one can tweak the definition of $L(G)$ and $\mathcal{L}(\Gamma)$ to a modified linking module $L'(G)$ and an element $\mathcal{L}'(\Gamma) \in L'(G)$ that is only well defined up to moves R1–R4 and an *even* number of R5 moves at each vertex. One can then define a map $\phi' \colon L'(G) \to \mathbb{Z}^C/C^*(\mathbb{Z}^{\mathrm{E}(G)})$ analogous to the above ϕ.

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