THE LIPSCHITZ CONTINUITY OF THE SOLUTION TO BRANCHED ROUGH DIFFERENTIAL EQUATIONS

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Abstract. Based on an isomorphism between Grossman Larson Hopf algebra and Tensor Hopf algebra, we apply a sub-Riemannian geometry technique to branched rough differential equations and obtain the explicit Lipschitz continuity of the solution with respect to the initial value, the vector field and the driving rough path.

1. Introduction

In his seminal paper [1], Lyons built the theory of rough paths. The theory gives a meaning to differential equations driven by highly oscillating signals and proves the existence, uniqueness and stability of the solution to differential equations. The theory has an embedded component in stochastic analysis, and has been successfully applied to differential equations driven by general stochastic processes [2, 3, 4, 5], the existence and smoothness of the density of solutions [6, 7], stochastic Taylor expansions [8], support theorem [9], large deviations theory [10] etc.

In Lyons' original framework [1], highly oscillating paths are lifted to geometric rough paths in a nilpotent Lie group. Geometric rough paths take values in a truncated group of characters of the shuffle Hopf algebra [11, Section 1.4] and satisfy an abstract integration by parts formula. Limits of continuous bounded variation paths in a rough path metric are geometric. For example, Brownian sample paths enhanced with Stratonovich iterated integrals are geometric rough paths. However, the geometric assumption can sometimes be restrictive. Itô iterated integrals do not satisfy the integration by parts formula and Itô Brownian rough paths are not geometric. Moreover, non-geometric rough paths appear naturally when solving stochastic partial differential equations [12].

To provide a natural framework for non-geometric rough paths, Gubinelli [12] introduced branched rough paths and proved the existence, uniqueness and continuity of the solution to branched rough differential equations. Branched rough paths take values in a truncated group of characters of Connes Kreimer Hopf algebra [13]. The multiplication of Connes Kreimer Hopf algebra is the free abelian multiplication of monomials of trees which does not impose the integration by parts formula. Branched rough paths can accomodate non-geometric stochastic integrals and Connes Kreimer Hopf algebra provides a natural algebraic setting for stochastic partial differential equations [12, 14, 15].

The stability of the solution to rough differential equations is a central result in rough path theory, commonly referred to as the Universal Limit Theorem [16, Theorem 5.3]. Based on the uniform decay of the differences between adjacent Picard iterations, Lyons [1, Theorem 4.1.1] proved the uniform continuity of the solution with respect to the driving geometric rough path. Through controlled paths [17], Gubinelli [12, Theorem 8.8] proved the Lipschitz continuity of the solution to branched rough differential equations with respect to the initial value and the driving rough path. Following the controlled

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paths approach, Friz and Zhang [18, Theorem 4.20] proved the Lipschitz continuity of the solution to differential equations driven by branched rough paths with jumps. Based on Davie's discrete approximation method [19] and by employing a sub-Riemannian geometry technique [8], Friz and Victoir [20, Theorem 10.26] proved the explicit Lipschitz continuity of the solution to differential equations driven by weak geometric rough paths over $\mathbb{R}^d$ with respect to the initial value, the vector field and the driving rough path.

In this paper, we will extend Friz and Victoir's approach and result [20, Theorem 10.26] to branched rough differential equations. Classically, the sub-Riemannian geometry technique only applies to geometric rough paths. Based on an isomorphism between Grossman Larson Hopf algebra and Tensor Hopf algebra [21, 22], Boedihardjo and Chevyrev [23] proved that branched rough paths are isomorphic to a class of Π-rough paths [24, 25]. A Π-rough path is an inhomogeneous geometric rough path, for which the regularities of the components of the underlying path are not necessarily the same. By applying a sub-Riemannian geometry technique to Π-rough paths, we prove in Theorem 2.3 the explicit Lipschitz dependence of the solution to branched rough differential equations.

Comparing with the current existing results [12, Theorem 8.8][18, Theorem 4.20], our result only requires that the vector field is $Lip\,(\gamma)$ for $\gamma > p$ (instead of $Lip\,([p]+1)$) and explicitly specifies the uniform Lipschitz continuity of the solution with respect to the initial value, the vector field and the driving branched rough path, with the constant only depending on p, γ, d (the roughness of the driving branched rough path, the regularity of the vector field and the dimension of the underlying driving path).

2. Notations

A rooted tree is a finite connected graph with no cycle and a special vertex called root. We call a rooted tree a tree. We assume trees are non-planar for which the children trees of each vertex are commutative. A forest is a commutative monomial of trees. The degree $|\tau|$ of a forest τ is given by the number of vertices in τ.

For the label set $\mathcal{L} := \{1, 2, \ldots, d\}$, an $\mathcal{L}$-labeled forest is a forest for which each vertex is attached with a label from $\mathcal{L}$. Let $\mathcal{T}_{\mathcal{L}}(\mathcal{F}_{\mathcal{L}})$ denote the set of $\mathcal{L}$-labeled trees (forests). Let $\mathcal{T}_{\mathcal{L}}^N(\mathcal{F}_{\mathcal{L}}^N)$ denote the subset of $\mathcal{T}_{\mathcal{L}}(\mathcal{F}_{\mathcal{L}})$ of degree $1, 2, \ldots, N$.

Let $G_{\mathcal{L}}^N$ denote the group of degree-N characters of $\mathcal{L}$-labeled Connes Kreimer Hopf algebra [13, p.214]. $a \in G_{\mathcal{L}}^N$ iff $a : \mathbb{R}\mathcal{F}_{\mathcal{L}}^N \to \mathbb{R}$ is an $\mathbb{R}$-linear map that satisfies

$$(a, \tau_1\tau_2) = (a, \tau_1)\,(a, \tau_2)$$

for every $\tau_1, \tau_2 \in \mathcal{F}_{\mathcal{L}}^N$, $|\tau_1| + |\tau_2| \leq N$, where $\tau_1\tau_2$ denotes the commutative multiplication of monomials of trees. The multiplication in $G_{\mathcal{L}}^N$ is induced by the coproduct of Connes Kreimer Hopf algebra based on admissible cuts [13, p.215]: for $a, b \in G_{\mathcal{L}}^N$ and $\tau \in \mathcal{F}_{\mathcal{L}}^N$,

$$(ab, \tau) := \sum_{(\tau)} \left(a, \tau_{(1)}\right)\left(b, \tau_{(2)}\right).$$

We equip $a \in G_{\mathcal{L}}^N$ with the norm:

$$\|a\| := \max_{\tau \in \mathcal{F}_{\mathcal{L}}^N} |(a, \tau)|^{\frac{1}{|\tau|}}.$$

Definition 2.1 (p-variation). For a topological group $(G, \|\cdot\|)$, suppose $X : [0, T] \to (G, \|\cdot\|)$ is continuous. For $0 \leq s \leq t \leq T$, denote

$$X_{s,t} := X_s^{-1} X_t.$$

For $p \geq 1$, define the p-variation of X on $[0,T]$ as

$$\|X\|_{p-var,[0,T]} := \sup_{D \subset [0,T]} \left(\sum_{k, t_k \in D} \left\| X_{t_k, t_{k+1}} \right\|^p \right)^{\frac{1}{p}},$$

where the supremum is taken over $D = \{t_k\}_{k=0}^n$, $0 = t_0 < t_1 < \cdots < t_n = T$, $n \geq 1$. Denote the set of continuous paths from $[0,T]$ to G with finite p-variation as $C^{p-var}([0,T], G)$.

For $p \geq 1$, let $[p]$ denote the largest integer which is less or equal to p.

Definition 2.2 (branched p-rough path). For $p \geq 1$, $X : [0,T] \to G_{\mathcal{L}}^{[p]}$ is a branched p-rough path if X is continuous and of finite p-variation.

For $\gamma > 0$, denote $\lfloor \gamma \rfloor := \max\{n \in \mathbb{N} \cup \{0\} \,|\, n < \gamma\}$ and denote $\{\gamma\} := \gamma - \lfloor \gamma \rfloor$. Suppose U and W are two Banach spaces. A function $f : U \to W$ is $Lip(\gamma)$ if

$$|f|_{Lip(\gamma)} := \left(\max_{k=0,1,\dots,\lfloor \gamma \rfloor} \left| D^k f \right|_\infty \right) \vee \left| D^{\lfloor \gamma \rfloor} f \right|_{\{\gamma\}-\text{Höl}} < \infty,$$

where $D^k f$ denotes the kth Fréchet derivative of f and

$$\begin{aligned} \left| D^k f \right|_\infty &:= \sup_{x \in U} \left| \left(D^k f \right)(x) \right|, \\ \left| D^{\lfloor \gamma \rfloor} f \right|_{\{\gamma\}-\text{Höl}} &:= \sup_{x,y \in U, x \neq y} \frac{\left| D^{\lfloor \gamma \rfloor} f(y) - D^{\lfloor \gamma \rfloor} f(x) \right|}{|y - x|^{\{\gamma\}}}. \end{aligned}$$

Let $L\left(\mathbb{R}^d, \mathbb{R}^e\right)$ denote the set of continuous linear mappings from $\mathbb{R}^d$ to $\mathbb{R}^e$. The following theorem is the main result of the current paper.

Theorem 2.3. *For $\gamma > p \geq 1$ and $i = 1,2$, suppose $f^i : \mathbb{R}^e \to L\left(\mathbb{R}^d, \mathbb{R}^e\right)$ are $Lip(\gamma)$ vector fields and $X^i : [0,T] \to G_{\mathcal{L}}^{[p]}$ are branched p-rough paths over $\mathbb{R}^d$. For $\xi^i \in \mathbb{R}^e$, $i = 1,2$, let y^i denote the unique solution of the branched rough differential equation:*

$$dy_t^i = f^i\left(y_t^i\right) dX_t^i, \ y_0^i = \xi^i.$$

Denote $\lambda := \max_{i=1,2} |f^i|_{Lip(\gamma)}$, $\omega(s,t) := \sum_{i=1,2} \|X^i\|_{p-var,[s,t]}^p$ and

$$\rho_{p-\omega;[0,T]}\left(X^1, X^2\right) := \max_{\tau \in \mathcal{F}_{\mathcal{L}}^{[p]}} \sup_{0 \leq s < t \leq T} \frac{\left| \left(X_{s,t}^1, \tau \right) - \left(X_{s,t}^2, \tau \right) \right|}{\omega(s,t)^{\frac{|\tau|}{p}}}.$$

Then there exists a constant $M > 0$ that only depends on γ, p, d such that

$$\begin{aligned} &\sup_{0 \leq s < t \leq T} \frac{\left| (y_t^1 - y_s^1) - (y_t^2 - y_s^2) \right|}{\omega(s,t)^{\frac{1}{p}}} \\ \leq\ & M\lambda \left(\left| \xi^1 - \xi^2 \right| + \lambda^{-1} \left| f^1 - f^2 \right|_{Lip(\gamma-1)} + \rho_{p-\omega;[0,T]}\left(X^1, X^2\right) \right) \exp\left(M\lambda^p \omega(0,T) \right). \end{aligned}$$

The existence and uniqueness of the solution when the vector field is $Lip(\gamma)$ for $\gamma > p$ follow from [26, Theorem 22]. The $\rho_{p-\omega;[0,T]}$ distance is consistent with the d_γ-Hölder distance defined by Gubinelli [12, p.710] where $\omega(s,t) = |t - s|$.

Based on an isomorphism between branched rough paths and a class of Π-rough paths [23], our proof relies on an inhomogeneous geodesic technique which extends the sub-Riemannian geometry for geometric rough paths [8, 20] to branched rough paths.

Comparing with the current existing results [12, Theorem 8.8] and [18, Theorem 4.20], our estimate only requires that the vector field is $Lip\,(\gamma)$ for $\gamma > p$ while not $Lip\,([p]+1)$. Moreover, our result specifies explicitly the Lipschitz dependence of the solution with respect to the initial value, the vector field and the driving branched rough path with the constant only depending on γ, p, d.

3. Proof

In [27], Grossman and Larson described several Hopf algebras associated with certain family of trees. By deleting the additional root, we call the Hopf algebra of non-planar forests with product [27, (3.1)] and coproduct [27, p.199] the Grossman Larson Hopf algebra. Based on Foissy [21, Section 8] and Chapoton [22], Grossman Larson algebra is freely generated by a collection of unlabeled trees. Denote this collection of trees as $\mathcal{B}$. Denote the $\mathcal{L}$-labeled version of $\mathcal{B}$ as $\mathcal{B}_{\mathcal{L}}$ with $\mathcal{L} = \{1, 2, \dots, d\}$.

Notation 3.1. Let $\mathcal{B}_{\mathcal{L}}^{[p]} = \{\nu_1, \nu_2, \dots, \nu_K\}$ denote the set of elements in $\mathcal{B}_{\mathcal{L}}$ of degree $1, \dots, [p]$.

Then K only depends on p, d.

Notation 3.2. Let $\mathcal{W}$ denote the set of finite sequences $k_1 \cdots k_m$ for $k_j \in \{1, 2, \dots, K\}$, $j = 1, 2, \dots, m$, including the empty sequence denoted as ϵ. For $k_1 \cdots k_m \in \mathcal{W}$, define its degree

$$\|k_1 \cdots k_m\| := |\nu_{k_1}| + \cdots + |\nu_{k_m}|$$

where $|\nu_j|$ denotes the number of vertices in ν_j and $\|\epsilon\| := 0$.

The set of infinite tensor series generated by $\mathcal{B}_{\mathcal{L}}^{[p]}$ with the operation of tensor product forms an algebra. An element a of the algebra can be represented as $a = \sum_{w \in \mathcal{W}} (a, w)\, w$ for $(a, w) \in \mathbb{R}$. For $n = 0, 1, 2, \dots$, the set $\sum_{w \in \mathcal{W}, \|w\| > n} c_w w$ for $c_w \in \mathbb{R}$ forms an ideal. Denote the quotient algebra as $\mathcal{A}^n$. Let $\mathfrak{G}$ denote the group of algebraic exponentials of Lie series generated by $\{1, 2, \dots, K\}$ ($\mathfrak{G}$ is a group based on Baker–Campbell–Hausdorff formula). Denote the group

$$\mathfrak{G}^n := \mathfrak{G} \cap \mathcal{A}^n$$

and denote the projection

$$\pi_n : \mathfrak{G} \to \mathfrak{G}^n.$$

We equip $a \in \mathfrak{G}^n$ with the norm

$$\|a\| := \sum_{w \in \mathcal{W}, 0 < \|w\| \le n} |(a, w)|^{\frac{1}{\|w\|}}.$$

$\mathfrak{G}^n$ is an inhomogeneous counterpart of the step-n free nilpotent Lie group [1, p.235, Theorem 2.1.1].

Notation 3.3. Suppose $x = \left(x^1, \dots, x^K\right) : [0, T] \to \mathbb{R}^K$ is a continuous bounded variation path. For $n = 0, 1, \dots$ and $0 \le s \le t \le T$, define $S_n(x)_{s,t} \in \mathfrak{G}^n$ as, for $k_1 \cdots k_m \in \mathcal{W}$, $\|k_1 \cdots k_m\| \le n$,

$$\left(S_n(x)_{s,t}, k_1 \cdots k_m\right) := \int \cdots \int_{s < u_1 < \cdots < u_m < t} dx_{u_1}^{k_1} \cdots dx_{u_m}^{k_m}$$

with $\left(S_n(x)_{s,t}, \epsilon\right) := 1$.

$S_n(x)$ is an inhomogeneous counterpart of the step-n signature [28, Definition 1.1]. The following Lemma is an inhomogeneous generalization of Proposition 7.64 [20].

Lemma 3.4. *For $i = 1, 2$, $C > 0$, $\delta > 0$ and an integer $n \geq 1$, suppose $h^i \in \mathfrak{G}^n$, $\|h^i\| \leq C$ and*

$$\max_{w \in \mathcal{W}, \|w\| \leq n} \left| \left(h^1 - h^2, w \right) \right| \leq \delta.$$

Then there exist $x^i \in C^{1-var}\left([0,1], \mathbb{R}^K\right), i = 1, 2$ such that

$$S_n\left(x^i\right)_{0,1} = h^i, \; i = 1, 2$$

and a constant $M = M(C, p, d, n) > 0$ such that

$$\max_{i=1,2} \left\| x^i \right\|_{1-var,[0,1]} \leq M \text{ and } \left\| x^1 - x^2 \right\|_{1-var,[0,1]} \leq \delta M.$$

Proof. In the following proof, the constant M may depend on C, p, d, n and its exact value may change.

Firstly, assume $(h^1, w) = (h^2, w) = 0$ for $w \in \mathcal{W}$, $\|w\| = 1, \ldots, n-1$. Then $h^i = 1 + l^i$ for $i = 1, 2$ with l^i a homogeneous element of degree n and $l^2 = l^1 + \delta m$ with $\|m\| \leq M$. Based on similar proof as that of [20, Theorem 7.32] and [20, Theorem 7.44], there exists $z \in C^{1-var}\left([0,1], \mathbb{R}^K\right)$ such that $S_n(z)_{0,1} = 1 + l^1 - m$ and $\|z\|_{1-var,[0,1]} \leq M$. Similarly, there exists $y = (y^i)_{i=1}^K \in C^{1-var}\left([0,1], \mathbb{R}^K\right)$ such that $S_n(y)_{0,1} = 1 + m$ and $\|y\|_{1-var,[0,1]} \leq M$. Let x^1 be the concatenation of z and y and let x^2 be the concatenation of z with $\widetilde{y} := \left((1+\delta)^{|\nu_i|/n} y^i\right)_{i=1}^K$. Since $n \geq |\nu_j|$ ($\mathfrak{G}^n$ does not involve j when $|\nu_j| > n$), we have $(1+\delta)^{|\nu_j|/n} - 1 \leq \delta$ and

$$\left\| x^1 - x^2 \right\|_{1-var,[0,2]} \leq \delta \|y\|_{1-var,[0,1]}.$$

The first case is proved.

General case: we provide an inductive proof. The case $n = 1$ follows from the first case. Assuming the statement holds for elements in $\mathfrak{G}^n$, we now prove that it holds for elements in $\mathfrak{G}^{n+1}$. By the inductive hypothesis, there exist continuous bounded variation paths $z^i : [0,1] \to \mathbb{R}^K$, $i = 1, 2$ such that $S_n(z^i)_{0,1} = \pi_n(h^i)$, $i = 1, 2$,

$$\max_{i=1,2} \left\| z^i \right\|_{1-var,[0,1]} \leq M \text{ and } \left\| z^1 - z^2 \right\|_{1-var,[0,1]} \leq \delta M.$$

Denote

$$k^i := b^i \otimes h^i \text{ with } b^i := S_{n+1}\left(\overleftarrow{z^i}\right), \; i = 1, 2,$$

where $\overleftarrow{z^i}$ denotes the time reversal of z^i. Then for $i = 1, 2$, $\|k^i\| \leq M$ and $(k^i, w) = 0$ for $w \in \mathcal{W}$, $\|w\| = 1, \ldots, n$. For $w \in \mathcal{W}$, $\|w\| = n+1$,

$$\left| \left(k^1 - k^2, w \right) \right| \leq \sum_{uv=w} \left(\left| \left(b^1, u \right) \right| \left| \left(h^1, v \right) - \left(h^2, v \right) \right| + \left| \left(b^1, u \right) - \left(b^2, u \right) \right| \left| \left(h^2, v \right) \right| \right),$$

where uv denotes the concatenation of u and v. Since iterated integrals are continuous in 1-variation of the underlying path, combined with the conditions on h^i, we have $|(k^1 - k^2, w)| \leq \delta M$ for $w \in \mathcal{W}$, $\|w\| = n+1$. Based on the first case, there exist continuous bounded variation paths $y^i : [0,1] \to \mathbb{R}^K, i = 1, 2$ such that

$$S_{n+1}\left(y^i\right)_{0,1} = k^i, \; i = 1, 2$$

and

$$\max_{i=1,2}\left\|y^{1}\right\|_{1-var,[0,1]}\leq M\text{ and }\left\|y^{1}-y^{2}\right\|_{1-var,[0,1]}\leq\delta M.$$

For $i=1,2$, let x^{i} be the concatenation of z^{i} with y^{i}. The proof is finished. $\square$

Based on [21, 22], Grossman Larson Hopf algebra is isomorphic as a Hopf algebra to the Tensor Hopf algebra generated by a collection of trees. By deleting the additional root, we assume Grossman Larson Hopf algebra with product [27, (3.1)] and coproduct [27, p.199] is a Hopf algebra of forests. Denote the degree-n truncated group of group-like elements in Grossman Larson Hopf algebra as $\mathcal{G}_{\mathcal{L}}^{n}$.

Notation 3.5. Denote the group isomorphism $\Phi:\mathfrak{G}^{[p]}\rightarrow\mathcal{G}_{\mathcal{L}}^{[p]}$.

Lemma 3.6. *For $i=1,2$, $C>0$ and $\delta>0$, suppose $g^{i}\in\mathcal{G}_{\mathcal{L}}^{[p]}$, $\left\|g^{i}\right\|\leq C$ and*

$$\max_{\tau\in\mathcal{F}_{\mathcal{L}}^{[p]}}\left|\left(g^{1}-g^{2},\tau\right)\right|\leq\delta.$$

Then there exist $x^{i}\in C^{1-var}\left(\left[0,1\right],\mathbb{R}^{K}\right)$, $i=1,2$ such that

$$\Phi\left(S_{[p]}\left(x^{i}\right)_{0,1}\right)=g^{i},\ i=1,2$$

and a constant $M=M\left(C,p,d\right)>0$ such that

$$\max_{i=1,2}\left\|x^{i}\right\|_{1-var,[0,1]}\leq M\text{ and }\left\|x^{1}-x^{2}\right\|_{1-var,[0,1]}\leq\delta M.$$

Proof. Denote $h^{i}:=\Phi^{-1}\left(g^{i}\right)$, $i=1,2$. By $\left\|g^{i}\right\|\leq C$, we have $\left\|h^{i}\right\|\leq M$, $i=1,2$ and

$$\sup_{w\in\mathcal{W},\|w\|\leq[p]}\left|\left(h^{1}-h^{2},w\right)\right|\leq M\max_{\tau\in\mathcal{F}_{\mathcal{L}}^{[p]}}\left|\left(g^{1}-g^{2},\tau\right)\right|\leq M\delta.$$

Then the statement holds based on Lemma 3.4. $\square$

For $a\in\mathcal{L}$, denote by $\bullet_{a}$ the tree that has one vertex and a label $a\in\mathcal{L}$ on the vertex. For $\mathcal{L}$-labeled trees $\left\{\tau_{i}\right\}_{i=1}^{k}$ and a label $a\in\mathcal{L}$, denote by $\left[\tau_{1}\cdots\tau_{k}\right]_{a}$ the labeled tree obtained by grafting the roots of $\left\{\tau_{i}\right\}_{i=1}^{k}$ to a new root with a label $a\in\mathcal{L}$ on the new root. Then $\left|\left[\tau_{1}\cdots\tau_{k}\right]_{a}\right|=\sum_{i=1}^{k}\left|\tau_{i}\right|+1$.

Notation 3.7. For sufficiently smooth $f=\left(f_{1},\ldots,f_{d}\right):\mathbb{R}^{e}\rightarrow L\left(\mathbb{R}^{d},\mathbb{R}^{e}\right)$, define $f:\mathcal{T}_{\mathcal{L}}\rightarrow\left(\mathbb{R}^{e}\rightarrow\mathbb{R}^{e}\right)$ inductively as, for $a\in\mathcal{L}$ and $\tau_{i}\in\mathcal{T}_{\mathcal{L}}$, $i=1,\ldots,k$,

$$f\left(\bullet_{a}\right):=f_{a}\text{ and }f\left(\left[\tau_{1}\cdots\tau_{k}\right]_{a}\right):=\left(d^{k}f_{a}\right)\left(f\left(\tau_{1}\right)\cdots f\left(\tau_{k}\right)\right)$$

where $d^{k}f_{a}$ denotes the k-th Fréchet derivative of f_{a}.

Suppose $x\in C^{1-var}\left(\left[0,T\right],\mathbb{R}^{K}\right)$, $f:\mathbb{R}^{e}\rightarrow L\left(\mathbb{R}^{K},\mathbb{R}^{e}\right)$ is $Lip\left(1\right)$ and $\xi\in\mathbb{R}^{e}$. Denote by

$$\pi_{f}\left(0,\xi;x\right)$$

the unique solution to the ODE

$$dy_{t}=f\left(y_{t}\right)dx_{t},\ y_{0}=\xi.$$

For $f_{j}:\mathbb{R}^{e}\rightarrow\mathbb{R}^{e}$, denote

$$\left|f_{j}\right|_{\infty}:=\sup_{y\in\mathbb{R}^{e}}\left|f_{j}\left(y\right)\right|.$$

For $y:\left[0,T\right]\rightarrow\mathbb{R}^{e}$ and $0\leq s\leq t\leq T$, denote

$$y_{s,t}:=y_{t}-y_{s}.$$

Proposition 3.8. *Assume that*

(i) $f=(f_1,\dots,f_K):\mathbb{R}^e\to L(\mathbb{R}^K,\mathbb{R}^e)$ *and* $\widetilde{f}=\left(\widetilde{f}_1,\dots,\widetilde{f}_K\right):\mathbb{R}^e\to L(\mathbb{R}^K,\mathbb{R}^e)$ *are* $Lip(1)$. *For* $j=1,\dots,K$, *denote*

$$M_j:=\max\left\{|f_j|_{Lip(1)},\left|\widetilde{f}_j\right|_{Lip(1)}\right\}.$$

(ii) $x=(x^1,\dots,x^K)$ *and* $\widetilde{x}=(\widetilde{x}^1,\dots,\widetilde{x}^K)$ *are in* $C^{1-var}([0,T],\mathbb{R}^K)$. *For* $j=1,\dots,K$, *denote*

$$l_j:=\max\left\{\left\|x^j\right\|_{1-var,[0,T]},\left\|\widetilde{x}^j\right\|_{1-var,[0,T]}\right\}.$$

(iii) $y_0,\widetilde{y}_0\in\mathbb{R}^e$ *are initial values.*

Denote $y=\pi_f(0,y_0;x)$ *and* $\widetilde{y}=\pi_{\widetilde{f}}(0,\widetilde{y}_0;\widetilde{x})$. *Then*

$$\begin{aligned}(3.1)\qquad &\sup_{t\in[0,T]}|y_{0,t}-\widetilde{y}_{0,t}|\\ &\le\sum_{j=1}^K\left(M_jl_j|y_0-\widetilde{y}_0|+M_j\left\|x^j-\widetilde{x}^j\right\|_{1-var,[0,T]}+l_j\left|f_j-\widetilde{f}_j\right|_\infty\right)\exp\left(2\sum_{j=1}^KM_jl_j\right)\end{aligned}$$

and

$$\begin{aligned}(3.2)\qquad &\sup_{t\in[0,T]}|y_t-\widetilde{y}_t|\\ &\le\left(|y_0-\widetilde{y}_0|+\sum_{j=1}^KM_j\left\|x^j-\widetilde{x}^j\right\|_{1-var,[0,T]}+\sum_{j=1}^Kl_j\left|f_j-\widetilde{f}_j\right|_\infty\right)\exp\left(2\sum_{j=1}^KM_jl_j\right).\end{aligned}$$

Proof. Without loss of generality, assume $x_0=\widetilde{x}_0=0$. Since

$$\int_0^tf_j(\widetilde{y}_r)\,d\left(x_r^j-\widetilde{x}_r^j\right)=f_j(\widetilde{y}_t)\left(x_t^j-\widetilde{x}_t^j\right)-\int_0^t\left(x_r^j-\widetilde{x}_r^j\right)df_j(\widetilde{y}_r),$$

we have

$$\begin{aligned}&|y_{0,t}-\widetilde{y}_{0,t}|\\ \le\ &|y_0-\widetilde{y}_0|\sum_{j=1}^KM_jl_j+\sum_{j=1}^KM_j\int_0^t|y_{0,r}-\widetilde{y}_{0,r}|\left|dx_r^j\right|+\sum_{j=1}^Kl_j\left|f_j-\widetilde{f}_j\right|_\infty\\ &+\left(1+\sum_{j=1}^KM_jl_j\right)\sum_{j=1}^KM_j\sup_{t\in[0,T]}\left|x_t^j-\widetilde{x}_t^j\right|.\end{aligned}$$

Since $x_0=\widetilde{x}_0=0$, we have $\sup_{t\in[0,T]}\left|x_t^j-\widetilde{x}_t^j\right|\le\|x^j-\widetilde{x}^j\|_{1-var,[0,T]}$. Based on Gronwall's Lemma, the first inequality holds. The second inequality can be proved similarly. □

Denote $I(x):=x$ for $x\in\mathbb{R}^e$. Recall that ϵ denotes the empty element in $\mathcal{W}$. For $f^i:\mathbb{R}^e\to L(\mathbb{R}^d,\mathbb{R}^e)$, $i=1,2$ and $\nu\in\mathcal{B}_{\mathcal{L}}^{[p]}$ in Notation 3.1, denote $f^i(\nu)$ as in Notation 3.7.

Notation 3.9. Suppose $f^i:\mathbb{R}^e\to L(\mathbb{R}^d,\mathbb{R}^e)$, $i=1,2$ are sufficiently smooth. For $k_1\cdots k_m\in\mathcal{W}$, define inductively

$$F_i^\epsilon:=I\text{ and }F_i^{k_1\cdots k_m}:=dF_i^{k_2\cdots k_m}\left(f^i(\nu_{k_1})\right),\ i=1,2,$$

where $dF_i^{k_2\cdots k_m}$ denotes the Fréchet derivative of $F_i^{k_2\cdots k_m}$.

The following simple Lemma is helpful when estimating the increments of functions.

Lemma 3.10. *For $i=1,2$, suppose $q^i:\mathbb{R}^e\to\mathbb{R}$ and $r^i:\mathbb{R}^e\to\mathbb{R}$. For $a,b\in\mathbb{R}^e$,*

$$\begin{aligned}
& \left(q^1r^1-q^2r^2\right)(a)-\left(q^1r^1-q^2r^2\right)(b)\\
= & \left(q^1\left(r^1-r^2\right)\right)(a)-\left(q^1\left(r^1-r^2\right)\right)(b)\\
& +\left(\left(q^1-q^2\right)r^2\right)(a)-\left(\left(q^1-q^2\right)r^2\right)(b)\\
=: & \ Q(a)-Q(b)+R(a)-R(b)
\end{aligned}$$

where $Q:=q^1\left(r^1-r^2\right)$ and $R:=\left(q^1-q^2\right)r^2$.

Lemma 3.11 and Lemma 3.12 below are generalizations of Lemma 10.23 [20] and Lemma 10.25 [20] respectively and apply to ODEs with inhomogeneous drivers. Recall $\mathcal{B}_{\mathcal{L}}^{[p]}=\{\nu_1,\nu_2,\ldots,\nu_K\}$ in Notation 3.1. Since K denotes the number of elements in $\mathcal{B}_{\mathcal{L}}^{[p]}$, K only depends on p,d.

Lemma 3.11. *Fix $\gamma>p\geq 1$.*

(i) Suppose $f^i:\mathbb{R}^e\to L\left(\mathbb{R}^d,\mathbb{R}^e\right)$, $i=1,2$ are $Lip(\gamma)$. Denote $\lambda:=\max_{i=1,2}\left|f^i\right|_{Lip(\gamma)}$.

(ii) For $i=1,2$, suppose $x^i=\left(x^{i,1},\ldots,x^{i,K}\right)$ and $\widetilde{x}^i=\left(\widetilde{x}^{i,1},\ldots,\widetilde{x}^{i,K}\right)$ are paths in $C^{1-var}\left([0,1],\mathbb{R}^K\right)$ such that

$$S_{[p]}\left(x^i\right)_{0,1}=S_{[p]}\left(\widetilde{x}^i\right)_{0,1},\ i=1,2.$$

(iii) For $C\geq 0$, $l\geq 0$ and $\delta\geq 0$, suppose for $j=1,\ldots,K$,

$$\begin{aligned}
\max_{i=1,2}\left\{\left\|x^{i,j}\right\|_{1-var,[0,1]},\left\|\widetilde{x}^{i,j}\right\|_{1-var,[0,1]}\right\} &\leq Cl^{|\nu_j|},\\
\max\left\{\left\|x^{1,j}-x^{2,j}\right\|_{1-var,[0,1]},\left\|\widetilde{x}^{1,j}-\widetilde{x}^{2,j}\right\|_{1-var,[0,1]}\right\} &\leq \delta Cl^{|\nu_j|}.
\end{aligned}$$

Denote vector fields $V^i:=\left(f^i(\nu_1),\ldots,f^i(\nu_K)\right)$, $i=1,2$. For $y_0^i\in\mathbb{R}^e$, $i=1,2$, denote $y^i:=\pi_{V^i}\left(0,y_0^i;x^i\right)$ and $\widetilde{y}^i:=\pi_{V^i}\left(0,y_0^i;\widetilde{x}^i\right)$, $i=1,2$. Then there exists a constant $M=M(C,\gamma,p,d)>0$ such that, when $\lambda l\leq 1$,

$$\begin{aligned}
& \left|\left(y_{0,1}^1-\widetilde{y}_{0,1}^1\right)-\left(y_{0,1}^2-\widetilde{y}_{0,1}^2\right)\right|\\
\leq & \ M(\lambda l)^\gamma\left(\left|y_0^1-y_0^2\right|+\delta+\lambda^{-1}\left|f^1-f^2\right|_{Lip(\gamma-1)}\right).
\end{aligned}$$

Proof. Without loss of generality, assume $\gamma\in(p,[p]+1]$ and denote $N:=[p]$. The constant M in the following proof may depend on C,γ,p,d and its exact value may change.

First case, assume $\widetilde{x}^1=\widetilde{x}^2=0$ and we want to estimate $\left|y_{0,1}^1-y_{0,1}^2\right|$. We assumed $S_N\left(x^i\right)_{0,1}=S_N\left(\widetilde{x}^i\right)_{0,1}$, $i=1,2$. In this case, $S_N\left(\widetilde{x}^i\right)_{0,1}=1$, $i=1,2$, so for any $k_1,\cdots,k_m\in\mathcal{B}_{\mathcal{L}}^N$, $\|k_1\cdots k_m\|\leq N$, we have

$$\int\cdots\int_{0<u_1<\cdots<u_m<1} dx_{u_1}^{i,k_1}\cdots dx_{u_m}^{i,k_m}=0.$$

By iteratively applying the fundamental theorem of calculus, for $i = 1, 2$,

$$
\begin{aligned}
y^i_{0,1} \quad = \quad & \sum_{\|k_1 \cdots k_m\| = N} \; \underset{0<u_1<\cdots<u_m<1}{\int \cdots \int} \left(F_i^{k_1 \cdots k_m}\left(y^i_{u_1}\right) - F_i^{k_1 \cdots k_m}\left(y^i_0\right)\right) dx^{i,k_1}_{u_1} \cdots dx^{i,k_m}_{u_m} \\
& + \sum_{\substack{\|k_1 \cdots k_m\| > N \\ \|k_2 \cdots k_m\| < N}} \; \underset{0<u_1<\cdots<u_m<1}{\int \cdots \int} F_i^{k_1 \cdots k_m}\left(y^i_{u_1}\right) dx^{i,k_1}_{u_1} \cdots dx^{i,k_m}_{u_m}.
\end{aligned}
$$

For $i = 1, 2$, denote $F_i^N := \left(F_i^w\right)_{w \in \mathcal{W}, \|w\| = N}$ with F_i^w in Notation 3.9 and denote $x^{i,N}_{u,1} := \left(x^{i,w}_{u,1}\right)_{w \in \mathcal{W}, \|w\| = N}$ where

$$
x^{i,k_1 \cdots k_m}_{u,1} := \underset{u<u_1<\cdots<u_m<1}{\int \cdots \int} dx^{i,k_1}_{u_1} \cdots dx^{i,k_m}_{u_m} \text{ for } k_1 \cdots k_m \in \mathcal{W}.
$$

Since $\lambda l \leq 1$, we have $\left|y^i_{0,\cdot}\right|_{\infty,[0,1]} \leq M \lambda l$, $i = 1, 2$. Separate the $Lip\left(\gamma - N + 1\right)$ term $\left(d^{N-1} f^i\right)\left(f^i\right)^{N-1}$ from F_i^N, $i = 1, 2$ (the rest terms are $Lip\left(2\right)$). Based on Lemma 3.10, by adapting the proof of Lemma 10.22 [20] and combining with assumption (iii), we have

$$
\begin{aligned}
& \left| \int_0^1 \left(F_1^N\left(y^1_u\right) - F_1^N\left(y^1_0\right)\right) dx^{1,N}_{u,1} - \int_0^1 \left(F_2^N\left(y^2_u\right) - F_2^N\left(y^2_0\right)\right) dx^{2,N}_{u,1} \right| \\
\leq \; & M\left(\lambda l\right)^N \left|y^1_{0,\cdot} - y^2_{0,\cdot}\right|_{\infty,[0,1]} \\
& + M\left(\lambda l\right)^\gamma \left(\left|y^1_0 - y^2_0\right| + \lambda^{-1}\left|f^1 - f^2\right|_{Lip(\gamma-1)} \right) \\
& + M\delta\left(\lambda l\right)^{N+1}.
\end{aligned} \tag{3.3}
$$

For $j = 1, \ldots, K$,

$$
\left|f^1\left(\nu_j\right) - f^2\left(\nu_j\right)\right|_\infty \leq M \lambda^{|\nu_j| - 1}\left|f^1 - f^2\right|_{Lip(\gamma-1)}.
$$

Since $\lambda l \leq 1$, based on (3.1), we have

$$
\left|y^1_{0,\cdot} - y^2_{0,\cdot}\right|_{\infty,[0,1]} \leq M \lambda l \left(\left|y^1_0 - y^2_0\right| + \delta + \lambda^{-1}\left|f^1 - f^2\right|_{Lip(\gamma-1)} \right).
$$

Putting the estimate into (3.3), we get

$$
\begin{aligned}
& \left| \int_0^1 \left(F_1^N\left(y^1_u\right) - F_1^N\left(y^1_0\right)\right) dx^{1,N}_{u,1} - \int_0^1 \left(F_2^N\left(y^2_u\right) - F_2^N\left(y^2_0\right)\right) dx^{2,N}_{u,1} \right| \\
\leq \; & M\left(\lambda l\right)^\gamma \left(\left|y^1_0 - y^2_0\right| + \delta + \lambda^{-1}\left|f^1 - f^2\right|_{Lip(\gamma-1)} \right).
\end{aligned} \tag{3.4}
$$

On the other hand, for $w \in \mathcal{W}$,

$$\begin{aligned}
&\underset{0<u_1<\cdots<u_m<1}{\int\cdots\int} \left(F_1^w\left(y_{u_1}^1\right) dx_{u,1}^{1,w} - F_2^w\left(y_{u_1}^2\right) dx_{u,1}^{2,w}\right) \\
= &\underset{0<u_1<\cdots<u_m<1}{\int\cdots\int} \left(F_1^w\left(y_{u_1}^1\right) - F_1^w\left(y_{u_1}^2\right)\right) dx_{u,1}^{1,w} \\
&+ \underset{0<u_1<\cdots<u_m<1}{\int\cdots\int} \left(F_1^w - F_2^w\right)\left(y_{u_1}^2\right) dx_{u,1}^{1,w} \\
&+ \underset{0<u_1<\cdots<u_m<1}{\int\cdots\int} F_2^w\left(y_{u_1}^2\right)\left(dx_{u,1}^{1,w} - dx_{u,1}^{2,w}\right).
\end{aligned}$$

Suppose $w = kw_1$, where $k \in \{1, \ldots, K\}$ and $w, w_1 \in \mathcal{W}$, $\|w\| > N$, $\|w_1\| < N$. Then $\max_{i=1,2}\left|F_i^w\right|_{Lip(1)} \leq M\lambda^{\|w\|}$ and $\left|F_1^w - F_2^w\right|_\infty \leq M\lambda^{\|w\|-1}\left|f^1 - f^2\right|_{Lip(\gamma-1)}$. Since $\lambda l \leq 1$, combined with (3.2) and assumption (iii), we have

$$\begin{aligned}
&\left|\underset{0<u_1<\cdots<u_m<1}{\int\cdots\int} \left(F_1^w\left(y_{u_1}^1\right) dx_{u,1}^{1,w} - F_2^w\left(y_{u_1}^2\right) dx_{u,1}^{2,w}\right)\right| \\
\leq\ & M(\lambda l)^{N+1}\left(\left|y_0^1 - y_0^2\right| + \delta + \lambda^{-1}\left|f^1 - f^2\right|_{Lip(\gamma-1)}\right).
\end{aligned} \tag{3.5}$$

Since we assumed that $\lambda l \leq 1$, combine (3.4) with (3.5), we have

$$\left|y_{0,1}^1 - y_{0,1}^2\right| \leq M(\lambda l)^\gamma\left(\left|y_0^1 - y_0^2\right| + \delta + \lambda^{-1}\left|f^1 - f^2\right|_{Lip(\gamma-1)}\right).$$

General case. For $i = 1, 2$, let $z^i := \overleftarrow{\widetilde{x}^i} \sqcup x^i$ be the concatenation of the time reversal of $\widetilde{x}^i$ with x^i. Reparametrize z^i to be from $[0,1]$ to $\mathbb{R}^K$. Based on the assumption (ii) and (iii), $S_{[p]}\left(z^i\right)_{0,1} = 1$, $i = 1, 2$ and $\max_{i=1,2}\left\|z^{i,j}\right\|_{1-var,[0,1]} \leq 2Cl^{|\nu_j|}$, $\left\|z^{1,j} - z^{2,j}\right\|_{1-var,[0,1]} \leq 2\delta Cl^{|\nu_j|}$, $j = 1, \ldots, K$. Since for $i = 1, 2$,

$$y_{0,1}^i - \widetilde{y}_{0,1}^i = \pi_{V^i}\left(0, \pi_{V^i}\left(0, y_0^i; \widetilde{x}^i\right)_1; z^i\right)_{0,1}.$$

Then the desired result follows by applying the first case to z^i, $i = 1, 2$ and combining with (3.2). $\square$

Lemma 3.12. *Fix $\gamma > p \geq 1$.*
(i) Suppose $f^i : \mathbb{R}^e \to L\left(\mathbb{R}^d, \mathbb{R}^e\right)$, $i = 1, 2$ are $Lip(\gamma)$. Denote $\lambda := \max_{i=1,2}\left|f^i\right|_{Lip(\gamma)}$.
(ii) For $i = 1, 2$, suppose $x^i = \left(x^{i,1}, \ldots, x^{i,K}\right) \in C^{1-var}\left([0,1], \mathbb{R}^K\right)$ and there exist constants $C \geq 0, \delta \geq 0$ and $l \geq 0$ such that for $j = 1, \ldots, K$,

$$\begin{aligned}
\max_{i=1,2}\left\|x^{i,j}\right\|_{1-var,[0,1]} &\leq Cl^{|\nu_j|}, \\
\left\|x^{1,j} - x^{2,j}\right\|_{1-var,[0,1]} &\leq \delta Cl^{|\nu_j|}.
\end{aligned}$$

Denote vector fields $V^i := \left(f^i(\nu_1), \ldots, f^i(\nu_K)\right)$, $i = 1, 2$. For $y_0^i, \widetilde{y}_0^i \in \mathbb{R}^e$, $i = 1, 2$, denote $y^i := \pi_{V^i}\left(0, y_0^i; x^i\right)$ and $\widetilde{y}^i := \pi_{V^i}\left(0, \widetilde{y}_0^i; x^i\right)$, $i = 1, 2$. Then there exists a constant

$M = M(C,\gamma,p,d) > 0$ *such that, when* $\lambda l \leq 1$,

$$\begin{aligned}
&\left|\left(y_{0,1}^1 - \widetilde{y}_{0,1}^1\right) - \left(y_{0,1}^2 - \widetilde{y}_{0,1}^2\right)\right| \\
\leq\ & M\lambda l \left|\left(y_0^1 - \widetilde{y}_0^1\right) - \left(y_0^2 - \widetilde{y}_0^2\right)\right| \\
&+ M\lambda l \left(\left|y_0^1 - \widetilde{y}_0^1\right| + \left|y_0^2 - \widetilde{y}_0^2\right|\right)\left(\left|\widetilde{y}_0^1 - \widetilde{y}_0^2\right| + \delta + \lambda^{-1}\left|f^1 - f^2\right|_{Lip(\gamma-1)}\right) \\
&+ M(\lambda l)^{\lfloor\gamma\rfloor}\left(\left|y_0^1 - \widetilde{y}_0^1\right| + \left|y_0^2 - \widetilde{y}_0^2\right|\right)^{\{\gamma\}}\left(\left|\widetilde{y}_0^1 - \widetilde{y}_0^2\right| + \delta + \lambda^{-1}\left|f^1 - f^2\right|_{Lip(\gamma-1)}\right) \\
&+ M\lambda l\delta\left|y_0^2 - \widetilde{y}_0^2\right|.
\end{aligned}$$

Proof. Assume $\gamma \in (p,[p]+1]$ and denote $N := [p] = \lfloor\gamma\rfloor$. The constant M in the following proof may depend on C,γ,p,d and its exact value may change.

Separate the $Lip(\gamma - N + 1)$ term $\left(d^{N-1}f^i\right)\left(f^i\right)^{N-1}$ from $\left\{f^i(\nu_j)\right\}_{j=1}^K$ (if it is one of $f^i(\nu_j)$, $j = 1,\ldots,K$, otherwise do nothing). Since $\lambda l \leq 1$, $\sum_{j=1}^K (\lambda l)^{|\nu_j|} \leq M\lambda l$. The term associated with $\left(d^{N-1}f^i\right)\left(f^i\right)^{N-1}$ contributes a factor that is comparable to $(\lambda l)^N$. Hence, based on Lemma 3.10, by adapting Lemma 10.22 [20],

$$\begin{aligned}
&\left|\left(y_{0,t}^1 - \widetilde{y}_{0,t}^1\right) - \left(y_{0,t}^2 - \widetilde{y}_{0,t}^2\right)\right| \\
\leq\ & \sum_{j=1}^K M\lambda^{|\nu_j|}\int_0^t \left|\left(y_{0,r}^1 - \widetilde{y}_{0,r}^1\right) - \left(y_{0,r}^2 - \widetilde{y}_{0,r}^2\right)\right|\left|dx_r^{1,j}\right| \\
&+ M\lambda l\left|\left(y_0^1 - \widetilde{y}_0^1\right) - \left(y_0^2 - \widetilde{y}_0^2\right)\right| \\
&+ M\lambda l\left(\sum_{i=1,2}\left|y^i - \widetilde{y}^i\right|_{\infty,[0,t]}\right)\left(\left|\widetilde{y}^1 - \widetilde{y}^2\right|_{\infty,[0,t]} + \lambda^{-1}\left|f^1 - f^2\right|_{Lip(\gamma-1)}\right) \\
&+ M(\lambda l)^N\left(\sum_{i=1,2}\left|y^i - \widetilde{y}^i\right|_{\infty,[0,t]}\right)^{\{\gamma\}}\left(\left|\widetilde{y}^1 - \widetilde{y}^2\right|_{\infty,[0,t]} + \lambda^{-1}\left|f^1 - f^2\right|_{Lip(\gamma-1)}\right) \\
&+ M\lambda l\delta\left|y^2 - \widetilde{y}^2\right|_{\infty,[0,t]}.
\end{aligned}$$

Since $\lambda l \leq 1$, based on (3.2), we have

$$\left|y^i - \widetilde{y}^i\right|_{\infty,[0,t]} \leq M\left|y_0^i - \widetilde{y}_0^i\right|,\ i = 1,2$$

and

$$\left|\widetilde{y}^1 - \widetilde{y}^2\right|_{\infty,[0,t]} \leq M\left(\left|\widetilde{y}_0^1 - \widetilde{y}_0^2\right| + \delta + \lambda^{-1}\left|f^1 - f^2\right|_{Lip(\gamma-1)}\right).$$

Based on Gronwall's Lemma and that $\lambda l \leq 1$, the proof is finished. □

Define the symmetry factor $\sigma : \mathcal{F}_{\mathcal{L}} \to \mathbb{N}$ inductively as $\sigma(\bullet_a) := 1$ and

$$\sigma\left(\tau_1^{n_1}\cdots\tau_k^{n_k}\right) = \sigma\left(\left[\tau_1^{n_1}\cdots\tau_k^{n_k}\right]_a\right) := n_1!\cdots n_k!\sigma(\tau_1)^{n_1}\cdots\sigma(\tau_k)^{n_k}$$

where $\tau_i \in \mathcal{T}_{\mathcal{L}}$, $i = 1,\ldots,k$ are different labeled trees (labels counted). Based on Proposition 2.3 [29], for a branched rough path $X \in C^{p-var}\left([0,T], G_{\mathcal{L}}^{[p]}\right)$, if define $\bar{X} : [0,T] \to \left(\mathcal{F}_{\mathcal{L}}^{[p]} \to \mathbb{R}\right)$ as, for $t \in [0,T]$ and $\tau \in \mathcal{F}_{\mathcal{L}}^{[p]}$,

$$\left(\bar{X}_t,\tau\right) := \frac{(X_t,\tau)}{\sigma(\tau)}, \tag{3.6}$$

then $\bar{X}$ takes values in the step-$[p]$ truncated group of group-like elements in Grossman Larson Hopf algebra (the truncated group is denoted as $\mathcal{G}_{\mathcal{L}}^{[p]}$). Moreover, based on Proposition 2.3 [29], for every $0 \leq s \leq t \leq T$ and $\tau \in \mathcal{F}_{\mathcal{L}}^{[p]}$,

$$\left(\bar{X}_{s,t}, \tau\right) = \frac{\left(X_{s,t}, \tau\right)}{\sigma\left(\tau\right)}. \tag{3.7}$$

We equip $a \in \mathcal{G}_{\mathcal{L}}^{[p]}$ with the norm

$$\|a\| := \max_{\tau \in \mathcal{F}_{\mathcal{L}}^{[p]}} \left|(a, \tau)\right|^{\frac{1}{|\tau|}}.$$

Proof of Theorem 2.3. For $i = 1, 2$, replace f^i by $\lambda^{-1} f^i$ and replace $\left(X_t^i, \tau\right)$ by $\lambda^{|\tau|}\left(X_t^i, \tau\right)$, $\tau \in \mathcal{F}_{\mathcal{L}}^{[p]}$. Then the solution to differential equations stays unchanged and $|f^i|_{Lip(\gamma)} \leq 1$, $i = 1, 2$. Suppose $\gamma \in (p, [p] + 1]$. Denote

$$N := [p] \text{ and } \delta := \rho_{p-\omega;[0,T]}\left(X^1, X^2\right).$$

The constant M in the following proof may depend on γ, p, d and its exact value may change.

Firstly suppose $\omega(0, T) \leq 1$. For $0 \leq s \leq t \leq T$, based on (3.7) and that $\sigma(\tau) \geq 1$, we have $\left\|\bar{X}_{s,t}^i\right\| \leq \omega(s,t)^{\frac{1}{p}}$, $i = 1, 2$, and for $\tau \in \mathcal{F}_{\mathcal{L}}^{[p]}$,

$$\left|\left(\bar{X}_{s,t}^1 - \bar{X}_{s,t}^2, \tau\right)\right| \leq \left|\left(X_{s,t}^1 - X_{s,t}^2, \tau\right)\right| \leq \delta\omega(s,t)^{\frac{|\tau|}{p}}.$$

Recall Φ in Notation 3.5 which denotes the isomorphism from a class of Π-rough paths to branched rough paths. Fix $[s, t] \subseteq [0, T]$. For $\tau \in \mathcal{F}_{\mathcal{L}}^{[p]}$, rescale $\left(\bar{X}_{s,t}^i, \tau\right)$ by $\omega(s,t)^{-|\tau|/p}$ and apply Lemma 3.6. Then there exist $x^{i,s,t} = \left(x^{i,s,t,1}, \ldots, x^{i,s,t,K}\right) \in C^{1-var}\left([s,t], \mathbb{R}^K\right)$, $i = 1, 2$ such that $\Phi\left(S_{[p]}\left(x^{i,s,t}\right)_{s,t}\right) = \bar{X}_{s,t}^i$, $i = 1, 2$ and for $j = 1, \ldots, K$,

$$\max_{i=1,2}\left\|x^{i,s,t,j}\right\|_{1-var,[s,t]} \leq M\omega(s,t)^{\frac{|\nu_j|}{p}}, \; i = 1, 2 \tag{3.8}$$

$$\left\|x^{1,s,t,j} - x^{2,s,t,j}\right\|_{1-var,[s,t]} \leq \delta M\omega(s,t)^{\frac{|\nu_j|}{p}}. \tag{3.9}$$

Let $y^{i,s,t} : [s, t] \to \mathbb{R}^e$ denote the unique solution of the ODE

$$dy_r^{i,s,t} = \sum_{j=1}^{K} f^i(\nu_j)\left(y_r^{i,s,t}\right) dx_r^{i,s,t,j}, \; y_s^{i,s,t} = y_s^i.$$

Denote

$$\begin{aligned} \Gamma_{s,t}^i &:= y_{s,t}^i - y_{s,t}^{i,s,t}, \; i = 1, 2 \\ \bar{\Gamma}_{s,t} &:= \Gamma_{s,t}^1 - \Gamma_{s,t}^2. \end{aligned}$$

Since we assumed $\omega(0, T) \leq 1$, by setting $\omega(0, T) = 1$ in Proposition 3.17 in [29], we have

$$\begin{aligned} \left|\Gamma_{s,t}^i\right| &\leq M\omega(s,t)^{\frac{[p]+1}{p}}, \; i = 1, 2 \\ \left|\bar{\Gamma}_{s,t}\right| &\leq M\omega(s,t)^{\frac{[p]+1}{p}}. \end{aligned} \tag{3.10}$$

In fact, based on the construction, $x^{i,s,t} \in C^{1-var}\left([s,t], \mathbb{R}^K\right)$ here may not be a geodesic associated with $\bar{X}_{s,t}^i$ in the sense of Definition 3.2 [29]. The estimate of Proposition 3.17

[29] applies, because $\Phi\left(S_{[p]}\left(x^{i,s,t}\right)_{s,t}\right)=\bar{X}^{i}_{s,t}$ and for $j=1,\ldots,K$, $\|x^{i,s,t,j}\|_{1-var,[s,t]}\leq M\omega\left(s,t\right)^{|\nu_{j}|/p}$ based on Lemma 3.6.

For $i=1,2$ and $0\leq s\leq t\leq u\leq T$, let $x^{i,s,t,u}\in C^{1-var}\left(\left[s,u\right],\mathbb{R}^{K}\right)$ denote the concatenation of $x^{i,s,t}$ with $x^{i,t,u}$. Denote by $y^{i,s,t,u}:[s,u]\rightarrow\mathbb{R}^{e}$ the solution of the ODE

$$dy^{i,s,t,u}_{r}=\sum_{j=1}^{K}f^{i}\left(\nu_{j}\right)\left(y^{i,s,t,u}_{r}\right)dx^{i,s,t,u,j}_{r},\ y^{i,s,t,u}_{s}=y^{i}_{s}.$$

For $i=1,2$, denote

$$A^{i}:=y^{i,s,t,u}_{s,u}-y^{i,s,u}_{s,u},\ B^{i}:=y^{i,s,t}_{t}+y^{i,t,u}_{t,u}-y^{i,s,t,u}_{u}$$

and denote

$$\bar{A}:=A^{1}-A^{2},\ \bar{B}:=B^{1}-B^{2}$$

so that

$$\bar{\Gamma}_{s,u}-\bar{\Gamma}_{s,t}-\bar{\Gamma}_{t,u}=\bar{A}+\bar{B}.$$

Denote

$$\bar{\delta}:=\delta+\lambda^{-1}\left|f^{1}-f^{2}\right|_{Lip(\gamma-1)}.$$

As $S_{[p]}\left(x^{i,s,t,u}\right)_{s,u}=S_{[p]}\left(x^{i,s,u}\right)_{s,u}$, $i=1,2$, based on (3.8) and (3.9), apply Lemma 3.11,

$$\left|\bar{A}\right|\leq M\omega\left(s,u\right)^{\frac{\gamma}{p}}\left(\left|y^{1}_{s}-y^{2}_{s}\right|+\bar{\delta}\right). \tag{3.11}$$

Denote vector fields $V^{i}:=\left(f^{i}\left(\nu_{1}\right),\ldots,f^{i}\left(\nu_{K}\right)\right)$, $i=1,2$. Based on Lemma 3.12,

$$\begin{aligned}
\left|\bar{B}\right| &= \Big|\left(\pi_{V^{1}}\left(t,y^{1}_{t};x^{1,t,u}\right)_{t,u}-\pi_{V^{1}}\left(t,y^{1}_{t}-\Gamma^{1}_{s,t};x^{1,t,u}\right)_{t,u}\right)\\
&\quad-\left(\pi_{V^{2}}\left(t,y^{2}_{t};x^{2,t,u}\right)_{t,u}-\pi_{V^{2}}\left(t,y^{2}_{t}-\Gamma^{2}_{s,t};x^{2,t,u}\right)_{t,u}\right)\Big|\\
&\leq M\omega\left(s,u\right)^{1/p}\left|\bar{\Gamma}_{s,t}\right|\\
&\quad+M\left(\omega\left(s,u\right)^{1/p}\left(\left|\Gamma^{1}_{s,t}\right|+\left|\Gamma^{2}_{s,t}\right|\right)+\omega\left(s,u\right)^{N/p}\left(\left|\Gamma^{1}_{s,t}\right|+\left|\Gamma^{2}_{s,t}\right|\right)^{\{\gamma\}}\right)\\
&\quad\times\left(\left|y^{1}_{t}-y^{2}_{t}\right|+\bar{\delta}\right)\\
&\quad+M\omega\left(s,u\right)^{1/p}\delta\left|\Gamma^{2}_{s,t}\right|.
\end{aligned}$$

Based on (3.10), $\left|\Gamma^{i}_{s,t}\right|\leq M\omega\left(s,t\right)^{\frac{[p]+1}{p}}$, $i=1,2$. Observe that $N+([p]+1)\{\gamma\}\geq\gamma$ and $[p]+2\geq\gamma$, we have

$$\left|\bar{B}\right|\leq M\omega\left(s,u\right)^{1/p}\left|\bar{\Gamma}_{s,t}\right|+M\omega\left(s,u\right)^{\frac{\gamma}{p}}\left(\left|y^{1}_{t}-y^{2}_{t}\right|+\bar{\delta}\right). \tag{3.12}$$

Since $1+M\omega\left(s,u\right)^{1/p}\leq\exp\left(M\omega\left(s,u\right)^{1/p}\right)$, combining (3.11) and (3.12), we obtain that when $\omega\left(0,T\right)\leq1$,

$$\begin{aligned}
\left|\bar{\Gamma}_{s,u}\right| &\leq \left|\bar{A}\right|+\left|\bar{B}\right|+\left|\bar{\Gamma}_{s,t}\right|+\left|\bar{\Gamma}_{t,u}\right|\\
&\leq \exp\left(M\omega\left(s,u\right)^{1/p}\right)\left(\left|\bar{\Gamma}_{s,t}\right|+\left|\bar{\Gamma}_{t,u}\right|\right)\\
&\quad+M\omega\left(s,u\right)^{\gamma/p}\left(\sup_{r\in[s,u]}\left|y^{1}_{r}-y^{2}_{r}\right|+\bar{\delta}\right).
\end{aligned} \tag{3.13}$$

Since $\left|f^1(\nu_j)-f^2(\nu_j)\right|_\infty \le M\lambda^{|\nu_j|-1}\left|f^1-f^2\right|_{Lip(\gamma-1)}$ for $j=1,\dots,K$ and $\omega(0,T)\le 1$, based on (3.1),

$$\left|y^{1,s,t}_{s,t}-y^{2,s,t}_{s,t}\right| \le M\left(\left|y^1_s-y^2_s\right|+\bar\delta\right)\omega(s,t)^{1/p}. \tag{3.14}$$

Combine (3.13), (3.14) and that $\left|\bar\Gamma_{s,t}\right|\le M\omega(s,t)^{\frac{[p]+1}{p}}$, based on Proposition 10.63 [20] (applying to the interval $[s,t]$), we have

$$\left|\bar\Gamma_{s,t}\right| \le M\left(\left|y^1_s-y^2_s\right|+\bar\delta\right)\omega(s,t)^{\gamma/p}\exp\left(M\omega(s,t)\right).$$

Hence, when $\omega(s,t)\le 1$,

$$\begin{aligned}\left|y^1_{s,t}-y^2_{s,t}\right| &\le \left|y^{1,s,t}_{s,t}-y^{2,s,t}_{s,t}\right|+\left|\bar\Gamma_{s,t}\right| \\ &\le M\left(\left|y^1_s-y^2_s\right|+\bar\delta\right)\omega(s,t)^{1/p}\exp\left(M\omega(s,t)\right).\end{aligned} \tag{3.15}$$

Suppose $\omega(0,T)>1$. When $\omega(s,t)\le 1$, the estimates above apply. When $\omega(s,t)>1$, divide $[s,t]=\cup_{i=0}^{n-1}[t_i,t_{i+1}]$ such that $\omega(t_i,t_{i+1})=1$, $i=0,\dots,n-2$ and $\omega(t_{n-1},t_n)\le 1$. By the super-additivity of ω (i.e. $\omega(s,t)+\omega(t,u)\le\omega(s,u)$ for $s\le t\le u$),

$$n=\sum_{i=0}^{n-2}\omega(t_i,t_{i+1})+1\le\omega(s,t)+1\le 2\omega(s,t). \tag{3.16}$$

Since $\omega(t_i,t_{i+1})\le 1$, $i=0,\dots,n-1$, based on (3.15), there exists $M_0>0$ such that

$$\left|y^1_{t_i,t_{i+1}}-y^2_{t_i,t_{i+1}}\right|\le M_0\left(\left|y^1_{t_i}-y^2_{t_i}\right|+\bar\delta\right)$$

and

$$\begin{aligned}\left|y^1_{t_i}-y^2_{t_i}\right| &\le \left|y^1_{t_{i-1}}-y^2_{t_{i-1}}\right|+\left|y^1_{t_{i-1},t_i}-y^2_{t_{i-1},t_i}\right| \\ &\le (1+M_0)\left|y^1_{t_{i-1}}-y^2_{t_{i-1}}\right|+M_0\bar\delta \\ &\le (1+M_0)^i\left|y^1_s-y^2_s\right|+M_0\left(\sum_{j=0}^{i-1}(1+M_0)^j\right)\bar\delta.\end{aligned}$$

Hence

$$\left|y^1_{t_i,t_{i+1}}-y^2_{t_i,t_{i+1}}\right|\le M_0(1+M_0)^i\left(\left|y^1_s-y^2_s\right|+\bar\delta\right)$$

and

$$\begin{aligned}&\left|y^1_{s,t}-y^2_{s,t}\right| \\ \le\ &\sum_{i=0}^{n-1}\left|y^1_{t_i,t_{i+1}}-y^2_{t_i,t_{i+1}}\right| \\ \le\ &\sum_{i=0}^{n-1}M_0(1+M_0)^i\left(\left|y^1_s-y^2_s\right|+\bar\delta\right) \\ \le\ &(1+M_0)^n\left(\left|y^1_s-y^2_s\right|+\bar\delta\right) \\ =\ &\exp\left(n\ln(1+M_0)\right)\left(\left|y^1_s-y^2_s\right|+\bar\delta\right) \\ \le\ &\left(\left|y^1_s-y^2_s\right|+\bar\delta\right)\exp\left(M\omega(s,t)\right)\end{aligned} \tag{3.17}$$

where in the last step we used (3.16). In particular, when $[s,t]=[0,s]$,

$$\begin{aligned}|y_s^1-y_s^2| &\le |y_0^1-y_0^2|+|y_{0,s}^1-y_{0,s}^2| \\ &\le 2\left(|y_0^1-y_0^2|+\bar{\delta}\right)\exp\left(M\omega\left(0,s\right)\right).\end{aligned}\tag{3.18}$$

Combining (3.17), (3.18) and that $\omega(s,t)\ge 1$, we have

$$|y_{s,t}^1-y_{s,t}^2|\le M\left(|y_0^1-y_0^2|+\bar{\delta}\right)\omega\left(s,t\right)^{1/p}\exp\left(M\omega\left(0,t\right)\right).$$

Combining (3.15), (3.18) and the super-additivity of ω, the same result holds when $\omega(s,t)\le 1$. Consequently, the proposed estimate holds as $\bar{\delta}:=\rho_{p-\omega,[0,T]}\left(X^1,X^2\right)+\lambda^{-1}|f^1-f^2|_{Lip(\gamma-1)}$. □

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