NORMALIZED SOLUTIONS FOR THE MASS SUPERCRITICAL p-KIRCHHOFF EQUATION WITH POTENTIAL

YUAN XU AND YONGYI LAN

ABSTRACT. In this paper, for given mass $c > 0$, we study the existence of normalized solutions to the following nonlinear p-Kirchhoff equation

$$\begin{cases} (a + b \int_{\mathbb{R}^N} |\nabla u|^p dx)(-\Delta_p u) - V(x)|u|^{p-2}u + \lambda|u|^{p-2}u = |u|^{q-2}u, & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^p dx = c^p, \end{cases}$$

where $a > 0$, $b > 0$, $1 < p < N < 2p$, $N \geq 2$, $p + \frac{2p^2}{N} < q < p^* = \frac{Np}{N-p}$ and $\lambda \in \mathbb{R}$ appears as a Lagrange multiplier. Firstly, we get a normalized solution to the equation with potential $V(x) \equiv 0$. Secondly, when $V(x) \geq 0$, and under some assumptions on V, we prove the existence of mountain pass solution with positive energy to the above equation.

YongYi Lan is the corresponding author.

2020 Mathematics Subject Classification. Primary: 35J35, 35B38, 35J92.

Key words and phrases. p-Kirchhoff equation; normalized solution; potential

1. Introduction

In this paper, we study the existence of normalized solutions to the following nonlinear Kirchhoff equation with potential

$$(a + b \int_{\mathbb{R}^N} |\nabla u|^p dx)(-\Delta_p u) - V(x)|u|^{p-2}u + \lambda|u|^{p-2}u = |u|^{q-2}u, \text{ in } \mathbb{R}^N, \tag{1.1}$$

where $a > 0$, $b > 0$, $1 < p < N < 2p$, $N \geq 2$, $p + \frac{2p^2}{N} < q < p^* = \frac{Np}{N-p}$. To begin with, we consider equation (1.1) with $V \equiv 0$. There has been a large number of studies on normalized solutions of Kirchhoff equation without potential. Many scholars have recently focused their attention to study the situation of p-Laplacian equation such as [3] and [6]. They considered the following p-Laplacian equation

$$-\Delta_p u = \lambda|u|^{p-2}u + \mu|u|^{q-2}u + |u|^{p^*-2}u, \quad \text{in } \mathbb{R}^N.$$

In [3], the authors obtained several existence results under $\mu > 0$ and other assumptions by using concentration compactness lemma, Schwarz rearrangement, Ekeland variational principle and mini-max theorems. In [6], the authors obtained the existence of ground state solution by virtue of truncation technique, and obtained multiplicity of normalized solutions in the purely L^p-subcritical case. In [11], the authors got a ground state solution to Eq.(1.1) and derived several asymptotic results on the obtained normalized solutions with $p = 2$. There has been less studies for normalized solutions of p-Kirchhoff equation. Inspired by [11], we prove equation (1.1) has a normalized solution if $V \equiv 0$ holds.

Then we study the existence of normalized solutions to the nonlinear Kirchhoff equation with potential. When $V \not\equiv 0$, there are many scholars studied the existence of normalized solutions of Eq.(1.1). In [2], the authors discussed the existence of solutions for a class of Kirchhoff equations with $p = 2$, and then they studied the behavior of the Palais-Smale sequences by splitting lemma. And some scholars studied the problem with $p = 2$ in [14] and [17]. In [14], the authors used a new concentration compactness type result to recover compactness in the

Sobolev critical case, and then they proved the existence of positive ground state solutions to the equation under an explicit assumption on V. In [17], the authors proved the existence of ground state normalized solution via variational methods. This paper is inspired by [4], which they studied the existence of normalized solutions for the p-Laplacian equation. We are interested in the existence of normalized solution of Eq.(1.1) and extend the results of [14]. We use the minimax method and Splitting lemma to study the existence of a mountain pass normalized solution of Eq.(1.1).

We define

$$S_c = \left\{u \in W^{1,p}(\mathbb{R}^N) : \|u\|_p = c\right\}$$

where $W^{1,p}(\mathbb{R}^N)$ is endowed with the usual norm $\|u\| = (\int_{\mathbb{R}^N} (|\nabla u|^p + |u|^p)dx)^{\frac{1}{p}}$ and $\| \|_q$ stands for the L^q-norm. Solutions to Eq.(1.1) are critical points of the energy functional $I : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ with

$$I(u) = \frac{a}{p}\|\nabla u\|_p^p + \frac{b}{2p}\|\nabla u\|_p^{2p} - \frac{1}{q}\|u\|_q^q - \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p dx$$

on the constraint S_c with a Lagrange multiplier $\lambda \in \mathbb{R}$. If u is a weak solution of (1.1), then

$$P(u) := (a+b\|\nabla u\|_p^p)\|\nabla u\|_p^p - \gamma_q\|u\|_q^q - \frac{N}{p}\int_{\mathbb{R}^N} V(x)|u|^p dx - \int_{\mathbb{R}^N} V(x)|u|^{p-2}u\nabla u\cdot x dx = 0$$

where $\gamma_q = \dfrac{N(q-p)}{pq}$.

For $u \in W^{1,p}(\mathbb{R}^N)$ and $s \in \mathbb{R}$, let $s * u = e^{\frac{Ns}{p}} u(e^s x) \in S_c$, we define the fiber map

$$\Psi_u(s) := I(s * u).$$

Then we introduce the following Pohozaev constrained set

$$P_c = \{u \in W^{1,p}(\mathbb{R}^N) : P(u) = 0\} \cap S_c.$$

Consider the decomposition of P_c into the disjoint union,

$$P_c = P_c^+ \cup P_c^0 \cup P_c^-,$$

where $P_c^+ = \{u \in S_c : \Psi_u'(0) = 0, \Psi_u''(0) > 0\}$.

2. Results

Our main results are the following:

Theorem 1 Assume that $N \geq 2$, $1 < p < N < 2p$, $p + \dfrac{2p^2}{N} < q < p^* = \dfrac{Np}{N-p}$ and $V \equiv 0$ holds, let $a > 0$, $b > 0$, $c > 0$, then equation (1.1) has a mountain pass solution u on S_c. In addition, u is a radial ground state solution.

Theorem 2 Assume that $N \geq 2$, $1 < p < N < 2p$, $p + \dfrac{2p^2}{N} < q < p^* = \dfrac{Np}{N-p}$ and $V \not\equiv 0$ holds, let $a > 0$, $b > 0$, $c > 0$, if fixed $\delta \in (0, a)$, we assume that

$$\|V\|_{N/p} \leq (a - \delta)S, \tag{2.1}$$

(2.2)

$$\begin{aligned}\|W\|_{N/(p-1)} < \min\Big\{&\frac{S^{\frac{p-1}{p}}}{p^2}[aY - N(q-2p)S^{-1}\|V\|_{N/p}],\\ &\frac{S^{\frac{p-1}{p}}}{Xp^2 + Zp}\Big(aXY - [XN(q-2p) + (N-p)Z]S^{-1}\|V\|_{N/p}\Big)\Big\},\end{aligned}$$

where

$X = Np - (N-p)q$, $Y = N(q-p) - p^2$, $Z = N(q-p)^2$, $W(x) := V(x) \cdot |x|$ and S denotes the Sobolev constant.

Then equation (1.1) has a mountain pass solution on S_c for every $c > 0$ with positive energy.

3. The proof of non-potential case

In this section, we study the structure of P_c and I to locate the position of critical points of $I|_{S_c}$ with $V \equiv 0$. To prove the Theorem 1, we mainly establish some preliminaries by showing the following definitions and lemmas.

3.1. **Preliminaries.** (1) $W_r^{1,p}(\mathbb{R}^N) = \left\{u \in W^{1,p}(\mathbb{R}^N) : u(x) = u(|x|)\right\}$ is equipped with the standard norm $\|\ \|$. $S_{c,r} = S_c \cap W_r^{1,p}(\mathbb{R}^N)$.

(2) Since $V \equiv 0$, we define

$$I_\infty(u) = \frac{a}{p}\|\nabla u\|_p^p + \frac{b}{2p}\|\nabla u\|_p^{2p} - \frac{1}{q}\|u\|_q^q$$

and

$$P_\infty(u) := (a + b\|\nabla u\|_p^p)\|\nabla u\|_p^p - \gamma_q\|u\|_q^q.$$

In this section, we abbreviate I_∞ and P_∞ as I and P.

(3) Gagliardo-Nirenberg inequality: there exists a constant $C_{N,p,q} > 0$ such that

$$\|u\|_q \leq C_{N,p,q}\|\nabla u\|_p^{\frac{N(q-p)}{pq}}\|u\|_p^{1-\frac{N(q-p)}{pq}}.$$

(4) We define

$$T(s) := \begin{cases} s, & \text{if } |s| \leq 1, \\ \dfrac{s}{|s|}, & \text{if } |s| > 1. \end{cases}$$

(5) Using a well known inequality found in [13, Lemma A.0.5], we know that

$$(|\eta|^{p-2}\eta - |\xi|^{p-2}\xi)\cdot(\eta - \xi) \geq \begin{cases} d_1|\eta - \xi|^p, & \text{if } p \geq 2, \\ d_2(|\eta| + |\xi|)^{p-2}|\eta - \xi|^2, & \text{if } 1 < p < 2, \end{cases} \tag{3.1}$$

where d_1, d_2 are positive constants.

Lemma 3.1([4 Lemma 2.1]) Let $N \geq 1$, $p > 1$ and $\{u_n\} \subset D^{1,p}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $D^{1,p}(\mathbb{R}^N)$, where $D^{1,p}$ denotes the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm $\|u\|_{D^{1,p}} := \|\nabla u\|_p$. Assume that for every $\varphi \in C_c^\infty(\mathbb{R}^N)$, there is

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\varphi(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\cdot\nabla T(u_n - u)dx = 0. \tag{3.2}$$

Then, up to a subsequence, $\nabla u_n \to \nabla u$ a.e. in $\mathbb{R}^N$.

In the proof, let $k \in \mathbb{N}_+$ and $\varphi \in C_c^\infty(\mathbb{R})$ satisfies

$$0 \leq \varphi \leq 1 \quad \varphi_k = 1 \text{ in } B_k \text{ and } \varphi_k = 0 \text{ in } B_{k+1}^c.$$

Lemma 3.2([7 Lemma 5.2]) Let ϕ be a C^1-functional on a complete connected C^1-Finsler manifold X and consider a homotopy-stable family $\mathcal{F}$ with an extended

closed boundary B. Set $m = m(\phi, \mathcal{F})$ and let F be a closed subset of X satisfying
(1) $(A \cap F) \setminus B \neq \emptyset$ for every $A \in \mathcal{F}$;
(2) $\sup \phi(B) \leq m \leq \inf \phi(F)$.
Then, for any sequence of sets $\{A_n\}$ in $\mathcal{F}$ such that $\lim\limits_{n\to\infty} \sup\limits_{A_n} \phi = m$, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} \phi(x_n) = m, \quad \lim_{n\to\infty} \|d\phi(x_n)\| = 0, \quad \lim_{n\to\infty} dist(x_n, F) = 0, \quad \lim_{n\to\infty} dist(x_n, A_n) = 0.$$

3.2. **Proof of Theorem 1.** Firstly, we give the compactness analysis of Palais-Smale sequences for $I|_{S_c}$.

Proposition 3.1 Let $a > 0$, $b > 0$, $c > 0$, $p + \dfrac{2p^2}{N} < q < p^*$ and $V \equiv 0$. Let $\{u_n\} \subset S_{c,r}$ be a Palais-Smale sequence for energy $I|_{S_c}$ level $m \neq 0$ with $P(u_n) \to 0$ as $n \to \infty$, then up to a subsequence $u_n \to u$ strongly in $W^{1,p}(\mathbb{R}^N)$ for some $u \in W^{1,p}(\mathbb{R}^N)$. Moreover, $u \in S_c$ and u is a radial solution to (1.1) for some $\lambda > 0$.
Proof. The proof is divided into three main steps.
(1) Boundedness of $\{u_n\}$ in $W^{1,p}(\mathbb{R}^N)$. If $p + \dfrac{2p^2}{N} < q < p^*$ and $V \equiv 0$, then $q\gamma_q > 2p$ and

$$I(u_n) = \frac{a}{p}\|\nabla u_n\|_p^p + \frac{b}{2p}\|\nabla u_n\|_p^{2p} - \frac{1}{q}\|u_n\|_q^q.$$

By $P(u_n) \to 0$, $a\|\nabla u_n\|_p^p + b\|\nabla u_n\|_p^{2p} - \gamma_q\|u_n\|_q^q + o_n(1) = 0$, where $o_n(1) \to 0$ as $n \to \infty$, then we have

$$I(u_n) - \frac{1}{2p}P(u_n) = \frac{a}{2p}\|\nabla u_n\|_p^p + (\frac{\gamma_q}{2p} - \frac{1}{q})\|u_n\|_q^q + o_n(1) \leq m + 1.$$

Hence, $\{u_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$ and $L^q(\mathbb{R}^N)$ by $q\gamma_q > 2p$.
(2) There exist Lagrange multipliers $\lambda_n \to \lambda \in \mathbb{R}$ and $\lambda > 0$. Since $W_r^{1,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$, we deduce that there exists $u \in W_r^{1,p}(\mathbb{R}^N)$ such that, up to a subsequence,

$$u_n \rightharpoonup u \text{ in } W_r^{1,p}(\mathbb{R}^N),\ u_n \to u \text{ in } L^q(\mathbb{R}^N) \text{ for } p < q < p^*,\ u_n \to u \text{ a.e. on } \mathbb{R}^N.$$

Since $\{u_n\}$ is a Palais-Smale sequence of $I|_{S_c}$, by the Lagrange multipliers rule, there exists $\lambda_n \in \mathbb{R}$ such that

$$\begin{aligned}(a+b\|\nabla u_n\|_p^p)\int_{\mathbb{R}^N}|\nabla u_n|^{p-2}\nabla u_n\nabla\psi dx+\lambda_n\int_{\mathbb{R}^N}|u_n|^{p-2}u_n\psi dx\\ =\int_{\mathbb{R}^N}|u_n|^{q-2}u_n\psi dx+o_n(1)\|\psi\|\end{aligned} \tag{3.3}$$

for every $\psi \in W^{1,p}(\mathbb{R}^N)$. In particular, take $\psi = u_n$, then

$$\lambda_n c^p = -a\|\nabla u_n\|_p^p - b\|\nabla u_n\|_p^{2p} + \|u_n\|_q^q + o_n(1).$$

The boundedness of $\{u_n\}$ in $W^{1,p}(\mathbb{R}^N)\cap L^q(\mathbb{R}^N)$ implies that $\lambda_n \to \lambda \in \mathbb{R}$, up to a subsequence.

Recalling that $P(u_n) \to 0$, by $\gamma_q < 1$ we have

$$\begin{aligned}\lambda c^p &= \lim_{n\to\infty}\{\lambda_n\|u_n\|_p^p\}\\ &= \lim_{n\to\infty}\{-a\|\nabla u_n\|_p^p - b\|\nabla u_n\|_p^{2p} + \|u_n\|_q^q\}\\ &= \lim_{n\to\infty}\{(1-\gamma_q)\|u_n\|_q^q\}\\ &= (1-\gamma_q)\|u\|_q^q \geq 0.\end{aligned}$$

Hence, we deduce that $\lambda \geq 0$, and $\lambda = 0$ if and only if $u \equiv 0$. If $\lambda_n \to 0$, we have $\|u_n\|_q^q \to 0$. Using again $P(u_n) \to 0$, we have $\lim_{n\to\infty}\{a\|\nabla u_n\|_p^p + b\|\nabla u_n\|_p^{2p}\} = 0$, then $I(u_n) \to 0$. There is a contradiction with $I(u_n) \to m \neq 0$ and thus $\lambda_n \to \lambda > 0$ and $u \not\equiv 0$.

(3) $u_n \to u$ in $W_r^{1,p}(\mathbb{R}^N)$. Firstly, we will show that

$$\nabla u_n \to \nabla u \ a.e. \text{ in } \mathbb{R}^N. \tag{3.4}$$

Since $u_n \rightharpoonup u \not\equiv 0$ in $W^{1,p}(\mathbb{R}^N)$, let $B := \lim_{n\to\infty}\|\nabla u_n\|_p^p$, then we get $B \geq \|\nabla u\|_p^p > 0$. By Egorov's theorem, then for every $\delta > 0$, there exists $F_\delta \subset supp\varphi$ such that $u_n \to u$ uniformly in F_δ and $m(supp\varphi \backslash F_\delta) < \delta$. Hence, $|u_n(x) - u(x)| \leq 1$ for all $x \in F_\delta$ as long as n sufficiently large.

Then, since $u_n \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^N)$, we have

$$
\begin{aligned}
&\limsup_{n\to\infty}\Big|(a+bB)\int_{\mathbb{R}^N}\varphi|\nabla u|^{p-2}\nabla u\cdot\nabla T(u_n-u)dx\Big|\\
&\le \limsup_{n\to\infty}\Big|(a+bB)\int_{F_\delta}\varphi|\nabla u|^{p-2}\nabla u\cdot\nabla T(u_n-u)dx\Big|\\
&\quad+\limsup_{n\to\infty}\Big|(a+bB)\int_{F_\delta^c}\varphi|\nabla u|^{p-2}\nabla u\cdot\nabla T(u_n-u)dx\Big|\\
&=\limsup_{n\to\infty}\Big|(a+bB)\int_{F_\delta^c}\varphi|\nabla u|^{p-2}\nabla u\cdot\nabla T(u_n-u)dx\Big|.
\end{aligned}
\tag{3.5}
$$

For every $\varepsilon>0$, by the definition of T,

$$
\Big|(a+bB)\int_{F_\delta^c}\varphi|\nabla u|^{p-2}\nabla u\cdot\nabla T(u_n-u)dx\Big|\le(a+bB)\int_{F_\delta^c}|\varphi||\nabla u|^{p-1}dx<C\delta,
$$

as long as δ sufficiently small, which implies

$$
\lim_{n\to\infty}\int_{\mathbb{R}^N}\varphi|\nabla u|^{p-2}\nabla u\cdot\nabla T(u_n-u)dx=0.
\tag{3.6}
$$

By Hölder inequality and the dominated convergence theorem,

$$
\begin{aligned}
&(a+bB)\Big|\int_{\mathbb{R}^N}\varphi|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla(T(u_n-u))dx\Big|\\
&\le\Big|\int_{\mathbb{R}^N}|u_n|^{q-2}u_n\cdot\varphi T(u_n-u)dx\Big|+\Big|\lambda_n\int_{\mathbb{R}^N}|u_n|^{p-2}u_n\cdot\varphi T(u_n-u)dx\Big|\\
&\quad+(a+bB)\Big|\int_{\mathbb{R}^N}|\nabla u_n|^{p-2}\nabla u_n\cdot T(u_n-u)\cdot\nabla\varphi dx\Big|+o_n(1)\\
&\le C\cdot\Big(\int_{\mathbb{R}^N}|\varphi T(u_n-u)|^q dx\Big)^{\frac{1}{q}}+C\cdot\Big(\int_{\mathbb{R}^N}|\varphi T(u_n-u)|^p dx\Big)^{\frac{1}{p}}\\
&\quad+C\cdot\Big(\int_{\mathbb{R}^N}|T(u_n-u)\cdot\nabla\varphi|^p dx\Big)^{\frac{1}{p}}+o_n(1)\\
&\le C\varepsilon
\end{aligned}
$$

which implies

$$
\lim_{n\to\infty}\int_{\mathbb{R}^N}\varphi|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla T(u_n-u)dx=0.
$$

By Lemma 3.1, (3.4) holds. Let $n \to \infty$, (3.3) implies that

$$(a + bB)\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla\psi dx + \lambda \int_{\mathbb{R}^N} |u|^{p-2}u\psi dx = \int_{\mathbb{R}^N} |u|^{q-2}u\psi dx. \tag{3.7}$$

That is, u is a weak solution of the equation $(a + bB)(-\Delta_p u) + \lambda|u|^{p-2}u = |u|^{q-2}u$. So we have the Pohozaev identity

$$Q(u) := (a + Bb)\|\nabla u\|_p^p - \gamma_q\|u\|_q^q = 0.$$

Let $v_n = u_n - u$ and by the Brezis-Lieb Lemma leads to

$$\|\nabla u_n\|_p^p = \|\nabla v_n\|_p^p + \|\nabla u\|_p^p \text{ and } \|u_n\|_q^q = \|v_n\|_q^q + \|u\|_q^q.$$

Then

$$Q(u_n) = (a+Bb)\|\nabla u_n\|_p^p-\gamma_q\|u_n\|_q^q = (a+Bb)\|\nabla v_n\|_p^p-\gamma_q\|v_n\|_q^q+(a+Bb)\|\nabla u\|_p^p-\gamma_q\|u\|_q^q.$$

We have $(a + Bb)\|\nabla v_n\|_p^p = \gamma_q\|v_n\|_q^q + o_n(1) \to 0$. Hence, $u_n \to u$ in $D^{1,p}$.
Take $\psi = u_n - u$ in (3.3) and (3.7), we obtain

$$\begin{aligned}
(a + bB)\int_{\mathbb{R}^N} &(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\nabla(u_n - u)dx \\
&+ \int_{\mathbb{R}^N} (\lambda_n|u_n|^{p-2}u_n - \lambda|u|^{p-2}u)(u_n - u)dx \\
&= \int_{\mathbb{R}^N} (|u_n|^{q-2}u_n - |u|^{q-2}u)(u_n - u)dx + o_n(1)\|u_n - u\|.
\end{aligned}$$

Now the first and the third integrals tend to 0. As a consequence,

$$\begin{aligned}
0 &= \lim_{n\to\infty}\{\int_{\mathbb{R}^N} (\lambda_n|u_n|^{p-2}u_n - \lambda|u|^{p-2}u)(u_n - u)dx\} \\
&= \lim_{n\to\infty} \lambda\{\int_{\mathbb{R}^N} (|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)dx\}.
\end{aligned} \tag{3.8}$$

For $1 < p < 2$, we deduce from (3.8) and (3.1) that

$$\begin{aligned}\Big(\int_{\mathbb{R}^N}|u_n-u|^p dx\Big)^{\frac{2}{p}} &= \Big(\int_{\mathbb{R}^N}\frac{|u_n-u|^p}{(|u_n|+|u|)^{p(2-p)/2}}\cdot(|u_n|+|u|)^{p(2-p)/2}dx\Big)^{\frac{2}{p}}\\ &\le \Big(\int_{\mathbb{R}^N}\frac{|u_n-u|^2}{(|u_n|+|u|)^{2-p}}dx\Big)\Big(\int_{\mathbb{R}^N}(|u_n|+|u|)^p dx\Big)^{\frac{2-p}{p}}\\ &\le C\int_{\mathbb{R}^N}(|u_n|^{p-2}u_n-|u|^{p-2}u)(u_n-u)dx \to 0.\end{aligned}$$

If $p \ge 2$, we have

$$\int_{\mathbb{R}^N}|u_n-u|^p dx \le C\int_{\mathbb{R}^N}(|u_n|^{p-2}u_n-|u|^{p-2}u)(u_n-u)dx \to 0.$$

The above limits lead to $u_n \to u$ in $L^p(\mathbb{R}^N)$. Hence, $u_n \to u$ in $W^{1,p}(\mathbb{R}^N)$.

□

Lemma 3.3 Let $\tilde{a},\ \tilde{b},\ \tilde{c},\ \tilde{q} \in (0,+\infty)$ and $f(t) = \tilde{a}t^p + \tilde{b}t^{2p} - \tilde{c}t^{\tilde{q}}$ for $t \ge 0$. If $\tilde{q} \in (2p,+\infty)$, $f(t)$ has a unique maximum point at a positive level on $[0,+\infty)$.

Proof. Direct calculations give

$$\begin{aligned}&f'(t) = t^{p-1}g(t) \text{ for } g(t) = p\tilde{a} + 2p\tilde{b}t^p - \tilde{q}\tilde{c}t^{\tilde{q}-p};\\ &g'(t) = t^{p-1}w(t) \text{ for } h(t) = 2p^2\tilde{b} - \tilde{q}(\tilde{q}-p)\tilde{c}t^{\tilde{q}-2p};\\ &h'(t) = -\tilde{q}(\tilde{q}-p)(\tilde{q}-2p)\tilde{c}t^{\tilde{q}-2p-1}.\end{aligned}$$

Since $\tilde{q} \in (2p,+\infty)$, then $h'(t) < 0$ for $t > 0$, we know that $h(t) \searrow$ on $[0,+\infty)$. The fact that $h(0) > 0$ and $h(+\infty) = -\infty$ imply that there exists unique $t_1 > 0$ such that $h(t_1) = 0$, $h(t) > 0$ if $t \in (0,t_1)$ and $h(t) < 0$ if $t \in (t_1,+\infty)$.

Consequently, $g(t) \nearrow$ on $[0,t_1)$ and $\searrow$ on $(t_1,+\infty)$. The fact that $g(0) > 0$ and $g(+\infty) = -\infty$ imply that there exists unique $t_2 > t_1$ such that $g(t_2) = 0$, $g(t) > 0$ if $t \in (0,t_2)$ and $g(t) < 0$ if $t \in (t_2,+\infty)$. We get $f'(t) > 0$ if $t \in (0,t_2)$ and $f'(t) < 0$ if $t \in (t_2,+\infty)$, which implies that $f(t) \nearrow$ on $[0,t_2)$ and $\searrow$ on $(t_2,+\infty)$. Since $f(0) = 0$, then $f(t)$ has a unique maximum point at t_2 and $f(t_2) > 0$.

□

Lemma 3.4 Let $a > 0$, $b > 0$, $c > 0$, $p + \dfrac{2p^2}{N} < q < p^*$ and $V \equiv 0$. For every $u \in S_c$, Ψ_u has a unique critical point $s_u \in \mathbb{R}$, which is a strict maximum point at a positive level. Moreover:
(1) $P_c = P_c^-$.
(2) Ψ_u is strictly decreasing on $(s_u, +\infty)$, and $s_u < 0 \Leftrightarrow P(u) < 0$.
(3) The maps $u \in S_c \to s_u \in \mathbb{R}$ are of class C^1.
Proof. Set

$$\Psi_u(s) = \frac{ae^{ps}}{p}\|\nabla u\|_p^p + \frac{be^{2ps}}{2p}\|\nabla u\|_p^{2p} - \frac{e^{\gamma_q \cdot qs}}{q}\|u\|_q^q$$

and

$$P(s * u) = ae^{ps}\|\nabla u\|_p^p + be^{2ps}\|\nabla u\|_p^{2p} - \gamma_q e^{\gamma_q \cdot qs}\|u\|_q^q.$$

Obviously, $\Psi_u'(s) = 0$ if and only if $P(s*u) = 0$, then $s*u \in P_c$. Clearly $\Psi_u(s) \to 0^+$ as $s \to -\infty$, and $\Psi_u(s) \to -\infty$ as $s \to +\infty$, for every $u \in S_c$. By Lemma 3.3, let $t = e^s$, $\tilde{a} = \dfrac{a}{p}\|\nabla u\|_p^p$, $\tilde{b} = \dfrac{b}{2p}\|\nabla u\|_p^{2p}$, $\tilde{c} = \dfrac{1}{q}\|u\|_q^q$ and $\tilde{q} = \gamma_q \cdot q$, then Ψ_u has a unique maximum point s_u at positive level. We assume that there exists $u \in P_c^0$, then $\Psi_u'(0) = a\|\nabla u\|_p^p + b\|\nabla u\|_p^{2p} - \gamma_q\|u\|_q^q = 0$ and $\Psi_u''(0) = ap\|\nabla u\|_p^p + 2pb\|\nabla u\|_p^{2p} - \gamma_q^2 \cdot q\|u\|_q^q = 0$. We deduce that

$$ap\|\nabla u\|_p^p = (2p\gamma_q - \gamma_q^2 q)\|u\|_q^q$$

if and only if $u \equiv 0$. This shows that $P_c^0 = \emptyset$. Since s_u is a maximum point of $\Psi_u(s)$, we have $\Psi_u''(s_u) \leq 0$. Hence, $s_u * u \in P_c^- \cap P_c^0$. Since $P_c^0 = \emptyset$, we deduce that $s_u * u \in P_c^-$. By Lemma 3.3, we have Ψ_u is strictly decreasing on $(s_u, +\infty)$. We observe that $\Psi_u'(s) < 0$ if and only if $s > s_u$. Since $P(u) = \Psi_u'(0)$, then if $s_u < 0$, $P(u) < 0$. Therefore, if $P(u) = \Psi_u'(0) < 0$, then $s_u < 0$.

We apply the implicit function theorem: we let $\Phi(s, u) = \Psi_u'(s)$, and observe that Φ is of class C^1 in the two variables $(s, u) \in \mathbb{R} \times S_c$, $\Phi(s_u, u) = 0$, and $\partial_s \Phi(s_u, u) = \Psi_u''(s_u) < 0$. Therefore, $u \in W^{1,p}(\mathbb{R}^N) \mapsto s_u$ is of class C^1.

□

Lemma 3.5 Let $a > 0$, $b > 0$, $c > 0$, $p + \frac{2p^2}{N} < q < p^*$ and $V \equiv 0$, then

$$m_{0,c} := \inf_{u \in P_c} I(u) > 0.$$

Proof. For $\forall\ u \in P_c$, we have $P(u) = 0$, then by Gagliardo-Nirenberg inequality

$$\begin{aligned} a\|\nabla u\|_p^p + b\|\nabla u\|_p^{2p} &= \gamma_q \|u\|_q^q \\ &\leq \gamma_q \cdot C_{N,p,q}^q \|\nabla u\|_p^{q\gamma_q} c^{q(1-\gamma_q)}. \end{aligned}$$

Since $q\gamma_q > 2p$, then $\inf_{u \in P_c} \|\nabla u\|_p \geq C > 0$. For $\forall\ u \in P_c$,

$$\begin{aligned} \inf_{u \in P_c} I(u) &= \inf_{u \in P_c} \{I(u) - \frac{1}{q\gamma_q} P(u)\} \\ &= \inf_{u \in P_c} \{(\frac{a}{p} - \frac{a}{q\gamma_q})\|\nabla u\|_p^p + (\frac{b}{2p} - \frac{b}{q\gamma_q})\|\nabla u\|_p^{2p}\} \\ &> 0. \end{aligned}$$

□

Lemma 3.6 Let $a > 0$, $b > 0$, $c > 0$, $p + \frac{2p^2}{N} < q < p^*$ and $V \equiv 0$. There exists $\rho > 0$ sufficiently small such that

$$0 < \sup_{u \in \bar{A}_\rho} I(u) < m_{0,c} \ \text{ and } \ u \in \bar{A}_\rho \Rightarrow I(u) > 0,\ P(u) > 0,$$

where $A_\rho := \{u \in S_c : \|\nabla u\|_p < \rho\}$.

Proof. By the Gagliardo-Nirenberg inequalities,

$$\begin{aligned} I(u) &= \frac{a}{p}\|\nabla u\|_p^p + \frac{b}{2p}\|\nabla u\|_p^{2p} - \frac{1}{q}\|u\|_q^q \\ &\geq \frac{a}{p}\|\nabla u\|_p^p + \frac{b}{2p}\|\nabla u\|_p^{2p} - \frac{1}{q} C_{N,p,q}^q \|\nabla u\|_p^{q\gamma_q} \cdot c^{q(1-\gamma_q)} \\ &\geq \frac{b}{2p}\|\nabla u\|_p^{2p} - \frac{1}{q} C_{N,p,q}^q \|\nabla u\|_p^{q\gamma_q} \cdot c^{q(1-\gamma_q)} \end{aligned}$$

and

$$P(u) \geq b\|\nabla u\|_p^{2p} - \gamma_q C_{N,p,q}^q \|\nabla u\|_p^{q\gamma_q} \cdot c^{q(1-\gamma_q)}.$$

Therefore, for any $u \in \bar{A}_\rho$ with ρ small enough, we have

$$I(u) > 0, \ P(u) > 0.$$

If necessary replacing ρ with a smaller quantity, we also have

$$I(u) \leq \frac{a}{p}\|\nabla u\|_p^p + \frac{b}{2p}\|\nabla u\|_p^{2p} < m_{0,c}, \ \forall u \in \bar{A}_\rho$$

since $m_{0,c} > 0$ by Lemma 3.5.

□

Proof of Theorem 1. Let $\rho > 0$ be defined by Lemma 3.6. For every $r \in \mathbb{R}$, define $I^r = \{u \in S_c : I(u) \leq r\}$. We consider the augmented functional $\tilde{I} : \mathbb{R} \times W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\tilde{I}(s,u) := I(s * u) = \frac{ae^{ps}}{p}\|\nabla u\|_p^p + \frac{be^{2ps}}{2p}\|\nabla u\|_p^{2p} - \frac{e^{\gamma_q \cdot qs}}{q}\|u\|_q^q$$

and the minimax class

$$\Gamma := \Big\{\gamma = (\alpha,\beta) \in C([0,1], \mathbb{R} \times S_c) : \gamma(0) \in (0, \bar{A}_\rho), \ \gamma(1) \in (0, I^0)\Big\}, \tag{3.9}$$

with associated minimax level

$$\delta_{0,c} = \inf_{\gamma\in\Gamma} \max_{(s,u)\in\gamma([0,1])} \tilde{I}(s,u).$$

Fixing $u \in S_c$, since $\|\nabla(s * u)\|_p \to 0^+$ as $s \to -\infty$, and $\Psi_u(s) \to -\infty$ as $s \to +\infty$, there exists $s_0 \ll -1$ and $s_1 \gg 1$ such that

$$\gamma_u : \tau \in [0,1] \mapsto (0, ((1-\tau)s_0 + \tau s_1) * u) \in \mathbb{R} \times S_c$$

is a path in Γ(the continuity follows from [6, Lemma 2.3]). Then $\delta_{0,c}$ is a real number.

For any $\gamma = (\alpha,\beta) \in \Gamma$, let us consider the function

$$P_\gamma : \tau \in [0,1] \longmapsto P(\alpha(\tau) * \beta(\tau)) \in \mathbb{R}.$$

We have $P_\gamma(0) = P(\alpha(0)*\beta(0)) > 0$, then we prove that $P_\gamma(1) = P(\alpha(1)*\beta(1)) < 0$: since $\Psi_{\beta(1)}(s) > 0$ for every $s \in (-\infty, s_{\beta(1)}]$ and $\Psi_{\beta(1)}(0) = I(\beta(1)) \leq 0$, it is necessary that $s_{\beta(1)} < 0$. By Lemma 3.4, we have $P_\gamma(1) < 0$. Moreover, P_γ is continuous by [6, Lemma 2.3], hence we deduce that there exists $\tau_\gamma \in (0,1)$ such that $P_\gamma(\tau_\gamma) = 0$, namely $\alpha(\tau_\gamma) * \beta(\tau_\gamma) \in P_c$, this implies that

$$\max_{\gamma([0,1])} \tilde{I} \geq \tilde{I}(\gamma(\tau_\gamma)) = I(\alpha(\tau_\gamma) * \beta(\tau_\gamma)) \geq \inf_{P_c} I = m_{0,c}. \tag{3.10}$$

Then, we have $\delta_{0,c} \geq m_{0,c}$.

In addition, if $u \in P_c$, then γ_u is a path in Γ with

$$I(u) = \max_{\gamma_u([0,1])} \tilde{I} \geq \delta_{0,c}$$

then we have the reverse inequality $m_{0,c} \geq \delta_{0,c}$. Then $m_{0,c} = \delta_{0,c}$. By Lemma 3.6, we infer that

$$\delta_{0,c} = m_{0,c} = \inf_{u \in P_c} I(u) > \sup_{(\bar{A}_\rho \cup I^0) \cap S_c} I = \sup_{(0,\bar{A}_\rho) \cup (0,I^0) \cap (\mathbb{R} \times S_c)} \tilde{I}. \tag{3.11}$$

In the following, we will apply Lemma 3.2 to achieve our result. For this purpose, let

$$X = \mathbb{R} \times S_c, \quad \mathcal{F} = \{\gamma([0,1]) : \gamma \in \Gamma\}, \quad B = (0, \bar{A}_\rho) \cup (0, I^0),$$

$$F = \{(s,u) \in \mathbb{R} \times S_c | \tilde{I} \geq \delta_{0,c}\}, \quad A = \gamma([0,1]), \quad A_n = \gamma_n([0,1]) = \gamma_n([0,1]) \times \{0\}.$$

We need to check that $\mathcal{F}$ is a homotopy stable family of compact subsets of X with extended closed boundary B and F satisfies the assumptions (1) and (2) in Lemma 3.2. In fact, for every $\gamma \in \Gamma$, since $\gamma(0) \in (0, \bar{A}_k)$ and $\gamma(1) \in (0, I_0)$, we have $\gamma(0), \gamma(1) \in B$. For any set A in $\mathcal{F}$ and any η in $C([0,1] \times X, X)$ satisfying $\eta(t,x) = x$ for all $(t,x) \in (\{0\} \times X) \cup \{[0,1] \times B\}$, there holds that $\eta(1,\gamma(0)) = \gamma(0)$, $\eta(1,\gamma(1)) = \gamma(1)$. Hence, we have that $\eta(\{1\} \times A) \in \mathcal{F}$. We have $A \cap F \neq \emptyset$ by (3.10) and $F \cap B = \emptyset$ by (3.11). Hence, we can deduce that the assumptions (1) and (2) in Lemma 3.2 are valid.

Therefore, taking any minimizing sequence $\{\gamma_n = (\alpha_n, \beta_n)\} \subset \Gamma_n$ for $\delta_{0,c}$ with the property that $\alpha_n \equiv 0$ and $\beta_n(\tau) \geq 0$ a.e. in $\mathbb{R}^N$ for every $\tau \in [0,1]$, there exists a

Palais-Smale sequence $\{(s_n, \omega_n)\} \subset \mathbb{R} \times S_c$ for $I|_{\mathbb{R}\times S_c}$ at level $\delta_{0,c}$, that is

$$\partial_s \tilde{I}(s_n, \omega_n) \to 0 \text{ and } \|\partial_u \tilde{I}(s_n, \omega_n)\|_{(T_{\omega_n} S_c)^*} \to 0 \text{ as } n \to \infty, \tag{3.12}$$

with the additional property that

$$|s_n| + dist(\omega_n, \beta_n[0,1]) \to 0 \text{ as } n \to \infty.$$

By the definition of $\tilde{I}$, the first condition in (3.12) reads $P(s_n * \omega_n) \to 0$, while the second condition gives

(3.13)

$$d\tilde{I}(s_n * \omega_n)[s_n * \varphi] = o_n(1)\|\varphi\| = o_n(1)\|s_n * \varphi\| \text{ as } n \to \infty, \text{ for every } \varphi \in T_{\omega_n} S_c.$$

Then let $u_n := s_n * \omega_n$. By Lemma 3.6, equation (3.13) establishes that $\{u_n\} \in S_c$ is a Palais-Smale sequence for $I|_{S_c}$ at level $\delta_{0,c} = m_{0,c}$ with $P(u_n) \to 0$. By Proposition 3.1, we have that $u \in S_c$ and u is a radial solution to (1.1) for some $\lambda > 0$ as $V \equiv 0$. In addition, u is a ground state solution by $\delta_{0,c} = m_{0,c}$.

4. The proof of potential case

4.1. **Preliminaries.** Throughout this section we will make the following assumptions on V:

$$V \geq 0 \text{ but } V \not\equiv 0, \tag{4.1}$$

and we define $W(x) := V(x) \cdot |x|$.
Then we define

$$I_{\infty,\lambda}(u) = \frac{a}{p}\|\nabla u\|_p^p + \frac{b}{2p}\|\nabla u\|_p^{2p} + \frac{\lambda}{p}\|u\|_p^p - \frac{1}{q}\|u\|_q^q$$

and

$$I_\lambda(u) = \frac{a}{p}\|\nabla u\|_p^p + \frac{b}{2p}\|\nabla u\|_p^{2p} + \frac{\lambda}{p}\|u\|_p^p - \frac{1}{q}\|u\|_q^q - \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p dx.$$

Lemma 4.1 Let $1 < p < N < 2p$, $p + \dfrac{2p^2}{N} < q < p^*$, $N \geq 2$, $V \in L^{N/p}(B_1)$ and $V \in L^{\tilde{r}}(\mathbb{R}^N \backslash B_1)$ for some $\tilde{r} \in [N/p, +\infty]$. If $\{u_n\}$ is a bounded PS sequence for I_λ

in $W^{1,p}(\mathbb{R}^N)$ and $u_n \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^N)$. Then, $\nabla u_n \to \nabla u$ a.e. in $\mathbb{R}^N$.

Proof. Since $\{u_n\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$, without loss of generality, there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, and $A \in \mathbb{R}$ such that

$$\|\nabla u_n\|_p^p \to A^p.$$

Then we define

$$I_{\lambda,A}(u) = \frac{1}{p}(a + bA^p)\|\nabla u\|_p^p + \frac{\lambda}{p}\|u\|_p^p - \frac{1}{q}\|u\|_q^q - \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p dx.$$

Since $u_n \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^N)$, we have $u_n \to u$ a.e. in $\mathbb{R}^N$. Similar to the proof of Proposition 3.1, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} \varphi|\nabla u|^{p-2}\nabla u \cdot \nabla T(u_n - u)dx = 0. \tag{4.2}$$

By Lemma 3.1, we just need to prove (3.2). Then, we need to prove

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} \varphi|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla T(u_n - u)dx = 0. \tag{4.3}$$

Since $\{u_n\}$ is a PS sequence for I_λ, then for every $\psi \in W^{1,p}(\mathbb{R}^N)$,

$$\begin{aligned}
\langle I'_{\lambda,A}(u_n), \psi\rangle &= (a + bA^p)\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\psi dx + \lambda\int_{\mathbb{R}^N} |u_n|^{p-2}u_n \cdot \psi dx \\
&\quad - \int_{\mathbb{R}^N} |u_n|^{q-2}u_n \cdot \psi dx - \int_{\mathbb{R}^N} V(x)|u_n|^{p-2}u_n \cdot \psi dx \\
&= \langle I'_\lambda(u_n), \psi\rangle + b(A^p - \|\nabla u_n\|_p^p)\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\psi dx \\
&= \langle I'_\lambda(u_n), \psi\rangle + o_n(1)\|\psi\|_{W^{1,p}}.
\end{aligned}$$

We obtain $\langle I'_{\lambda,A}(u_n), \psi\rangle \to 0$ as $n \to \infty$, then

$$\begin{aligned}
(a + bA^p)\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\psi dx &= -\lambda\int_{\mathbb{R}^N} |u_n|^{p-2}u_n \cdot \psi dx + \int_{\mathbb{R}^N} |u_n|^{q-2}u_n \cdot \psi dx \\
&\quad + \int_{\mathbb{R}^N} V(x)|u_n|^{p-2}u_n \cdot \psi dx + o_n(1)\|\psi\|_{W^{1,p}}.
\end{aligned}$$

Let $\psi = \varphi T(u_n - u)$, then

(4.4)

$$\begin{aligned}
&\limsup_{n\to\infty} \left|(a + bA^p) \int_{\mathbb{R}^N} \varphi |\nabla u_n|^{p-2} \nabla u_n \nabla T(u_n - u) dx\right| \\
&\le \limsup_{n\to\infty} \Big((a + bA^p) \int_{\mathbb{R}^N} |\nabla u_n|^{p-1} |T(u_n - u)\nabla\varphi| dx + |\lambda| \int_{\mathbb{R}^N} |u_n|^{p-1} |\varphi T(u_n - u)| dx \\
&+ \int_{\mathbb{R}^N} |u_n|^{q-1} |\varphi T(u_n - u)| dx + \int_{\mathbb{R}^N} |V(x)||u_n|^{p-1} |\varphi T(u_n - u)| dx\Big).
\end{aligned}$$

We know

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |V(x)||u_n|^{p-1} \cdot |\varphi T(u_n - u)| dx \le \limsup_{n\to\infty} \int_{F_\delta^c} |V(x)||u_n|^{p-1} \cdot |\varphi| dx.$$

Since $1 < p < N < 2p$, by Hölder inequality and take δ sufficiently small, for every $\varepsilon > 0$, we have

$$\limsup_{n\to\infty} \int_{F_\delta^c \bigcap B_1} |V(x)||u_n|^{p-1} \cdot |\varphi| dx \le \limsup_{n\to\infty} \|V\|_{L^{\frac{N}{p}}(B_1)} \|u_n\|_{p^*}^{p-1} \|\varphi\|_{L^{\frac{Np}{N-p}}(F_\delta^c)} < \varepsilon$$

and

$$\limsup_{n\to\infty} \int_{F_\delta^c \bigcap B_1^c} |V(x)||u_n|^{p-1} \cdot |\varphi| dx \le \limsup_{n\to\infty} \|V\|_{L^{\tilde{r}}(B_1^c)} \|u_n\|_{p^*}^{p-1} \|\varphi\|_{L^{\frac{Np\tilde{r}}{Np(\tilde{r}-1)-\tilde{r}(N-p)(p-1)}}(F_\delta^c)} < \varepsilon.$$

Therefore, we obtain

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |V(x)||u_n|^{p-1} \cdot |\varphi T(u_n - u)| dx \le C\varepsilon.$$

Similarly, we have

$$\limsup_{n\to\infty} (a + bA^p) \int_{\mathbb{R}^N} |\nabla u_n|^{p-1} \cdot |T(u_n - u)\nabla\varphi| dx \le C\varepsilon,$$

$$\limsup_{n\to\infty} |\lambda| \int_{\mathbb{R}^N} |u_n|^{p-1} \cdot |\varphi T(u_n - u)| dx \le C\varepsilon$$

and

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{q-1} \cdot |\varphi T(u_n - u)| dx \le C\varepsilon.$$

Hence, by (4.4), we obtain

$$\limsup_{n\to\infty} \left|(a + bA^p) \int_{\mathbb{R}^N} \varphi |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla T(u_n - u) dx\right| \le C\varepsilon$$

which implies

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\varphi|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla T(u_n-u)dx=0.$$

□

Similarly, we define

$$I_A(u):=\frac{1}{p}(a+bA^p)\|\nabla u\|_p^p-\frac{1}{p}\int_{\mathbb{R}^N}V(x)|u|^pdx-\frac{1}{q}\|u\|_q^q,$$

$$I_{\infty,A}(u):=\frac{1}{p}(a+bA^p)\|\nabla u\|_p^p-\frac{1}{q}\|u\|_q^q,$$

$$I_{\infty,\lambda,A}(u):=\frac{1}{p}(a+bA^p)\|\nabla u\|_p^p+\frac{\lambda}{p}\|u\|_p^p-\frac{1}{q}\|u\|_q^q,$$

$$P_A(u):=(a+bA^p)\|\nabla u\|_p^p-\gamma_q\|u\|_q^q-\frac{N}{p}\int_{\mathbb{R}^N}V(x)|u|^pdx-\int_{\mathbb{R}^N}V(x)|u|^{p-2}u\nabla u\cdot xdx$$

and

$$P_{\infty,A}(u):=(a+bA^p)\|\nabla u\|_p^p-\gamma_q\|u\|_q^q.$$

Remark 4.1 ([4] Remark 2.1) If $\{u_n\}$ is a PS sequence for I_λ in $W^{1,p}(\mathbb{R}^N)$ and $u_n\rightharpoonup u$ in $W^{1,p}(\mathbb{R}^N)$. Then, by Lemma 4.1 and weak convergence, u is a solution of (1.1).

Lemma 4.2 Let $1<p<N<2p$, and V satisfies the assumptions of Lemma 4.1. Assume $\{u_n\}$ is a bounded PS sequence for I_λ in $W^{1,p}(\mathbb{R}^N)$, and $u_n\rightharpoonup u$ in $W^{1,p}(\mathbb{R}^N)$. Let $v_n=u_n-u$. Then, $\{v_n\}$ is a PS sequence for $I_{\infty,\lambda,A}$.

Proof. Since $u_n\rightharpoonup u$, we have $v_n\rightharpoonup 0$ in $W^{1,p}(\mathbb{R}^N)$, $v_n\to 0$ in $L^p_{loc}(\mathbb{R}^N)$, $L^q_{loc}(\mathbb{R}^N)$ and a.e. in $\mathbb{R}^N$. Then we prove that as $n\to\infty$,

$$\int_{\mathbb{R}^N}V(x)|v_n|^pdx\to 0.$$

Now, set

$$\int_{\mathbb{R}^N}V(x)|v_n|^pdx=\int_{B_1}V(x)|v_n|^pdx+\int_{B_1^c}V(x)|v_n|^pdx.$$

Firstly, we assume $\tilde{r} < +\infty$. Since $\dfrac{N}{N-p}$ be the conjugate exponent of $\dfrac{N}{p}$, by Hölder inequality, we have

$$\int_{B_1} V(x)|v_n|^p dx \le \|V\|_{L^{\frac{N}{p}}(B_1)} \||v_n|^p\|_{L^{\frac{N}{N-p}}(B_1)}.$$

Since $\{|v_n|^p\}$ is bounded in $L^{N/(N-p)}(B_1)$ and $V \in L^{N/p}(B_1)$, we have $|v_n|^p \rightharpoonup 0$ in $L^{N/(N-p)}(B_1)$, hence $\int_{B_1} V(x)|v_n|^p dx \to 0$ as $n \to \infty$. Similarly, let $\tilde{r}'$ be the conjugate exponent of $\tilde{r}$. Since $V \in L^{\tilde{r}}(B_1^c)$, we have $\int_{B_1^c} V(x)|v_n|^p dx \to 0$ as $n \to \infty$.

Next, we assume $\tilde{r} = +\infty$. By $V \in L^{N/p}(B_1)$, it is not difficult to prove that $\int_{B_1} V(x)|v_n|^p dx \to 0$ as $n \to \infty$. Since $v_n \to 0$ in $L^p_{loc}(\mathbb{R}^N)$, then for every $R > 1$,

$$\begin{aligned}
\limsup_{n\to\infty} \Big| \int_{B_1^c} V(x)|v_n|^p dx \Big| &= \limsup_{n\to\infty} \Big| \int_{B_R^c} V(x)|v_n|^p dx + \int_{B_R \setminus B_1} V(x)|v_n|^p dx \Big| \\
&= \limsup_{n\to\infty} \Big| \int_{B_R^c} V(x)|v_n|^p dx \Big| \le C \sup_{B_R^c} |V|,
\end{aligned}$$

which implies $\int_{B_1^c} V(x)|v_n|^p dx \to 0$ as $n \to \infty$. To sum up, we obtain $\int_{\mathbb{R}^N} V(x)|v_n|^p dx \to 0$. Then

$$\int_{\mathbb{R}^N} V(x)|u_n|^p dx \to \int_{\mathbb{R}^N} V(x)|u|^p dx. \tag{4.5}$$

Since $\{u_n\}$ is a PS sequence for I_λ, there exists $m \in \mathbb{R}$ such that

$$I_\lambda(u_n) \to m \ \text{ and } \ \|I'_\lambda(u_n)\| \to 0 \ \text{ in } \ W^{-1,p}(\mathbb{R}^N) \ \text{ as } \ n \to \infty.$$

By Brézis-Lieb lemma and Lemma 4.1, we have

$$I_\lambda(u_n) = I_{\lambda,A}(u) + I_{\infty,\lambda,A}(v_n) - \frac{b}{2p}A^{2p} + o_n(1),$$

which implies

$$I_{\infty,\lambda,A}(v_n) \to m + \frac{b}{2p}A^{2p} - I_{\lambda,A}(u) \ \text{ as } \ n \to \infty. \tag{4.6}$$

Finally, we prove that

$$\|I'_{\infty,\lambda,A}(v_n)\| \to 0 \ \text{ in } \ W^{-1,p}(\mathbb{R}^N).$$

We just need to prove that for every $\psi \in W^{1,p}(\mathbb{R}^N)$,

$$\langle I'_{\infty,\lambda,A}(v_n), \psi\rangle = o_n(1)\|\psi\|_{W^{1,p}},$$

that is

$$(a+bA^p)\int_{\mathbb{R}^N} |\nabla v_n|^{p-2}\nabla v_n \nabla\psi dx + \int_{\mathbb{R}^N} \lambda |v_n|^{p-2} v_n \psi dx - \int_{\mathbb{R}^N} |v_n|^{q-2} v_n \psi dx = o_n(1)\|\psi\|_{W^{1,p}}. \tag{4.7}$$

By Hölder inequality,

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^N} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla v_n|^{p-2}\nabla v_n - |\nabla u|^{p-2}\nabla u)\cdot \nabla\psi dx\Big| \\
&\le \Big(\int_{\mathbb{R}^N} \Big||\nabla u_n|^{p-2}\nabla u_n - |\nabla v_n|^{p-2}\nabla v_n - |\nabla u|^{p-2}\nabla u)\Big|^{\frac{p}{p-1}} dx\Big)^{\frac{p-1}{p}} \|\nabla\psi\|_p \\
&\le \Big(\int_{\mathbb{R}^N} \Big||\nabla u_n|^{p-2}\nabla u_n - |\nabla v_n|^{p-2}\nabla v_n - |\nabla u|^{p-2}\nabla u)\Big|^{\frac{p}{p-1}} dx\Big)^{\frac{p-1}{p}} \|\psi\|_{W^{1,p}}.
\end{aligned}$$

From [12, Lemma 3.2], we know

$$\int_{\mathbb{R}^N} \Big||\nabla u_n|^{p-2}\nabla u_n - |\nabla v_n|^{p-2}\nabla v_n - |\nabla u|^{p-2}\nabla u\Big|^{\frac{p}{p-1}} dx = o_n(1)$$

which implies

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla v_n|^{p-2}\nabla v_n \cdot \nabla\psi dx = &\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\psi dx \\
&- \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \cdot \nabla\psi dx + o_n(1)\|\psi\|_{W^{1,p}}.
\end{aligned} \tag{4.8}$$

Similarly, we have

$$\int_{\mathbb{R}^N} |v_n|^{p-2} v_n \cdot \psi dx = \int_{\mathbb{R}^N} |u_n|^{p-2} u_n \cdot \psi dx - \int_{\mathbb{R}^N} |u|^{p-2} u \cdot \psi dx + o_n(1)\|\psi\|_{W^{1,p}} \tag{4.9}$$

and

(4.10)
$$\int_{\mathbb{R}^N} |v_n|^{q-2}v_n \cdot \psi dx = \int_{\mathbb{R}^N} |u_n|^{q-2}u_n \cdot \psi dx - \int_{\mathbb{R}^N} |u|^{q-2}u \cdot \psi dx + o_n(1)\|\psi\|_{W^{1,p}}.$$

Now, we use (4.5), (4.8), (4.9), (4.10), Lemma 4.1 and the fact that $\{u_n\}$ is a PS sequence for I_λ, we obtain (4.7).

□

Lemma 4.3 Let $1 < p < N < 2p$, and assume that
(i)$V \in L^{N/p}(B_1)$ and $V \in L^{\tilde{r}}(\mathbb{R}^N \backslash B_1)$ for some $\tilde{r} \in [N/p, +\infty]$,
(ii)in case $\tilde{r} = +\infty$, V satisfies $V(x) \to 0$ as $|x| \to +\infty$,
(iii)$\lambda > 0$.
If $\{u_n\}$ is a bounded PS sequence for I_λ in $W^{1,p}(\mathbb{R}^N)$, and $u_n \rightharpoonup u$ but not strongly, then there exist an integer $k \geq 1$, k nontrivial solutions $\omega^1, \ldots, \omega^k \in W^{1,p}(\mathbb{R}^N)$ to the equation

$$(a + bA^p)(-\Delta_p \omega) + \lambda|\omega|^{p-2}\omega = |\omega|^{q-2}\omega, \tag{4.11}$$

and k sequence $\{y_n^j\} \subset \mathbb{R}^N$, $1 \leq j \leq k$, such that $|y_n^j| \to +\infty$ as $n \to \infty$, $|y_n^{j_1} - y_n^{j_2}| \to +\infty$ for $j_1 \neq j_2$ as $n \to \infty$, and

$$u_n = u + \sum_{j=1}^{k} \omega^j(\cdot - y_n^j) + o_n(1) \ \text{ in } \ W^{1,p}(\mathbb{R}^N). \tag{4.12}$$

Moreover, we have

$$\|u_n\|_p^p = \|u\|_p^p + \sum_{j=1}^{k} \|\omega^j\|_p^p + o_n(1), \tag{4.13}$$

$$A^p = \|\nabla u\|_p^p + \sum_{j=1}^{k} \|\nabla \omega^j\|_p^p \tag{4.14}$$

and

$$I_\lambda(u_n) = I_{\lambda,A}(u) + \sum_{j=1}^{k} I_{\infty,\lambda,A}(\omega^j) - \frac{b}{2p}A^{2p} + o_n(1). \tag{4.15}$$

Proof. Let $u_n^1 = u_n - u$. Then $u_n^1 \rightharpoonup 0$ in $W^{1,p}(\mathbb{R}^N)$, $u_n \to u$ in $L^p_{loc}(\mathbb{R}^N)$, $L^q_{loc}(\mathbb{R}^N)$ and a.e. in $\mathbb{R}^N$. Similar to the proof of Lemma 4.2, we can prove that

$$\int_{\mathbb{R}^N} V(x)|u_n^1|^p dx \to 0 \ \text{ as } \ n \to \infty. \tag{4.16}$$

Since $u_n \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^N)$ but not strongly, then there is

$$\liminf_{n\to\infty} \|u_n^1\|^p > 0.$$

By Lemma 4.2, we know $\{u_n^1\}$ is a PS sequence for $I_{\infty,\lambda,A}$, hence

$$(a + bA^p)\|\nabla u_n^1\|_p^p + \lambda\|u_n^1\|_p^p = \|u_n^1\|_q^q + o_n(1),$$

Since $\lambda > 0$,

$$\liminf_{n\to\infty} \|u_n^1\|_q^q > 0.$$

Let us decompose $\mathbb{R}^N$ into N-dimensional unit hypercubes Q_i and set

$$l_n = \sup_{i\in\mathbb{N}_+} \|u_n^1\|_{L^q(Q_i)}.$$

Since $u_n^1 \in L^p(\mathbb{R}^N)$, for any $\varepsilon > 0$, there exist R such that $\int_{R^c} |u_n^1|^p dx < \varepsilon$. Then there exist limited $i_n \in \mathbb{N}_+$ such that l_n can be attained. We claim that

$$\liminf_{n\to\infty} l_n > 0.$$

Since

$$\|u_n^1\|_q^q = \sum_{i=1}^{\infty} \|u_n^1\|_{L^q(Q_i)}^q \le l_n^{q-p} \sum_{i=1}^{\infty} \|u_n^1\|_{L^q(Q_i)}^p \le C l_n^{q-p} \sum_{i=1}^{\infty} \|\nabla u_n^1\|_{L^p(Q_i)}^p \le C l_n^{q-p}.$$

Then, since $\liminf_{n\to\infty} \|u_n^1\|^p > 0$, we have $\liminf_{n\to\infty} l_n > 0$.
Let y_n^1 be the center of Q_{i_n} and

$$v_n^1 := u_n^1(\cdot + y_n^1),$$

then $\{v_n^1\}$ is a PS sequence for $I_{\infty,\lambda,A}$ and there exists $\omega^1 \in W^{1,p}(\mathbb{R}^N)\backslash\{0\}$ such that $v_n^1 \rightharpoonup \omega^1$ in $W^{1,p}(\mathbb{R}^N)$. By weak convergence, we know ω^1 satisfies (4.11). Since $u_n^1 \rightharpoonup 0$ in $W^{1,p}(\mathbb{R}^N)$ implies that $\{y_n^1\}$ is unbounded, we assume that $|y_n^1| \to +\infty$ as $n \to \infty$.Moreover, by (4.5), Brézis-Lieb Lemma and Lemma 4.1, we have

$$u_n = u + u_n^1 = u + v_n^1(\cdot - y_n^1) = u + \omega^1(\cdot - y_n^1) + [v_n^1(\cdot - y_n^1) - \omega^1(\cdot - y_n^1)],$$

$$\|u_n\|_p^p = \|u\|_p^p + \|\omega^1\|_p^p + \|v_n^1 - \omega^1\|_p^p + o_n(1),$$

$$A^p = \|\nabla u\|_p^p + \|\nabla\omega^1\|_p^p + \|\nabla(v_n^1 - \omega^1)\|_p^p + o_n(1),$$

and

$$I_\lambda(u_n) = I_{\lambda,A}(u) + I_{\infty,\lambda,A}(\omega^1) + I_{\infty,\lambda,A}(v_n^1 - \omega^1) - \frac{b}{2p}A^{2p} + o_n(1).$$

Now, set

$$u_n^2 = v_n^1(\cdot - y_n^1) - \omega^1(\cdot - y_n^1),$$

and iterate the above procedure. To complete the proof, we just need to prove that the iteration will be ended in finite steps. Suppose the iteration will not be ended, then we have

$$\|\nabla u_n\|_p^p \geq \sum_{j=1}^{\infty} \|\nabla\omega^j\|_p^p.$$

By Lemma 4.4, there exists a constant C such that $\|\nabla\omega^j\|_p^p \geq C$, then we have $\|\nabla u_n\|_p = +\infty$, which is an absurd.

□

Lemma 4.4 Let $\omega \in W^{1,p}(\mathbb{R}^N)$ be a non-trivial solution of

$$(a + bA^p)(-\Delta_p\omega) + \lambda|\omega|^{p-2}\omega = |\omega|^{q-2}\omega$$

for some $\lambda > 0$. Then there exists a constant C depending on N, p, q, and λ such that

$$\|\omega\|_{W^{1,p}} \geq C.$$

Proof. By the Pohozaev identity, we have

$$(a+bA^p)\|\nabla\omega\|_p^p = \frac{N(q-p)}{pq}\|\omega\|_q^q,$$

which together with

$$(a+bA^p)\|\nabla\omega\|_p^p + \lambda\|\omega\|_p^p = \|\omega\|_q^q$$

implies

$$\lambda\|\omega\|_p^p = \frac{pq-N(q-p)}{N(q-p)}(a+bA^p)\|\nabla\omega\|_p^p.$$

By Gagliardo-Nirenberg inequality, we have

$$\frac{pq}{N(q-p)}(a+bA^p)\|\nabla\omega\|_p^p = \|\omega\|_q^q \leq C^q_{N,p,q}\|\nabla\omega\|_p^{\frac{N(q-p)}{p}}\|\omega\|_p^{q-\frac{N(q-p)}{p}} = C\|\nabla\omega\|_p^q.$$

Hence, since $q > p$, $\|\omega\|_{W^{1,p}} \geq C(N,p,q,\lambda)$.

□

Define

$$Z_c := \{\upsilon \in S_c : \exists\lambda > 0, s.t.(a+b\|\nabla\upsilon\|_p^p)(-\Delta_p\upsilon) + \lambda|\upsilon|^{p-2}\upsilon = |\upsilon|^{q-2}\upsilon\} \tag{4.17}$$

and

$$m_c := \inf_{\upsilon\in Z_c} I_\infty(\upsilon). \tag{4.18}$$

Recalling Theorem 1, we have $m_c > 0$ and m_c can be achieved by some $\upsilon \in Z_c$.

Lemma 4.5 m_c is decreasing on $(0,+\infty)$.

Proof. Fix $\rho_2 > \rho_1 > 0$, let $u := u_{\rho_1} \in P^\infty_{\rho_1}$ and satisfy $I_\infty(u) = m_{\rho_1}$. Set $\upsilon(x) = (\frac{\rho_1}{\rho_2})^{\frac{N-p}{p}}u(\frac{\rho_1}{\rho_2}x)$ for $x \in \mathbb{R}$. Then $\|\upsilon\|_p = \rho_2$. Let $s * u = e^{\frac{N}{p}s}u(e^s x)$,

$$I_\infty(s*u) = \frac{ae^{ps}}{p}\|\nabla u\|_p^p + \frac{be^{2ps}}{2p}\|\nabla u\|_p^{2p} - \frac{e^{\frac{N(q-p)}{p}s}}{q}\|u\|_q^q.$$

Since $q > p + \dfrac{2p^2}{N}$, we have $I_\infty(s*\upsilon) \to -\infty$ as $s \to +\infty$ and $I_\infty(s*\upsilon) \to 0^+$ as $s \to -\infty$. Then, there exists $s_0 < 0$ such that $I(s_0 * \upsilon) > 0$ for $s < s_0$. Hence, there exists $s_\upsilon \in \mathbb{R}$ such that $I_\infty(s_\upsilon * \upsilon) = \max_{s\in\mathbb{R}} I_\infty(s * \upsilon)$ and $s_\upsilon * \upsilon \in P^\infty_{\rho_2}$. Moreover,

$$\|\nabla(s_\upsilon * \upsilon)\|_p^p = \|\nabla(s_\upsilon * u)\|_p^p, \quad \|s_\upsilon * \upsilon\|_q^q = (\frac{\rho_2}{\rho_1})^{\frac{Np-q(N-p)}{p}}\|s_\upsilon * u\|_q^q.$$

For any $q \in (p + \frac{2p^2}{N}, \frac{Np}{N-p})$, denote

$$\phi_q(s_\upsilon * u) = \frac{1}{q}e^{\frac{N(q-p)}{p}s_\upsilon}[1 - (\frac{\rho_2}{\rho_1})^{\frac{Np-q(N-p)}{p}}]\|u\|_q^q.$$

Then, $\phi_q(s_\upsilon * u) < 0$ for $q \in (p + \frac{2p^2}{N}, \frac{Np}{N-p})$. Thus, we can deduce that

$$m_{\rho_2} \le I_\infty(s_\upsilon * \upsilon) = I_\infty(s_\upsilon * u) + \phi_q(s_\upsilon * u) < I_\infty(s_\upsilon * u) < I_\infty(u) = m_{\rho_1},$$

which indicates m_c is decreasing on $(0, \infty)$.

□

4.2. **Proof of Theorem 2.** For some fixed $\delta \in (0, a)$, we give following assumptions on V and W.

$$\|V\|_{N/p} \le (a - \delta)S, \tag{4.19}$$

$$\|W\|_{N/(p-1)} < \frac{S^{\frac{p-1}{p}}}{p^2}\Big(aY - N(q - 2p)S^{-1}\|V\|_{N/p}\Big), \tag{4.20}$$

and

$$\|W\|_{N/(p-1)} < \frac{S^{\frac{p-1}{p}}}{Xp^2 + Zp}(aXY - [XN(q - 2p) + (N - p)Z]S^{-1}\|V\|_{N/p}), \tag{4.21}$$

where

$$X = Np - (N - p)q, \quad Y = N(q - p) - p^2 \;\; and \;\; Z = N(q - p)^2.$$

Firstly, we prove that the functional I has a mountain pass geometry.

Lemma 4.6 For every $u \in S_c$,

$$I(u) \ge \frac{\delta}{p}\|\nabla u\|_p^p - \frac{1}{q}C^q_{N,p,q}c^{q-\frac{N(q-p)}{p}}\|\nabla u\|_p^{\frac{N(q-p)}{p}}.$$

Proof. By Gagliardo-Nirenberg inequality, we have

$$\|u\|_q^q \le C_{N,p,q}^q c^{q-\frac{N(q-p)}{p}} \|\nabla u\|_p^{\frac{N(q-p)}{p}}.$$

And by the Hölder inequality and Sobolev inequality, we have

$$\int_{\mathbb{R}^N} V(x)|u|^p dx \le \|V\|_{\frac{N}{p}} \|u\|_{p^*}^p \le S^{-1} \|V\|_{\frac{N}{p}} \|\nabla u\|_p^p.$$

Hence, by (4.19)

$$\begin{aligned} I(u) &= \frac{a}{p}\|\nabla u\|_p^p + \frac{b}{2p}\|\nabla u\|_p^{2p} - \frac{1}{q}\|u\|_q^q - \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p dx \\ &\ge \frac{1}{p}(a - S^{-1}\|V\|_{\frac{N}{p}})\|\nabla u\|_p^p - \frac{1}{q} C_{N,p,q}^q c^{q-\frac{N(q-p)}{p}} \|\nabla u\|_p^{\frac{N(q-p)}{p}} \\ &\ge \frac{\delta}{p}\|\nabla u\|_p^p - \frac{1}{q} C_{N,p,q}^q c^{q-\frac{N(q-p)}{p}} \|\nabla u\|_p^{\frac{N(q-p)}{p}}. \end{aligned}$$

□

Lemma 4.7 For every $u \in S_c$,

$$\lim_{s\to-\infty} \|\nabla(s * u)\|_p = 0, \quad \lim_{s\to+\infty} \|\nabla(s * u)\|_p = +\infty, \tag{4.22}$$

$$\lim_{s\to-\infty} I(s * u) = 0, \quad \lim_{s\to+\infty} I(s * u) = -\infty. \tag{4.23}$$

Proof. Since

$$\|\nabla(s * u)\|_p^p = \int_{\mathbb{R}^N} |\nabla(e^{\frac{N}{p}s} u(e^s x))|^p dx = e^{ps} \|\nabla u\|_p^p.$$

It is obvious to obtain (4.22), then we prove (4.23). By Hölder inequality

$$\int_{\mathbb{R}^N} V(x)|s * u|^p dx \le \|V\|_{N/p} \|s * u\|_{p^*}^p = e^{ps} \|V\|_{N/p} \|u\|_{p^*}^p \to 0$$

as $s \to -\infty$. Moreover, since $q > p$, we have

$$\|s * u\|_q^q = e^{\frac{q-p}{p}Ns} \|u\|_q^q \to 0$$

as $s \to -\infty$. Then,

$$\lim_{s\to-\infty} I(s * u) = \lim_{s\to-\infty} I_\infty(s * u) - \frac{1}{p} \lim_{s\to-\infty} \int_{\mathbb{R}^N} V(x)|s * u|^p dx = 0.$$

Since $V(x) \geq 0$, we have

$$\lim_{s\to+\infty} I(s*u) \leq \lim_{s\to+\infty} I_\infty(s*u).$$

Since $q > p + \dfrac{2p^2}{N}$,

$$\lim_{s\to+\infty} I_\infty(s*u) = \lim_{s\to+\infty}\left(ae^{ps}\|\nabla u\|_p^p + be^{2ps}\|\nabla u\|_p^{2p} - e^{\frac{q-p}{p}Ns}\|u\|_q^q\right) \to -\infty.$$

as $s \to +\infty$.

□

For every $r \in \mathbb{R}$ and $R > 0$, define $I^r = \{u \in S_c : I(u) \leq r\}$ and

$$M_R = \inf\{I(u) : u \in S_c, \|\nabla u\|_p = R\}.$$

From Lemma 4.6 and 4.7, it is easy to know that $I^0 \neq \emptyset$ and there exist $\tilde{R} > R_0 > 0$ such that for all $0 < R \leq R_0$, $0 < M_R < M_{\tilde{R}}$. Thus, we can construct a min-max structure

$$\Gamma := \{\gamma \in C([0,1], \mathbb{R}\times S_c) : \gamma(0) \in (0, A_{R_0}), \gamma(1) \in (0, I^0)\} \tag{4.24}$$

with associated min-max level

$$m_{V,c} = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \tilde{I}(\gamma(t)) > 0, \tag{4.25}$$

where

$$A_R = \{u \in S_c : \|\nabla u\|_p \leq R\},\ \ \tilde{I}(s,u) = I(s*u).$$

Then, we prove I has a bounded PS sequence.

Lemma 4.8 There exists a bounded PS sequence $\{u_n\}$ for $I|_{S_c}$ at the level $m_{V,c}$ that is

$$I(u_n) \to m_{V,c} \ \text{ and } \ \|I'(u_n)\|_{(T_{u_n}S_c)^*} \to 0 \ \text{ as } \ n \to \infty, \tag{4.26}$$

such that

$$a\|\nabla u_n\|_p^p + b\|\nabla u_n\|_p^{2p} - \frac{N(q-p)}{pq}\|u_n\|_q^q - \frac{N}{p}\int_{\mathbb{R}^N} V(x)|u_n|^p dx - \int_{\mathbb{R}^N} V(x)|u_n|^{p-2}u_n\nabla u_n \cdot x dx \to 0. \tag{4.27}$$

Moreover, the related Lagrange multipliers of $\{u_n\}$

$$\lambda_n = -\frac{1}{c^p}I'(u_n)u_n, \tag{4.28}$$

converges to $\lambda > 0$ as $n \to \infty$.

Proof. The proof is divided into three steps.

(1) Existence of PS sequence. Let

$$X = \mathbb{R} \times S_c, \quad \mathcal{F} = \Gamma \quad \text{and} \quad B = (0, A_{R_0}) \cup (0, I^0).$$

By [7, Definition 5.1], Γ is a homotopy-stable family of compact subsets of $\mathbb{R}\times S_c$ with extend closed boundary $(0, A_{R_0}) \cup (0, I^0)$. Let

$$\phi = \tilde{I}(s,u) \quad \text{and} \quad F = \{(s,u) \in \mathbb{R} \times S_c : \tilde{I}(s,u) \geq m_{V,c}\}.$$

Similar to the proof of Theorem 1, we can deduce that the assumptions (1) and (2) in Lemma 3.2 are valid. We consider the sequence

$$\{A_n\} := \{\gamma_n\} = \{(\alpha_n, \beta_n)\} \subset \Gamma$$

such that $\lim\limits_{n\to\infty} \sup\limits_{A_n} \phi = m_{V,c}$. (We may assume that $\alpha_n = 0$, if not, replacing $\{(\alpha_n, \beta_n)\}$ with $\{(0, \alpha_n * \beta_n)\}$). By Lemma 3.2, there exist a sequence $\{(s_n, v_n)\}$ for $I|_{S_c}$ at the level $m_{V,c}$ such that

$$\partial_s \tilde{I}(s_n, v_n) \to 0 \quad \text{and} \quad \|\partial \tilde{I}(s_n, v_n)\|_{(T_{v_n}S_c)^*} \to 0 \quad \text{as} \quad n \to \infty.$$

Moreover, by Lemma 3.2 we have

$$|s_n| + dist_{W^{1,p}}(v_n, \beta_n([0,1])) \to 0 \quad \text{as} \quad n \to \infty$$

which implies $s_n \to 0$ as $n \to \infty$. Therefore, we have

$$I(s_n * v_n) = \tilde{I}(s_n, v_n) \to m_{V,c} \quad \text{as} \quad n \to \infty$$

and

$$I'(s_n * v_n)(s_n * \psi) = \partial \tilde{I}_u(s_n, v_n)\psi = o_n(1)\|\psi\| = o_n(1)\|s_n * \psi\|$$

for every $\psi \in T_{v_n}S_c$. Let $\{u_n\} = \{(s_n * v_n)\}$, then $\{u_n\}$ is a PS sequence for $I|_{S_c}$ at the level $m_{V,c}$. Differentiating $\tilde{I}$ with respect to s, we obtain (4.27).

(2) Boundedness of $\{u_n\}$ in $W^{1,p}(\mathbb{R}^N)$. Set

$$a_n := \|\nabla u_n\|_p^p, \quad b_n := \|u_n\|_q^q,$$

$$c_n := \int_{\mathbb{R}^N} V(x)|u_n|^p dx \ \text{ and } \ d_n := \int_{\mathbb{R}^N} V(x)|u_n|^{p-2}u_n \nabla u_n \cdot x dx.$$

By (4.26), we have

$$\frac{a}{p}a_n + \frac{b}{2p}a_n^2 - \frac{1}{q}b_n - \frac{1}{p}c_n = m_{V,c} + o_n(1), \tag{4.29}$$

and by (4.27), we have

$$(a + ba_n)a_n - \frac{N(q-p)}{pq}b_n - \frac{N}{p}c_n - d_n = o_n(1) \tag{4.30}$$

which implies

$$a\frac{N(q-p)-p^2}{Np(q-p)} \cdot a_n + b\frac{N(q-p)-2p^2}{2Np(q-p)} \cdot a_n^2 - \frac{q-2p}{p(q-p)}c_n + \frac{p}{N(q-p)}d_n = m_{V.c} + o_n(1).$$

Since

$$|c_n| \leq S^{-1}\|V\|_{N/p}a_n, \quad \text{and} \quad |d_n| \leq S^{-\frac{p-1}{p}}\|W\|_{N/(p-1)}a_n, \tag{4.31}$$

we get

$$\Big(aY - N(q-2p)S^{-1}\|V\|_{N/p} - p^2 S^{-\frac{p-1}{p}}\|W\|_{N/(p-1)}\Big)a_n + b\frac{N(q-p)-2p^2}{2} \cdot a_n^2 \leq Np(q-p)m_{V,c} + o_n(1).$$

By $q > p + \dfrac{2p^2}{N}$ and assumption (4.20), $\{a_n\}$ is bounded and

$$a_n \leq \frac{Np(q-p)m_{V,c}}{aY - N(q-2p)S^{-1}\|V\|_{N/p} - p^2 S^{-\frac{p-1}{p}}\|W\|_{N/(p-1)}} + o_n(1). \tag{4.32}$$

(3) $\lambda > 0$. By (4.26), there exist λ_n such that

$$I'(u_n)\psi + \lambda_n \int_{\mathbb{R}^N} |u_n|^{p-2}u_n\psi dx = o_n(1)\|\psi\|_{W^{1,p}},$$

and hence we can choose λ_n as (4.28). Since a_n is bounded, so b_n, c_n, d_n and λ_n are also bounded, we assume that they converge to a_0, b_0, c_0, d_0 and λ respectively. By (4.29), (4.30), (4.31) and (4.32), we have

$$\begin{aligned}
\lambda c^p &= -\lim_{n\to\infty} I'(u_n)u_n = -a\cdot a_0 - b\cdot a_0^2 + b_0 + c_0 \\
&= \frac{pX}{Y}m_{V,c} + \frac{b}{2}\cdot\frac{X}{Y}a_0^2 - \frac{(N-p)(q-p)}{Y}c_0 - \frac{qp-p^2}{Y}d_0 \\
&\geq \frac{pX}{Y}m_{V,c} - \frac{(N-p)(q-p)}{Y}S^{-1}\|V\|_{N/p}a_0 - \frac{qp-p^2}{Y}S^{-\frac{p-1}{p}}\|W\|_{N/(p-1)}a_0 \\
&\geq \frac{p}{Y}\Big(X - \frac{Z(N-p)S^{-1}\|V\|_{N/p} + ZpS^{-\frac{p-1}{p}}\|W\|_{N/(p-1)}}{aY - N(q-2p)S^{-1}\|V\|_{N/p} - p^2S^{-\frac{p-1}{p}}\|W\|_{N/(p-1)}}\Big)m_{V,c} \\
&= \frac{p}{Y}\frac{aXY - [XN(q-2p)+Z(N-p)]S^{-1}\|V\|_{N/p} - p[pX+Z]S^{-\frac{p-1}{p}}\|W\|_{N/(p-1)}}{aY - N(q-2p)S^{-1}\|V\|_{N/p} - p^2S^{-\frac{p-1}{p}}\|W\|_{N/(p-1)}}m_{V,c}.
\end{aligned}$$

Therefore, assumption (4.21) implies $\lambda > 0$.

□

Lemma 4.9 For every $c > 0$, there holds $m_{V,c} < m_c$.

Proof. By Theorem 1, there exist $\upsilon_c \in Z_c$ such that $I_\infty(\upsilon_c) = m_c$. By Lemma 4.7, let

$$\gamma(t) = (0, [(1-t)h_0 + th_1] * \upsilon_c),$$

where $h_0 << -1$ such that $\|\nabla(h_0 * \upsilon_c)\|_p < R_0$ and $h_1 >> 1$ such that $I(h_1 * \upsilon_c) < 0$. Then, $\gamma \in \Gamma$ and $\max_{t\in[0,1]} \tilde{I}(\gamma(t)) \geq m_{V,c}$. Since $V(x) \geq 0$,

$$\begin{aligned}
\max_{t\in[0,1]} \tilde{I}(\gamma(t)) &= \max_{t\in[0,1]} I([(1-t)h_0 + th_1] * \upsilon_c) \\
&< \max_{t\in[0,1]} I_\infty\big([(1-t)h_0 + th_1] * \upsilon_c\big) \\
&\leq \max_{s\in\mathbb{R}} I_\infty(s * \upsilon_c) = m_c,
\end{aligned}$$

we have $m_{V,c} < m_c$.

$\square$

Proof of Theorem 2. Now, let us prove the existence result. By Lemma 4.8, there exist PS sequence $\{u_n\}$ for $I|_{S_c}$ at level $m_{V,c}$ and $u \in W^{1,p}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^N)$ and $\lambda_n \to \lambda > 0$ as $n \to \infty$.

Then, we prove $u_n \to u$ in $W^{1,p}(\mathbb{R}^N)$. It is not difficult to prove that $\{u_n\}$ is a PS sequence for I_λ at level $m_{V,c}+\frac{\lambda}{p}c^p$. Suppose that $\{u_n\}$ does not strongly convergence to u in $W^{1,p}(\mathbb{R}^N)$. By Lemma 4.3, there exists $k \in \mathbb{N}_+$ and $y_n^j \in \mathbb{R}^N (1 \le j \le k)$ such that

$$u_n = u + \sum_{j=1}^{k} \omega^j(\cdot - y_n^j) + o_n(1) \ \text{ in } \ W^{1,p}(\mathbb{R}^N),$$

where ω^j satisfies

$$(a + bA^p)(-\Delta_p \omega) + \lambda|\omega|^{p-2}\omega = |\omega|^{q-2}\omega.$$

We claim that

$$I_A(u) - \frac{bA^p}{2p}\|\nabla u\|_p^p \ge 0. \tag{4.33}$$

Since $I'_\lambda(u_n) \to 0$ gives that $I'_{\lambda,A}(u) = 0$, then $P_A(u) = 0$, by assumption (4.20),

$$\begin{aligned}
&I_A(u) - \frac{bA^p}{2p}\|\nabla u\|_p^p - \frac{p}{N(q-p)}P_A(u)\\
&= \Big(a\frac{N(q-p)-p^2}{Np(q-p)} + b\frac{N(q-p)-2p^2}{2Np(q-p)}A^p\Big)\|\nabla u\|_p^p - \frac{q-2p}{p(q-p)}\int_{\mathbb{R}^N} V(x)|u|^p dx\\
&\quad + \frac{p}{N(q-p)}\int_{\mathbb{R}^N} V(x)|u|^{p-2}u\nabla u \cdot x dx\\
&\ge \frac{1}{Np(q-p)}\Big(aY - N(q-2p)S^{-1}\|V\|_{N/p} - p^2 S^{-\frac{p-1}{p}}\|W\|_{N/(p-1)}\Big)\|\nabla u\|_p^p\\
&\ge 0.
\end{aligned}$$

Then by (4.14), we have

$$\begin{aligned}P^{\infty}(\omega^j) &= a\|\nabla\omega^j\|_p^p + b\|\nabla\omega^j\|_p^{2p} - \gamma_q\|\omega^j\|_q^q \\ &< a\|\nabla\omega^j\|_p^p + bA^p\|\nabla\omega^j\|_p^p - \gamma_q\|\omega^j\|_q^q = P_A^{\infty}(\omega^j) = 0.\end{aligned}$$

By Lemma 3.4, we derive $s_j := s_{\omega^j} \leq 0$ such that $s_{\omega^j} * \omega^j \in P^{\infty}_{\|\omega^j\|_p} = \{u \in W^{1,p}(\mathbb{R}^N) : P_\infty(u) = 0\} \cap S_{\|\omega^j\|_p}$. Then, we claim

$$I_{\infty,A}(\omega^j) - \frac{bA^p}{2p}\|\nabla\omega^j\|_p^p \geq m_c. \tag{4.34}$$

Since $\|\omega^j\|_p^p \leq c^p$, by Lemma 4.5

$$\begin{aligned}&I_{\infty,A}(\omega^j) - \frac{bA^p}{2p}\|\nabla\omega^j\|_p^p - \frac{p}{N(q-p)}P_{\infty,A}(\omega^j) \\ &= a\frac{N(q-p)-p^2}{Np(q-p)}\|\nabla\omega^j\|_p^p + b\frac{N(q-p)-2p^2}{2Np(q-p)}A^p\|\nabla\omega^j\|_p^p \\ &\geq a\frac{N(q-p)-p^2}{Np(q-p)}\|\nabla\omega^j\|_p^p + b\frac{N(q-p)-2p^2}{2Np(q-p)}\|\nabla\omega^j\|_p^{2p} \\ &\geq a\frac{N(q-p)-p^2}{Np(q-p)}e^{ps_j}\|\nabla\omega^j\|_p^p + b\frac{N(q-p)-2p^2}{2Np(q-p)}e^{2ps_j}\|\nabla\omega^j\|_p^{2p} \\ &= I_\infty(s_j * \omega^j) - \frac{p}{N(q-p)}P_\infty(s_j * \omega^j) \\ &= I_\infty(s_j * \omega^j) \geq m_{\|\omega^j\|_p} \geq m_c.\end{aligned}$$

Let $n \to \infty$, by (4.15), we have

$$\begin{aligned}m_{V,c} + \frac{\lambda}{p}c^p &= I_{\lambda,A}(u) + \sum_{j=1}^{k} I_{\infty,\lambda,A}(\omega^j) - \frac{b}{2p}A^{2p} \\ &= I_A(u) - \frac{b}{2p}A^{2p} + \sum_{j=1}^{k} I_{\infty,A}(\omega^j) + \frac{\lambda}{p}\|u\|_p^p + \frac{\lambda}{p}\sum_{j=1}^{k}\|\omega^j\|_p^p.\end{aligned}$$

By (4.13), we know

$$c^p = \lim_{n\to\infty}\|u_n\|_p^p = \|u\|_p^p + \sum_{j=1}^{k}\|\omega^j\|_p^p.$$

Hence, by (4.14), (4.33) and (4.34)

$$
\begin{aligned}
m_{V,c} &= I_A(u) - \frac{b}{2p}A^{2p} + \sum_{j=1}^{k} I_{\infty,A}(\omega^j) \\
&\geq \frac{bA^p}{2p}(\|\nabla u\|_p^p + \sum_{j=1}^{k} \|\nabla \omega^j\|_p^p) - \frac{b}{2p}A^{2p} + km_c \\
&\geq m_c,
\end{aligned}
$$

which is a contradiction with Lemma 4.9. By strong convergence, we know $u \in S_c$ satisfies Eq.(1.1).

Acknowledgments

The authors thank the referee for valuable suggestions.

This paper is supported by Natural Science Foundation of Fujian Province (No.2022J01339).

References

[1] T. Bartsch, N. Soave: Multiple normalized solutions for a competing system of Schrödinger equations, Calc. Var. Partial Differ. Equ. **58** (2019), 22.
[2] L. Cai, F.B. Zhang: Normalized Solutions of Mass Supercritical Kirchhoff Equation with Potential, J. Geom. Anal. **33** (2023), 107.
[3] S.B. Deng, Q.R. Wu: Normalized solutions for p-Laplacian equation with critical Sobolev exponent and mixed nonlinearities, arXiv: 2306.06709.
[4] S.B. Deng, Q.R. Wu: Normalized solutions for p-Laplacian equations with potential, arXiv: 2310.10510.
[5] S. De Valeriola, M. Willem: On some quasilinear critical problems, Adv. Nonlinear Stud. **9** (2009), 825-836.
[6] X.J. Feng, Y.H. Li: Normalized solutions for some quasilinear elliptic equation with critical Sobolev exponent, arXiv: 2306.10207.
[7] N. Ghoussoub: Duality and perturbation methods in critical point theory, Cambridge University Press, UK, 1993.
[8] L. Jeanjean: Existence of solutions with prescribed norm for semilinear elliptic equations. Nonlinear Anal. **28** (1997), 1633-1659.
[9] Kang, J.C., Tang, C.L.: Normalized solutions for the nonlinear Schrödinger equation with potential and combined nonlinearities, Nonlinear Anal. **246**(2024), 113581.
[10] G. Kirchhoff: Mechanik, Teubner, Leipzig, 1883.
[11] G.B. Li, X. Luo, T. Yang: Normalized solutions to a class of Kirchhoff equations with Sobolev critical exponent, Ann. Fenn. Math. **47** (2022), 895-925.

[12] C. Mercuri, M. Willem: A global compactness result for the p-Laplacian involving critical nonlinearities, Discrete Cont. Dyn-A. **28** (2010), 469-493.

[13] I. Peral: Multiplicity of Solutions for the p-Laplacian, in: Lecture notes for the Second School of Nonlinear Functional Analysis and Applications to Differential Equations, Internationl Centre of Theoretical Physics, Trieste (Italia) (1997)

[14] T. Rong, F.Y. Li: Normalized solutions to the mass supercritical Kirchhoff-type equation with non-trapping potential, J. Math. Phys. **64** (2023), 081501.

[15] N. Soave: Normalized ground states for the NLS equation with combined nonlinearities, J. Differ. Equ. **269** (2020), 6941-6987.

[16] N. Soave: Normalized ground states for the NLS equation with combined nonlinearities: The Sobolev critical case, J. Funct. Anal. **279** (2020), 108610.

[17] Q. Wang, A.X. Qian: Normalized Solutions to the Kirchhoff equation with Potential Term: Mass Super-Critical Case, B. Malays. Math. Sci. So. **46** (2023), 77.

[18] X.Y. Zhen, Y.M. Zhang: Existence and uniqueness of normalized solutions for the Kirchhoff equation, Appl. Math. Lett. **74** (2017), 52-59.

Yuan Xu

E-mail address:xy1759383550@163.com

YongYi Lan

E-mail address:lanyongyi@jmu.edu.cn

Jimei University, School of Sciences, Jimei, Xiamen 361005, Fujian, China