BF PATH INTEGRALS FOR ELLIPTIC CURVES AND p-ADIC L-FUNCTIONS

JEEHOON PARK AND JUNYEONG PARK

ABSTRACT. We prove an arithmetic path integral formula for the inverse p-adic absolute values of the p-adic L-functions of elliptic curves over the rational numbers with good ordinary reduction at an odd prime p based on the Iwasawa main conjecture and Mazur's control theorem. This is an elliptic curve analogue of [2].

CONTENTS

1. INTRODUCTION

The arithmetic BF theory for number fields and abelian varieties was introduced in [3] to show the philosophy of arithmetic gauge theory which indicates that the path integral of the physical theory is closely related to the L-function of the relevant number theory. The arithmetic BF theory for cyclotomic fields led to [2], which proves an arithmetic path integral formula for the inverse p-adic absolute values of Kubota-Leopoldt p-adic L-functions at roots of unity (a precise connection between Kubota-Leopoldt p-adic L-function and the arithmetic BF path integral of cyclotomic fields up to p-adic units); the formula in [2] adds a small step toward such philosophy. Then a natural question is to prove a similar path integral formula for the p-adic L-function of elliptic curves with ordinary good reduction at p; the aim of this article is to resolve this question.

1.1. **The statement of the main theorem.** Let $E/\mathbb{Q}$ be an elliptic curve with semistable reduction at all places. Let p be an odd prime where E has good ordinary reduction. For $n \geq 0$, denote

$$K_n := \mathbb{Q}(\zeta_{p^{n+1}}), \quad X_n := \operatorname{Spec} \mathbb{Z}[\zeta_{p^{n+1}}]$$

where $\zeta_{p^{n+1}}$ is a primitive p^{n+1}-th root of unity. Also, we simply denote $K := K_0$ so that

$$\Gamma := \operatorname{Gal}(K_\infty/K) \cong \mathbb{Z}_p.$$

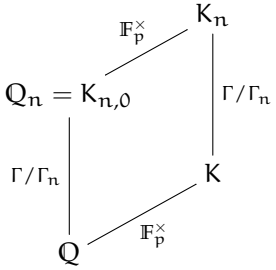
2020 *Mathematics Subject Classification.* Primary 11M41, 11R23; Secondary 81T45
Key words and phrases: The BF theory, BF path integrals, p-adic L-functions of elliptic curves

Denote $\Gamma_n \subseteq \Gamma$ the subgroup of index p^n. On the other hand, let

$$\omega : \mathrm{Gal}(K/\mathbb{Q}) \longrightarrow \mathbb{Z}_p^\times (\cong \mathrm{Gal}(K_\infty/\mathbb{Q}))$$

be the Teichmüller character and let $(\cdot)_r$ be the ω^r-isotypic component of a Γ-module with $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{F}_p^\times$-action. Then $\mathbb{Q}_n := K_{n,0} \subseteq K_n$ is the subfield fixed under ω so that we have the following diagram of field extensions:

(1.1)



Note that $\Gamma/\Gamma_n \cong \mathbb{Z}/p^n\mathbb{Z}$. We also denote

$$Y_n := \mathrm{Spec}\, \mathcal{O}_{\mathbb{Q}_n}.$$

Denote $\mathcal{E}$ the Néron model of E over $\mathbb{Z}$. For $m \geq 1$ we use the following notation:

$$\mathcal{T}^m(\mathrm{Spec}\, \mathcal{O}) := H^1_{\mathrm{fppf}}(\mathrm{Spec}\, \mathcal{O}, \mathcal{E}[p^m]) \times H^1_{\mathrm{fppf}}(\mathrm{Spec}\, \mathcal{O}, \mathcal{E}[p^m])$$

where $\mathcal{O}$ is the ring of integers of a number field and we view $\mathcal{E}[p^m]$ as sheaves in the flat topology. Denote $\mathcal{E}^0 \subseteq \mathcal{E}$ the identity component and $\Phi_\mathcal{E}$ the group of connected components. Then, as in [3, p. 1305], we have an exact sequence of fppf sheaves:

$$0 \longrightarrow \mathcal{E}^0 \longrightarrow \mathcal{E} \longrightarrow \Phi_\mathcal{E} \longrightarrow 0$$

Note that if $\mathbb{Z} \hookrightarrow \mathcal{O}$ is ramified only at $p \in \mathbb{Z}$, then $\mathcal{E} \otimes_\mathbb{Z} \mathcal{O}$ is the Néron model of E over $\mathcal{O}$ and the order[1] of $\Phi_{\mathcal{E}\otimes_\mathbb{Z}\mathcal{O}}$ is the same as the order of $\Phi_\mathcal{E}$ (Lemma 2.1).

We briefly recall the definition of the Tate-Shafarevich group $Ш(K_n, E)$ for each $n \in \mathbb{N} \cup \{\infty\}$:

$$Ш(K_n, E) := \ker\left(H^1(K_n, E(\overline{K}_n)) \longrightarrow \prod_v H^1(K_{n,v}, E(\overline{K}_{n,v})) \right)$$

where v runs over all primes of K_n. The Selmer group for each $n \in \mathbb{N} \cup \{\infty\}$ is defined so that it fits into the following exact sequence:

$$0 \longrightarrow E(K_n) \otimes_\mathbb{Z} \mathbb{Q}/\mathbb{Z} \longrightarrow \mathrm{Sel}(K_n, E) \longrightarrow Ш(K_n, E) \longrightarrow 0\ .$$

We refer to [6, chapter 2] for detailed definition. Then we make the following assumptions:

- the Selmer group $\mathrm{Sel}(K_n, E)$ is finite,
- the order of $\Phi_\mathcal{E}$ is relatively prime to p, and
- $E[p]$ is irreducible as a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation.

[1] Given a Néron model $\mathcal{A}$ over $\mathcal{O}$ of an abelian variety A over $\mathrm{Frac}(\mathcal{O})$, we have

$$\Phi_\mathcal{A} = \bigoplus_{s \in \mathrm{Spec}\, \mathcal{O}} i_{s*} \Phi_{\mathcal{A}_s}$$

where each $\Phi_{\mathcal{A}_s}$ is a finite étale group of connected component of $\mathcal{A}_s$ over the residue field $\kappa(s)$ (see [4, Proposition B.2] for example). Then the order of $\Phi_\mathcal{A}$ is defined to be the sum of orders of $\Phi_{\mathcal{A}_s}$.

Note that by the first assumption, $Ш(K_n, E)$ becomes finite (Lemma 2.2) and hence

$$Ш(K_n, E)[p^m] = Ш(K_n, E)[p^{2m}] \quad \text{for all sufficiently large } m.$$

The second assumption is needed to prove a correspondence between the flat cohomology and the Selmer group (Lemma 2.1). The third assumption is needed for the integrality of the p-adic L-function of E (i.e. $g_E(t) \in \mathbb{Z}_p[[t]]$; see (1.2)).

With the first two assumptions, one can define a 3-dimensional arithmetic BF theory [3]. The input data of such theory consists of

- (spacetime) the scheme $Y_n = \mathrm{Spec}(\mathcal{O}_{\mathbb{Q}_n})$
- (space of fields) the space $\mathcal{F}^m(Y_n)$
- (action functional) an arithmetic BF-functional (see (2.1) and (3.1)):

$$BF : \mathcal{F}^m(Y_n) \longrightarrow \frac{1}{p^m}\mathbb{Z}/\mathbb{Z}$$

for each $n \geq 0$ and $m \geq 1$.

Then the output of the theory is the following arithmetic path integral:

$$\sum_{(a,b)\in\mathcal{F}^m(Y_n)} \exp(2\pi i BF(a,b)).$$

For the reason that the above sum is called an 'arithmetic path integral', we refer to [2, Section 1.3]. By Lemma 2.1, this sum becomes finite if we assume that $\mathrm{Sel}(\mathbb{Q}_n, E[p^\infty])$ is finite. This follows from our finite assumption on $\mathrm{Sel}(K_n, E)$ by Lemma 2.2. Since we will define and use the BF-functional over X_n, we have assumed that $\mathrm{Sel}(K_n, E)$ is finite. In section 3, we will explain how to get the path integral over Y_n from the path integral over X_n.

Now we briefly review the p-adic L-function of E. Since E has good ordinary reduction at p, there exists a power series

$$g_E(t) \in \mathbb{Z}_p[[t]] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \tag{1.2}$$

which represents the p-adic L-function $L_p(E/\mathbb{Q}, s)$ of an elliptic curve (see [11] and [6, p. 459]). Under the assumption that $E[p]$ is an irreducible $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module, $g_E(t) \in \mathbb{Z}_p[[t]]$ holds. Let $\alpha_p, \beta_p \in \overline{\mathbb{Q}}$ be defined by $\alpha_p + \beta_p = a_p$ and $\alpha_p\beta_p = p$, where $a_p = 1 + p - \tilde{E}(\mathbb{F}_p)$. Then p does not divide a_p, which means p splits in $\mathbb{Q}(\alpha_p, \beta_p)$. Let $\tau(\chi) \in \overline{\mathbb{Q}}$ be the Gauss sum for a Dirichlet character χ. By the modularity of $E/\mathbb{Q}$, the L-value $L(E/\mathbb{Q}, \chi, 1)$ is defined and $L(E/\mathbb{Q}, \chi, 1)/\Omega_E$ is known to be algebraic by a theorem of Shimura, where $\Omega_E = \int_{E(\mathbb{R})} \frac{dx}{y}$. Let us fix a topological generator $\gamma_0 \in \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$. If χ is viewed as a faithful character of $\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$ with $n \geq 1$, then the conductor of χ is p^{n+1} and $\zeta = \chi(\gamma_0)$ is a p^n-th root of unity. Now the interpolation property of $g_E(t)$ is given by

$$g_E(0) = \frac{(1 - \beta_p p^{-1})^2 L(E/\mathbb{Q}, 1)}{\Omega_E}$$

$$g_E(\zeta - 1) = \frac{(\beta_p)^{n+1} L(E/\mathbb{Q}, \chi, 1)}{\tau(\chi)\Omega_E}$$

for $n \geq 1$. Sometimes $g_E(t)$ is called an analytic p-adic L-function of E, while there is the notion of an algebraic p-adic L-function of E defined by the generator of the characteristic ideal of the Pontrygin dual of the Selmer group $\mathrm{Sel}(\mathbb{Q}_\infty, E[p^\infty])$ which is a torsion $\mathbb{Z}_p[[t]]$-module.[2] The Iwasawa main conjecture [6, Conjecture

[2]The Selmer groups for each $n \in \mathbb{N} \cup \{\infty\}$ fit into the following exact sequences:

$$0 \longrightarrow E(\mathbb{Q}_n)[p^\infty] \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \mathrm{Sel}(\mathbb{Q}_n, E[p^\infty]) \longrightarrow Ш(\mathbb{Q}_n, E)[p^\infty] \longrightarrow 0\ .$$

We refer to [6, chapter 2] for detailed definition.

4.16], which is now a theorem by Skinner-Urban [13], asserts that they are the same.

Let $|\cdot|_p$ be the p-adic absolute value on the algebraic closure $\overline{\mathbb{Q}}_p$ normalized by $|p|_p = p^{-1}$. Now we can state our main theorem.

Theorem 1.1. *For each $v \in Y_n$, let $c_v^{(p)}(E)$ be the highest power of p dividing the Tamagawa factor $c_v(E)$ for E at v. Then the following arithmetic path integral formula holds.*

$$\left| \prod_{\zeta^{p^n}=1} g_E(\zeta - 1) \right|_p^{-1}$$
$$= \frac{\left|\widetilde{E}(\mathbb{F}_p)[p^\infty]\right|^2}{|E(\mathbb{Q}_n)[p^\infty]|} \cdot \prod_{\substack{v \in Y_n \\ v \nmid p, v \mid N_E}} c_v^{(p)}(E) \cdot \lim_{m \to \infty} \sum_{(a,b) \in \mathcal{F}^m(Y_n)} \exp(2\pi i BF(a,b))$$

where $\widetilde{E}$ is the reduction of E at p, and N_E is the conductor of E.

We prove this theorem in section 4. For the proof, we first derive a path integral formula (Lemma 4.1) in a more general context, using Mazur's control theorem [6, Theorem 4.1] and the Iwasawa main conjecture. Then we analyze the "error term" of Mazur's control theorem following the method of [5].

1.2. **Open question.** An interesting open question is to enlarge the space of fields $\mathcal{F}^m(Y_n)$ or modify the BF-functional so that we can obtain a path integral formula for the L-value $\prod_{\zeta^{p^n}=1} g_E(\zeta - 1)$ itself, which amounts to remove the p-adic absolute value from the formula in Theorem 1.1 incorporating p-adic unit information.

1.3. **Acknowledgement.** Jeehoon Park was supported by the National Research Foundation of Korea (NRF-2021R1A2C1006696) and the National Research Foundation of Korea grant (NRF-2020R1A5A1016126) funded by the Korea government (MSIT). Junyeong Park was supported by Samsung Science and Technology Foundation under Project Number SSTF-BA2001-02, the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (RS-2024-00341372), and the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (RS-2024-00449679).

2. The BF-functional for elliptic curves

We now recall the definition of the BF-functional in [3, p.1303]. By [10, Corollary 3.4], we have a perfect pairing[3]:

$$\cup : H^1_{fppf}(X_n, \mathcal{E}[p^m]) \times H^2_{fppf}(X_n, \mathcal{E}[p^m]) \longrightarrow H^3_{fppf}(X_n, \mathbb{G}_m)[p^m] \ .$$

together with an isomorphism as in [10, p. 252]:

$$\mathrm{inv} : H^3_{fppf}(X_n, \mathbb{G}_m) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$$

which restricts to

$$\mathrm{inv} : H^3_{fppf}(X_n, \mathbb{G}_m)[p^m] \xrightarrow{\sim} \frac{1}{p^m}\mathbb{Z}/\mathbb{Z} \ .$$

Finally, let

$$\delta : H^1_{fppf}(X_n, \mathcal{E}[p^m]) \longrightarrow H^2_{fppf}(X_n, \mathcal{E}[p^m])$$

[3]By [10, p.220], we have $H^\bullet_{fppf,c}(X_n, -) \cong H^\bullet_{fppf}(X_n, -)$.

be the Bockstein map coming from the exact sequence:

$$0 \longrightarrow \mathcal{E}[p^m] \longrightarrow \mathcal{E}[p^{2m}] \xrightarrow{p^m} \mathcal{E}[p^m] \longrightarrow 0 .$$

Combining all these, we define the arithmetic BF-functional as follows:

$$\mathrm{BF} : \mathcal{F}^m(X_n) \longrightarrow \frac{1}{p^m}\mathbb{Z}/\mathbb{Z} \qquad (a, b) \longmapsto \mathrm{inv}(a \cup \delta b) \tag{2.1}$$

Lemma 2.1. *Let $\mathcal{O}$ be the ring of integers of a number field such that $\mathbb{Z} \hookrightarrow \mathcal{O}$ is ramified only at $p \in \mathbb{Z}$. Assume that the order of $\Phi_{\mathcal{E}}$ is relatively prime to p.*

(1) $\mathcal{E} \otimes_{\mathbb{Z}} \mathcal{O}$ is the Néron model of E over $\mathcal{O}$.
(2) The order of $\Phi_{\mathcal{E}\otimes_{\mathbb{Z}}\mathcal{O}}$ is the same as the order of $\Phi_{\mathcal{E}}$.
(3) We have the following isomorphisms:

$$H^1_{\mathrm{fppf}}(\mathrm{Spec}\,\mathcal{O}, \mathcal{E}[p^m]) \cong \mathrm{Sel}(\mathrm{Frac}(\mathcal{O}), E[p^m])$$

$$H^1_{\mathrm{fppf}}(\mathrm{Spec}\,\mathcal{O}, \mathcal{E})[p^m] \cong Ш(\mathrm{Frac}(\mathcal{O}), E)[p^m]$$

where $\mathrm{Frac}(\mathcal{O})$ *is the field of fractions of* $\mathcal{O}$.

Proof. (1) follows because the étale base change of a Néron model is still a Néron model and our E has good ordinary reduction at p.

(2) Since $\mathbb{Z} \hookrightarrow \mathcal{O}$ is ramified only at $p \in \mathbb{Z}$, the number of connected component may vary at the primes in $\mathcal{O}$ lying over p. Since E has good ordinary reduction at p, our $\mathcal{E}$ is always connected at these primes.

(3) By (1) and (2), the first isomorphism comes from [3, Lemma A.2] and the second from [3, Lemma A.3]. □

Lemma 2.2. *With the assumptions so far, the following holds.*

(1) $E(K_n)$ and $Ш(K_n, E)$ are finite.
(2) $\mathrm{Sel}(\mathbb{Q}_n, E)$ is finite. Hence $E(\mathbb{Q}_n)$, $Ш(\mathbb{Q}_n, E)$, and $\mathrm{Sel}(\mathbb{Q}_n, E[p^\infty])$ become finite.

Proof. For a number field F we have the following exact sequence ([6, chapter 2]):

$$0 \longrightarrow E(F) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \longrightarrow \mathrm{Sel}(F, E) \longrightarrow Ш(F, E) \longrightarrow 0 . \tag{2.2}$$

(1) Taking $F = K_n$ in (2.2), we immediately conclude that $Ш(K_n, E)$ is finite and $E(K_n) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$, i.e., $E(K_n)$ is a torsion abelian group. Since $E(K_n)$ is a finitely generated abelian group by the Mordell-Weil theorem, we conclude that $E(K_n)$ is finite.

(2) From the inflation-restriction sequence for group cohomology, we get

$$\ker\left(\mathrm{Sel}(\mathbb{Q}_n, E) \longrightarrow \mathrm{Sel}(K_n, E)^{\mathrm{Gal}(K_n/\mathbb{Q}_n)} \right) \subseteq H^1(\mathrm{Gal}(K_n/\mathbb{Q}_n), E(K_n)).$$

Note that $E(K_n)_{\mathrm{tors}} = E(K_n)$ by (1). Therefore, $\mathrm{Sel}(\mathbb{Q}_n, E)$ is finite, which immediately says that $\mathrm{Sel}(\mathbb{Q}_n, E[p^\infty])$ is finite. By the same argument as in the proof (1), we conclude that $E(\mathbb{Q}_n)$ and $Ш(\mathbb{Q}_n, E)$ are finite. □

Proposition 2.3. *For every n and every sufficiently large m, we have*

$$\sum_{(a,b)\in\mathcal{F}^m(X_n)} \exp(2\pi i \mathrm{BF}(a, b)) = |\mathrm{Sel}(K_n, E[p^m])| \left| \frac{E(K_n)}{p^m E(K_n)} \right|$$

Proof. The assertion follows from [3, Section 3] by noting that E is self-dual. □

3. Isotypic components of the BF-functional

The $\mathrm{Gal}(K_\infty/\mathbb{Q})$-action on X_n induces by functoriality a $\mathrm{Gal}(K_\infty/\mathbb{Q})$-module structure on $H^\bullet_{\mathrm{fppf}}(X_n, \mathcal{E}[p^m])$. Since $p-1$ is relatively prime to p, the $\mathrm{Gal}(K/\mathbb{Q})$-action on $H^\bullet_{\mathrm{fppf}}(X_n, \mathcal{E}[p^m])$ is semisimple. Denote

$$\mathcal{F}^m_r(X_n) := H^1_{\mathrm{fppf}}(X_n, \mathcal{E}[p^m])_r \times H^1_{\mathrm{fppf}}(X_n, \mathcal{E}[p^m])_{-r} \tag{3.1}$$

so that we have

$$\mathcal{F}^m(X_n) = \bigoplus_{r=0}^{p-2} \mathcal{F}^m_r(X_n).$$

We will show that $\mathcal{F}^m_0(X_n) = \mathcal{F}^m(Y_n)$.

By functoriality, $\cup$ and δ are ω^r-equivariant. Hence they restrict to

$$\cup : H^1_{\mathrm{fppf}}(X_n, \mathcal{E}[p^m])_r \times H^2_{\mathrm{fppf}}(X_n, \mathcal{E}[p^m])_s \longrightarrow H^3_{\mathrm{fppf}}(X_n, \mathbb{G}_m)_{r+s}$$

$$\delta : H^1_{\mathrm{fppf}}(X_n, \mathcal{E}[p^m])_r \longrightarrow H^2_{\mathrm{fppf}}(X_n, \mathcal{E}[p^m])_r \ .$$

Since the $\mathrm{Gal}(K/\mathbb{Q})$-action on

$$\mathrm{inv} : H^3_{\mathrm{fppf}}(X_n, \mathbb{G}_m) \xrightarrow{\sim} \frac{1}{p^m}\mathbb{Z}/\mathbb{Z}$$

is trivial, $H^3_{\mathrm{fppf}}(X_n, \mathbb{G}_m)_{r+s} \neq 0$ if and only if $r+s \equiv 0 \bmod p-1$ so the BF-functional (2.1) splits into

$$\sum_{r=0}^{p-2} \mathrm{BF}_r : \bigoplus_{r=0}^{p-2} \mathcal{F}^m_r(X_n) \longrightarrow \frac{1}{p^m}\mathbb{Z}/\mathbb{Z} \ .$$

Therefore, we have

$$\sum_{(a,b)\in\mathcal{F}^m(X_n)} \exp(2\pi i \mathrm{BF}(a,b)) = \prod_{r=0}^{p-2} \sum_{(a,b)\in\mathcal{F}^m_r(X_n)} \exp(2\pi i \mathrm{BF}(a,b))$$

Proposition 3.1. *For every n and every sufficiently large m, we have*

$$\sum_{(a,b)\in\mathcal{F}^m_r(X_n)} \exp(2\pi i \mathrm{BF}(a,b)) = |\mathrm{Sel}(K_n, E[p^m])_r| \left| \left(\frac{E(K_n)}{p^m E(K_n)} \right)_{-r} \right|$$

Proof. If $\delta b \neq 0$, then the sum over $a \in H^1_{\mathrm{fppf}}(X_n, \mathcal{E}[p^m])_{-r}$ becomes

$$\sum_a \exp(2\pi i \mathrm{BF}(a,b)) = \sum_a \exp(2\pi i \cdot \mathrm{inv}(a \cup \delta b)) = 0.$$

On the other hand, if $\delta b = 0$, then $\exp(2\pi i \mathrm{BF}(a,b)) = 1$. Since δ is ω-equivariant,

$$(\ker\delta)_{-r} = H^1_{\mathrm{fppf}}(X_n, \mathcal{E}[p^m])_{-r} \cap \ker\delta.$$

Combining these, we get

$$\sum_{(a,b)\in\mathcal{F}^m_r(X_n)} \exp(2\pi i \mathrm{BF}(a,b)) = \left| H^1_{\mathrm{fppf}}(X_n, \mathcal{E}[p^m])_r \right| |(\ker\delta)_{-r}|.$$

For each n and sufficiently large m, we have a factorization coming from [3, p.1304]:

$$\begin{array}{ccc} H^1_{fppf}(X_n, \mathcal{E}[p^m]) & \xrightarrow{\delta} & H^2_{fppf}(X_n, \mathcal{E}[p^m]) \\ & \searrow \quad \nearrow & \\ & H^1_{fppf}(X_n, \mathcal{E})[p^m] & \end{array}$$

where the surjection fits into the Kummer sequence:

$$0 \longrightarrow \frac{E(K_n)}{p^m E(K_n)} \longrightarrow H^1_{fppf}(X_n, \mathcal{E}[p^m]) \longrightarrow H^1_{fppf}(X_n, \mathcal{E})[p^m] \longrightarrow 0$$

Consequently,

$$(\ker \delta)_{-r} = \left| \left(\frac{E(K_n)}{p^m E(K_n)} \right)_{-r} \right|$$

The other factor is determined from (3) of Lemma 2.1. □

We conclude this section by realizing Proposition 3.1 as an arithmetic path integral on Y_n. For the reader's convenience, we begin with a lemma.

Lemma 3.2. *Let m be a positive integer.*

(1) For each scheme S, the canonical map is an isomorphism:

$$H^\bullet_{\text{ét}}(S, \mathcal{E}[p^m]_S) \xrightarrow{\sim} H^\bullet_{fppf}(S, \mathcal{E}[p^m]_S)\,.$$

(2) For each $n \geq 0$, the canonical map is an isomorphism:

$$H^\bullet_{fppf}\left(Y_n, \mathrm{Res}_{X_n/Y_n}(\mathcal{E}[p^m]_{X_n})\right) \xrightarrow{\sim} H^\bullet_{fppf}(X_n, \mathcal{E}[p^m]_{X_n})$$

where Res *denotes the Weil restriction.*

Proof. (1) By [9, Theorem III.3.9], the canonical map for $\mathcal{E}$:

$$H^\bullet_{\text{ét}}(S, \mathcal{E}_S) \xrightarrow{\sim} H^\bullet_{fppf}(S, \mathcal{E}_S)$$

is an isomorphism. Hence the assertion follows from applying the five lemma to the long exact sequence associated to the following exact sequence of fppf sheaves:

$$0 \longrightarrow \mathcal{E}[p^m]_S \longrightarrow \mathcal{E}_S \xrightarrow{p^m} \mathcal{E}_S \longrightarrow 0\,.$$

(2) By Lemma 2.1 and [1, Proposition 7.6.6], the Weil restriction $\mathrm{Res}_{X_n/Y_n}(\mathcal{E}_{X_n})$ is a Néron model of its generic fiber:

$$\left(\mathrm{Res}_{X_n/Y_n}(\mathcal{E}_{X_n})\right)_{Q_n} \cong \mathrm{Res}_{K_n/Q_n}(E_{K_n}).$$

Since K_n/Q_n is separable, the right hand side is an abelian variety over Q_n (cf. [8, p.178]). Hence we have the following commutative square of abelian groups:

$$\begin{array}{ccc} H^\bullet_{\text{ét}}\left(Y_n, \mathrm{Res}_{X_n/Y_n}(\mathcal{E}_{X_n})\right) & \xrightarrow{\sim} & H^\bullet_{fppf}\left(Y_n, \mathrm{Res}_{X_n/Y_n}(\mathcal{E}_{X_n})\right) \\ \downarrow \wr & & \downarrow \\ H^\bullet_{\text{ét}}(X_n, \mathcal{E}_{X_n}) & \xrightarrow{\sim} & H^\bullet_{fppf}(X_n, \mathcal{E}_{X_n}) \end{array}$$

where the rows are isomorphisms by [10, Proposition III.0.4d] again, and the left column is an isomorphism by [14, Tag 03QP] because $X_n \to Y_n$ is finite.

On the other hand, by [7, Proposition 3.19], we have

$$\left|\Phi_{\mathrm{Res}_{X_n/Y_n}(\mathcal{E}_{X_n})}\right| = \left|\Phi_{\mathcal{E}_{X_n}}\right|$$

which are relatively prime to p by assumption. Hence the following sequence of fppf sheaves is exact (note that the Weil restriction is left exact):

$$0 \longrightarrow \mathrm{Res}_{X_n/Y_n}(\mathcal{E}[p^m]_{X_n}) \longrightarrow \mathrm{Res}_{X_n/Y_n}(\mathcal{E}_{X_n}) \xrightarrow{p^m} \mathrm{Res}_{X_n/Y_n}(\mathcal{E}_{X_n}) \longrightarrow 0\,.$$

Therefore the assertion follows from applying the five lemma to the associated cohomology long exact sequence. □

Since $(\mathcal{O}_{K_n})_0 = \mathcal{O}_{Q_n}$, the ω^r-isotypic decomposition of $\mathcal{O}_{K_n}$ becomes

$$\mathcal{O}_{K_n} = \mathcal{O}_{Q_n} \oplus \bigoplus_{r=1}^{p-2} (\mathcal{O}_{K_n})_r$$

where each factor is canonically an $\mathcal{O}_{Q_n}$-module. Hence, from Lemma 3.2, we get

$$\begin{aligned} H^\bullet_{\mathrm{fppf}}(X_n, \mathcal{E}[p^m]) &\cong H^\bullet_{\mathrm{fppf}}(X_n, \mathcal{E}[p^m] \otimes_{\mathbb{Z}} \mathcal{O}_{K_n}) \\ &\cong H^\bullet_{\mathrm{fppf}}\left(Y_n, \mathrm{Res}_{X_n/Y_n}(\mathcal{E}[p^m] \otimes_{\mathbb{Z}} \mathcal{O}_{K_n})\right) \\ &\cong H^\bullet_{\mathrm{fppf}}(Y_n, \mathcal{E}[p^m]) \oplus \bigoplus_{r=1}^{p-2} H^\bullet_{\mathrm{fppf}}(Y_n, \mathcal{E}[p^m] \otimes_{\mathbb{Z}} (\mathcal{O}_{K_n})_r) \end{aligned}$$

because taking the $\mathrm{Gal}(K/Q)$-invariant of abelian p-groups is an exact functor. Consequently, (3.1) can be rewritten as

$$\mathcal{F}^m_r(X_n) \cong H^1_{\mathrm{fppf}}(Y_n, \mathcal{E}[p^m] \otimes_{\mathbb{Z}} (\mathcal{O}_{K_n})_r) \times H^1_{\mathrm{fppf}}(Y_n, \mathcal{E}[p^m] \otimes_{\mathbb{Z}} (\mathcal{O}_{K_n})_{-r}).$$

Moreover, we have the corresponding Bockstein map:

$$\delta : H^1_{\mathrm{fppf}}(Y_n, \mathcal{E}[p^m] \otimes_{\mathbb{Z}} (\mathcal{O}_{K_n})_{-r}) \longrightarrow H^2_{\mathrm{fppf}}(Y_n, \mathcal{E}[p^m] \otimes_{\mathbb{Z}} (\mathcal{O}_{K_n})_{-r})\,.$$

Therefore, Proposition 3.1 can be rewritten as an arithmetic path integral on Y_n as desired. In particular, we have $\mathcal{F}^m_0(X_n) = \mathcal{F}^m(Y_n)$ and hence (for sufficiently large m)

$$\sum_{(a,b)\in\mathcal{F}^m(Y_n)} \exp(2\pi i \mathrm{BF}(a,b)) = |\mathrm{Sel}(Q_n, E[p^m])| \left|\frac{E(Q_n)}{p^m E(Q_n)}\right|. \tag{3.2}$$

4. The proof of the main theorem

In this section, we prove Theorem 1.1. We begin with a lemma. Define abelian groups A_n and B_n via the following exact sequence:

$$0 \longrightarrow A_n \longrightarrow \mathrm{Sel}(Q_n, E[p^\infty]) \longrightarrow \mathrm{Sel}(Q_\infty, E[p^\infty])^{\Gamma_n} \longrightarrow B_n \longrightarrow 0\,,$$

where $\mathrm{Sel}(F, E[p^\infty])$ is the Selmer group over F associated to $E[p^\infty]$. By Mazur's control theorem [6, Theorem 4.1], A_n and B_n are finite p-groups whose orders are bounded as $n \to \infty$.

Lemma 4.1. *For each $n \geq 0$, the BF functional satisfies the following formula:*

$$\left|\prod_{\zeta^{p^n}=1} g_E(\zeta - 1)\right|_p^{-1} = \frac{1}{|E(Q_n)[p^\infty]|}\frac{|B_n|}{|A_n|}\lim_{m\to\infty}\sum_{(a,b)\in\mathcal{F}^m(Y_n)} \exp(2\pi i \mathrm{BF}(a,b)).$$

Proof. The proof is based on the Iwasawa main conjecture [6, Conjecture 4.16] and Mazur's control theorem.

Since $E(\mathbb{Q}_n)$ is finite by lemma 2.2, the following composition is an isomorphism for every sufficiently large m:

$$E(\mathbb{Q}_n)[p^\infty] = E(\mathbb{Q}_n)[p^m] \hookrightarrow E(\mathbb{Q}_n) \longrightarrow \frac{E(\mathbb{Q}_n)}{p^m E(\mathbb{Q}_n)}\ .$$

Consequently, for sufficiently large m,

$$\left|\frac{E(\mathbb{Q}_n)}{p^m E(\mathbb{Q}_n)}\right| = |E(\mathbb{Q}_n)[p^\infty]|.$$

Hence (3.2) can be rewritten as

$$\sum_{(a,b)\in\mathcal{F}^m(Y_n)} \exp(2\pi i BF(a,b)) = |Sel(\mathbb{Q}_n, E[p^m])| \cdot |E(\mathbb{Q}_n)[p^\infty]|.$$

Denote $(\cdot)^\vee$ the Pontryagin dual of abelian groups and let

$$V_n := Sel(\mathbb{Q}_n, E[p^\infty])^\vee.$$

Choose a topological generator $\gamma \in \Gamma$, which gives a topological ring isomorphism

$$\Lambda := \mathbb{Z}_p[[t]] \xrightarrow{\sim} \mathbb{Z}_p[[\Gamma]] \qquad t \longmapsto \gamma - 1$$

Along this isomorphism, we can write

$$\left(Sel(\mathbb{Q}_\infty, E[p^\infty])^{\Gamma_n}\right)^\vee \cong \frac{V_\infty}{((t+1)^{p^n}-1)V_\infty}.$$

Note that V_∞ is a torsion Λ-module by [5, Theorem 1.5]. Moreover, $g_E(t) \in \Lambda$ by [6, p. 459], and V_∞ has no nonzero pseudo-null Λ-submodule by [5, Proposition 4.15]. Applying the structure theorem for Λ-modules [6, Theorem 3.1] to V_∞ (cf. [12, Lemma 4]) and using [6, Exercise 3.7], we get

$$\left|\frac{V_\infty}{((t+1)^{p^n}-1)V_\infty}\right| = \left|\frac{\Lambda}{(g_E(t),(t+1)^p-1)}\right| = u \cdot \prod_{\zeta^{p^n}=1} g_E(\zeta-1), \quad u \in \mathbb{Z}_p^\times.$$

This in turn gives the following equalities:

$$\begin{aligned} |Sel(\mathbb{Q}_n, E[p^\infty])| &= |V_n| \\ &= \frac{|A_n^\vee|}{|B_n^\vee|}\left|\frac{V_\infty}{((t+1)^{p^n}-1)V_\infty}\right| = \frac{|A_n|}{|B_n|} \cdot u \cdot \prod_{\zeta^{p^n}=1} g_E(\zeta-1). \end{aligned}$$

Since $Sel(\mathbb{Q}_n, E[p^\infty])$, A_n, and B_n are abelian p-groups, we conclude that

$$\frac{|B_n|}{|A_n|}|Sel(\mathbb{Q}_n, E[p^\infty])| = \left|\prod_{\zeta^{p^n}=1} g_E(\zeta-1)\right|_p^{-1},$$

which finishes the proof. □

Let us return to the proof of Theorem 1.1. It remains to determine $|A_n|$ and $|B_n|$ in Lemma 4.1. We begin with the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Sel(\mathbb{Q}_n, E[p^\infty]) & \longrightarrow & H^1(\mathbb{Q}_n, E[p^\infty]) & \longrightarrow & \mathcal{G}_E(\mathbb{Q}_n) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle s_n} & & \downarrow{\scriptstyle h_n} & & \downarrow{\scriptstyle g_n} & & \\ 0 & \longrightarrow & Sel(\mathbb{Q}_\infty, E[p^\infty])^{\Gamma_n} & \longrightarrow & H^1(\mathbb{Q}_\infty, E[p^\infty])^{\Gamma_n} & \longrightarrow & \mathcal{G}_E(\mathbb{Q}_\infty) & & \end{array}$$

where $\mathcal{G}_E$ is essentially defined to be the cokernel of $\mathrm{Sel}(-, E[p^\infty]) \hookrightarrow H^1(-, E[p^\infty])$. See [5, pp.451–452] for the precise description of this diagram. By our assumptions and the proof of [5, Lemma 3.1], we have

$$|\ker h_n| = |E(\mathbb{Q}_n)[p^\infty]|.$$

From the snake lemma together with [5, Lemma 3.2], we get

$$\frac{|B_n|}{|A_n|} = \frac{|\operatorname{coker} s_n|}{|\ker s_n|} = \frac{|\ker g_n|}{|\ker h_n|} = \frac{|\ker g_n|}{|E(\mathbb{Q}_m)[p^\infty]|}.$$

Since $\Gamma_n \cong \mathbb{Z}_p$, we may apply [5, Lemma 4.7] to the above diagram. Using [5, Lemma 3.3], [5, Lemma 3.4], and [5, Proposition 4.8], we get

$$|\ker g_n| = \left|\widetilde{E}(\mathbb{F}_p)[p^\infty]\right|^2 \cdot \prod_{\substack{v \in Y_n \\ v \nmid p, v \mid N_E}} c_v^{(p)}(E).$$

Note that $\mathbb{Q}_n/\mathbb{Q}$ is totally ramified at p so the unique prime in Y_n lying over p has the residue field $\mathbb{F}_p$. This concludes the proof of Theorem 1.1.

REFERENCES

[1] S. Bosch, W. Lütkebohmert, M. Raynaud: *Néron models*, Springer-Verlag (1990).
[2] M. Carlson, C. Hee-Joong, D. Kim, M. Kim, J. Park, H. Yoo: *Path integrals and* p*-adic* L*-functions*, Bull. London Math. Soc., Volume **56** (2024), Issue 6, Pages 1951-1966.
[3] M. Carlson, M. Kim,: *A note on abelian arithmetic BF-theory*, Bull. London Math. Soc., **54** (2022) 1299–1307.
[4] K. Česnavičius: *Selmer groups as flat cohomology groups*, J. Ramanujan Math. Soc. **31**, No.1 (2016) 31–61.
[5] R. Greenberg: *Iwasawa theory for elliptic curves*, Arithmetic theory of elliptic curves (Cetraro, 1997), Lecture Notes in Math., 1716 (1999) 51–144.
[6] R. Greenberg: *Introduction to Iwasawa theory for elliptic curves*, Arithmetic algebraic geometry (Park City, UT, 1999), IAS/Park City Math. Ser., Vol. **9** (2001) 407–464.
[7] D. Lorenzini: *Torsion and Tamagawa numbers*, Ann. Inst. Fourier **61**, no. 5 (2011), 1995-2037.
[8] J. S. Milne: *On the arithmetic of abelian varieties*, Invent. Math. **17** (1972), 177-190.
[9] J. S. Milne: *Étale cohomology*, Princeton University Press (1980).
[10] J. S. Milne: *Arithmetic duality theorems*, BookSurge, LLC, Charleston, SC, 2nd ed. (2006).
[11] B. Mazur, J. Tate, J. Teitelbaum: *On* p*-adic analogues of the conjectures of Birch and Swinnerton-Dyer*, Invent. Math. **84** (1986), 1-48.
[12] C. Skinner: *Lectures on the Iwasawa theory of elliptic curves*, Lecture notes for the Arizona Winter school, `https://swc-math.github.io/aws/2018/2018SkinnerNotes.pdf`.
[13] C. Skinner, E. Urban: *The Iwasawa Main Conjectures for* GL_2, Invent. math. **195**, 1-277 (2014).
[14] The stacks project authors: *Stacks project*, `https://stacks.math.columbia.edu` (2018).

JEEHOON PARK: QSMS, SEOUL NATIONAL UNIVERSITY, 1 GWANAK-RO, GWANAK-GU, SEOUL, SOUTH KOREA 08826
Email address: `jpark.math@gmail.com`

JUNYEONG PARK: DEPARTMENT OF MATHEMATICS EDUCATION, CHONNAM NATIONAL UNIVERSITY, 77, YONGBONG-RO, BUK-GU, GWANGJU 61186, KOREA
Email address: `junyeongp@gmail.com`