Bézout type theorem for higher order objects of groups

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Abstract

We introduce "higher order objects" of groups, which are given by subgroup products and coset products. We provide a geometric framework for describing them, and deduce various combinatorial formulas for finite groups, notably *intersection formula* and *index formula*, analogous to Bézout's theorem in algebraic geometry and Poincaré–Hopf index theorem in differential topology; the classical formula on the orders of double cosets is a special case of our intersection formula. A central role is played by *synergies*, which are a counterpart of structure sheaves over algebraic varieties.

Contents

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1 Introduction

This work has two motivations. The first one arose from our naive observation that a formula in group theory and a formula in algebraic geometry take similar forms after rewriting: they are a formula for a double coset HaK of a finite group G (where $a \in G$, H and K are subgroups of G) and a formula for plane algebraic curves C and D in the complex projective plane $\mathbb{P}^2$ with all intersection points having the same tangency (i.e. the same intersection multiplicities). Explicitly these two formulas are as follows (below, for a finite set S, $|S|$ denotes the number of its elements):

Group theory	Algebraic geometry
$\lvert H\rvert\lvert K\rvert = m\lvert H \cap K^a\rvert$	$\deg C \deg D = n\lvert C \cap D\rvert$
where $m := \lvert HaK\rvert$,	where n is the common intersection multiplicity,
$K^a := aKa^{-1}$	$\deg C$, $\deg D$ are the degrees of C, D

Here:

(i) The formula on the group theory side is a rewriting of the classical formula for the orders of double cosets: $|HaK| = \dfrac{|H||K|}{|H \cap K^a|}$ ([5] p.6).

(ii) The formula on the algebraic geometry side is a special case of Bézout's theorem ([1] p.140) about the intersections of two plane algebraic curves in the complex projective plane $\mathbb{P}^2$.

The study of intersections of two plane algebraic curves is a genesis of intersection theory in algebraic geometry [2]. We wondered that the similarity of the above two formulas suggests a hidden 'geometry' in group theory, together with intersection theory as in algebraic geometry. This is our first motivation.

The second motivation arose from another naive observation — two motivations merge to form a backbone of the present work. Lagrange's theorem tells that for any subgroup H of a finite group G, its order $|H|$ divides $|G|$. As a consequence, if H is proper, then $|H| \leq |G|/2$. This means that proper subgroups of G are quite small in G. A question then occurred to us: "Even if all subgroups are described, can we really say that G is completely understood? — they are merely small parts of G. What is going on the region between $|G|/2$ and $|G|$?" This motivates us to introduce *higher order objects* of G, which are special subsets of G defined as analogs of polynomials. We then wonder whether there is some 'geometry' behind them, just as algebraic geometry lies behind commutative algebras. However, groups in general are noncommutative, and accordingly higher order objects are very complicated and tough to treat. It is thus hard to have a perspective for them. Of course this does not mean that there is no geometry behind them. In fact, in this paper we reveal it. To that end, we introduce *upper structures* on higher order objects, which are considered as a counterpart of structure sheaves in algebraic geometry, and in terms of which we dig out the hidden geometry — for instance, an intersection formula analogous to Bézout's theorem in algebraic geometry is formulated and proved.

In what follows, unless otherwise mentioned, groups are not necessarily finite. The main actors in group theory are group elements $a \in G$, subgroups H, cosets aH, or more generally, double cosets KaH. In our viewpoint, a is a scalar (degree 0) and H is of degree 1, accordingly aH is of degree 1 and KaH is of degree 2. *Group theory, in principle, treats objects of degree less than* 2. This is like a situation in linear algebra, where at most degree 2 objects are treated: linear equations (degree 1) and inner product (degree 2). Beyond linear algebra, algebraic geometry treats polynomials of higher degrees, whose zero sets provide various spaces, bringing rich geometry. In our context, "higher order objects" of a group are *sects* (subgroup products)

$$H_1 H_2 \cdots H_n = \{h_1 h_2 \cdots h_n \,:\, h_i \in H_i \ (i = 1, 2, \ldots, n)\}$$

and *clans* (coset products)

$$a_1 H_1 a_2 H_2 \cdots a_n H_n = \{a_1 h_1 a_2 h_2 \cdots a_n h_n \,:\, h_i \in H_i \ (i = 1, 2, \ldots, n)\},$$

where n is called the *degree* (or *length*). Sects and clans are analogs of polynomials $x_1 x_2 \cdots x_n$ and $c_1 x_1 c_2 x_2 \cdots c_n x_n$ with c_i scalar.

Linear algebra	Classical group theory
scalar (deg 0)	group element (deg 0)
linear equation (deg 1)	subgroup, coset (deg 1)
inner product (deg 2)	double coset (deg 2)
Algebraic geometry	**Higher group theory**
equation of higher degree	sect, clan (deg n)

As the degrees of sects and clans grow, they become extremely complicated. What should be a starting point for their study? What should be studied for them? What should be "good theorems" for them? These are to be contemplated from scratch. To find a right way to proceed, note first that there are two milestones in the road from classical algebraic geometry to modern algebraic geometry (or scheme theory): One is a rigorous formulation of Bézout's theorem concerning intersections of plane algebraic curves. Another milestone (though related to the first one) is the introduction of "upper structures" on algebraic varieties, that is, *sheaves*, via which many invariants of algebraic varieties are defined. In our work, there are corresponding milestones in the road from classical group theory to higher group theory: Bézout type theorem for finite sects and finite clans, and the introduction of "upper structures" on sects and clans. These two milestones are not independent but closely related.

For an algebraic variety X, its upper structure is given by the *structure sheaf* $\mathcal{O}_X \to X$. For a sect $H_1 H_2 \cdots H_n$, its upper structure is a *synergy* defined by

$$\pi : H_1 \times H_2 \times \cdots \times H_n \to H_1 H_2 \cdots H_n, \quad \pi(h_1, h_2, \ldots, h_n) = h_1 h_2 \cdots h_n.$$

Here while $H_1 \times H_2 \times \cdots \times H_n$ is simple, in general the map π is complicated, and accordingly fibers $\pi^{-1}(x)$ ($x \in H_1H_2\cdots H_n$) are complicated. Yet each fiber $\pi^{-1}(x)$ has an algebraic interpretation that it consists of the *expressions* of x: say that $x = h_1h_2\cdots h_n = h_1'h_2'\cdots h_n' = h_1''h_2''\cdots h_n'' = \cdots$ with $h_i, h_i', h_i'' \in H_i$ ($i = 1, 2, \ldots, n$), then

$$\pi^{-1}(x) = \{(h_1, h_2, \ldots, h_n),\ (h_1', h_2', \ldots, h_n'),\ (h_1'', h_2'', \ldots, h_n''),\ \ldots\}.$$

An advantage of synergies is thus that they enable us to turn algebraic problems to geometric ones, and vice versa.

Remark 1.1. Synergies are also defined for clans. Besides, clans have other upper structures called *telergies*. See §2 for details.

To think geometrically, we often regard the synergy π as a geometric object like a fibration in algebraic geometry, by viewing the fibers $\pi^{-1}(x)$ as varying with a parameter $x \in H_1H_2\cdots H_n$. In fact, the following correspondence is our leading principle — our viewpoint to regard synergies as 'fibrations' on sects reflects our former works [8], [9] on complex geometry.

Algebraic geometry	Higher group theory
algebraic variety X	sect $H_1H_2\cdots H_n$
structure sheaf $\mathcal{O}_X \to X$	synergy $H_1 \times H_2 \times \cdots \times H_n \to H_1H_2\cdots H_n$
cohomologies	combinatorial formulas
Euler number $\chi(X)$	order product $\lvert H_1\rvert\lvert H_2\rvert\cdots\lvert H_n\rvert$

Table 1.1: Correspondence between objects

The introduction of synergies is the launch point of higher group theory. In fact, based on them, various combinatorial formulas are formulated and shown. cf. classical algebraic geometry vs. modern algebraic geometry — the former is concerned with algebraic varieties themselves only, whereas the latter is concerned with ringed spaces, i.e. algebraic varieties equipped with structure sheaves [1], and based on sheaves, cohomologies and various invariants of algebraic varieties are introduced.

In addition to sects and clans, we introduce more general objects — *guilds*. A guild is a subset of G of the form

$$HAK = \{hak : h \in H, a \in A, k \in K\}, \tag{1.1}$$

where H and K are subgroups of G, but A is allowed to be an *any* subset of G. Note that sects and clans are special cases of guilds:

(i) For a sect $H_1H_2\cdots H_n$, take $H := H_1$, $A := H_2H_3\cdots H_{n-1}$ and $K := H_n$. Then HAK is the sect.

(ii) For a *monic* clan $H_1a_2H_2a_3H_3\cdots a_nH_n$ (i.e. a_1 is the identity e), take $H := H_1$, $A := a_2H_2a_3H_3\cdots a_{n-1}H_{n-1}a_n$ and $K := H_n$. Then HAK is the monic clan.

(iii) For a general clan $a_1H_1a_2H_2a_3H_3\cdots a_nH_n$, first regard it as a monic clan $H_0a_1H_1a_2H_2\cdots a_nH_n$ with $H_0 := \{e\}$ (the identity subgroup). Then as in (ii), it is regarded as a guild.

From (iii), $\{\text{clans of } G\} \subset \{\text{guilds of } G\}$, so the following holds:

$$\{\text{sects of } G\} \subset \{\text{clans of } G\} \subset \{\text{guilds of } G\}. \tag{1.2}$$

In particular, properties of guilds are possessed by sects and clans.

Remark 1.2. A major advantage to use guilds is the clarification of argument by avoiding complicated notations of sects and clans. This however does *not* mean that sects and clans are unnecessary. In fact some properties and formulas of sects and clans do *not* hold for general guilds, and to deduce them, direct investigation of sects and clans is inevitable.

For guilds, we deduce an *intersection formula* (analogous to Bézout's theorem in algebraic geometry). To that end, we first show that any guild HAK admits a double coset decomposition (Lemma 4.8), say $HAK = \coprod_{i\in I} Ha^{(i)}K$, where $a^{(i)} \in A$ $(i \in I)$ are representatives of (H,K)-double cosets in HAK. Next we define a *guild synergy* $\psi : H\times A\times K \to HAK$ by $(h,a,k) \mapsto hak$, and for each double coset $Ha^{(i)}K\,(\subset HAK)$, we define its *multiplicity* $m^{(i)}$ with respect to ψ (Definition 6.3). Then the following holds:

Intersection formula (Theorem 6.5) *For a finite guild HAK,*

$$|H||A||K| = \sum_{i\in I} m^{(i)}|H\cap K^{a^{(i)}}|, \tag{1.3}$$

where $K^{a^{(i)}} := a^{(i)}K{a^{(i)}}^{-1}$ (the conjugation by $a^{(i)}$).

Our intersection formula has an analog in algebraic geometry. For plane curves C and D in the projective plane $\mathbb{P}^2$, say the zero sets of polynomials f and g respectively, we define $\deg C$ and $\deg D$ to be the degrees of f and g. Then Bézout's theorem [1] p.140 states that $\deg C \deg D$ *is equal to the sum of intersection multiplicities at the points of $C\cap D$.* We shall restate this. Say that the intersection multiplicities appearing herein are m_i $(i = 1,2,\dots,l)$. For each m_i, denote by $(C\cap D)_i$ the set of intersection points with intersection multiplicity m_i, and by $|(C\cap D)_i|$ the number of its points. Then Bézout's theorem is restated as

$$\deg C \deg D = \sum_{i=1}^{l} m_i|(C\cap D)_i|, \tag{1.4}$$

which is analogous to (1.3); note that in the particular case that HAK is a double coset HaK, the classical formula $|H||K| = m|H\cap K^a|$ with $m := |HaK|$ is an analog of the special case of Bézout's theorem such that all multiplicities m_i are the same; write it as n, then $\deg C \deg D = n|C\cap D|$.

Algebraic geometry	Higher group theory
$\mathbb{P}^2$: complex projective plane	HAK: guild of a group G
$C, D \subset \mathbb{P}^2$: plane curves	$H, K \subset HAK$: subgroups of G
$(C \cap D)_i$: intersection with intersection multiplicity m_i	$H \cap K^{a^{(i)}}$: intersection (twisted by $a^{(i)}$) with multiplicity $m^{(i)}$ (of $Ha^{(i)}K$)
$\deg C$, $\deg D$	$\lvert H\rvert, \lvert K\rvert$
Bézout's theorem $\deg C \deg D = \sum_{i=1}^{l} m_i \lvert(C \cap D)_i\rvert$ Special case: If all multiplicities are n, then $\deg C \deg D = n\lvert C \cap D\rvert$	Intersection formula $\lvert H\rvert\lvert A\rvert\lvert K\rvert = \sum_{i \in I} m^{(i)} \lvert H \cap K^{a^{(i)}}\rvert$ Special case: If HAK is a double coset HaK, then $\lvert H\rvert\lvert K\rvert = m\lvert H \cap K^a\rvert$ with $m := \lvert HaK\rvert$

Note: While the ambient $\mathbb{P}^2$ of C and D is independent of C and D, the ambient HAK of H and K depends on A, and accordingly $|A|$ and $a^{(i)} \in A$ emerge in our intersection formula.

Remark 1.3. In spite of the correspondence in the above table, algebraic geometry and higher group theory have an essential difference. In algebraic geometry, there exists a duality between algebra (commutative rings) and geometry (zero sets). In contrast, in higher group theory, algebra and geometry are *not* even separated — a sect $H_1 H_2 \cdots H_n$ of a group G is an algebraic object as an analog of a polynomial, but at the same time it is a geometric object as a subset of G.

Specializations of intersection formula In the case that the middle part A of a guild HAK is a singleton, say $A = \{a\}$ for some $a \in G$, the guild is a double coset HaK. Its double coset decomposition is trivial (i.e. itself) and its multiplicity is just $|HaK|$ (see Example 6.9). Noting that $|A| = |\{a\}| = 1$, the intersection formula (1.3) reads as $|H||K| = |HaK||H \cap K^a|$, that is,

$$|HaK| = \frac{|H||K|}{|H \cap K^a|} \quad \text{(in particular } |HK| = \frac{|H||K|}{|H \cap K|} \text{ if } a \text{ is the identity).} \tag{1.5}$$

This is nothing but the classical order formula of the double coset HaK.

The intersection formula (1.3) is further specialized to sects and clans of arbitrary lengths ≥ 2 (below, I, $m^{(i)}$ and $a^{(i)}$ depend on sects and clans):

(i) For $H := H_1$, $A := H_2 H_3 \cdots H_{n-1}$ and $K := H_n$ (where $A = \{e\}$ if $n = 2$), the intersection formula (1.3) reads as

$$|H_1||H_2 H_3 \cdots H_{n-1}||H_n| = \sum_{i \in I} m^{(i)} |H_1 \cap H_n^{a^{(i)}}|. \tag{1.6}$$

(ii) For $H := H_1$, $A := a_2 H_2 a_3 H_3 \cdots a_{n-1} H_{n-1} a_n$ and $K := H_n$ (where $A = \{a_2\}$ if $n = 2$), the intersection formula (1.3) reads as

$$|H_1||a_2 H_2 a_3 H_3 \cdots a_{n-1} H_{n-1} a_n||H_n| = \sum_{i \in I} m^{(i)} |H_1 \cap H_n^{a^{(i)}}|. \tag{1.7}$$

There is a finer version of formula (1.6) for a *permutative* sect $H_1H_2\cdots H_n$, i.e. $H_iH_j = H_jH_i$ for any $i,j\in\{1,2,\ldots,n\}$ (e.g. any sect in an abelian group):

"Partitioned" intersection formula (Theorem 7.3) *If $H_1H_2\cdots H_n$ $(n\geq 3)$ is a finite permutative sect, then for any partition of it into three parts*

$$\underbrace{H_1H_2\cdots H_p}\,\underbrace{H_{p+1}H_{p+2}\cdots H_q}\,\underbrace{H_{q+1}H_{q+2}\cdots H_n}, \tag{1.8}$$

the following holds:

$$\begin{aligned}&|H_1H_2\cdots H_p||H_{p+1}H_{p+2}\cdots H_q||H_{q+1}H_{q+2}\cdots H_n|\\ &\quad=\sum_{i\in I} m^{(i)}|(H_1H_2\cdots H_p)\cap(H_{q+1}H_{q+2}\cdots H_n)^{a^{(i)}}|,\end{aligned} \tag{1.9}$$

where I, $m^{(i)}$ and $a^{(i)}$ depend on the partition.

The notion of permutativity of a clan is also defined (Definition 7.11), and for permutative clans, a partitioned intersection formula also holds (Theorem 7.15).

Other formulas

Besides the above formulas, sects and clans also have their *own* formulas, which do not arise as specializations of the intersection formula (1.3). For instance, the following holds:

"Factorized" intersection formula (Theorem 9.12):
For any finite sect $H_1H_2\cdots H_n$ $(n\geq 2)$, the following holds:

$$|H_1||H_2|\cdots|H_n| = \sum_{i\in I}\mu^{(i)}|H_1\cap H_n^{a^{(i)}}|, \tag{1.10}$$

where $\mu^{(i)}$ is the absolute multiplicity *of $H_1a^{(i)}H_n$* (Definition 9.10).

This formula is generalized to clans (Theorem 11.12). Moreover there are many other formulas. Distinguished ones are "Index theorems for clans", analogous to Poincaré–Hopf index theorem [3] p.134 in differential topology that states: *For a vector field V on an oriented closed manifold M with isolated zeros, say $p_j\in M$ $(j\in J)$, we denote by $\mathrm{Ind}_{p_j}(V)$ their indices and by $\chi(M)$ the Euler number of M. Then*

$$\chi(M)=\sum_{j\in J}\mathrm{Ind}_{p_j}(V). \tag{1.11}$$

This kind of theorems hold for finite clans. In fact we show the following:

Index theorem I (Theorem 12.4) *For a finite clan $a_1H_1a_2H_2\cdots a_nH_n$ $(n\geq 2)$, let $a_1H_1a_2H_2\cdots a_nH_n=\coprod_{j\in J}b^{(j)}H_n$ be its coset decomposition* (Definition 12.1)*, and for each $b^{(j)}H_n$ let $\nu^{(j)}$ be its absolute multiplicity* (Definition 12.3)*. Then*

$$|H_1||H_2|\cdots|H_n|=\sum_{j\in J}\nu^{(j)}. \tag{1.12}$$

In terms of the analogy of this with (1.11), $|H_1||H_2|\cdots|H_n|$ is considered as the 'Euler number' of the clan $a_1H_1a_2H_2\cdots a_nH_n$. This is an evidence to support our leading principle (see Table 1.1).

Besides Index theorem I, we also show the following:

Index theorem II (Theorem 13.5) *Under the assumption in Index theorem I, for each $b^{(j)}H_n$ let $r^{(j)}$ be its planet index* (Definition 12.3). *Then*

$$|H_1||H_2|\cdots|H_{n-1}| = \sum_{j\in J} r^{(j)}. \tag{1.13}$$

At first it seemed that higher order objects (sects and clans) are anarchy, but actually *not* — as this paper shows, they are under control in terms of their upper structures (synergies). *The advantage of higher group theory is, like scheme theory in algebraic geometry, to enable one to* think geometrically*, and it provides geometric perspective for groups — just as scheme theory provides geometric perspective for commutative rings.*

In addition to the results obtained in this paper, there are many other aspects of higher group theory, which will be described in our subsequent papers. Besides, motivated by our former works on subgroup posets [10], [11] and [4], we are also concerned with "the sect poset of a group" (i.e. the set of sects of a group with partial order given by inclusion relation). The study of sect posets is another research direction in higher group theory, which will be discussed in our later paper.

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2 Preparation

"Higher order objects" of a group G are *sects* (products of finite numbers of subgroups) and *clans* (products of finite numbers of cosets). Explicitly for subgroups H_i $(i = 1, 2, \ldots, n)$ of G, the following subset of G is a sect of *degree* (or *length*) n:

$$H_1H_2\cdots H_n = \{h_1h_2\cdots h_n \,:\, h_i \in H_i \ (i = 1, 2, \ldots, n)\}. \tag{2.1}$$

Similarly for cosets a_iH_i of G ($a_i \in G$, $i = 1, 2, \ldots, n$), the following subset of G is a clan of *degree* (or *length*) n:

$$a_1H_1a_2H_2\cdots a_nH_n = \{a_1h_1a_2h_2\cdots a_nh_n \,:\, h_i \in H_i \ (i = 1, 2, \ldots, n)\}. \tag{2.2}$$

Precisely speaking this is a *left* clan. A *right* clan is a product of right cosets, that is, of the form

$$H_1a_1H_2a_2\cdots H_na_n = \{h_1a_1h_2a_2\cdots h_na_n \,:\, h_i \in H_i \ (i = 1, 2, \ldots, n)\}. \tag{2.3}$$

Besides, a *biclan* is defined as a product of double cosets, that is, of the form

$$(H_1a_1K_1)(H_2a_2K_2)\cdots(H_na_nK_n). \tag{2.4}$$

The simplest sect, clan, and biclan are respectively a subgroup H_1, a coset a_1H_1, and a double coset $H_1a_1K_1$. Thus sects, clans, and biclans are generalizations of fundamental objects in classical group theory.

Remark 2.1. Sects also appear in our other works [12, 13, 14], where based on sects, we developed a theory of prime factorizations of groups.

For later use, we describe fundamental properties of sects.

Lemma 2.2. *For a sect $H_1H_2\cdots H_n$ of a group G, the following holds:*

$$H_1H_2\cdots H_n \supset H_i \quad (i = 1, 2, \ldots, n).$$

Proof. Let e be the identity of G. Then $H_1H_2\cdots H_n \supset ee\cdots eH_ie\cdots e = H_i$.□

Corollary 2.3. *A sect $H_1H_2\cdots H_n$ is finite if and only if H_i $(i = 1, 2, \ldots, n)$ are finite; in this case $|H_1H_2\cdots H_n| \leq |H_1||H_2|\cdots|H_n|$.*

Proof. $\Longrightarrow$: This is immediate as H_i $(i = 1, 2, \ldots, n)$ are subsets of $H_1H_2\cdots H_n$ (Lemma 2.2).

$\Longleftarrow$: The sect $H_1H_2\cdots H_n$ consists of $h_1h_2\cdots h_n$ $(h_i \in H_i)$. Thus if H_i $(i = 1, 2, \ldots, n)$ are finite, then $|H_1H_2\cdots H_n| \leq |H_1||H_2|\cdots|H_n|$, so $H_1H_2\cdots H_n$ is finite. □

It may happen that a sect is a subgroup. We give a criterion.

Lemma 2.4. *Let G be a group. Then the following hold:*

(1) *For subgroups of H and K of G, if $HK = KH$, then HK is a subgroup of G.*

(2) *For subgroups $H_1, H_2, \ldots, H_n$ $(n \geq 2)$ of G, if $H_iH_j = H_jH_i$ for any $i, j \in \{1, 2, \ldots, n\}$, then $H_1H_2\cdots H_n$ is a subgroup of G. Moreover for any subset $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$, $H_{i_1}H_{i_2}\cdots H_{i_k}$ is a subgroup of G.*

Proof. (1): It suffices to show (i) $e \in HK$, (ii) $hk \in HK \Longrightarrow (hk)^{-1} \in HK$, and (iii) $h_1k_1, h_2k_2 \in HK \Longrightarrow h_1k_1h_2k_2 \in HK$.

(i): From $e \in H, K$, we have $ee \in HK$, that is, $e \in HK$.

(ii): Note that $(hk)^{-1} = k^{-1}h^{-1} \in KH$. Here $KH = HK$ by assumption, so $(hk)^{-1} \in HK$.

(iii): Note that $h_1k_1h_2k_2 \in HKHK$. It thus suffices to show $HKHK = HK$. First from $HK = KH$, we have $HKHK = HHKK$. Here $HH = H$ and $KK = K$ (closedness under multiplication of subgroups). Hence $HKHK = HK$.

(2): We show that $H_1H_2 \cdots H_n$ is a subgroup of G by induction on n. If $n = 2$, the assertion holds by (1). If the assertion is valid for $n-1$, then $H_1H_2 \cdots H_{n-1}$ is a subgroup of G. Setting $H := H_1H_2 \cdots H_{n-1}$ and $K := H_n$, we write $H_1H_2 \cdots H_n$ as a product HK of two subgroups of G. Here $HK = KH$ by repeated application of the assumption (e.g. if $n = 4$, then $HK = H_1H_2H_3H_4 = H_1H_2H_4H_3 = H_1H_4H_2H_3 = H_4H_1H_2H_3 = KH$). Hence HK (i.e. $H_1H_2 \cdots H_n$) is a subgroup of G by (1). This completes the induction step. By the same argument, any $H_{i_1}H_{i_2} \cdots H_{i_k}$ is a subgroup of G. □

Remark 2.5. The subgroup $H_1H_2 \cdots H_n$ in Lemma 2.4 (2) is called a *factorized group.*

We next introduce "upper structures" on sects:

Definition 2.6. For a sect $H_1H_2 \cdots H_n$, its *synergy* is defined by

$$\begin{gathered} \pi : H_1 \times H_2 \times \cdots \times H_n \longrightarrow H_1H_2 \cdots H_n, \\ \pi(h_1, h_2, \ldots, h_n) := h_1h_2 \cdots h_n. \end{gathered} \tag{2.5}$$

Observation 2.7. Suppose that a sect $H_1H_2 \cdots H_n$ $(n \geq 2)$ of a group G is equal to a subgroup K of G:

$$H_1H_2 \cdots H_n = K \quad \text{(as subsets of } G\text{)}. \tag{2.6}$$

Even so, their synergies are distinct: $H_1 \times H_2 \times \cdots \times H_n \to H_1H_2 \cdots H_n$ and $K \to K$ (the identity map). This is analogous to a situation in scheme theory: the zero sets $x = 0$ and $x^n = 0$ $(n \geq 2)$ in $\mathbb{C}$ are both the origin — *the underlying spaces are the same, but their upper structures* (*structure sheaves*) *are different*, as the rings $\mathbb{C}[x]/(x)$ and $\mathbb{C}[x]/(x^n)$ are different. *In our context, $H_1H_2 \cdots H_n = K$ means that 'underlying spaces' are the same, but their upper structures* (*synergies*) *are different.* Synergies thus give finer informations than sects themselves.

We shall describe synergies for the case $n = 2$:

Lemma 2.8. *Let $\pi : H_1 \times H_2 \to H_1H_2$ be a synergy. Then for $h_1h_2 \in H_1H_2$, the following hold:*

(1) $\pi^{-1}(h_1h_2) = \{(h_1\alpha, \alpha^{-1}h_2) : \alpha \in H_1 \cap H_2\}$.

(2) *For distinct $\alpha, \beta \in H_1 \cap H_2$, $(h_1\alpha, \alpha^{-1}h_2) \neq (h_1\beta, \beta^{-1}h_2)$.*

(3) *The elements of $\pi^{-1}(h_1h_2)$ are in one-to-one correspondence with those of $H_1 \cap H_2$ via $\alpha \in H_1 \cap H_2 \longleftrightarrow (h_1\alpha, \alpha^{-1}h_2) \in \pi^{-1}(h_1h_2)$.*

Proof. (1): Let $(h_1', h_2') \in H_1 \times H_2$. Then $(h_1', h_2') \in \pi^{-1}(h_1h_2)$ precisely when $h_1'h_2' = h_1h_2$, that is, $h_1^{-1}h_1' = h_2{h_2'}^{-1}$. Here $h_1^{-1}h_1' \in H_1$ and $h_2{h_2'}^{-1} \in H_2$. Thus $h_1^{-1}h_1' = h_2{h_2'}^{-1} \in H_1 \cap H_2$. We write this element of $H_1 \cap H_2$ as α. Then $h_1' = h_1\alpha$ and $h_2' = \alpha^{-1}h_2$, that is, $(h_1', h_2') = (h_1\alpha, \alpha^{-1}h_2)$. Conversely for any $(h_1\alpha, \alpha^{-1}h_2) \in H_1 \times H_2$ where $\alpha \in H_1 \cap H_2$, we have $(h_1\alpha, \alpha^{-1}h_2) \in \pi^{-1}(h_1h_2)$, as $h_1\alpha \cdot \alpha^{-1}h_2 = h_1h_2$.

(2): Otherwise $(h_1\alpha, \alpha^{-1}h_2) = (h_1\beta, \beta^{-1}h_2)$, but then $h_1\alpha = h_1\beta$, so $\alpha = \beta$, which contradicts the assumption.

(3): This follows from (1) and (2). □

Remark 2.9. In the case $n \geq 3$, for a synergy $\pi : H_1 \times H_2 \times \cdots \times H_n \to H_1H_2 \cdots H_n$, its fibers $\pi^{-1}(x)$ $(x \in H_1H_2 \cdots H_n)$ are much more complicated than those in the case $n = 2$. They will be described in our subsequent paper.

We restate Lemma 2.8 (3) as follows:

Corollary 2.10. *For a synergy $\pi : H_1 \times H_2 \to H_1H_2$, any fiber $\pi^{-1}(x)$ $(x \in H_1H_2)$ is, as a set, bijective to a subgroup $H_1 \cap H_2$.* (Note: In general "bijective" is not "isomorphic". In fact a fiber $\pi^{-1}(x)$ is generally not a group.)

We consider the case that H_1H_2 is finite (or equivalently H_1 and H_2 are finite by Corollary 2.3). Then the orders $|H_1H_2|$, $|H_1 \times H_2|$ $(= |H_1||H_2|)$ and $|H_1 \cap H_2|$ are finite. Moreover they are related — in fact the following classical formula holds, which we here prove *in terms of a synergy.*

Formula 2.11. *For a finite sect H_1H_2, we have* $|H_1H_2| = \dfrac{|H_1||H_2|}{|H_1 \cap H_2|}$.

Proof. Let $\pi : H_1 \times H_2 \to H_1H_2$ be the synergy of H_1H_2, and decompose $H_1 \times H_2$ into the fibers of π:

$$H_1 \times H_2 = \coprod_{x \in H_1H_2} \pi^{-1}(x). \tag{2.7}$$

Here all fibers $\pi^{-1}(x)$ $(x \in H_1H_2)$ are bijective to $H_1 \cap H_2$ (Corollary 2.10). Thus $|H_1 \times H_2| = |H_1 \cap H_2||H_1H_2|$, that is, $|H_1||H_2| = |H_1 \cap H_2||H_1H_2|$. Hence $|H_1H_2| = |H_1||H_2| / |H_1 \cap H_2|$. □

We turn to a clan $a_1H_1a_2H_2 \cdots a_nH_n$. As for sects, its *synergy* (more precisely *clan synergy*) is defined by

$$\begin{gathered} \pi : a_1H_1 \times a_2H_2 \times \cdots \times a_nH_n \longrightarrow a_1H_1a_2H_2 \cdots a_nH_n, \\ \pi(a_1h_1, a_2h_2, \ldots, a_nh_n) := a_1h_1a_2h_2 \cdots a_nh_n. \end{gathered} \tag{2.8}$$

Convention 2.12. We later also use the following modified version of clan synergy (Definition 11.1 (c)) replacing $a_{n-1}H_{n-1}$ with $a_{n-1}H_{n-1}a_n$, and a_nH_n with H_n:

$$\begin{gathered} \pi' : a_1H_1 \times a_2H_2 \times \cdots \times a_{n-1}H_{n-1}a_n \times H_n \longrightarrow a_1H_1a_2H_2 \cdots a_nH_n, \\ \pi'(a_1h_1, a_2h_2, \ldots, a_{n-1}h_{n-1}a_n, h_n) := a_1h_1a_2h_2 \cdots a_{n-1}h_{n-1}a_nh_n. \end{gathered}$$

There is still another "upper structure" on the clan. To clarify the subsequent construction, we adopt *representative-free expressions*: a_iH_i is denoted by C_i, and $a_1H_1a_2H_2\cdots a_nH_n$ is denoted by $C_1C_2\cdots C_n$. We then choose representatives of the cosets $C_1, C_2, \ldots, C_n$, say $r_i \in C_i$ $(i = 1, 2, \ldots, n)$. The 'plasma state' of the clan $C_1C_2\cdots C_n$ is then given by

$$\{r_1\} \times H_1 \times \{r_2\} \times H_2 \times \cdots \times \{r_n\} \times H_n. \tag{2.9}$$

We define another "upper structure" on the clan — a *telergy* — by

$$\begin{aligned} \eta :\ \{r_1\} \times H_1 \times \{r_2\} \times H_2 \times \cdots \times \{r_n\} \times H_n &\longrightarrow r_1H_1r_2H_2\cdots r_nH_n, \\ \eta(r_1, h_1, r_2, h_2, \ldots, r_n, h_n) &:= r_1h_1r_2h_2\cdots r_nh_n. \end{aligned} \tag{2.10}$$

Here noting that $r_iH_i = C_i$ $(i = 1, 2, \ldots, n)$, we have

$$r_1H_1r_2H_2\cdots r_nH_n = C_1C_2\cdots C_n, \tag{2.11}$$

so the telergy is

$$\eta :\ \{r_1\} \times H_1 \times \{r_2\} \times H_2 \times \cdots \times \{r_n\} \times H_n \longrightarrow C_1C_2\cdots C_n. \tag{2.12}$$

Remark 2.13. While the synergy of the clan $C_1C_2\cdots C_n$ is unique, there are many telergies of the clan, depending on the choices of representatives $r_1, r_2, \ldots, r_n$ of the cosets $C_1, C_2, \ldots, C_n$.

We next consider a biclan $(H_1a_1K_1)(H_2a_2K_2)\cdots(H_na_nK_n)$ of a group G, which is a product of double cosets: $a_i \in G$, and H_i and K_i $(i = 1, 2, \ldots, n)$ are subgroups of G. We shall define its synergy and telergy. To that end, we use representative-free expressions: each double coset $H_ia_iK_i$ is denoted by D_i and the biclan is denoted by $D_1D_2\cdots D_n$. Then the *biclan synergy* is defined by

$$\begin{gathered} \pi :\ D_1 \times D_2 \times \cdots \times D_n \longrightarrow D_1D_2\cdots D_n, \\ \pi(x_1, x_2, \ldots, x_n) := x_1x_2\cdots x_n. \end{gathered} \tag{2.13}$$

Next to define the biclan telergy, for each D_i we take a representative $q_i \in D_i$; so $D_i = H_iq_iK_i$. The 'plasma state' of the biclan $D_1D_2\cdots D_n$ is then given by

$$(H_1 \times \{q_1\} \times K_1) \times (H_2 \times \{q_2\} \times K_2) \times \cdots \times (H_n \times \{q_n\} \times K_n). \tag{2.14}$$

We define the *biclan telergy* of $D_1D_2\cdots D_n$ by

$$\begin{aligned} &\eta :\ (H_1 \times \{q_1\} \times K_1) \times (H_2 \times \{q_2\} \times K_2) \times \cdots \times (H_n \times \{q_n\} \times K_n) \\ &\qquad\qquad \longrightarrow (H_1q_1K_1)(H_2q_2K_2)\cdots(H_nq_nK_n), \\ &\quad \eta\big((h_1, q_1, k_1), (h_2, q_2, k_2), \ldots, (h_n, q_n, k_n)\big) \\ &\qquad\qquad := (h_1q_1k_1)(h_2q_2k_2)\cdots(h_nq_nk_n). \end{aligned} \tag{2.15}$$

Note that since $H_i q_i K_i = D_i$ $(i = 1, 2, \ldots, n)$, we have

$$(H_1 q_1 K_1)(H_2 q_2 K_2) \cdots (H_n q_n K_n) = D_1 D_2 \cdots D_n, \tag{2.16}$$

so the biclan telergy is

$$\begin{aligned} \eta : \ &(H_1 \times \{q_1\} \times K_1) \times (H_2 \times \{q_2\} \times K_2) \times \cdots \times (H_n \times \{q_n\} \times K_n) \\ &\longrightarrow D_1 D_2 \cdots D_n. \end{aligned} \tag{2.17}$$

Remark 2.14. While the synergy of the biclan $D_1 D_2 \cdots D_n$ is unique, there are many telergies of the biclan, depending on the choices of representatives $q_1, q_2, \ldots, q_n$ of the double cosets $D_1, D_2, \ldots, D_n$.

We give an example of a telergy:

Example 2.15. Let HaK be a double coset of a group G, where $a \in G$ and H and K are subgroups of G. We regard HaK as a biclan (of length 1). Its telergy

$$\eta : H \times \{a\} \times K \longrightarrow HaK, \quad (h, a, k) \longmapsto hak \tag{2.18}$$

is specifically called a *double coset telergy.* We point out that even for the description of a synergy, a double coset telergy emerges (Remark 5.6).

Lemma 2.16. *Let $\eta : H \times \{a\} \times K \to HaK$ be a double coset telergy. Then for $hak \in HaK$ $(h \in H, k \in K)$, the following hold:*

(1) $\eta^{-1}(hak) = \bigl\{(h\beta, a, a^{-1}\beta^{-1}ak) \,:\, \beta \in H \cap K^a\bigr\}$*, where* $K^a := aKa^{-1}$.

(2) *For distinct* $\beta, \gamma \in H \cap K^a$, $(h\beta, a, a^{-1}\beta^{-1}ak) \neq (h\gamma, a, a^{-1}\gamma^{-1}ak)$.

(3) *The elements of $\eta^{-1}(hak)$ are in one-to-one correspondence with those of $H \cap K^a$ via $\beta \in H \cap K^a \longleftrightarrow (h\beta, a, a^{-1}\beta^{-1}ak) \in \eta^{-1}(hak)$.*

Proof. (1): Let $(h', a, k') \in H \times \{a\} \times K$. Then $(h', a, k') \in \eta^{-1}(hak)$ precisely when $h'ak' = hak$. This is rewritten as $h^{-1}h' = a(kk'^{-1})a^{-1}$. Here

$$\begin{cases} h^{-1}h' \in H, \\ kk'^{-1} \in K, \text{ so } a(kk'^{-1})a^{-1} \in aKa^{-1} \, (= K^a). \end{cases}$$

Hence $h^{-1}h' = a(kk'^{-1})a^{-1} \in H \cap K^a$. Denote this element of $H \cap K^a$ by β. Then (i) $h^{-1}h' = \beta$ and (ii) $a(kk'^{-1})a^{-1} = \beta$. From (i), $h' = h\beta$ and from (ii), $kk'^{-1} = a^{-1}\beta a$, i.e. $k' = a^{-1}\beta^{-1}ak$. Thus $(h', a, k') = (h\beta, a, a^{-1}\beta^{-1}ak)$. Namely any element of $\eta^{-1}(hak)$ is of the form $(h\beta, a, a^{-1}\beta^{-1}ak)$ for some $\beta \in H \cap K^a$. Hence $\eta^{-1}(hak) \subset \bigl\{(h\beta, a, a^{-1}\beta^{-1}ak) \,:\, \beta \in H \cap K^a\bigr\}$. It remains to show

$$\eta^{-1}(hak) \supset \bigl\{(h\beta, a, a^{-1}\beta^{-1}ak) \,:\, \beta \in H \cap K^a\bigr\}. \tag{2.19}$$

First for *any* $\beta \in H \cap K^a$, note that $(h\beta, a, a^{-1}\beta^{-1}ak) \in H \times \{a\} \times K$. Indeed $h\beta \in H$ is obvious and $a^{-1}\beta^{-1}ak \in K$ is seen as follows: From $\beta \in K^a$, we have $\beta^{-1} \in K^a = aKa^{-1}$, so $a^{-1}\beta^{-1}a \in K$, thus $a^{-1}\beta^{-1}ak \in K$. Next $(h\beta, a, a^{-1}\beta^{-1}ak) \in \eta^{-1}(hak)$, as $\eta(h\beta, a, a^{-1}\beta^{-1}ak) = h\beta \cdot a \cdot a^{-1}\beta^{-1}ak = hak$. This implies (2.19).

(2): Otherwise $(h\beta, a, a^{-1}\beta^{-1}ak) = (h\gamma, a, a^{-1}\gamma^{-1}ak)$, but then $h\beta = h\gamma$, so $\beta = \gamma$, which contradicts the assumption. (3) follows from (1) and (2). □

We restate Lemma 2.16 (3) as follows:

Corollary 2.17. *For a double coset telergy $\eta : H \times \{a\} \times K \to HaK$, any fiber $\eta^{-1}(x)$ $(x \in HaK)$ is, as a set, bijective to a subgroup $H \cap K^a$, where $K^a := aKa^{-1}$.* (Note: In general "bijective" is not "isomorphic". In fact a fiber $\eta^{-1}(x)$ is generally not a group.)

3 Technical results

We show some technical results used later.

Lemma 3.1. *Let HaK be a double coset of a group G, where $a \in G$, and H and K are subgroups of G. Then the following hold:*

(A.1) $HaK \cong HK^a$ (*bijective*).

(A.2) $H \cap K^a$ *and* $H^{a^{-1}} \cap K$ *are conjugate* (*so bijective*).

Moreover if HaK is finite, then the following hold:

(B.1) H, K, $H^{a^{-1}}$ *and* K^a *are finite.*

(B.2) $|H \cap K^a| = |H^{a^{-1}} \cap K|$.

(B.3) $|HaK| = |HK^a| = \dfrac{|H||K|}{|H \cap K^a|} = \dfrac{|H||K|}{|H^{a^{-1}} \cap K|}$.

Proof. (A.1): First $HaK = HaKa^{-1}a = HK^a a$. Here $HK^a a \cong HK^a$ (indeed a translation $x \in HK^a \mapsto xa \in HK^a a$ gives a bijection). Consequently $HaK \cong HK^a$.

(A.2): $a^{-1}(H \cap K^a)a = H^{a^{-1}} \cap K$.

(B.1): If HaK is finite, then HK^a is finite (as $HaK \cong HK^a$ by (A.1)). Since $H, K^a \subset HK^a$, the finiteness of HK^a implies that H and K^a are finite. Consequently their conjugates $H^{a^{-1}}$ and K are finite.

(B.2): This follows from (A.2).

(B.3): First by (A.1), $|HaK| = |HK^a|$. Next by Formula 2.11, $|HK^a| = \dfrac{|H||K^a|}{|H \cap K^a|}$. Here $|K^a| = |K|$ (as K^a and K are conjugate), so $|HK^a| = \dfrac{|H||K|}{|H \cap K^a|}$. Here $|H \cap K^a| = |H^{a^{-1}} \cap K|$ by (B.2). Thus $\dfrac{|H||K|}{|H \cap K^a|} = \dfrac{|H||K|}{|H^{a^{-1}} \cap K|}$. □

Any clan $a_1H_1a_2H_2\cdots a_nH_n$ of a group G may be expressed as a product of a sect and an element of G. To see this, we first write it as follows:

$$\begin{aligned} &a_1H_1a_2H_2\cdots a_nH_n \\ &\quad = a_1H_1a_1^{-1}(a_1a_2)H_2(a_1a_2)^{-1}(a_1a_2a_3)H_3(a_1a_2a_3)^{-1}\cdots \\ &\qquad \cdots(a_1a_2\cdots a_n)H_n(a_1a_2\cdots a_n)^{-1}(a_1a_2\cdots a_n). \end{aligned} \tag{3.1}$$

Set $H_i' := (a_1a_2\cdots a_i)H_i(a_1a_2\cdots a_i)^{-1}$, $i = 1, 2, \ldots, n$ and $g := a_1a_2\cdots a_n$. Then (3.1) is written as

$$a_1H_1a_2H_2\cdots a_nH_n = H_1'H_2'\cdots H_n'g. \tag{3.2}$$

Here $g \in G$ and any H_i' $(i = 1, 2, \ldots, n)$ is a subgroup of G (because H_i' is conjugate to H_i). Thus $H_1'H_2'\cdots H_n'$ is a sect, and the following is obtained:

Lemma 3.2. *Any clan $a_1H_1a_2H_2\cdots a_nH_n$ of a group G may be expressed as a product of a sect and an element of G as in* (3.2)*.*

We keep the above notation.

Claim 3.3. *The following equivalences hold:*

$$\begin{aligned} a_1H_1a_2H_2\cdots a_nH_n \textit{ is finite } &\overset{\text{(a)}}{\iff} H_1, H_2, \ldots, H_n \textit{ are finite} \\ &\overset{\text{(b)}}{\iff} a_1H_1, a_2H_2, \ldots, a_nH_n \textit{ are finite} \\ &\overset{\text{(c)}}{\iff} a_1H_1 \times a_2H_2 \times \cdots \times a_nH_n \textit{ is finite.} \end{aligned}$$

Proof. (b) is from $a_iH_i \cong H_i$ (bijective). (c) is trivial. (a) is confirmed as follows:

$$\begin{aligned} &a_1H_1a_2H_2\cdots a_nH_n \text{ is finite} \\ &\iff H_1'H_2'\cdots H_n'g \text{ is finite} \quad \text{(from (3.2))} \\ &\iff H_1'H_2'\cdots H_n' \text{ is finite} \quad \text{(see Remark 3.4 below)} \\ &\iff H_1', H_2', \ldots, H_n' \text{ are finite} \quad \text{(by Corollary 2.3)} \\ &\iff H_1, H_2, \ldots, H_n \text{ are finite} \quad \text{(as } H_i' \text{ is conjugate to } H_i\text{).} \end{aligned}$$

□

Remark 3.4. $H_1'H_2'\cdots H_n'$ and $H_1'H_2'\cdots H_n'g$ are bijective under the right multiplication by g.

Note next the following:

Claim 3.5. *In the case of finite $a_1H_1a_2H_2\cdots a_nH_n$, the following hold:*

(1) $|a_1H_1a_2H_2\cdots a_nH_n| = |H_1'H_2'\cdots H_n'|$.

(2) $|a_1H_1a_2H_2\cdots a_nH_n| \leq |H_1||H_2|\cdots|H_n|$.

Proof. (1): This is confirmed as follows:

$$\begin{aligned}|a_1H_1a_2H_2\cdots a_nH_n| &= |H_1'H_2'\cdots H_n'g| \quad \text{from (3.2)}\\ &= |H_1'H_2'\cdots H_n'| \quad \text{from Remark 3.4.}\end{aligned} \tag{3.3}$$

(2): Note first that $|H_1'H_2'\cdots H_n'| \le |H_1'||H_2'|\cdots|H_n'|$ by Corollary 2.3. Here $|H_i'| = |H_i|$ $(i = 1, 2, \ldots, n)$, because H_i' is conjugate to H_i. Thus $|H_1'H_2'\cdots H_n'| \le |H_1||H_2|\cdots|H_n|$. This with (1) yields (2). □

We summarize Claim 3.3 and Claim 3.5 (2) as follows:

Proposition 3.6. *For a clan $a_1H_1a_2H_2\cdots a_nH_n$, the following equivalences hold:*

$$\begin{aligned} a_1H_1a_2H_2\cdots a_nH_n \text{ is finite} &\iff H_1, H_2, \ldots, H_n \text{ are finite}\\ &\iff a_1H_1, a_2H_2, \ldots, a_nH_n \text{ are finite}\\ &\iff a_1H_1 \times a_2H_2 \times \cdots \times a_nH_n \text{ is finite.}\end{aligned}$$

Moreover in this case, $|a_1H_1a_2H_2\cdots a_nH_n| \le |H_1||H_2|\cdots|H_n|$.

4 Guilds and their synergies

Let A be an arbitrary subset of a group G, and H and K be subgroups of G. Then the subset of G given by

$$HAK := \{hak : h \in H, a \in A, k \in K\} \tag{4.1}$$

is called a *guild* of G. Its *synergy* (more precisely, *guild synergy*) is given by

$$\psi : H \times A \times K \longrightarrow HAK, \quad \psi(h, a, k) := hak. \tag{4.2}$$

Remark 4.1. In the particular case that A is a singleton $\{a\}$ where $a \in G$, this is a double coset telergy $\psi : H \times \{a\} \times K \longrightarrow HaK$.

A distinguished property of a guild HAK is that it admits a *biaction* — a both-sided action with left H-action and right K-action:

$$\begin{aligned}&H \curvearrowright HAK \curvearrowleft K, \quad \text{where for } u \in H \text{ and } v \in K,\\ &hak \in HAK \longmapsto u \cdot hak \cdot v := (uh)a(kv) \in HAK.\end{aligned} \tag{4.3}$$

Definition 4.2. A clan $a_1H_1a_2H_2a_3H_3\cdots a_nH_n$ is *monic* if a_1 is the identity, that is, of the form $H_1a_2H_2a_3H_3\cdots a_nH_n$.

Note that sects are special cases of monic clans. Indeed a sect $H_1H_2\cdots H_n$ is a specialization of a monic clan $H_1a_2H_2a_3H_3\cdots a_nH_n$ by setting $a_2 = a_3 = \cdots = a_n = e$ (the identity). Thus

$$\{\text{sects of } G\} \subset \{\text{monic clans of } G\}. \tag{4.4}$$

Note next that a sect, a monic clan, and a biclan are special cases of guilds:

(I) A sect $H_1H_2\cdots H_n$ is a guild HAK with $H := H_1$, $A := H_2H_3\cdots H_{n-1}$ and $K := H_n$.

(II) A monic clan $H_1a_2H_2a_3H_3\cdots a_nH_n$ is a guild HAK with $H := H_1$, $A := a_2H_2a_3H_3\cdots a_{n-1}H_{n-1}a_n$ and $K := H_n$.

(III) A biclan $(H_1a_1K_1)(H_2a_2K_2)\cdots(H_na_nK_n)$ (a product of double cosets) is a guild HAK with $H := H_1$, $A := (a_1K_1)(H_2a_2K_2)\cdots(H_na_n)$ and $K := K_n$.

From (II), $\{$monic clans of $G\} \subset \{$guilds of $G\}$. This with (4.4) gives

$$\{\text{sects of } G\} \subset \{\text{monic clans of } G\} \subset \{\text{guilds of } G\}. \tag{4.5}$$

Note that a monic clan $H_1a_2H_2a_3H_3\cdots a_nH_n$ is rewritten as follows (below, $H_iH_i = H_i$ follows from the closedness of H_i under multiplication):

$$\begin{aligned} H_1a_2H_2a_3H_3\cdots a_nH_n &= H_1a_2H_2H_2a_3H_3H_3\cdots a_nH_n \quad \text{by } H_iH_i = H_i \\ &= (H_1a_2H_2)(H_2a_3H_3)\cdots(H_{n-1}a_nH_n). \end{aligned} \tag{4.6}$$

Here the last one is a biclan, so the following is obtained:

Lemma 4.3. *A monic clan is equal to a biclan. In fact*

$$H_1a_2H_2a_3H_3\cdots a_nH_n = (H_1a_2H_2)(H_2a_3H_3)\cdots(H_{n-1}a_nH_n).$$

Consequently, monic clans are considered as special cases of biclans:

$$\{\text{monic clans of } G\} \subset \{\text{biclans of } G\}. \tag{4.7}$$

Conversely, biclans are considered as special cases of clans. Indeed a biclan $(H_1a_1K_1)(H_2a_2K_2)\cdots(H_na_nK_n)$ may be regarded as a coset product, i.e. a clan as follows:

$$H_1(a_1K_1)H_2(a_2K_2)\cdots H_n(a_nK_n). \tag{4.8}$$

Thus $\{$biclans of $G\} \subset \{$clans of $G\}$. This with (4.7) yields

$$\{\text{monic clans of } G\} \subset \{\text{biclans of } G\} \subset \{\text{clans of } G\}. \tag{4.9}$$

Here the following actually holds:

Claim 4.4. $\{$*monic clans of* $G\} = \{$*clans of* $G\}$.

Proof. First note that $\{$monic clans of $G\} \subset \{$clans of $G\}$, because a monic clan $H_1a_2H_2\cdots a_nH_n$ is equal to a clan $a_1H_1a_2H_2\cdots a_nH_n$ with $a_1 = e$ (the identity). Conversely $\{$monic clans of $G\} \supset \{$clans of $G\}$, because a clan $a_1H_1a_2H_2\cdots a_nH_n$ is equal to a monic clan $H_0a_1H_1a_2H_2\cdots a_nH_n$ with $H_0 = \{e\}$ (the identity subgroup). □

By Claim 4.4, "$\subset$" in (4.9) are actually "$=$", thus the following holds:

Proposition 4.5. *For any group G,*

$$\{\textit{monic clans of } G\} = \{\textit{biclans of } G\} = \{\textit{clans of } G\}. \tag{4.10}$$

This combined with (4.5) yields the following:

Corollary 4.6. *The following inclusion relations hold:*

$$\begin{aligned}\{\textit{sects of } G\} \subset \{\textit{monic clans of } G\} &= \{\textit{biclans of } G\} = \{\textit{clans of } G\} \\ &\subset \{\textit{guilds of } G\}.\end{aligned}$$

Remark 4.7. Note that for a clan $a_1H_1a_2H_2a_3H_3\cdots a_nH_n$ and its cutoff biclan $H_1a_2H_2a_3H_3\cdots a_nH_n$, in general

$$a_1H_1a_2H_2a_3H_3\cdots a_nH_n \neq H_1a_2H_2a_3H_3\cdots a_nH_n, \tag{4.11}$$

but they are bijective — in fact the former clan is a 'translation' of the latter clan by the left multiplication of a_1. In particular in the finite case, we have

$$|a_1H_1a_2H_2a_3H_3\cdots a_nH_n| = |H_1a_2H_2a_3H_3\cdots a_nH_n|. \tag{4.12}$$

Proposition 4.5 ensures that we may interchangeably use clans, monic clans, and biclans depending on situations — they have their own advantages.

We now return to guilds.

Lemma 4.8. *Any guild HAK admits a double coset decomposition:*

$$HAK = \coprod_{i\in I} Ha^{(i)}K \quad \textit{with } a^{(i)} \in A \ (i \in I). \tag{4.13}$$

Proof. First we write $HAK = \bigcup_{a\in A} HaK$. Here by the basic property of double cosets (two double cosets are either disjoint or identical), for $a, b \in A$ either $HaK \cap HbK = \emptyset$ or $HaK = HbK$. Therefore among HaK ($a \in A$), taking representatives $Ha^{(i)}K$ ($a^{(i)} \in A, i \in I$), we obtain the double coset decomposition $HAK = \coprod_{i\in I} Ha^{(i)}K$. □

Remark 4.9. Geometrically, the double coset decomposition (4.13) is the orbit decomposition of HAK under the biaction $H \curvearrowright HAK \curvearrowleft K$ in (4.3).

Lemma 4.10. *If a guild HAK is finite, then H, A and K are finite.*

Proof. Let e be the identity. Noting that $e \in H, K$, we have $HAK \supset eAe = A$. The finiteness of HAK then implies that A is finite. Note that to show that H and K are finite, we cannot apply this argument (because in general $e \notin A$, in which case generally $HAK \not\supset Hee = H$ and $HAK \not\supset eeK = K$). Instead consider the double coset decomposition $HAK = \coprod_{i\in I} Ha^{(i)}K$. The finiteness of HAK implies that $Ha^{(i)}K$ is finite. Here $Ha^{(i)}K \supset ea^{(i)}K = a^{(i)}K$, so the coset $a^{(i)}K$ is finite, thus K is finite. Similarly $Ha^{(i)}K \supset Ha^{(i)}e = Ha^{(i)}$, so the coset $Ha^{(i)}$ is finite, thus H is finite. □

By definition, a guild HAK consists of hak $(h \in H, a \in A, k \in K)$. Thus, in the case that H, A and K are finite, we have $|HAK| \leq |H||A||K|$, so HAK is finite. We restate this result combined with Lemma 4.10 as follows:

Corollary 4.11. *A guild HAK is finite if and only if H, A and K are finite; moreover in this case $|HAK| \leq |H||A||K|$.*

While a guild HAK has its double coset decomposition $HAK = \coprod_{i \in I} Ha^{(i)}K$, the direct product $H \times A \times K$ also has a decomposition:

$$H \times A \times K = \coprod_{b \in A} H \times \{b\} \times K, \tag{4.14}$$

with *no* need to take representatives. Note that in general $H \times A \times K$ is not a group.

Definition 4.12. The decomposition (4.14) is called the *satellite decomposition* of $H \times A \times K$ with each $H \times \{b\} \times K$ a *satellite*.

Observation 4.13. As for the biaction $H \curvearrowright HAK \curvearrowleft K$ on HAK (see (4.3)), $H \times A \times K$ also admits a biaction:

$$\begin{aligned} &H \curvearrowright H \times A \times K \curvearrowleft K, \quad \text{where for } u \in H \text{ and } v \in K, \\ &(h, a, k) \in H \times A \times K \longmapsto u \cdot (h, a, k) \cdot v := (uh, a, kv) \in H \times A \times K. \end{aligned} \tag{4.15}$$

The orbit decomposition of $H \times A \times K$ under this biaction is nothing but the satellite decomposition (4.14).

While HAK is a guild, $H \times A \times K$ is also regarded as a guild. To see this, noting that $H \times A \times K$ is a subset of a direct product group $G' := H \times G \times K$, we set

$$H' := H \times \{e\} \times \{e\} \quad \text{and} \quad K' := \{e\} \times \{e\} \times K, \tag{4.16}$$

which are subgroups of G'. We also set $A' := \{e\} \times A \times \{e\}$, which is a subset of G'. Then $H \times A \times K$ is rewritten as $H'A'K'$, being a guild of G'. This confirms the following:

Lemma 4.14. *Let HAK be a guild of a group G. Then $H \times A \times K$ is a guild of a direct product group $H \times G \times K$.*

In the notation above Lemma 4.14, $b \in A$ is expressed as $(e, b, e) \in A'$, accordingly each satellite $H \times \{b\} \times K$ in the satellite decomposition (4.14) is expressed as $H'(e, b, e)K'$, which is a double coset of the group G'. Thus the following holds:

Lemma 4.15. *The satellite decomposition* (4.14) *of $H \times A \times K$ is the double coset decomposition of $H \times A \times K$ in the direct product group $H \times G \times K$.*

Remark 4.16. The biaction $H \curvearrowright H \times A \times K \curvearrowleft K$ given by (4.15) may be also alternatively defined *inside* the group $H \times G \times K$ under the identification of H with $H \times \{e\} \times \{e\}$ and K with $\{e\} \times \{e\} \times K$. Namely it is regarded as a biaction

$$H \times \{e\} \times \{e\} \curvearrowright H \times A \times K \curvearrowleft \{e\} \times \{e\} \times K$$

given by

$$\begin{aligned}(h, a, k) \in H \times A \times K \longmapsto (u, e, e) \cdot (h, a, k) \cdot (e, e, v) \\ := (uh, a, kv) \in H \times A \times K.\end{aligned} \tag{4.17}$$

Supplement: Extended synergy

For a guild HAK of a group G, let $\psi : H \times A \times K \to HAK$, $\psi(h, a, k) := hak$ be its synergy. Then the *extended synergy* is defined by

$$\Psi : H \times G \times K \to HGK, \qquad \Psi(h, g, k) := hgk. \tag{4.18}$$

Here $HGK = G$ (from $G \supset HGK \supset eGe = G$). Thus, setting $G' := H \times G \times K$ (a direct product group), we have $\Psi : G' \to G$, which is a map between groups G' and G, i.e. a *group map* (but generally not a group homomorphism). Now $H \times A \times K$ is a guild of G' (Lemma 4.14), so $\psi : H \times A \times K \to HAK$ is a map between guilds, i.e. a *guild map*. We take

- $HAK = \coprod_{i \in I} Ha^{(i)}K$: the double coset decomposition of HAK, and
- $H \times A \times K = \coprod_{b \in A} H \times \{b\} \times K$: the satellite decomposition of $H \times A \times K$, which is also the double coset decomposition of $H \times A \times K$ (Lemma 4.15).

Then ψ is regarded as a map

$$\psi : \coprod_{b \in A} H \times \{b\} \times K \longrightarrow \coprod_{i \in I} Ha^{(i)}K, \tag{4.19}$$

mapping a double coset (a satellite) $H \times \{b\} \times K$ to a double coset HbK ($= Ha^{(i)}K$ for some $i \in I$).

5 Satellites and fiber theorem

Let HAK be a guild of a group G, where H and K are subgroups of G and A be a subset of G.

Definition 5.1. For each double coset $Ha^{(i)}K$ in the double coset decomposition $HAK = \coprod_{i \in I} Ha^{(i)}K$, we adopt the following terms:

(1) A satellite $H \times \{b\} \times K$ mapped to $Ha^{(i)}K$ under the synergy $\psi : H \times A \times K \to HAK$ is said to be *over* $Ha^{(i)}K$; this condition is equivalent to $HbK = Ha^{(i)}K$.

(2) The number of satellites over $Ha^{(i)}K$ is called the *satellite index* of $Ha^{(i)}K$, and denoted by $s^{(i)}$ (possibly infinite — but later we mainly focus on the finite case). By convention, for any $x \in Ha^{(i)}K$, $s^{(i)}$ is also called the satellite index of x.

Note the following:

Lemma 5.2. *For a finite guild HAK, any satellite index $s^{(i)}$ is finite.*

Proof. Note first that A is finite by Corollary 4.11. Consequently the number of all satellites $H \times \{b\} \times K$ ($b \in A$) is finite. In particular the number $s^{(i)}$ of satellites over $Ha^{(i)}K$ is finite. □

Note next the following:

Lemma 5.3. *Let HAK be a guild, let $\psi : H \times A \times K \to HAK$ be its synergy and let $HAK = \coprod_{i\in I} Ha^{(i)}K$ be its double coset decomposition. For each $Ha^{(i)}K$, let $s^{(i)}$ be its satellite index and let $H \times \{b^{(ij)}\} \times K$ ($j \in J^{(i)}$) be the satellites over $Ha^{(i)}K$. Then the following hold:*

(1) $\psi^{-1}(Ha^{(i)}K) = \coprod_{j\in J^{(i)}} H \times \{b^{(ij)}\} \times K.$

(2) *Suppose that HAK is finite; so H, K and $s^{(i)}$ are finite (Corollary 4.11 and Lemma 5.2). Then $|\psi^{-1}(Ha^{(i)}K)| = s^{(i)}|H||K|$.*

Proof. (1) is just from the definition of "satellites over $Ha^{(i)}K$".

(2): From (1), we have $|\psi^{-1}(Ha^{(i)}K)| = \sum_{j\in J^{(i)}} |H \times \{b^{(ij)}\} \times K|$. Here $|H \times \{b^{(ij)}\} \times K| = |H||K|$ (constant independent of j) and $|J^{(i)}| = s^{(i)}$ (by the definition of $s^{(i)}$). Thus $|\psi^{-1}(Ha^{(i)}K)| = s^{(i)}|H||K|$. □

We keep the notation of Lemma 5.3. Pulling back the double coset decomposition $HAK = \coprod_{i\in I} Ha^{(i)}K$ via the guild synergy $\psi : H \times A \times K \to HAK$, we obtain

$$\psi^{-1}(HAK) = \coprod_{i\in I} \psi^{-1}(Ha^{(i)}K). \tag{5.1}$$

Here each $\psi^{-1}(Ha^{(i)}K)$ consists of the satellites over $Ha^{(i)}K$, that is, as in Lemma 5.3 (1),

$$\psi^{-1}(Ha^{(i)}K) = \coprod_{j\in J^{(i)}} H \times \{b^{(ij)}\} \times K. \tag{5.2}$$

We consider the restriction of ψ to this:

$$\psi : \coprod_{j\in J^{(i)}} H \times \{b^{(ij)}\} \times K \longrightarrow Ha^{(i)}K. \tag{5.3}$$

Denote the further restriction of ψ to $H \times \{b^{(ij)}\} \times K$ by $\psi^{(ij)}$. Then for each $x \in Ha^{(i)}K$, the fiber $\psi^{-1}(x)$ is, from (5.3), expressed as a disjoint union:

$$\psi^{-1}(x) = \coprod_{j\in J^{(i)}} {\psi^{(ij)}}^{-1}(x). \tag{5.4}$$

Convention 5.4. Let X_λ ($\lambda \in \Lambda$) be a collection of a set X (i.e. $X_\lambda = X$), where the cardinality $|\Lambda|$ of Λ may be infinite. Then $\coprod_{\lambda\in\Lambda} X_\lambda$ is called a disjoint union of $|\Lambda|$ *copies of* X.

Theorem 5.5 (Fiber theorem for guild synergies). *Let HAK be a guild, let $\psi : H \times A \times K \to HAK$ be its synergy and let $HAK = \coprod_{i\in I} Ha^{(i)}K$ be its double coset decomposition. For each $Ha^{(i)}K$, let $H \times \{b^{(ij)}\} \times K$ ($j \in J^{(i)}$) be the satellites over it and denote the cardinality of $J^{(i)}$ by $s^{(i)}$* (i.e. *the satellite index of $Ha^{(i)}K$*)*. Then the following hold:*

(1) *Denote by $\psi^{(ij)} : H \times \{b^{(ij)}\} \times K \to Hb^{(ij)}K \,(= Ha^{(i)}K)$ the restriction of ψ to $H \times \{b^{(ij)}\} \times K$. Then for any $x \in Ha^{(i)}K$, the fiber ${\psi^{(ij)}}^{-1}(x)$ is, as a set, bijective to $H \cap K^{a^{(i)}}$ (regardless of the choice of $x \in Ha^{(i)}K$), where $K^{a^{(i)}} := a^{(i)}K{a^{(i)}}^{-1}$.*

(2) *For any $x \in Ha^{(i)}K$, the fiber $\psi^{-1}(x)$ is a disjoint union:*

$$\psi^{-1}(x) = \coprod_{j\in J^{(i)}} {\psi^{(ij)}}^{-1}(x) \ \textit{with}\ {\psi^{(ij)}}^{-1}(x) \cong H \cap K^{a^{(i)}} \ (\textit{bijective as sets}).$$

(3) *For any $x \in Ha^{(i)}K$, the fiber $\psi^{-1}(x)$ is bijective to the disjoint union of $s^{(i)}$ copies of $H \cap K^{a^{(i)}}$ (regardless of the choice of $x \in Ha^{(i)}K$).*

(4) *All fibers $\psi^{-1}(x)$ ($x \in Ha^{(i)}K$) are bijective — in fact to the disjoint union of $s^{(i)}$ copies of $H \cap K^{a^{(i)}}$.*

Proof. (1): Since $H \times \{b^{(ij)}\} \times K$ is over $Ha^{(i)}K$, we have (by definition)

$$Hb^{(ij)}K = Ha^{(i)}K. \tag{5.5}$$

Noting that

$$\begin{aligned} \psi^{(ij)} :\ H \times \{b^{(ij)}\} \times K &\longrightarrow Hb^{(ij)}K \,(= Ha^{(i)}K) \\ (h, b^{(ij)}, k) &\longmapsto hb^{(ij)}k \end{aligned} \tag{5.6}$$

is a double coset telergy (Example 2.15), we apply Corollary 2.17 to $\psi^{(ij)}$, obtaining

$${\psi^{(ij)}}^{-1}(x) \cong H \cap K^{b^{(ij)}} \ \text{(bijective as sets)}. \tag{5.7}$$

We shall rewrite $K^{b^{(ij)}}$. Note first that from (5.5), $b^{(ij)} = ha^{(i)}k$ for some $h \in H$ and $k \in K$. Then

$$\begin{aligned} K^{b^{(ij)}} &:= b^{(ij)}K{b^{(ij)}}^{-1} = ha^{(i)}kK(ha^{(i)}k)^{-1} = ha^{(i)}kKk^{-1}{a^{(i)}}^{-1}h^{-1} \\ &= ha^{(i)}K{a^{(i)}}^{-1}h^{-1} \quad \text{as } kKk^{-1} = K \\ &= hK^{a^{(i)}}h^{-1}. \end{aligned}$$

Hence $K^{b^{(ij)}} = hK^{a^{(i)}}h^{-1}$. Then

$$\begin{aligned} H \cap K^{b^{(ij)}} &= H \cap hK^{a^{(i)}}h^{-1} = hHh^{-1} \cap hK^{a^{(i)}}h^{-1} \quad \text{as } hHh^{-1} = H \\ &= h(H \cap K^{a^{(i)}})h^{-1}. \end{aligned}$$

So $H \cap K^{b^{(ij)}}$ is conjugate to $H \cap K^{a^{(i)}}$. In particular $H \cap K^{b^{(ij)}} \cong H \cap K^{a^{(i)}}$ (bijective). This with (5.7) yields ${\psi^{(ij)}}^{-1}(x) \cong H \cap K^{a^{(i)}}$ (bijective).

(2) and (3): First we have (see (5.4))

$$(*) \quad \psi^{-1}(x) = \coprod_{j \in J^{(i)}} {\psi^{(ij)}}^{-1}(x).$$

Here ${\psi^{(ij)}}^{-1}(x) \cong H \cap K^{a^{(i)}}$ by (1), and so (2) is confirmed. The cardinality of $J^{(i)}$ is by definition equal to $s^{(i)}$. Thus the right hand side of $(*)$ is bijective to a disjoint union of $s^{(i)}$ copies of $H \cap K^{a^{(i)}}$; so (3) is confirmed. (4) is immediate from (3). $\square$

Remark 5.6. Notice that even for the description of the synergy ψ, a telergy — the double coset telergy $\psi^{(ij)}$ — emerges.

In the case that HAK is finite, from Theorem 5.5 (3) we have $|\psi^{-1}(x)| = s^{(i)}|H \cap K^{a^{(i)}}|$, which is independent of the choice of $x \in Ha^{(i)}K$ but depends only on i. We formalize this as follows:

Theorem 5.7 (Fiber order formula). *Let HAK be a finite guild, let $\psi : H \times A \times K \to HAK$ be its synergy and let $HAK = \coprod_{i \in I} Ha^{(i)}K$ be its double coset decomposition. For each $Ha^{(i)}K$, let $s^{(i)}$ be its satellite index. Then the order $|F^{(i)}|$ of the fiber $F^{(i)} := \psi^{-1}(x)$ over* any *$x \in Ha^{(i)}K$ is given by*

$$|F^{(i)}| = s^{(i)}|H \cap K^{a^{(i)}}|. \tag{5.8}$$

Example 5.8. We consider the simplest case that A is a singleton, i.e. $A = \{a\}$ for some $a \in G$. Then the double coset decomposition of HAK is itself: $HAK = HaK$; for consistency with the above notation, we shall denote a as $a^{(1)}$. For the synergy $H \times \{a^{(1)}\} \times K \to Ha^{(1)}K$, the satellite $H \times \{a^{(1)}\} \times K$ is trivially a unique satellite over $Ha^{(1)}K$, so the satellite index $s^{(1)}$ of $Ha^{(1)}K$ is 1. Then by the fiber order formula, $|F^{(1)}| = s^{(1)}|H \cap K^{a^{(1)}}| = |H \cap K^{a^{(1)}}|$. In the case $a^{(1)} = e$ (i.e. HAK is a sect HK), this reduces to $|F^{(1)}| = |H \cap K|$.

Example 5.9. Let $D_4 = \langle a, b : a^4 = b^2 = 1, bab^{-1} = a^3 \rangle$ be the dihedral group of degree 4 (order 8) — the identity e is denoted by 1 with the intention of saving space in Figure 5.1 (which illustrates this example). Take order 2 cyclic subgroups $H_1 := \langle b \rangle$, $H_2 := \langle ab \rangle$ and $H_3 := \langle a^2 b \rangle$, and consider a sect $H_1H_2H_3$. Then take its synergy $\psi : H_1 \times H_2 \times H_3 \to H_1H_2H_3$ and its double coset decomposition $H_1H_2H_3 = H_1a^{(1)}H_3 \amalg H_1a^{(2)}H_3$ with $a^{(1)} = 1$ and $a^{(2)} = ab$. Note the following:

- As illustrated in Figure 5.1,
  $H_1 \times \{1\} \times H_3$ is a unique satellite over $H_1a^{(1)}H_3$, so $s^{(1)} = 1$.
  $H_1 \times \{ab\} \times H_3$ is a unique satellite over $H_1a^{(2)}H_3$, so $s^{(2)} = 1$.

- $H_1 \cap H_3 = \{1\}$, so $|H_1 \cap H_3| = 1$.
  $H_1 \cap H_3^{a^{(2)}} = H_1$, so $|H_1 \cap H_3^{a^{(2)}}| = |H_1| = 2$.

Then applying the fiber order formula (5.8) to the case that $H := H_1$, $A := H_2$ and $K := H_3$, we obtain the following:

(1) $|F^{(1)}| = s^{(1)}|H_1 \cap H_3| = 1$ for any fiber $F^{(1)}$ of ψ over $H_1a^{(1)}H_3$.

(2) $|F^{(2)}| = s^{(2)}|H_1 \cap H_3^{a^{(2)}}| = 2$ for any fiber $F^{(2)}$ of ψ over $H_1a^{(2)}H_3$.

Thus a fiber of the synergy $\psi : H_1 \times H_2 \times H_3 \to H_1H_2H_3$ consists of either one or two elements (see Figure 5.1).

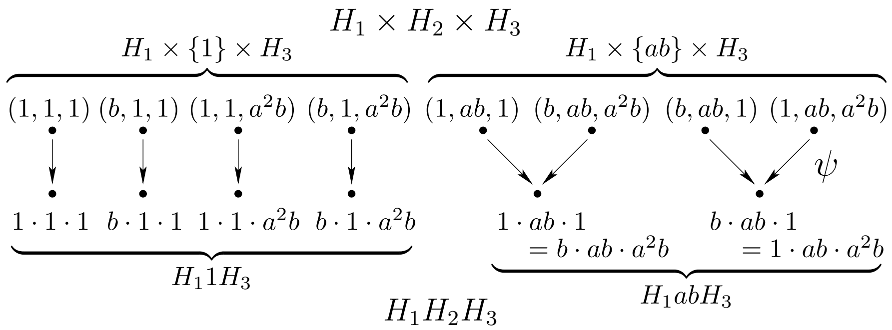

Figure 5.1: The synergy ψ in Example 5.9: To save space, the identity e is denoted by 1.

6 Order formulas and intersection formula

We begin with the following:

Lemma 6.1. *Let HAK be a guild, let $\psi : H \times A \times K \to HAK$ be its synergy and let $HAK = \coprod_{i \in I} Ha^{(i)}K$ be its double coset decomposition.*

(1) *Take the fiber $F^{(i)} := \psi^{-1}(x)$ over any $x \in Ha^{(i)}K$. Then $\psi^{-1}(Ha^{(i)}K) \cong F^{(i)} \times Ha^{(i)}K$ (bijective as sets).*

(2) *Suppose that HAK is finite. Then $|\psi^{-1}(Ha^{(i)}K)| = |F^{(i)}||Ha^{(i)}K|$.*

Proof. (1): First we express $\psi^{-1}(Ha^{(i)}K)$ as a disjoint union of fibers:

$$\psi^{-1}(Ha^{(i)}K) = \coprod_{y \in Ha^{(i)}K} \psi^{-1}(y). \tag{6.1}$$

Here $\psi^{-1}(y) \cong F^{(i)}$ (bijective as sets) by Theorem 5.5 (4). Thus

$$\begin{aligned}\psi^{-1}(Ha^{(i)}K) &\cong \coprod_{y \in Ha^{(i)}K} F^{(i)} \\ &\cong F^{(i)} \times Ha^{(i)}K.\end{aligned} \tag{6.2}$$

(2): This is immediate from (1). $\square$

Remark 6.2. Suppose that HAK is a finite guild. We have two expressions of $|\psi^{-1}(Ha^{(i)}K)|$:

(i) $|\psi^{-1}(Ha^{(i)}K)| = s^{(i)}|H||K|$ (Lemma 5.3 (2)), where $s^{(i)}$ is the satellite index of $Ha^{(i)}K$.

(ii) $|\psi^{-1}(Ha^{(i)}K)| = |F^{(i)}||Ha^{(i)}K|$ (Lemma 6.1 (2)), where $F^{(i)} := \psi^{-1}(x)$ is the fiber over any $x \in Ha^{(i)}K$.

By equating the right hand sides of (i) and (ii), we have

$$|F^{(i)}||Ha^{(i)}K| = s^{(i)}|H||K|. \tag{6.3}$$

Here by Lemma 3.1 (B.3), $|Ha^{(i)}K| = \dfrac{|H||K|}{|H \cap K^{a^{(i)}}|}$, so $|F^{(i)}| = s^{(i)}|H \cap K^{a^{(i)}}|$. This reproves Theorem 5.7.

For a finite guild HAK, let $HAK = \coprod_{i \in I} Ha^{(i)}K$ be its double coset decomposition; note that the finiteness of HAK implies that every $Ha^{(i)}K$ is finite. For each $Ha^{(i)}K$, its satellite index is denoted by $s^{(i)}$; this is also finite by Lemma 5.2. By convention, for any $x \in Ha^{(i)}K$, $s^{(i)}$ is also called the satellite index of x. Then:

Definition 6.3. The totality of satellite indices of all $x \in Ha^{(i)}K$, that is, $m^{(i)} := s^{(i)}|Ha^{(i)}K|$ is called the *multiplicity* of $Ha^{(i)}K$.

Lemma 6.4. *For a finite guild HAK, let $\psi : H \times A \times K \to HAK$ be its synergy and let $HAK = \coprod_{i \in I} Ha^{(i)}K$ be its double coset decomposition. For each $Ha^{(i)}K$, let $m^{(i)}$ be its multiplicity. Then*

$$|\psi^{-1}(Ha^{(i)}K)| = m^{(i)}|H \cap K^{a^{(i)}}|. \tag{6.4}$$

Proof. By Lemma 6.1 (2), $|\psi^{-1}(Ha^{(i)}K)| = |F^{(i)}||Ha^{(i)}K|$. Substituting $|F^{(i)}| = s^{(i)}|H \cap K^{a^{(i)}}|$ (Theorem 5.7) into this yields the following:

$$|\psi^{-1}(Ha^{(i)}K)| = s^{(i)}|H \cap K^{a^{(i)}}||Ha^{(i)}K|.$$

Since $m^{(i)} := s^{(i)}|Ha^{(i)}K|$, we obtain $|\psi^{-1}(Ha^{(i)}K)| = m^{(i)}|H \cap K^{a^{(i)}}|$. $\square$

Theorem 6.5 (Intersection formula). *Let HAK be a finite guild and let $HAK = \coprod_{i\in I} Ha^{(i)}K$ be its double coset decomposition. For each $Ha^{(i)}K$, let $m^{(i)}$ be its multiplicity. Then*

$$|H||A||K| = \sum_{i\in I} m^{(i)}|H \cap K^{a^{(i)}}|. \tag{6.5}$$

Proof. Under the synergy $\psi : H \times A \times K \to HAK$, the pullback of the double coset decomposition $HAK = \coprod_{i\in I} Ha^{(i)}K$ gives a decomposition:

$$H \times A \times K = \coprod_{i\in I} \psi^{-1}(Ha^{(i)}K). \tag{6.6}$$

Thus $|H \times A \times K| = \sum\limits_{i\in I} |\psi^{-1}(Ha^{(i)}K)|$, that is,

$$|H||A||K| = \sum_{i\in I} |\psi^{-1}(Ha^{(i)}K)|. \tag{6.7}$$

Here $|\psi^{-1}(Ha^{(i)}K)| = m^{(i)}|H \cap K^{a^{(i)}}|$ by Lemma 6.4. Hence $|H||A||K| = \sum_{i\in I} m^{(i)}|H \cap K^{a^{(i)}}|$. This confirms the assertion. □

Remark 6.6. We point out the following correspondence between objects in algebraic geometry and those in higher group theory used in the above discussion:

Algebraic geometry	Higher group theory
algebraic variety X	guild HAK
structure sheaf $\mathcal{O}_X \to X$	guild synergy $H \times A \times K \to HAK$
Zariski topology on X	double coset decomposition of HAK

We shall give an example for (6.5).

Example 6.7. As in Example 5.9, for the dihedral group

$$D_4 = \langle a, b : a^4 = b^2 = 1,\, bab^{-1} = a^3\rangle,$$

take order 2 cyclic subgroups $H_1 := \langle b\rangle$, $H_2 := \langle ab\rangle$ and $H_3 := \langle a^2b\rangle$, and consider a sect $H_1H_2H_3$; its double coset decomposition is $H_1H_2H_3 = H_1a^{(1)}H_3 \amalg H_1a^{(2)}H_3$ with $a^{(1)} = 1$ and $a^{(2)} = ab$. Note that $s^{(1)} = s^{(2)} = 1$ (Example 5.9) and $|H_1a^{(i)}H_3| = \dfrac{|H_1||H_3|}{|H_1 \cap H_3^{a^{(i)}}|}$ (Lemma 3.1 (B.3)), where

$$\begin{cases} H_1 \cap H_3^{a^{(1)}} = \{1\} & \text{(from } a^{(1)} = 1 \text{ and } H_1 \cap H_3 = \{1\}), \\ H_1 \cap H_3^{a^{(2)}} = H_1 & \text{(from } H_1 = H_3^{a^{(2)}}), \end{cases} \tag{6.8}$$

thus

$$\begin{cases} |H_1a^{(1)}H_3| = \dfrac{|H_1||H_3|}{|H_1 \cap H_3|} = |H_1||H_3| = 4, \\ |H_1a^{(2)}H_3| = \dfrac{|H_1||H_3|}{|H_1 \cap H_3^{a^{(2)}}|} = |H_3| = 2. \end{cases} \tag{6.9}$$

Hence

$$\begin{cases} m^{(1)} = s^{(1)}|H_1 a^{(1)} H_3| = 1 \cdot 4 = 4, \\ m^{(2)} = s^{(2)}|H_1 a^{(2)} H_3| = 1 \cdot 2 = 2. \end{cases} \tag{6.10}$$

Accordingly

$$\begin{cases} m^{(1)}|H_1 \cap H_3^{a^{(1)}}| = m^{(1)}|H_1 \cap H_3| = 4 \cdot 1 = 4, \\ m^{(2)}|H_1 \cap H_3^{a^{(2)}}| = m^{(2)}|H_1| = 2 \cdot 2 = 4. \end{cases} \tag{6.11}$$

Now the intersection formula (6.5) applied to the current case is

$$|H_1||H_2||H_3| = m^{(1)}|H_1 \cap H_3^{a^{(1)}}| + m^{(2)}|H_1 \cap H_3^{a^{(2)}}|, \tag{6.12}$$

which from the above data reads as $2 \cdot 2 \cdot 2 = 4 + 4$.

Comparison with Bézout's theorem

Our intersection formula (6.5) has an analog in algebraic geometry. First note that an intersection point p of two plane curves in the projective plane $\mathbb{P}^2$ is not merely set-theoretic, but scheme-theoretic — formally a *multiple* point mp, i.e. p equipped with *multiplicity* m (more precisely, *intersection multiplicity* — the order of tangency). For a plane curve C defined by a polynomial f (i.e. C is the zero set of f), we define $\deg C$ to be the degree of f. Then:

Bézout's theorem ([1] p.140). *For plane curves C and D in the projective plane $\mathbb{P}^2$, $\deg C \deg D$ is equal to the sum of intersection multiplicities at the points of $C \cap D$.*

We shall restate this more explicitly. Say that the set of intersection multiplicities at the points of $C \cap D$ is $\{m_1, m_2, \ldots, m_l\}$. For each m_i $(i = 1, 2, \ldots, l)$, denote by $(C \cap D)_i$ the set of intersection points with intersection multiplicity m_i, and by $|(C \cap D)_i|$ the number of its points. Then Bézout's theorem is restated as

$$\deg C \deg D = \sum_{i=1}^{l} m_i |(C \cap D)_i|. \tag{6.13}$$

This is analogous to our intersection formula (6.5).

Remark 6.8. In spite of the above analogy, there is a significant difference between algebraic geometry and higher group theory. The former is based upon a duality between algebra (commutative rings) and geometry (zero sets). In contrast, in the latter, algebra and geometry are *not* separated — for instance, a sect $H_1 H_2 \cdots H_n$ of a group G is both algebraic (as an analog of a polynomial) and geometric (as a subset of G); on the other hand, in algebraic geometry a polynomial $f(x_1, x_2, \ldots, x_n)$ is merely a symbol, and only *a posteriori* its zero set is a geometric object (an affine variety in the projective space $\mathbb{P}^n$).

Specializations to sects and clans

The intersection formula (6.5) for guilds is specialized to sects and clans. We first consider the case of length 2, i.e. double cosets:

Example 6.9. A guild HAK with $A = \{a\}$ for some $a \in G$ is a double coset HaK (a clan of length 2; or a sect of length 2 if $a = e$). For consistency, we write a as $a^{(1)}$. Consider the case of finite HaK. As in Example 5.8, $s^{(1)} = 1$, so $m^{(1)} = s^{(1)}|HaK| = |HaK|$. Thus the intersection formula $|H||K| = m^{(1)}|H \cap K^a|$ is given by $|H||K| = |HaK||H \cap K^a|$, that is,

$$|HaK| = \frac{|H||K|}{|H \cap K^a|}, \tag{6.14}$$

which is the classical order formula of the double coset HaK.

We next consider the cases of sects and clans of length greater than 2.

Example 6.10. To a finite sect $H_1H_2\cdots H_n$ ($n \geq 3$), we apply (6.5) to the case that $H := H_1$, $A := H_2H_3\cdots H_{n-1}$ and $K := H_n$. Then

$$|H_1||H_2H_3\cdots H_{n-1}||H_n| = \sum_{i\in I} m^{(i)}|H_1 \cap H_n^{a^{(i)}}|. \tag{6.15}$$

Example 6.11. To a finite monic clan $H_1a_2H_2\cdots a_nH_n$ ($n \geq 3$), we apply (6.5) to the case that $H := H_1$, $A := a_2H_2a_3H_3\cdots a_{n-1}H_{n-1}a_n$ and $K := H_n$. Then

$$|H_1||a_2H_2a_3H_3\cdots a_{n-1}H_{n-1}a_n||H_n| = \sum_{i\in I} m^{(i)}|H_1 \cap H_n^{a^{(i)}}|. \tag{6.16}$$

Remark 6.12. For any subset S of a group G, note that S and xSy for any $x, y \in G$ are bijective under $s \in S \mapsto xsy \in xSy$. In particular if S is finite, then $|xSy| = |S|$. Applying this to the case that $x := a_2$, $y := a_n$ and $S := H_2a_3H_3\cdots a_{n-1}H_{n-1}$ ($n \geq 3$) in (6.16), we have

$$|a_2H_2a_3H_3\cdots a_{n-1}H_{n-1}a_n| = |H_2a_3H_3\cdots a_{n-1}H_{n-1}|. \tag{6.17}$$

Thus for a finite monic clan $H_1a_2H_2\cdots a_nH_n$ ($n \geq 3$), the following holds:

$$|H_1||H_2a_3H_3\cdots a_{n-1}H_{n-1}||H_n| = \sum_{i\in I} m^{(i)}|H_1 \cap H_n^{a^{(i)}}|. \tag{6.18}$$

7 Partitioned intersection formulas

We begin with the following:

Definition 7.1. A sect $H_1H_2\cdots H_n$ of a group G is said to be *permutative* if $H_iH_j = H_jH_i$ for any $i, j \in \{1, 2, \ldots, n\}$. cf. commutative.

For example, when G is an abelian group, all sects of G are permutative.

The following is just a restatement of Lemma 2.4 (2):

Lemma 7.2. *If $H_1H_2\cdots H_n$ is a permutative sect of a group G, then for any $\{i_1,i_2,\ldots,i_k\}\subset\{1,2,\ldots,n\}$, the sect $H_{i_1}H_{i_2}\cdots H_{i_k}$ is a subgroup of G.*

Let $H_1H_2\cdots H_n$ $(n\geq 3)$ be a permutative sect of a group G. Consider any partition of it into three parts H, A and K:

$$\underbrace{H_1H_2\cdots H_p}_{H}\underbrace{H_{p+1}H_{p+2}\cdots H_q}_{A}\underbrace{H_{q+1}H_{q+2}\cdots H_n}_{K}. \tag{7.1}$$

Then H and K are subgroups of G (Lemma 7.2), so HAK gives a guild expression of $H_1H_2\cdots H_n$. Now suppose that HAK is finite. Applying Theorem 6.5 to HAK then gives the following result:

Theorem 7.3 ("Partitioned" intersection formula for sects).
If $H_1H_2\cdots H_n$ $(n\geq 3)$ is a finite permutative sect, then for any partition (7.1) *of $H_1H_2\cdots H_n$,*

$$\begin{aligned}|H_1H_2\cdots H_p||H_{p+1}H_{p+2}\cdots H_q||H_{q+1}H_{q+2}\cdots H_n|\\ =\sum_{i\in I}m^{(i)}|(H_1H_2\cdots H_p)\cap(H_{q+1}H_{q+2}\cdots H_n)^{a^{(i)}}|,\end{aligned} \tag{7.2}$$

where $I, m^{(i)}$ and $a^{(i)}$ are as in Theorem 6.5 applied to HAK in (7.1)*.*

Note that if G is abelian, then $a^{(i)}$ in the formula (7.2) is unnecessary because $(H_{q+1}H_{q+2}\cdots H_n)^{a^{(i)}}=H_{q+1}H_{q+2}\cdots H_n$; indeed

$$\begin{aligned}(H_{q+1}H_{q+2}\cdots H_n)^{a^{(i)}} &:= a^{(i)}H_{q+1}H_{q+2}\cdots H_n{a^{(i)}}^{-1}\\ &= H_{q+1}H_{q+2}\cdots H_n \quad \text{as } G \text{ is abelian.}\end{aligned} \tag{7.3}$$

This holds for more general groups — Dedekind groups. Here:

Definition 7.4. A group is said to be *Dedekind* if any subgroup of it is normal. In particular any abelian group is Dedekind.

Lemma 7.5. *Let G be a Dedekind group. Then for any subgroups H and K of G, we have $HK=KH$.*

Proof. This is confirmed as follows:

$$\begin{aligned}HK &= \bigcup_{k\in K}Hk=\bigcup_{k\in K}kH \quad \text{as } Hk=kH \text{ by the Dedekindness of } G\\ &= KH.\end{aligned}$$

□

The following is immediate from Lemma 7.5:

Lemma 7.6. *Any sect $H_1H_2\cdots H_n$ of a Dedekind group G is permutative; so it is a subgroup of G by Lemma 7.2.*

Consequently for any finite sect $H_1H_2\cdots H_n$ of a Dedekind group G, the formula (7.2) is applicable. Moreover therein $a^{(i)}$ is unnecessary. In fact the following holds:

Claim 7.7. $(H_{q+1}H_{q+2}\cdots H_n)^{a^{(i)}} = H_{q+1}H_{q+2}\cdots H_n$.

Proof. Note first that

$$\begin{aligned}(H_{q+1}H_{q+2}\cdots H_n)^{a^{(i)}} &:= a^{(i)}(H_{q+1}H_{q+2}\cdots H_n){a^{(i)}}^{-1}\\ &= a^{(i)}H_{q+1}{a^{(i)}}^{-1}a^{(i)}H_{q+2}{a^{(i)}}^{-1}\cdots a^{(i)}H_n{a^{(i)}}^{-1}\\ &= H_{q+1}^{a^{(i)}}H_{q+2}^{a^{(i)}}\cdots H_n^{a^{(i)}}.\end{aligned}$$

To rewrite the last expression, note that since G is Dedekind, for any $a\in G$ and any subgroup H of G we have $aHa^{-1}=H$, that is, $H^a=H$. In particular $H_j^{a^{(i)}}=H_j$ $(j=q+1,q+2,\ldots,n)$. Thus

$$H_{q+1}^{a^{(i)}}H_{q+2}^{a^{(i)}}\cdots H_n^{a^{(i)}} = H_{q+1}H_{q+2}\cdots H_n. \tag{7.4}$$

This confirms the assertion. □

We summarize the above results as follows:

Corollary 7.8. *Let $H_1H_2\cdots H_n$ $(n\geq 3)$ be a finite sect of a Dedekind group. Then for any partition of* (7.1),

$$\begin{aligned}&|H_1H_2\cdots H_p||H_{p+1}H_{p+2}\cdots H_q||H_{q+1}H_{q+2}\cdots H_n|\\ &\qquad=\sum_{i\in I} m^{(i)}|(H_1H_2\cdots H_p)\cap(H_{q+1}H_{q+2}\cdots H_n)|,\end{aligned} \tag{7.5}$$

where I and $m^{(i)}$ are as in Theorem 6.5 applied to HAK in (7.1).

Observation 7.9. For a Dedekind group, all sects are equal to subgroups (Lemma 7.6). Thus a Dedekind group contains essentially *no* higher order objects — in this sense, any Dedekind group is *of first order in sects.* Therefore from the viewpoint of higher group theory, sects of a Dedekind group seem uninteresting — but actually *not*: *A finite sect of a Dedekind group admits various combinatorial formulas as in* (7.5) *depending on its partition.*

Remark 7.10. *Even if a sect $H_1H_2\cdots H_n$ $(n\geq 2)$ of a group G is equal to a subgroup K of G, their synergies are distinct*:

$$\begin{cases} H_1\times H_2\times\cdots\times H_n\to H_1H_2\cdots H_n,\\ K\to K \quad \text{(the identity map)}.\end{cases} \tag{7.6}$$

The former synergy carries higher order information, which, in the case of finite $H_1H_2\cdots H_n$, results in our various formulas for it.

Permutative clans and intersection formulas

We next generalize the above results for permutative sects to permutative clans. Here:

Definition 7.11. A clan $a_1H_1a_2H_2\cdots a_nH_n$ of a group G is *permutative* if $H_iH_j = H_jH_i$ and $a_iH_j = H_ja_i$ for any $i, j \in \{1, 2, \ldots, n\}$.

Whenever we need to distinguish this concept and the permutativity of a sect, we say *clan-permutative* and *sect-permutative*. The following is clear from the definition of clan-permutativity:

Lemma 7.12. *If a clan $a_1H_1a_2H_2\cdots a_nH_n$ is clan-permutative, then the sect $H_1H_2\cdots H_n$ is sect-permutative; so it is a subgroup by Lemma 7.2.*

Note next the following:

Lemma 7.13. (1) *If a clan $a_1H_1a_2H_2\cdots a_nH_n$ of a group G is permutative, then*

$$a_1H_1a_2H_2\cdots a_nH_n = a_1a_2\cdots a_nH_1H_2\cdots H_n, \tag{7.7}$$

and the right hand side is a coset. Moreover for any

$$\{i_1, i_2, \ldots, i_k\}, \{j_1, j_2, \ldots, j_k\} \subset \{1, 2, \ldots, n\},$$

$a_{i_1}H_{j_1}a_{i_2}H_{j_2}\cdots a_{i_k}H_{j_k} = a_{i_1}a_{i_2}\cdots a_{i_k}H_{j_1}H_{j_2}\cdots H_{j_k}$, *and the right hand side is a coset.*

(2) *In* (1), *if $a_1H_1a_2H_2\cdots a_nH_n$ is finite, then*

$$|a_1H_1a_2H_2\cdots a_nH_n| = |H_1H_2\cdots H_n|, \tag{7.8}$$

and moreover $|a_{i_1}H_{j_1}a_{i_2}H_{j_2}\cdots a_{i_k}H_{j_k}| = |H_{j_1}H_{j_2}\cdots H_{j_k}|$.

Proof. (1): First by the clan-permutativity, (7.7) holds. Its right hand side is a coset as $a_1a_2\cdots a_n \in G$ and $H_1H_2\cdots H_n$ is a subgroup of G (by Lemma 7.12). This confirms the first assertion. The second assertion follows from the first one applied to the permutative clan $a_{i_1}H_{j_1}a_{i_2}H_{j_2}\cdots a_{i_k}H_{j_k}$.

(2): Note first that for any finite coset aH, we have $|aH| = |H|$. Thus the assertion is immediate from (1). □

For later use, we simplify the first assertion of Lemma 7.13 (1) as follows:

Corollary 7.14. *Any permutative clan is equal to a coset.*

Let $a_1H_1a_2H_2\cdots a_nH_n$ $(n \geq 3)$ be a permutative clan of a group G. We partition it into three parts as follows:

$$\underbrace{a_1H_1a_2H_2\cdots a_pH_p}\,\underbrace{a_{p+1}H_{p+1}a_{p+2}H_{p+2}\cdots a_qH_q}\,\underbrace{a_{q+1}H_{q+1}a_{q+2}H_{q+2}\cdots a_nH_n}. \tag{7.9}$$

Using the clan-permutativity, we rewrite this. Move factors $H_1, H_2, \dots, H_p$ to the left and factors $H_{q+1}, H_{q+2}, \dots, H_{n-1}$ to the right (next to H_n), and partition the resulting clan into three parts H, A and K as follows:

$$\begin{aligned} &a_1H_1a_2H_2\cdots a_nH_n \\ &= \underbrace{H_1H_2\cdots H_p}_{H}\underbrace{a_1a_2\cdots a_pa_{p+1}H_{p+1}a_{p+2}H_{p+2}\cdots a_qH_qa_{q+1}a_{q+2}\cdots a_n}_{A} \\ &\quad\times \underbrace{H_{q+1}H_{q+2}\cdots H_{n-1}H_n}_{K}. \end{aligned} \tag{7.10}$$

Note that H and K are subgroups of G (Lemma 7.2), so HAK gives a guild expression of $a_1H_1a_2H_2\cdots a_nH_n$. Suppose now that HAK is finite. Then applying Theorem 6.5 to HAK yields the following, where we set $x := a_1a_2\cdots a_{p+1}$ and $y := a_{q+1}a_{q+2}\cdots a_n$:

$$\begin{aligned} &|H_1H_2\cdots H_p||xH_{p+1}a_{p+2}H_{p+2}\cdots a_qH_qy||H_{q+1}H_{q+2}\cdots H_n| \\ &\qquad = \sum_{i\in I} m^{(i)}|(H_1H_2\cdots H_p)\cap(H_{q+1}H_{q+2}\cdots H_n)^{a^{(i)}}|. \end{aligned} \tag{7.11}$$

Here (see Remark 6.12),

$$|xH_{p+1}a_{p+2}H_{p+2}\cdots a_qH_qy| = |H_{p+1}H_{p+2}\cdots H_q|. \tag{7.12}$$

We thus obtain the following:

Theorem 7.15 ("Partitioned" intersection formula for clans).
If $a_1H_1a_2H_2\cdots a_nH_n$ $(n \geq 3)$ is a finite permutative clan, then for any partition of it into three parts of the form (7.9)*, letting I, $m^{(i)}$ and $a^{(i)}$ be as in Theorem 6.5 applied to HAK in* (7.10)*, we have*

$$\begin{aligned} &|H_1H_2\cdots H_p||H_{p+1}H_{p+2}\cdots H_q||H_{q+1}H_{q+2}\cdots H_n| \\ &\qquad = \sum_{i\in I} m^{(i)}|(H_1H_2\cdots H_p)\cap(H_{q+1}H_{q+2}\cdots H_n)^{a^{(i)}}|. \end{aligned} \tag{7.13}$$

We shall consider clans of Dedekind groups. Note first the following:

Proposition 7.16. *Let G be a Dedekind group. Then the following hold:*

(1) *Any sect of G is permutative; so it is equal to a subgroup of G by Lemma 7.2.*

(2) *Any clan of G is permutative; so it is equal to a coset of G by Lemma 7.13 (1).*

Proof. (1): See Lemma 7.6.

(2): Let $a_1H_1a_2H_2\cdots a_nH_n$ be a clan of G. Then by the Dedekindness of G, $a_iH_j = H_ja_i$ for any $i, j \in \{1, 2, \dots, n\}$, and moreover by Lemma 7.6, $H_iH_j = H_jH_i$ for any $i, j \in \{1, 2, \dots, n\}$. Hence $a_1H_1a_2H_2\cdots a_nH_n$ is permutative. □

For a Dedekind group, any clan $a_1H_1a_2H_2\cdots a_nH_n$ is permutative (Proposition 7.16 (2)), and so, in the case of finite $a_1H_1a_2H_2\cdots a_nH_n$, the formula (7.13) is applicable to it; moreover therein $a^{(i)}$ is unnecessary, because $(H_{q+1}H_{q+2}\cdots H_n)^{a^{(i)}} = H_{q+1}H_{q+2}\cdots H_n$ (see Claim 7.7). We summarize these results as follows:

Corollary 7.17. *Let $a_1H_1a_2H_2\cdots a_nH_n$ $(n \geq 3)$ be a finite clan of a Dedekind group. Then for any partition of it into three parts of the form* (7.9)*, letting I and $m^{(i)}$ be as in Theorem 6.5 applied to HAK in* (7.10)*, we have*

$$\begin{aligned}|H_1H_2\cdots H_p||H_{p+1}H_{p+2}\cdots H_q||H_{q+1}H_{q+2}\cdots H_n|\\ = \sum_{i\in I} m^{(i)}|(H_1H_2\cdots H_p)\cap(H_{q+1}H_{q+2}\cdots H_n)|.\end{aligned}\tag{7.14}$$

Observation 7.18. Proposition 7.16 (2) implies that a Dedekind group contains essentially *no* higher order clans. In this sense, any Dedekind group is *of first order in clans*. This suggests that clans of a Dedekind group are uninteresting from the viewpoint of higher group theory — but actually *not*: *In fact, a finite clan of a Dedekind group admits various combinatorial formulas as in* (7.14) *depending on its partition.*

Remark 7.19. *Even if a clan $a_1H_1a_2H_2\cdots a_nH_n$ $(n \geq 2)$ of a group G is equal to a coset bK of G, their synergies are distinct*:

$$\begin{cases} a_1H_1 \times a_2H_2 \times \cdots \times a_nH_n \to a_1H_1a_2H_2\cdots a_nH_n, \\ bK \to bK \quad \text{(the identity map)}. \end{cases}\tag{7.15}$$

The former synergy carries higher order information, which, in the case of finite $a_1H_1a_2H_2\cdots a_nH_n$, results in our various formulas for it.

8 Boosters and their local bijectivity

For a finite sect $H_1H_2\cdots H_n$, the following formula holds (see (6.15)):

$$|H_1||H_2H_3\cdots H_{n-1}||H_n| = \sum_{i\in I} m^{(i)}|H_1 \cap H_n^{a^{(i)}}|.\tag{8.1}$$

For $n = 2, 3$, the left hand side is $|H_1||H_2|$ (by convention) and $|H_1||H_2||H_3|$. We then ask:

For $n \geq 4$, does there exist a formula for $|H_1||H_2|\cdots|H_n|$?

The answer is YES, but its deduction is beyond the scope of guilds, and requires introducing more concepts — for instance, in addition to the guild synergy, a further "upper structure" (lying above the guild synergy).

First recall that in deducing (8.1), we used the guild expression HAK with $H = H_1$, $A = H_2H_3 \cdots H_{n-1}$ and $K = H_n$, but here we use the notation $H_1H_2 \ldots H_n$ itself. Then two basic ingredients in deducing (8.1) are as follows:

- the guild synergy:

$$\begin{aligned} \psi : \ H_1 \times H_2H_3 \cdots H_{n-1} \times H_n &\longrightarrow H_1H_2H_3 \cdots H_{n-1}H_n, \\ (h_1, h_2h_3 \cdots h_{n-1}, h_n) &\longmapsto h_1h_2h_3 \cdots h_{n-1}h_n, \end{aligned} \tag{8.2}$$

- the satellite decomposition:

$$H_1 \times H_2H_3 \cdots H_{n-1} \times H_n = \coprod_{b \in H_2H_3 \cdots H_{n-1}} H_1 \times \{b\} \times H_n, \tag{8.3}$$

where each $H_1 \times \{b\} \times H_n$ is a satellite. These two ingredients are *not* enough for deducing a formula for $|H_1||H_2| \cdots |H_n|$. In addition, we need the following "upper structure" on $H_1 \times H_2H_3 \cdots H_{n-1} \times H_n$ and the decomposition of $H_1 \times H_2 \times \cdots \times H_n$:

- the *booster*:

$$\begin{aligned} \varphi : H_1 \times H_2 \times \cdots \times H_n &\longrightarrow H_1 \times H_2H_3 \cdots H_{n-1} \times H_n, \\ (h_1, h_2, h_3, \ldots, h_{n-1}, h_n) &\longmapsto (h_1, h_2h_3 \cdots h_{n-1}, h_n), \end{aligned} \tag{8.4}$$

- the *planet decomposition*:

$$H_1 \times H_2 \times \cdots \times H_n = \coprod_{(c_2, c_3, \ldots, c_{n-1}) \in H_2 \times H_3 \times \cdots \times H_{n-1}} H_1 \times \{(c_2, c_3, \ldots, c_{n-1})\} \times H_n, \tag{8.5}$$

where each $H_1 \times \{(c_2, c_3, \ldots, c_{n-1})\} \times H_n$ is called a *planet*.

Observation 8.1. Using the booster φ, we may express the sect synergy $\pi : H_1 \times H_2 \times \cdots \times H_n \to H_1H_2 \cdots H_n$ as a composition $\pi = \psi \circ \varphi$:

$$\pi : \ H_1 \times H_2 \times \cdots \times H_n \xrightarrow{\varphi} H_1 \times H_2H_3 \cdots H_{n-1} \times H_n \xrightarrow{\psi} H_1H_2 \cdots H_n. \tag{8.6}$$

We may thus say that the booster φ *lies above* the guild synergy ψ.

Remark 8.2. The booster φ is considered as a 'partial synergy' contracting $H_2 \times H_3 \times \cdots \times H_{n-1}$ to $H_2H_3 \cdots H_{n-1}$.

Definition 8.3. For a satellite $S := H_1 \times \{b\} \times H_n$, we say that a planet $P := H \times \{(c_2, c_3, \ldots, c_{n-1})\} \times H_n$ is *over* S if the booster φ maps P to S, i.e. $\varphi(P) = S$; this is the case exactly when $c_2c_3 \cdots c_{n-1} = b$. The restriction φ_P of φ to P, that is, $\varphi_P : P \to S$ is called a *planet booster* of S.

The following holds:

Lemma 8.4. *For any planet $P := H \times \{(c_2, c_3, \dots, c_{n-1})\} \times H_n$, the planet booster*

$$\begin{aligned}\varphi_P :\ & H_1 \times \{(c_2, c_3, \dots, c_{n-1})\} \times H_n \longrightarrow H_1 \times \{b\} \times H_n, \\ & \big(h_1, (c_2, c_3, \dots, c_{n-1}), h_n\big) \longmapsto (h_1, c_2c_3 \cdots c_{n-1}, h_n) \ \Big(= (h_1, b, h_n)\Big)\end{aligned}$$

is bijective.

Proof. The surjectivity of φ_P: It suffices to show that for any $(h_1, b, h_n) \in H_1 \times \{b\} \times H_n$, there exists an element of $H_1 \times \{(c_2, c_3, \dots, c_{n-1})\} \times H_n$ that is mapped to (h_1, b, h_n) under φ_P. This is clear; just take

$$\big(h_1, (c_2, c_3, \dots, c_{n-1}), h_n\big) \in H_1 \times \{(c_2, c_3, \dots, c_{n-1})\} \times H_n. \tag{8.7}$$

The injectivity of φ_P: For two elements

$$\big(h_1, (c_2, c_3, \dots, c_{n-1}), h_n\big) \quad \text{and} \quad \big(k_1, (c_2, c_3, \dots, c_{n-1}), k_n\big)$$

of $H_1 \times \{(c_2, c_3, \dots, c_{n-1})\} \times H_n$, we have

$$\begin{aligned}&\varphi_P\big(h_1, (c_2, c_3, \dots, c_{n-1}), h_n\big) = \varphi_P\big(k_1, (c_2, c_3, \dots, c_{n-1}), k_n\big) \\ &\quad\iff (h_1, c_2c_3 \cdots c_{n-1}, h_n) = (k_1, c_2c_3 \cdots c_{n-1}, k_n) \\ &\quad\iff h_1 = k_1,\ h_n = k_n \\ &\quad\iff \big(h_1, (c_2, c_3, \dots, c_{n-1}), h_n\big) = \big(k_1, (c_2, c_3, \dots, c_{n-1}), k_n\big).\end{aligned}$$

This confirms the injectivity of φ_P. □

Remark 8.5. By Lemma 8.4, the booster φ is *locally* bijective — bijective on each planet P. Thus φ is considered as an analog of an étale map in topology, which is a local homeomorphism.

In contrast to the bijectivity of φ_P, the restriction ψ_S of the guild synergy ψ to a satellite $S = H_1 \times \{b\} \times H_n$, that is,

$$\psi_S :\ H_1 \times \{b\} \times H_n \to H_1 b H_n \tag{8.8}$$

is surjective but not necessarily bijective.

Remark 8.6. Note that (8.8) is a double coset telergy. Thus by Corollary 2.17, all fibers $\psi_S^{-1}(x)$ $(x \in H_1 b H_n)$ are bijective to a subgroup $H_1 \cap H_n^b$, where $H_n^b := bH_nb^{-1}$. This is an analog of a *submersion* in differential topology, in that all fibers of a surjective submersion $f : M \to N$ between smooth manifolds M and N with M compact are diffeomorphic to a submanifold of M (Ehresmann's fibration theorem).

For brevity, denote an element $(c_2, c_3, \dots, c_{n-1})$ of $H_2 \times H_3 \times \cdots \times H_{n-1}$ by $\vec{c}$, and accordingly the planet decomposition (8.5) by

$$H_1 \times H_2 \times \cdots \times H_n = \coprod_{\vec{c} \in H_2 \times H_3 \times \cdots \times H_{n-1}} H_1 \times \{\vec{c}\} \times H_n. \tag{8.9}$$

Let $H_1 H_2 \cdots H_n = \coprod_{i \in I} H_1 a^{(i)} H_n$, $(a^{(i)} \in H_2 H_3 \cdots H_{n-1})$ be the double coset decomposition.

Convention 8.7. The guild synergy ψ in (8.2), the booster φ in (8.4) and the sect synergy $\pi = \psi \circ \varphi$ are regarded as maps between disjoint unions in terms of the planet decomposition, the satellite decomposition and the double coset decomposition:

$$\psi : \coprod_{b \in H_2 H_3 \cdots H_{n-1}} H_1 \times \{b\} \times H_n \longrightarrow \coprod_{i \in I} H_1 a^{(i)} H_n, \tag{8.10}$$

$$\varphi : \coprod_{\vec{c} \in H_2 \times H_3 \times \cdots \times H_{n-1}} H_1 \times \{\vec{c}\} \times H_n \longrightarrow \coprod_{b \in H_2 H_3 \cdots H_{n-1}} H_1 \times \{b\} \times H_n, \tag{8.11}$$

$$\pi : \coprod_{\vec{c} \in H_2 \times H_3 \times \cdots \times H_{n-1}} H_1 \times \{\vec{c}\} \times H_n \longrightarrow \coprod_{i \in I} H_1 a^{(i)} H_n. \tag{8.12}$$

Lemma 8.8. *Let*

$$\varphi : \coprod_{\vec{c} \in H_2 \times H_3 \times \cdots \times H_{n-1}} H_1 \times \{\vec{c}\} \times H_n \longrightarrow \coprod_{b \in H_2 H_3 \cdots H_{n-1}} H_1 \times \{b\} \times H_n.$$

be the booster (8.11)*. For a satellite $S := H_1 \times \{b\} \times H_n$, denote by p_S the cardinality of the set of planets over S. Then the following hold:*

(1) *The preimage $\varphi^{-1}(S)$ is bijective to a disjoint union of p_S copies of S.*

(2) *For any subset T of S, its preimage $\varphi^{-1}(T)$ is bijective to a disjoint union of p_S copies of T.*

Proof. (1): We write $\varphi^{-1}(S)$ as a disjoint union:

$$\varphi^{-1}(S) = \coprod_{k \in K_S} P^{(k)}, \tag{8.13}$$

where $P^{(k)}$ $(k \in K_S)$ are the planets over S; the cardinality of K_S is by definition p_S. Now denote the restriction of φ to $P^{(k)}$ by $\varphi^{(k)} : P^{(k)} \to S$ (which is the planet booster of $P^{(k)}$). This is a bijection (Lemma 8.4), so $P^{(k)} \cong S$ (bijective), thus (8.13) implies that $\varphi^{-1}(S)$ is bijective to a disjoint union of p_S copies of S.

(2): We keep the notation in the proof of (1). Setting $Q^{(k)} := {\varphi^{(k)}}^{-1}(T)$, then we have

$$\varphi^{-1}(T) = \coprod_{k \in K_S} Q^{(k)} \quad \text{(the 'restriction' of (8.13))}. \tag{8.14}$$

Since $\varphi^{(k)} : P^{(k)} \to S$ is a bijection (Lemma 8.4), its restriction $\varphi^{(k)} : Q^{(k)} \to T$ is also a bijection, so $Q^{(k)} \cong T$ (bijective). Then (8.14) implies that $\varphi^{-1}(T)$ is bijective to a disjoint union of p_S copies of T. $\square$

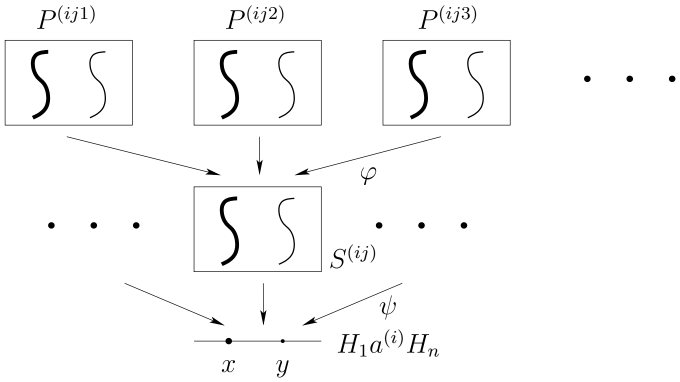

Figure 8.1: $S^{(ij)}$ is a satellite over a double coset $H_1 a^{(i)} H_n$, and each $P^{(ijk)}$ is a planet over $S^{(ij)}$. Curves illustrated inside $S^{(ij)}$ (resp. $P^{(ijk)}$) are fibers over $x, y \in H_1 a^{(i)} H_n$ under ψ (resp. $\varphi \circ \psi\,(= \pi)$). While φ is an analog of an étale map (Remark 8.5), ψ is an analog of a submersion (Remark 8.6).

Remark 8.9. The booster φ is an analog of an étale map (Remark 8.5) and the guild synergy ψ is an analog of a submersion (Remark 8.6), as illustrated in Figure 8.1. Then the factorization (8.6) of the sect synergy into the booster and the guild synergy is comparable to the *Stein factorization* in algebraic geometry — the factorization of a proper surjective map into a finite covering and a proper surjective map with connected fibers.

9 Factorized intersection formula for sects

For simplicity, we adopt the following:

Notation 9.1. Write a sect $H_1 H_2 \cdots H_n$ $(n \geq 2)$ as HAK (*guild expression*) with

$$\begin{aligned} & H := H_1, \quad A := H_2 H_3 \cdots H_{n-1}, \quad K := H_n, \\ & \text{and } \ \widetilde{A} := H_2 \times H_3 \times \cdots \times H_{n-1}. \\ & \textit{Convention}\text{: If } n = 2, \text{ then } A = \widetilde{A} = \{e\}. \end{aligned} \tag{9.1}$$

Then

$$HAK = H_1 H_2 \cdots H_n, \qquad H \times A \times K = H_1 \times H_2 H_3 \cdots H_{n-1} \times H_n,$$
$$H \times \widetilde{A} \times K = H_1 \times H_2 \times H_3 \times \cdots \times H_{n-1} \times H_n.$$

Accordingly

$$\begin{cases} \varphi : H \times \widetilde{A} \times K \longrightarrow H \times A \times K: \quad \text{the booster,} \\ \psi : H \times A \times K \longrightarrow HAK: \quad \text{the guild synergy,} \\ \pi = \psi \circ \varphi : H \times \widetilde{A} \times K \longrightarrow HAK: \quad \text{the sect synergy.} \end{cases} \tag{9.2}$$

Note that by Corollary 2.3,

$$HAK\,(= H_1 H_2 \cdots H_n) \text{ is finite } \iff H_1, H_2, \ldots, H_n \text{ are finite.} \tag{9.3}$$

We claim that

$$HAK\,(= H_1 H_2 \cdots H_n) \text{ is finite } \iff H, A \text{ and } K \text{ are finite.} \tag{9.4}$$

In fact the finiteness of $HAK\,(= H_1 H_2 \cdots H_n)$ is equivalent to the finiteness of $H_1\,(= H), H_2, \ldots, H_n\,(= K)$ by (9.3). Here the finiteness of $H_2, H_3, \ldots, H_{n-1}$ is equivalent to the finiteness of $H_2 H_3 \cdots H_{n-1}\,(= A)$ by Corollary 2.3. Hence (9.4) holds. Note moreover that the finiteness of $H_2, H_3, \ldots, H_{n-1}$ is also equivalent to the finiteness of $H_2 \times H_3 \times \cdots \times H_{n-1}\,(= \widetilde{A})$. Namely the following holds:

$$\begin{aligned} A \text{ is finite } &\iff H_2, H_3, \ldots, H_{n-1} \text{ are finite} \\ &\iff \widetilde{A} \text{ is finite.} \end{aligned} \tag{9.5}$$

Besides, we claim that

$$HAK \text{ is finite } \iff H \times A \times K \text{ is finite.} \tag{9.6}$$

In fact the finiteness of HAK is equivalent to the finiteness of H, A and K by (9.4), which is further equivalent to the finiteness of $H \times A \times K$.

We summarize the above equivalences as follows:

Proposition 9.2. *For a sect HAK with $H := H_1$, $A := H_2 H_3 \cdots H_{n-1}$, $K := H_n$ and $\widetilde{A} := H_2 \times H_3 \times \cdots \times H_{n-1}$, the following equivalences hold:*

$$\begin{aligned} HAK \textit{ is finite } &\iff H,\ A \textit{ and } K \textit{ are finite} \\ &\iff H,\ \widetilde{A} \textit{ and } K \textit{ are finite} \\ &\iff H \times \widetilde{A} \times K \textit{ is finite} \\ &\iff H \times A \times K \textit{ is finite.} \end{aligned} \tag{9.7}$$

Now we take the double coset decomposition of a sect HAK:

$$HAK = \coprod_{i \in I} H a^{(i)} K, \quad (a^{(i)} \in A). \tag{9.8}$$

For each double coset $Ha^{(i)}K$, we write $\psi^{-1}(Ha^{(i)}K)$ as a disjoint union:

$$\psi^{-1}(Ha^{(i)}K) = \coprod_{j \in J^{(i)}} S^{(ij)}, \tag{9.9}$$

where $S^{(ij)}$ ($j \in J^{(i)}$) are the satellites over $Ha^{(i)}K$. Let $s^{(i)}$ denote the number of satellites over $Ha^{(i)}K$ (or the cardinality of $J^{(i)}$ if it is infinite), i.e. $s^{(i)}$ is the *satellite index* of $Ha^{(i)}K$. By convention, for any $x \in Ha^{(i)}K$, this number is also called the satellite index of x. Next for each satellite $S^{(ij)}$ over $Ha^{(i)}K$, we write $\varphi^{-1}(S^{(ij)})$ as a disjoint union:

$$\varphi^{-1}(S^{(ij)}) = \coprod_{k \in K^{(ij)}} P^{(ijk)}, \tag{9.10}$$

where $P^{(ijk)}$ ($k \in K^{(ij)}$) are the planets over $S^{(ij)}$.

Definition 9.3. (1) For each satellite $S^{(ij)}$, the number of planets $P^{(ijk)}$ ($k \in K^{(ij)}$) over $S^{(ij)}$ (or the cardinality of $K^{(ij)}$ if it is infinite) is the *planet index of the satellite* $S^{(ij)}$, and denoted by $p^{(ij)}$.

(2) For each double coset $Ha^{(i)}K$, the number of planets over $Ha^{(i)}K$ (or the cardinality of the set of planets over $Ha^{(i)}K$ if it is infinite) is the the *planet index of the double coset* $Ha^{(i)}K$, and denoted by $q^{(i)}$.

Before proceeding, we recall a basic fact on cardinalities of sets.

Fact 9.4. For a set A, let $\mathrm{Card}(A)$ denote its cardinality. Then for a disjoint union $S = \coprod_{\lambda \in \Lambda} A_\lambda$, the cardinality of S is the *cardinal sum* ([7] p.38) of the cardinalities of A_λ ($\lambda \in \Lambda$):

$$\mathrm{Card}(S) = \sum_{\lambda \in \Lambda} \mathrm{Card}(A_\lambda) \quad \text{(cardinal sum)}. \tag{9.11}$$

Here if A_λ ($\lambda \in \Lambda$) and Λ are finite sets, then the right hand side is just a finite sum of natural numbers.

We return to our context. Fix a double coset $Ha^{(i)}K$ and let $S^{(ij)}$ ($j \in J^{(i)}$) be the satellites over it. For each satellite $S^{(ij)}$, let $\mathcal{P}^{(ij)}$ be the *set* of planets over it. Then:

Claim 9.5. *The set $\mathcal{Q}^{(i)}$ of planets over $Ha^{(i)}K$ is a disjoint union of $\mathcal{P}^{(ij)}$ ($j \in J^{(i)}$):*

$$\mathcal{Q}^{(i)} = \coprod_{j \in J^{(i)}} \mathcal{P}^{(ij)}. \tag{9.12}$$

Proof. Note first that

$$\begin{aligned}
(\text{the planets over } Ha^{(i)}K) &= (\psi \circ \varphi)^{-1}(Ha^{(i)}K) \quad \text{by definition} \\
&= \varphi^{-1}\big(\psi^{-1}(Ha^{(i)}K)\big) \\
&= \varphi^{-1}\Big(\coprod_{j \in J^{(i)}} S^{(ij)}\Big) = \coprod_{j \in J^{(i)}} \varphi^{-1}(S^{(ij)}) \\
&= \coprod_{j \in J^{(i)}} (\text{the planets over } S^{(ij)}).
\end{aligned}$$

Hence

$$\text{(the planets over } Ha^{(i)}K) = \coprod_{j\in J^{(i)}} \text{(the planets over } S^{(ij)}). \tag{9.13}$$

Here noting that planets are disjoint, we obtain $\mathcal{Q}^{(i)} = \coprod_{j\in J^{(i)}} \mathcal{P}^{(ij)}$. $\square$

From (9.12), $\mathrm{Card}(\mathcal{Q}^{(i)}) = \sum_{j\in J^{(i)}} \mathrm{Card}(\mathcal{P}^{(ij)})$. Here by definition,

$$\begin{cases} \mathrm{Card}(\mathcal{Q}^{(i)}) = q^{(i)} \text{ (the planet index of } Ha^{(i)}K), \\ \mathrm{Card}(\mathcal{P}^{(ij)}) = p^{(ij)} \text{ (the planet index of } S^{(ij)}). \end{cases}$$

Accordingly $q^{(i)} = \sum\limits_{j\in J^{(i)}} p^{(ij)}$. We formalize this as follows:

Lemma 9.6. *For a sect HAK with $H := H_1$, $A := H_2H_3\cdots H_{n-1}$ and $K := H_n$, take its double coset decomposition $HAK = \coprod_{i\in I} Ha^{(i)}K$ and for each double coset $Ha^{(i)}K$, let $S^{(ij)}$ $(j \in J^{(i)})$ be the satellites over it. Then the planet index $q^{(i)}$ of $Ha^{(i)}K$ and the planet indices $p^{(ij)}$ of $S^{(ij)}$ $(j \in J^{(i)})$ are related as follows:*

$$q^{(i)} = \sum_{j\in J^{(i)}} p^{(ij)} \quad \textit{(cardinal sum; see Fact 9.4)}. \tag{9.14}$$

For the guild synergy $\psi : H \times A \times K \to HAK$, we denote its restriction to a satellite $S^{(ij)}$ by $\psi^{(ij)}$, which maps $S^{(ij)}$ to $Ha^{(i)}K$:

$$\psi^{(ij)} : S^{(ij)} \longrightarrow Ha^{(i)}K. \tag{9.15}$$

By Theorem 5.5 (2) (therein $S^{(ij)}$ is denoted by $H \times \{b^{(ij)}\} \times K$), the following holds: For $x \in Ha^{(i)}K$,

$$\psi^{-1}(x) = \coprod_{j\in J^{(i)}} {\psi^{(ij)}}^{-1}(x) \ \text{ with } {\psi^{(ij)}}^{-1}(x) \cong H \cap K^{a^{(i)}} \text{ (bijective as sets)}. \tag{9.16}$$

So irrespective of $x \in Ha^{(i)}K$, its fiber $\psi^{-1}(x)$ is, as a set, bijective to the disjoint union of $s^{(i)}$ copies of $H \cap K^{a^{(i)}}$. For simplicity, we set $F^{(ij)} := {\psi^{(ij)}}^{-1}(x)$. Then (9.16) is rewritten as

$$\psi^{-1}(x) = \coprod_{j\in J^{(i)}} F^{(ij)} \ \text{ with } F^{(ij)} \cong H \cap K^{a^{(i)}} \text{ (bijective as sets)}. \tag{9.17}$$

Convention 9.7. In what follows, unless otherwise stated, “bijective” means “bijective as sets”.

Note the following:

Claim 9.8. *For $F^{(ij)}$ in* (9.17)*, its preimage $\varphi^{-1}(F^{(ij)})$ under the booster φ is bijective to a disjoint union of $p^{(ij)}$ copies of $H \cap K^{a^{(i)}}$.*

Proof. First $\varphi^{-1}(F^{(ij)})$ is bijective to a disjoint union of $p^{(ij)}$ copies of $F^{(ij)}$ by Lemma 8.8 (2) (note that p_S therein is now $p^{(ij)}$). Here $F^{(ij)} \cong H \cap K^{a^{(i)}}$ (see (9.17)), thus the assertion follows. $\square$

We keep the above notation and let $\pi = \psi \circ \varphi : H \times \widetilde{A} \times K \to HAK$ be the sect synergy of HAK (see (9.2)). Any fiber $\pi^{-1}(x)$ ($x \in Ha^{(i)}K$) is rewritten as follows:

$$\begin{aligned}
\pi^{-1}(x) = (\psi \circ \varphi)^{-1}(x) = \varphi^{-1}\Big(\psi^{-1}(x)\Big) &= \varphi^{-1}\Big(\coprod_{j \in J^{(i)}} F^{(ij)}\Big) \quad \text{by (9.17)} \\
&= \coprod_{j \in J^{(i)}} \varphi^{-1}(F^{(ij)}) \\
&\cong \coprod_{j \in J^{(i)}} \Big(\text{disjoint union of } p^{(ij)} \text{ copies of } H \cap K^{a^{(i)}}\Big) \quad \text{by Claim 9.8} \\
&= \text{disjoint union of } \sum_{j \in J^{(i)}} p^{(ij)} \text{ copies of } H \cap K^{a^{(i)}}.
\end{aligned}$$

Here $\sum_{j \in J^{(i)}} p^{(ij)} = q^{(i)}$ (see (9.14)), so $\pi^{-1}(x)$ is bijective to the disjoint union of $q^{(i)}$ copies of $H \cap K^{a^{(i)}}$. We formalize this as follows:

Theorem 9.9 (Fiber theorem for sect synergies). *Write a sect as HAK (as in Notation 9.1). Let $HAK = \coprod_{i \in I} Ha^{(i)}K$ be its double coset decomposition and let $\pi : H \times \widetilde{A} \times K \to HAK$ be its sect synergy. For each $Ha^{(i)}K$, let $q^{(i)}$ be its planet index. Then for any $x \in Ha^{(i)}K$, the fiber $\pi^{-1}(x)$ is, as a set, bijective to a disjoint union of $q^{(i)}$ copies of $H \cap K^{a^{(i)}}$; consequently in the case of finite HAK,*

$$|\pi^{-1}(x)| = q^{(i)} |H \cap K^{a^{(i)}}| \quad \text{(fiber order formula)}. \tag{9.18}$$

We keep the notation of Theorem 9.9, and focus on finite HAK. We claim that

$$|\pi^{-1}(Ha^{(i)}K)| = q^{(i)} |H \cap K^{a^{(i)}}| |Ha^{(i)}K|. \tag{9.19}$$

Indeed from $\pi^{-1}(Ha^{(i)}K) = \coprod_{x \in Ha^{(i)}K} \pi^{-1}(x)$, we have

$$\begin{aligned}
|\pi^{-1}(Ha^{(i)}K)| &= \sum_{x \in Ha^{(i)}K} |\pi^{-1}(x)| \\
&= \sum_{x \in Ha^{(i)}K} q^{(i)} |H \cap K^{a^{(i)}}| \qquad \text{by (9.18)} \\
&= q^{(i)} |H \cap K^{a^{(i)}}| |Ha^{(i)}K|.
\end{aligned}$$

Definition 9.10. We set $\mu^{(i)} := q^{(i)}|Ha^{(i)}K|$, which is called the *absolute multiplicity* of $Ha^{(i)}K$. cf. the multiplicity $m^{(i)} := s^{(i)}|Ha^{(i)}K|$ of $Ha^{(i)}K$ (Definition 6.3).

Using the absolute multiplicity $\mu^{(i)}$, we rewrite (9.19) to obtain the following:

Corollary 9.11. *In the notation of Theorem 9.9, if HAK is finite, then*

$$|\pi^{-1}(Ha^{(i)}K)| = \mu^{(i)}|H \cap K^{a^{(i)}}|. \tag{9.20}$$

Under the above preparation, for a finite sect $H_1H_2\cdots H_n$ we deduce a "factorized" intersection formula to express $|H_1||H_2|\cdots|H_n|$. As usual, we write $H_1H_2\cdots H_n$ as HAK with $H := H_1$, $A := H_2H_3\cdots H_{n-1}$, $K := H_n$, and $\widetilde{A} := H_2 \times H_3 \times \cdots \times H_{n-1}$. Take the double coset decomposition $HAK = \coprod_{i\in I} Ha^{(i)}K$, $(a^{(i)} \in A)$. Pulling back this via the sect synergy $\pi : H\times\widetilde{A}\times K \to HAK$ gives

$$\pi^{-1}(HAK) = \coprod_{i\in I} \pi^{-1}(Ha^{(i)}K). \tag{9.21}$$

Here the left hand side is $H \times \widetilde{A} \times K$ (as π is surjective), so

$$H \times \widetilde{A} \times K = \coprod_{i\in I} \pi^{-1}(Ha^{(i)}K). \tag{9.22}$$

Here since HAK is finite, H, $\widetilde{A}$ and K are finite (Proposition 9.2). Then from (9.22), the following holds:

$$|H \times \widetilde{A} \times K| = \sum_{i\in I} |\pi^{-1}(Ha^{(i)}K)|. \tag{9.23}$$

Here $|H \times \widetilde{A} \times K| = |H||\widetilde{A}||K|$ and $|\pi^{-1}(Ha^{(i)}K)| = \mu^{(i)}|H \cap K^{a^{(i)}}|$ (Corollary 9.11). Thus

$$|H||\widetilde{A}||K| = \sum_{i\in I} \mu^{(i)}|H \cap K^{a^{(i)}}|. \tag{9.24}$$

Substituting $H := H_1$, $\widetilde{A} := H_2 \times H_3 \times \cdots \times H_{n-1}$ and $K := H_n$ into this yields

$$|H_1||H_2 \times H_3 \times \cdots \times H_{n-1}||H_n| = \sum_{i\in I} \mu^{(i)}|H_1 \cap H_n^{a^{(i)}}|. \tag{9.25}$$

Here $|H_2 \times H_3 \times \cdots \times H_{n-1}| = |H_2||H_3|\cdots|H_{n-1}|$, so

$$|H_1||H_2|\cdots|H_n| = \sum_{i\in I} \mu^{(i)}|H_1 \cap H_n^{a^{(i)}}|. \tag{9.26}$$

We formalize this as follows:

Theorem 9.12 (Factorized intersection formula for sects).
For a finite sect $H_1H_2\cdots H_n$ ($n \geq 2$), let $H_1H_2\cdots H_n = \coprod_{i\in I} H_1a^{(i)}H_n$ be its double coset decomposition and for each $H_1a^{(i)}H_n$, let $\mu^{(i)}$ be its absolute multiplicity. Then

$$|H_1||H_2|\cdots|H_n| = \sum_{i\in I} \mu^{(i)}|H_1 \cap H_n^{a^{(i)}}|. \tag{9.27}$$

Here:

Lemma 9.13. *When $n = 2, 3$,* (9.27) *coincides with the following intersection formula in Example 6.10:*

$$|H_1||H_2H_3\cdots H_{n-1}||H_n| = \sum_{i\in I} m^{(i)}|H_1 \cap H_n^{a^{(i)}}|. \tag{9.28}$$

Proof. For $n = 2, 3$, the left hand side of (9.28) is $|H_1||H_2|$ (by convention) and $|H_1||H_2||H_3|$. In fact for $n = 2$, $A = \widetilde{A} = \{e\}$ (by convention) and for $n = 3$, $A = \widetilde{A} = H_2$; in particular for $n = 2, 3$, $H \times A \times K = H \times \widetilde{A} \times K$ and so $\mu^{(i)} = m^{(i)}$ (the absolute multiplicities coincide with the multiplicities). Thus two formulas (9.27) and (9.28) coincide for $n = 2, 3$. □

10 Permutations of factors of sects

A permutation of factors of a sect $H_1H_2\cdots H_n$ ($n \geq 2$) of a group G generally results in another sect: for a permutation $\sigma \in \mathfrak{S}_n$, in general

$$H_{\sigma(1)}H_{\sigma(2)}\cdots H_{\sigma(n)} \neq H_1H_2\cdots H_n \quad \text{(as subsets of } G). \tag{10.1}$$

In the finite case, it may even occur that

$$|H_{\sigma(1)}H_{\sigma(2)}\cdots H_{\sigma(n)}| \neq |H_1H_2\cdots H_n|. \tag{10.2}$$

However we always have

$$|H_{\sigma(1)}||H_{\sigma(2)}|\cdots|H_{\sigma(n)}| = |H_1||H_2|\cdots|H_n|. \tag{10.3}$$

Now we apply Theorem 9.12 to the sect $H_{\sigma(1)}H_{\sigma(2)}\cdots H_{\sigma(n)}$ and denote the resulting (9.27) as

$$|H_{\sigma(1)}||H_{\sigma(2)}|\cdots|H_{\sigma(n)}| = \sum_{i\in I_\sigma} \mu_\sigma^{(i)}|H_{\sigma(1)} \cap H_{\sigma(n)}^{a_\sigma^{(i)}}|. \tag{10.4}$$

Here the right hand side *a priori* depends on the permutation σ, but *a posteriori* not — in fact the left hand side does not depend on σ; see (10.3).

Permutative sects

For a sect $H_1H_2\cdots H_n$ ($n\geq 2$), take an arbitrary partition

$$\underbrace{H_1H_2\cdots H_{i_1}}\underbrace{H_{i_1+1}H_{i_1+2}\cdots H_{i_2}}\underbrace{H_{i_2+1}H_{i_2+2}\cdots H_{i_3}}\cdots\underbrace{H_{i_k+1}H_{i_k+2}\cdots H_n}. \tag{10.5}$$

Setting

$$\begin{aligned}\mathcal{H}_1 := H_1H_2\cdots H_{i_1},\ \mathcal{H}_2 := H_{i_1+1}H_{i_1+2}\cdots H_{i_2},\\ \ldots,\mathcal{H}_{k+1} := H_{i_k+1}H_{i_k+2}\cdots H_n,\end{aligned} \tag{10.6}$$

we write (10.5) as $\mathcal{H}_1\mathcal{H}_2\cdots\mathcal{H}_{k+1}$. Suppose now that $H_1H_2\cdots H_n$ is permutative. Then $\mathcal{H}_j$ ($j=1,2,\ldots,k+1$) are subgroups of G (Lemma 7.2), so $\mathcal{H}_1\mathcal{H}_2\cdots\mathcal{H}_{k+1}$ is a sect. In the case of finite $H_1H_2\cdots H_n$ ($=\mathcal{H}_1\mathcal{H}_2\cdots\mathcal{H}_{k+1}$), we may apply Theorem 9.12 to the finite sect $\mathcal{H}_1\mathcal{H}_2\cdots\mathcal{H}_{k+1}$ — say that

$$\begin{cases}\mathcal{H}_1\mathcal{H}_2\cdots\mathcal{H}_{k+1}=\coprod_{j\in J}\mathcal{H}_1b^{(j)}\mathcal{H}_{k+1}: & \text{double coset decomposition,}\\ \nu^{(j)}: \ \text{absolute multiplicity of } \mathcal{H}_1b^{(j)}\mathcal{H}_{k+1}, & \end{cases} \tag{10.7}$$

then we obtain the following:

Theorem 10.1 (Intersection formula for permutative sects). *Let $H_1H_2\cdots H_n$ ($n\geq 2$) be a finite permutative sect. Then for any partition*

$$\underbrace{H_1H_2\cdots H_{i_1}}_{\mathcal{H}_1}\underbrace{H_{i_1+1}H_{i_1+2}\cdots H_{i_2}}_{\mathcal{H}_2}\cdots\underbrace{H_{i_k+1}H_{i_k+2}\cdots H_n}_{\mathcal{H}_{k+1}},$$

letting J, $b^{(j)}$ and $\nu^{(j)}$ be as in (10.7)*, we have*

$$|\mathcal{H}_1||\mathcal{H}_2|\cdots|\mathcal{H}_{k+1}|=\sum_{j\in J}\nu^{(j)}|\mathcal{H}_1\cap\mathcal{H}_{k+1}^{b^{(j)}}|. \tag{10.8}$$

That is,

$$\begin{aligned}&|H_1H_2\cdots H_{i_1}||H_{i_1+1}H_{i_1+2}\cdots H_{i_2}|\cdots|H_{i_k+1}H_{i_k+2}\cdots H_n|\\ &\qquad=\sum_{j\in J}\nu^{(j)}|(H_1H_2\cdots H_{i_1})\cap(H_{i_k+1}H_{i_k+2}\cdots H_n)^{b^{(j)}}|.\end{aligned} \tag{10.9}$$

11 Factorized intersection formula for clans

For finite sects, we deduced the factorized intersection formula (9.27). In this section, more generally we deduce it for finite clans. Note that it suffices to derive it for finite *monic* clans, i.e. of the form $H_1a_2H_2\cdots a_nH_n$, because any clan $a_1H_1a_2H_2\cdots a_nH_n$ may be regarded as a monic clan $H_0a_1H_1a_2H_2\cdots a_nH_n$ with $H_0=\{e\}$ (the identity subgroup).

For a monic clan $H_1a_2H_2\cdots a_nH_n$ ($n\geq 2$), we introduce fundamental materials corresponding to those used in deducing (9.27):

Definition 11.1. (a) The *guild synergy* is given by

$$\begin{aligned}\psi :\ & H_1 \times a_2H_2a_3H_3 \cdots a_{n-1}H_{n-1}a_n \times H_n \to H_1a_2H_2a_3H_3 \cdots a_{n-1}H_{n-1}a_nH_n,\\ & (h_1, a_2h_2a_3h_3 \cdots a_{n-1}h_{n-1}a_n, h_n) \longmapsto h_1a_2h_2a_3h_3 \cdots a_{n-1}h_{n-1}a_nh_n.\end{aligned}$$

(b) The *booster* is given by

$$\begin{aligned}\varphi :\ & H_1 \times a_2H_2 \times \cdots \times a_{n-1}H_{n-1}a_n \times H_n\\ & \qquad \longrightarrow H_1 \times a_2H_2a_3H_3 \cdots a_{n-1}H_{n-1}a_n \times H_n,\\ & (h_1, a_2h_2, a_3h_3, \ldots, a_{n-1}h_{n-1}a_n, h_n) \longmapsto (h_1, a_2h_2a_3h_3 \cdots a_{n-1}h_{n-1}a_n, h_n).\end{aligned}$$

(c) The *clan synergy* (precisely speaking, the modified one in Convention 2.12) is given by

$$\begin{aligned}\pi :\ & H_1 \times a_2H_2 \times \cdots \times a_{n-1}H_{n-1}a_n \times H_n \longrightarrow H_1a_2H_2 \cdots a_{n-1}H_{n-1}a_nH_n,\\ & (h_1, a_2h_2, \ldots, a_{n-1}h_{n-1}a_n, h_n) \longmapsto h_1a_2h_2 \cdots a_{n-1}h_{n-1}a_nh_n.\end{aligned}$$

Note that $\pi = \psi \circ \varphi$.

Remark 11.2. The original clan synergy is given by (see (2.8))

$$\pi :\ H_1 \times a_2H_2 \times \cdots \times a_{n-1}H_{n-1} \times a_nH_n \longrightarrow H_1a_2H_2 \cdots a_{n-1}H_{n-1}a_nH_n,$$

which however does *not* fit our current purpose, as $\pi \neq \psi \circ \varphi$.

To simplify notation, we use the following:

Notation 11.3. Write a monic clan $H_1a_2H_2 \cdots a_{n-1}H_{n-1}a_nH_n$ ($n \geq 2$) as HBK (*guild expression*) with

$$\begin{aligned}& H := H_1, \quad B := a_2H_2a_3H_3 \cdots a_{n-1}H_{n-1}a_n, \quad K := H_n,\\ & \text{and } \ \widetilde{B} := a_2H_2 \times a_3H_3 \times \cdots \times a_{n-1}H_{n-1}a_n,\\ & \text{where note that if } n = 2 \text{ then } B = \widetilde{B} = \{a_2\}.\end{aligned} \tag{11.1}$$

Then

$$\begin{aligned}& \psi :\ H \times B \times K \longrightarrow HBK: \quad \text{the guild synergy},\\ & \varphi :\ H \times \widetilde{B} \times K \longrightarrow H \times B \times K: \quad \text{the booster},\\ & \pi = \psi \circ \varphi :\ H \times \widetilde{B} \times K \longrightarrow HBK: \quad \text{the clan synergy}.\end{aligned} \tag{11.2}$$

Note that from Proposition 3.6,

$$HBK\ (= H_1a_2H_2 \cdots a_nH_n) \text{ is finite } \iff H_1, a_2H_2, \ldots, a_nH_n \text{ are finite.} \tag{11.3}$$

We shall modify the right hand side. First note that

$$a_{n-1}H_{n-1} \text{ is finite } \iff a_{n-1}H_{n-1}a_n \text{ is finite,} \tag{11.4}$$

as $a_{n-1}H_{n-1}$ and $a_{n-1}H_{n-1}a_n$ are bijective under the right multiplication by a_n. Similarly

$$a_nH_n \text{ is finite} \iff H_n \text{ is finite.} \tag{11.5}$$

Hence (11.3) is modified as follows:

$$\begin{aligned} &HBK\,(= H_1a_2H_2\cdots a_nH_n) \text{ is finite} \\ &\quad\iff H_1, a_2H_2, a_3H_3, \ldots, a_{n-1}H_{n-1}a_n, H_n \text{ are finite.} \end{aligned} \tag{11.6}$$

Note here that

$$\begin{aligned} &a_2H_2, a_3H_3, \ldots, a_{n-1}H_{n-1}a_n \text{ are finite} \\ &\quad\iff a_2H_2 \times a_3H_3 \times \cdots \times a_{n-1}H_{n-1}a_n\,(= \widetilde{B}) \text{ is finite.} \end{aligned} \tag{11.7}$$

Thus (11.6) is simplified as follows:

$$\begin{aligned} &HBK\,(= H_1a_2H_2\cdots a_nH_n) \text{ is finite} \\ &\quad\iff H_1\,(= H), \widetilde{B}, H_n\,(= K) \text{ are finite.} \end{aligned} \tag{11.8}$$

Note next the following from Proposition 3.6:

$$\begin{aligned} &a_2H_2a_3H_3\cdots a_{n-1}H_{n-1} \text{ is finite} \\ &\quad\iff a_2H_2 \times a_3H_3 \times \cdots \times a_{n-1}H_{n-1} \text{ is finite.} \end{aligned} \tag{11.9}$$

Multiplying this by a_n from the right yields the following:

$$\begin{aligned} &a_2H_2a_3H_3\cdots a_{n-1}H_{n-1}a_n \text{ is finite} \\ &\quad\iff a_2H_2 \times a_3H_3 \times \cdots \times a_{n-1}H_{n-1}a_n \text{ is finite.} \end{aligned} \tag{11.10}$$

That is,

$$B \text{ is finite} \iff \widetilde{B} \text{ is finite.} \tag{11.11}$$

This and (11.8) yield

$$\begin{aligned} HBK \text{ is finite} &\iff H,\ B \text{ and } K \text{ are finite} \\ &\iff H,\ \widetilde{B} \text{ and } K \text{ are finite.} \end{aligned} \tag{11.12}$$

Here

$$\begin{cases} H,\ B \text{ and } K \text{ are finite} \iff H \times B \times K \text{ is finite,} \\ H,\ \widetilde{B} \text{ and } K \text{ are finite} \iff H \times \widetilde{B} \times K \text{ is finite.} \end{cases} \tag{11.13}$$

Hence we obtain the following:

Proposition 11.4. *For a monic clan HBK with $H := H_1$, $K := H_n$, $B := a_2H_2a_3H_3\cdots a_{n-1}H_{n-1}a_n$ and $\widetilde{B} := a_2H_2 \times a_3H_3 \times \cdots \times a_{n-1}H_{n-1}a_n$, the following equivalences hold:*

$$\begin{aligned} HBK \textit{ is finite} &\iff H,\ B \textit{ and } K \textit{ are finite} \\ &\iff H,\ \widetilde{B} \textit{ and } K \textit{ are finite} \\ &\iff H \times \widetilde{B} \times K \textit{ is finite} \\ &\iff H \times B \times K \textit{ is finite.} \end{aligned} \tag{11.14}$$

Now we consider the following decompositions:

- $H \times B \times K = \coprod_{b \in B} H \times \{b\} \times K$: the *satellite decomposition*, where each component $H \times \{b\} \times K$ is a *satellite*.
- $H \times \widetilde{B} \times K = \coprod_{c \in \widetilde{B}} H \times \{c\} \times K$: the *planet decomposition*, where each $H \times \{c\} \times K$ is a *planet*.

A satellite (resp. a planet) is denoted by S (resp. P). If P is over S, i.e. $\varphi(P) = S$, then the restriction $\varphi_P : P \to S$ of the booster φ to P is called a *planet booster*. The following holds:

Lemma 11.5. *Any planet booster $\varphi_P : P \to S$ is bijective.* (Thus the booster φ is *locally* bijective — bijective on each planet P.)

Proof. This is shown by the same argument in the proof of Lemma 8.4. □

Lemma 11.6. *Let $\varphi : H \times \widetilde{B} \times K \to H \times B \times K$ be the booster* (11.2). *For a satellite $S\,(\subset H \times B \times K)$, denote by p_S the cardinality of the set of planets over S. Then the following hold:*

(1) *The preimage $\varphi^{-1}(S)$ is bijective to a disjoint union of p_S copies of S.*

(2) *For any subset T of S, its preimage $\varphi^{-1}(T)$ is bijective to a disjoint union of p_S copies of T.*

Proof. This is shown by the same argument in the proof of Lemma 8.8 (modulo replacing Lemma 8.4 with Lemma 11.5). □

As before, we write a monic clan $H_1 a_2 H_2 a_3 H_3 \cdots a_n H_n$ $(n \geq 2)$ as HBK, where

$$H := H_1, \quad B := a_2 H_2 a_3 H_3 \cdots a_{n-1} H_{n-1} a_n, \quad K := H_n. \tag{11.15}$$

Take its double coset decomposition $HBK = \coprod_{i \in I} Ha^{(i)}K$.

Definition 11.7. (1) For a double coset $Ha^{(i)}K$, a satellite S is *over* $Ha^{(i)}K$ if the guild synergy ψ maps it to $Ha^{(i)}K$, i.e. $\psi(S) = Ha^{(i)}K$. The number of satellites over $Ha^{(i)}K$ (or the cardinality of the set of satellites over $Ha^{(i)}K$ if it is infinite) is the *satellite index* of $Ha^{(i)}K$, and denoted by $s^{(i)}$.

(2) For each double coset $Ha^{(i)}K$, let $S^{(ij)}$ $(j \in J^{(i)})$ denote the satellites over it. Then for each satellite $S^{(ij)}$, a planet P is *over* $S^{(ij)}$ if the booster φ maps it to $S^{(ij)}$, i.e. $\varphi(P) = S^{(ij)}$. The number of planets over $S^{(ij)}$ (or the cardinality of the set of planets over $S^{(ij)}$ if it is infinite) is the *planet index of the satellite* $S^{(ij)}$, and denoted by $p^{(ij)}$.

(3) For a double coset $Ha^{(i)}K$, a planet P is *over* $Ha^{(i)}K$ if the clan synergy π maps it to $Ha^{(i)}K$, i.e. $\pi(P) = Ha^{(i)}K$. The number of planets over $Ha^{(i)}K$ (or the cardinality of the set of planets over $Ha^{(i)}K$ if it is infinite) is the *planet index of the double coset* $Ha^{(i)}K$, and denoted by $q^{(i)}$.

The following holds — the proof is the same as that of Lemma 9.6.

Lemma 11.8. *For a monic clan HBK (see (11.15)), take its double coset decomposition $HBK = \coprod_{i\in I} Ha^{(i)}K$ and for each double coset $Ha^{(i)}K$, let $S^{(ij)}$ $(j \in J^{(i)})$ be the satellites over it. Then the planet index $q^{(i)}$ of $Ha^{(i)}K$ and the planet indices $p^{(ij)}$ of $S^{(ij)}$ $(j \in J^{(i)})$ are related as follows:*

$$q^{(i)} = \sum_{j\in J^{(i)}} p^{(ij)} \quad (\textit{cardinal sum; see Fact 9.4}). \tag{11.16}$$

Theorem 11.9 (Fiber theorem for clan synergies). *Write a monic clan $H_1a_2H_2a_3H_3\cdots a_nH_n$ $(n \geq 2)$ as HBK (see (11.15)), and take*

$$\begin{cases} HBK = \coprod_{i\in I} Ha^{(i)}K\text{: } \textit{its double coset decomposition}, \\ \pi : H \times \widetilde{B} \times K \to HBK\text{: } \textit{its clan synergy.} \end{cases}$$

For each $Ha^{(i)}K$, let $q^{(i)}$ be its planet index. Then for any $x \in Ha^{(i)}K$, the fiber $\pi^{-1}(x)$ is, as a set, bijective to a disjoint union of $q^{(i)}$ copies of $H \cap K^{a^{(i)}}$; consequently in the case of finite HBK,

$$|\pi^{-1}(x)| = q^{(i)}|H \cap K^{a^{(i)}}| \quad \text{(fiber order formula)}. \tag{11.17}$$

Proof. This is shown by the same argument in the proof of Theorem 9.9 (modulo replacing Lemma 8.8 with Lemma 11.6). □

We keep the notation of Theorem 11.9, and focus on finite HBK. We claim that

$$|\pi^{-1}(Ha^{(i)}K)| = q^{(i)}|H \cap K^{a^{(i)}}||Ha^{(i)}K|. \tag{11.18}$$

Indeed from $\pi^{-1}(Ha^{(i)}K) = \coprod_{x\in Ha^{(i)}K} \pi^{-1}(x)$, we have

$$\begin{aligned} |\pi^{-1}(Ha^{(i)}K)| &= \sum_{x\in Ha^{(i)}K} |\pi^{-1}(x)| \\ &= \sum_{x\in Ha^{(i)}K} q^{(i)}|H \cap K^{a^{(i)}}| \qquad \text{by (11.17)} \\ &= q^{(i)}|H \cap K^{a^{(i)}}||Ha^{(i)}K|. \end{aligned}$$

Definition 11.10. We set $\mu^{(i)} := q^{(i)}|Ha^{(i)}K|$, which is called the *absolute multiplicity* of $Ha^{(i)}K$. cf. the multiplicity $m^{(i)} := s^{(i)}|Ha^{(i)}K|$ of $Ha^{(i)}K$ (Definition 6.3).

Then (11.18) is written as $|\pi^{-1}(Ha^{(i)}K)| = \mu^{(i)}|H \cap K^{a^{(i)}}|$. We formalize this as follows:

Corollary 11.11. *In the notation of Theorem 11.9, if HBK is finite, then*

$$|\pi^{-1}(Ha^{(i)}K)| = \mu^{(i)}|H \cap K^{a^{(i)}}|. \tag{11.19}$$

Under the above preparation, we deduce a "factorized" intersection formula for a finite monic clan HBK. First pull back its double coset decomposition $HBK = \coprod_{i\in I} Ha^{(i)}K$ via the clan synergy $\pi : H \times \widetilde{B} \times K \to HBK$:

$$\pi^{-1}(HBK) = \coprod_{i\in I} \pi^{-1}(Ha^{(i)}K). \tag{11.20}$$

Here the left hand side is $H \times \widetilde{B} \times K$ (as π is surjective), so

$$H \times \widetilde{B} \times K = \coprod_{i\in I} \pi^{-1}(Ha^{(i)}K). \tag{11.21}$$

Hence

$$|H \times \widetilde{B} \times K| = \sum_{i\in I} |\pi^{-1}(Ha^{(i)}K)|. \tag{11.22}$$

Here $|H \times \widetilde{B} \times K| = |H||\widetilde{B}||K|$ and $|\pi^{-1}(Ha^{(i)}K)| = \mu^{(i)}|H \cap K^{a^{(i)}}|$ (Corollary 11.11). Thus

$$|H||\widetilde{B}||K| = \sum_{i\in I} \mu^{(i)}|H \cap K^{a^{(i)}}|. \tag{11.23}$$

Here since $H := H_1$, $\widetilde{B} := a_2H_2 \times a_3H_3 \times \cdots \times a_{n-1}H_{n-1}a_n$ and $K := H_n$, the left hand side is rewritten as follows:

$$\begin{aligned} |H||\widetilde{B}||K| &= |H_1||a_2H_2 \times a_3H_3 \times \cdots \times a_{n-1}H_{n-1}a_n||H_n| \\ &= |H_1||a_2H_2||a_3H_3|\cdots|a_{n-1}H_{n-1}a_n||H_n| \\ &= |H_1||H_2||H_3|\cdots|H_{n-1}||H_n|. \end{aligned} \tag{11.24}$$

Accordingly (11.23) is rewritten as $|H_1||H_2|\cdots|H_n| = \sum\limits_{i\in I} \mu^{(i)}|H_1 \cap H_n^{a^{(i)}}|$. We thus reach the following:

Theorem 11.12 (Factorized intersection formula for monic clans). *For a finite monic clan $H_1a_2H_2a_3H_3\cdots a_nH_n$ ($n \geq 2$), let*

$$H_1a_2H_2a_3H_3\cdots a_nH_n = \coprod_{i\in I} H_1a^{(i)}H_n \tag{11.25}$$

be its double coset decomposition, and for each $H_1a^{(i)}H_n$ let $\mu^{(i)}$ be its absolute multiplicity. Then

$$|H_1||H_2|\cdots|H_n| = \sum_{i\in I} \mu^{(i)}|H_1 \cap H_n^{a^{(i)}}|. \tag{11.26}$$

Here:

Lemma 11.13. *When $n = 2, 3$,* (11.26) *coincides with the following intersection formula in Example 6.11:*

$$|H_1||a_2H_2a_3H_3 \cdots a_{n-1}H_{n-1}a_n||H_n| = \sum_{i \in I} m^{(i)}|H_1 \cap H_n^{a^{(i)}}|. \tag{11.27}$$

Proof. First note that for $n = 2, 3$, the left hand side of (11.27) is equal to $|H_1||H_2|$ and $|H_1||H_2||H_3|$. Indeed for $n = 2$, the left hand side $|H_1||\{a_2\}||H_2|$ is equal to $|H_1||H_2|$ (as $|\{a_2\}| = 1$) and for $n = 3$, the left hand side $|H_1||a_2H_2a_3||H_3|$ is equal to $|H_1||H_2||H_3|$ (as $|a_2H_2a_3| = |H_2|$; see Remark 6.12). Note next that for $n = 2$, $B = \widetilde{B} = \{a_2\}$ and for $n = 3$, $B = \widetilde{B} = a_2H_2a_3$. In particular for $n = 2, 3$, we have $H \times B \times K = H \times \widetilde{B} \times K$ and so $\mu^{(i)} = m^{(i)}$ (the absolute multiplicities coincide with the multiplicities). Hence two formulas (11.26) and (11.27) coincide for $n = 2, 3$. □

12 Index theorem for clans, I

Poincaré–Hopf index theorem [3] p.134 states: *For a vector field on an oriented closed manifold with isolated zeros, the sum of indices of zeros is equal to the Euler number of the manifold.* This kind of theorems hold for clans. They are Index theorems I, II to be shown below.

We begin with preparation. For a finite *monic* clan $H_1a_2H_2 \cdots a_nH_n$, we deduced the formula (11.26). We may actually apply this formula to *any* finite clans by 'monicizing' them, i.e. regarding a clan $a_1H_1a_2H_2 \cdots a_nH_n$ as a monic clan $H_0a_1H_1a_2H_2 \cdots a_nH_n$ with $H_0 = \{e\}$, for which we take the double coset decomposition:

$$H_0a_1H_1a_2H_2 \cdots a_nH_n = \coprod_{j \in J} H_0b^{(j)}H_n. \tag{12.1}$$

For each $H_0b^{(j)}H_n$, let $\nu^{(j)}$ be its absolute multiplicity. Then by (11.26), we have $|H_0||H_1||H_2| \cdots |H_n| = \sum_{j \in J} \nu^{(j)}|H_0 \cap H_n^{b^{(j)}}|$. Here from $H_0 = \{e\}$, we have $|H_0| = 1$ and $|H_0 \cap H_n^{b^{(j)}}| = 1$, so

$$|H_1||H_2| \cdots |H_n| = \sum_{j \in J} \nu^{(j)}. \tag{12.2}$$

Note that since H_0 is the identity subgroup, the (H_0, H_n)-double coset decomposition (12.1) is actually a decomposition into left H_n-cosets:

$$a_1H_1a_2H_2 \cdots a_nH_n = \coprod_{j \in J} b^{(j)}H_n. \tag{12.3}$$

Definition 12.1. The decomposition (12.3) is called the (*left* H_n-)*coset decomposition* of the clan $a_1H_1a_2H_2 \cdots a_nH_n$.

Observation 12.2. While a monic clan $H_1a_2H_2\cdots a_nH_n$ admits a biaction $H_1 \curvearrowright H_1a_2H_2\cdots a_nH_n \curvearrowleft H_n$ (see (4.3)), a non-monic clan $a_1H_1a_2H_2\cdots a_nH_n$ merely admits a right H_n-action $a_1H_1a_2H_2\cdots a_nH_n \curvearrowleft H_n$; the orbit decomposition under this action is nothing but the coset decomposition (12.3).

We adopt the following:

Definition 12.3. For a double coset, its associated quantities such as satellite index, planet index (Definition 11.7 (1), (3)) and absolute multiplicity (Definition 11.10) are specialized to those of cosets. Namely for a double coset HaK, if $H = \{e\}$ or $K = \{e\}$, then the above quantities are called those of the *coset* aK or Ha.

We then formalize the results obtained above as follows:

Theorem 12.4 (Index theorem I). *For a finite clan $a_1H_1a_2H_2\cdots a_nH_n$ ($n \geq 2$), let $a_1H_1a_2H_2\cdots a_nH_n = \coprod_{j\in J} b^{(j)}H_n$ be its coset decomposition, and for each $b^{(j)}H_n$ let $\nu^{(j)}$ be its absolute multiplicity. Then*

$$|H_1||H_2|\cdots|H_n| = \sum_{j\in J} \nu^{(j)}. \tag{12.4}$$

Here observe that while $\nu^{(j)}$ depends on the clan $a_1H_1a_2H_2\cdots a_nH_n$, the sum $\sum_{j\in J}\nu^{(j)}$ depends only on $H_1, H_2, \ldots, H_n$, but not on $a_1, a_2, \ldots, a_n$ as seen from (12.4).

Remark 12.5. In terms of the analogy of (12.4) with Poincaré–Hopf index theorem, $|H_1||H_2|\cdots|H_n|$ is considered as the 'Euler number' of the clan $a_1H_1a_2H_2\cdots a_nH_n$.

Now combining (12.4) with (11.26) yields $\sum\limits_{i\in I}\mu^{(i)}|H_1\cap H_n^{a^{(j)}}| = \sum\limits_{j\in J}\nu^{(j)}$. We formalize this as follows:

Theorem 12.6. *Let $a_1H_1a_2H_2\cdots a_nH_n$ ($n \geq 2$) be a finite clan. Take its coset decomposition $a_1H_1a_2H_2\cdots a_nH_n = \coprod_{j\in J} b^{(j)}H_n$, and for each $b^{(j)}H_n$ let $\nu^{(j)}$ be its absolute multiplicity. Next for the monic clan $H_1a_2H_2\cdots a_nH_n$, take its double coset decomposition $H_1a_2H_2\cdots a_nH_n = \coprod_{i\in I} H_1a^{(i)}H_n$, and for each $H_1a^{(i)}H_n$ let $\mu^{(i)}$ be its absolute multiplicity. Then*

$$\sum_{i\in I}\mu^{(i)}|H_1\cap H_n^{a^{(i)}}| = \sum_{j\in J}\nu^{(j)}. \tag{12.5}$$

For a monic clan $H_0a_1H_1a_2H_2\cdots a_nH_n$ with $H_0 = \{e\}$, we trivially have

$$H_0a_1H_1a_2H_2\cdots a_nH_n = a_1H_1a_2H_2\cdots a_nH_n. \tag{12.6}$$

In contrast, for a monic clan $H_1a_2H_2\cdots a_nH_n$, in general

$$H_1a_2H_2\cdots a_nH_n \neq a_1H_1a_2H_2\cdots a_nH_n. \tag{12.7}$$

In fact the right hand side is a translation of the left hand side by a_1 from the left — but, for this very reason, we have

$$|H_1 a_2 H_2 \cdots a_n H_n| = |a_1 H_1 a_2 H_2 \cdots a_n H_n|. \tag{12.8}$$

Lemma 12.7. *For a finite clan $a_1 H_1 a_2 H_2 \cdots a_n H_n$, the following hold:*

(1) $|a_1 H_1 a_2 H_2 \cdots a_n H_n| = |H_1 a_2 H_2 \cdots a_n H_n|$ (already shown in (12.8)).

(2) *The number of distinct left H_n-cosets in $a_1 H_1 a_2 H_2 \cdots a_n H_n$ is equal to that of distinct left H_n-cosets in $H_1 a_2 H_2 \cdots a_n H_n$.*

(3) *Denote the number in* (2) *by ℓ. Then $|a_1 H_1 a_2 H_2 \cdots a_n H_n| = \ell |H_n|$.*

(4) *$|H_n|$ divides $|a_1 H_1 a_2 H_2 \cdots a_n H_n|$.*

Proof. (2): Let $a_1 H_1 a_2 H_2 \cdots a_n H_n = \coprod_{j \in J} b^{(j)} H_n$ be the H_n-coset decomposition of $a_1 H_1 a_2 H_2 \cdots a_n H_n$. Then $H_1 a_2 H_2 \cdots a_n H_n = \coprod_{j \in J} a_1^{-1} b^{(j)} H_n$ is the H_n-coset decomposition of $H_1 a_2 H_2 \cdots a_n H_n$, so (2) holds. (Note: The translation $a_1 H_1 a_2 H_2 a_3 H_3 \cdots a_n H_n \to H_1 a_2 H_2 a_3 H_3 \cdots a_n H_n$ by the left multiplication of a_1^{-1} is a bijection, inducing a one-to-one correspondence between the H_n-cosets in these clans.)

(3): From $|a_1 H_1 a_2 H_2 \cdots a_n H_n| = \sum_{j \in J} |b^{(j)} H_n|$ together with $|b^{(j)} H_n| = |H_n|$, we have $|a_1 H_1 a_2 H_2 \cdots a_n H_n| = \sum_{j \in J} |H_n| = |J||H_n|$. By definition, $|J|$ is ℓ, so the assertion is confirmed.

(4): This is immediate from (3). □

We restate statements in Lemma 12.7 as follows:

Corollary 12.8. *For a finite clan $a_1 H_1 a_2 H_2 \cdots a_n H_n$, the order $|H_n|$ divides $|a_1 H_1 a_2 H_2 \cdots a_n H_n|$ $(= |H_1 a_2 H_2 \cdots a_n H_n|)$. Moreover letting ℓ denote the number of distinct H_n-cosets in $a_1 H_1 a_2 H_2 \cdots a_n H_n$ (or in $H_1 a_2 H_2 \cdots a_n H_n$), then we have*

$$\ell = \frac{|a_1 H_1 a_2 H_2 \cdots a_n H_n|}{|H_n|} \quad \left(= \frac{|H_1 a_2 H_2 \cdots a_n H_n|}{|H_n|} \right). \tag{12.9}$$

When $n = 2$, (12.9) gives $\ell = \dfrac{|H_1 a_2 H_2|}{|H_2|}$. On the other hand, $|H_1 a_2 H_2| = \dfrac{|H_1||H_2|}{|H_1 \cap H_2^{a_2}|}$ by Lemma 3.1 (B.3). Thus $\ell = \dfrac{|H_1|}{|H_1 \cap H_2^{a_2}|}$. This confirms the following:

Formula 12.9. *For a finite clan $a_1 H_1 a_2 H_2$ (or $H_1 a_2 H_2$), let ℓ be the number of distinct left H_2-cosets in it. Then*

$$\ell = \frac{|H_1|}{|H_1 \cap H_2^{a_2}|}. \tag{12.10}$$

We give an example. For simplicity a double coset $H_1 a_2 H_2$ is denoted by HaK.

Example 12.10. For $G = \mathfrak{S}_4$ (the symmetric group of degree 4), take $H := \langle h \rangle$ (order 4, generated by a cyclic permutation $h := (1\,2\,3\,4)$), $a := (1\,2)$ (transposition) and $K := \langle k \rangle$ (order 2, generated by $k := (1\,4)(2\,3)$). Noting that $H = \{e, h, h^2, h^3\}$, we have $HaK = eaK \cup haK \cup h^2aK \cup h^3aK$, that is,

$$HaK = aK \cup haK \cup h^2aK \cup h^3aK. \tag{12.11}$$

Here $h^2a = ak$ (Remark 12.11 (1) below), so $h^2aK = akK$, that is, $h^2aK = aK$. Multiplying this by h from the left yields $h^3aK = haK$. Moreover $haK \neq aK$ (Remark 12.11 (2)). Hence (12.11) reduces to

$$HaK = aK \amalg haK \quad (\text{with } aK = h^2aK \text{ and } haK = h^3aK). \tag{12.12}$$

Thus the number of left K-cosets in HaK is 2, i.e. $\ell = 2$. Next note that $K^a := aKa^{-1} = \langle aka^{-1} \rangle = \langle h^2 \rangle$ (Remark 12.11 (3)). Thus $K^a \subset H = \langle h \rangle$, so $H \cap K^a = K^a$. Here $K^a \cong \mathbb{Z}_2$, so $|H \cap K^a| = 2$. Hence $|H|/|H \cap K^a| = 4/2 = 2$. Therefore $\ell = |H|/|H \cap K^a|$ (cf. (12.10)).

Remark 12.11. In the notation of Example 12.10,

$$(*) \qquad h^2 = (1\,2\,3\,4)^2 = (1\,3)(2\,4) = (2\,4)(1\,3).$$

Moreover the following hold:

(1) $h^2a = ak = (1\,4\,2\,3)$. Indeed

$$\begin{aligned} ak &= (1\,2)(1\,4)(2\,3) = (1\,4\,2\,3), \\ h^2a &= (1\,2\,3\,4)^2(1\,2) = (1\,3)(2\,4)(1\,2) \quad \text{by } (*) \\ &= (1\,4\,2\,3). \end{aligned}$$

(2) $aK \neq haK$. Indeed

$$aK = \{a, ak\} = \{(1\,2), (1\,4\,2\,3)\}, \quad haK = \{ha, hak\} = \{(1\,3\,4), (2\,4\,3)\}.$$

(3) $aka^{-1} = h^2$; equivalently $ak = h^2a$, which is already shown in (1).

13 Index theorem for clans, II

Let $a_1H_1a_2H_2 \cdots a_nH_n$ be a finite clan of a group G. Recall that in deducing the index formula (12.4), letting $H_0 = \{e\}$ (the identity subgroup of G), at first we formally considered a monic clan $H_0a_1H_1a_2H_2 \cdots a_nH_n$, and then took its (H_0, H_n)-double coset decomposition, which is actually the left H_n-coset decomposition of $a_1H_1a_2H_2 \cdots a_nH_n$. For the convenience of later discussion, *without using* H_0 *at all*, we reformulate the deduction procedure of the index formula (12.4).

Convention 13.1. *We identify*

$$H_0 \times \big(a_1 H_1 a_2 H_2 \cdots a_{n-1} H_{n-1} a_n\big) \times H_n$$

with

$$\big(a_1 H_1 a_2 H_2 \cdots a_{n-1} H_{n-1} a_n\big) \times H_n \tag{13.1}$$

via a bijection $(e, \beta, h_n) \mapsto (\beta, h_n)$, *where* $\beta \in a_1 H_1 a_2 H_2 \cdots a_{n-1} H_{n-1} a_n$. Accordingly:

- The satellite decomposition

$$H_0 \times \big(a_1 H_1 a_2 H_2 \cdots a_{n-1} H_{n-1} a_n\big) \times H_n = \coprod_{\beta \in a_1 H_1 a_2 H_2 \cdots a_{n-1} H_{n-1} a_n} H_0 \times \{\beta\} \times H_n$$

is identified with

$$\big(a_1 H_1 a_2 H_2 \cdots a_{n-1} H_{n-1} a_n\big) \times H_n = \coprod_{\beta \in a_1 H_1 a_2 H_2 \cdots a_{n-1} H_{n-1} a_n} \{\beta\} \times H_n, \tag{13.2}$$

and each $\{\beta\} \times H_n$ is called a satellite.

- The guild synergy

$$\begin{aligned} \psi :\ & H_0 \times \big(a_1 H_1 a_2 H_2 \cdots a_{n-1} H_{n-1} a_n\big) \times H_n \\ & \longrightarrow H_0 \big(a_1 H_1 a_2 H_2 \cdots a_{n-1} H_{n-1} a_n\big) H_n \end{aligned}$$

is identified with

$$\begin{aligned} \psi :\ & \big(a_1 H_1 a_2 H_2 \cdots a_{n-1} H_{n-1} a_n\big) \times H_n \\ & \longrightarrow \big(a_1 H_1 a_2 H_2 \cdots a_{n-1} H_{n-1} a_n\big) H_n. \end{aligned} \tag{13.3}$$

In response to the above convention, the following is adopted:

Convention 13.2. *We identify*

$$H_0 \times \big(a_1 H_1 \times a_2 H_2 \times \cdots \times a_{n-1} H_{n-1} a_n\big) \times H_n$$

with

$$\big(a_1 H_1 \times a_2 H_2 \times \cdots \times a_{n-1} H_{n-1} a_n\big) \times H_n \tag{13.4}$$

via a bijection $(e, \gamma, h_n) \mapsto (\gamma, h_n)$, *where* $\gamma \in a_1 H_1 \times a_2 H_2 \times \cdots \times a_{n-1} H_{n-1} a_n$. Accordingly:

- The planet decomposition

$$H_0 \times \big(a_1 H_1 \times a_2 H_2 \times \cdots \times a_{n-1} H_{n-1} a_n\big) \times H_n = \coprod_{\gamma \in a_1 H_1 \times a_2 H_2 \times \cdots \times a_{n-1} H_{n-1} a_n} H_0 \times \{\gamma\} \times H_n$$

is identified with

$$\big(a_1 H_1 \times a_2 H_2 \times \cdots \times a_{n-1} H_{n-1} a_n\big) \times H_n = \coprod_{\gamma \in a_1 H_1 \times a_2 H_2 \times \cdots \times a_{n-1} H_{n-1} a_n} \{\gamma\} \times H_n, \tag{13.5}$$

and each $\{\gamma\} \times H_n$ is called a planet.

- The booster

$$\begin{aligned}\varphi :\ & H_0 \times \big(a_1H_1 \times a_2H_2 \times \cdots \times a_{n-1}H_{n-1}a_n\big) \times H_n \\ & \longrightarrow H_0 \times \big(a_1H_1a_2H_2 \cdots a_{n-1}H_{n-1}a_n\big) \times H_n\end{aligned}$$

is identified with

$$\begin{aligned}\varphi :\ & \big(a_1H_1 \times a_2H_2 \times \cdots \times a_{n-1}H_{n-1}a_n\big) \times H_n \\ & \longrightarrow \big(a_1H_1a_2H_2 \cdots a_{n-1}H_{n-1}a_n\big) \times H_n.\end{aligned} \tag{13.6}$$

- The clan synergy

$$\begin{aligned}\pi = \psi \circ \varphi :\ & H_0 \times \big(a_1H_1 \times a_2H_2 \times \cdots \times a_{n-1}H_{n-1}a_n\big) \times H_n \\ & \longrightarrow H_0\big(a_1H_1a_2H_2 \cdots a_{n-1}H_{n-1}a_n\big)H_n\end{aligned}$$

is identified with (below, ψ and φ are replaced with the simplified versions (13.3) and (13.6))

$$\begin{aligned}\pi = \psi \circ \varphi :\ & \big(a_1H_1 \times a_2H_2 \times \cdots \times a_{n-1}H_{n-1}a_n\big) \times H_n \\ & \longrightarrow \big(a_1H_1a_2H_2 \cdots a_{n-1}H_{n-1}a_n\big)H_n.\end{aligned} \tag{13.7}$$

Remark 13.3. (1) When $n = 2$,

- the booster $\varphi : a_1H_1a_2 \times H_2 \to a_1H_1a_2 \times H_2$ in (13.6) is the identity map; so the clan synergy $\pi = \psi \circ \varphi : a_1H_1a_2 \times H_2 \to a_1H_1a_2H_2$ in (13.7) coincides with the guild synergy $\psi : a_1H_1a_2 \times H_2 \to a_1H_1a_2H_2$ in (13.3), and
- the planet decomposition (13.5) is a decomposition of $a_1H_1a_2 \times H_2$, coinciding with the satellite decomposition (13.2).

In particular, *when $n = 2$, "absolute multiplicity" coincides with "multiplicity".*

(2) For $n \geq 3$, the booster φ is *not* the identity map, and the clan synergy $\pi\,(= \psi \circ \varphi)$ is different from the guild synergy ψ. Similarly the planet decomposition is different from the satellite decomposition.

Deduction of another index formula

To simplify discussion, we first consider the case $n = 2$, i.e. $a_1H_1a_2H_2$; this is denoted by $uHvK$ for simplicity. We proceed as follows:

Step 1. Take the coset decomposition $uHvK = \coprod_{j \in J} b^{(j)}K$.

Step 2. Take the satellite decomposition $uHv \times K = \coprod_{\beta \in uHv} \{\beta\} \times K$, where each $\{\beta\} \times K$ is a satellite. Note that in the current case $n = 2$, there is *no* need to consider the planet decomposition, as it coincides with the satellite decomposition (Remark 13.3 (1)).

Step 3. Take the guild synergy $\psi : uHv \times K \to uHvK$ given by $(\beta, k) \in uHv \times K \mapsto \beta k \in uHvK$. For a coset $b^{(j)}K$, a satellite $\{\beta\} \times K$ is *over* it if $\beta K = b^{(j)}K$. For each $b^{(j)}K$, let $t^{(j)}$ be its satellite index, i.e. the number of satellites over $b^{(j)}K$ and let $l^{(j)}$ be its multiplicity, i.e. $l^{(j)} := t^{(j)}|b^{(j)}K|$. Here $|b^{(j)}K| = |K|$ (regardless of $b^{(j)}$), so $l^{(j)} = t^{(j)}|K|$. In the current case $n = 2$, the multiplicity $l^{(j)}$ coincides with the absolute multiplicity $\nu^{(j)}$ of $b^{(j)}K$ (Remark 13.3 (1)). Thus $\nu^{(j)} = t^{(j)}|K|$. Summation of this over j then gives

$$\sum_{j\in J} \nu^{(j)} = (\sum_{j\in J} t^{(j)})|K|. \tag{13.8}$$

On the other hand, $|H||K| = \sum_{j\in J} \nu^{(j)}$ by applying (12.4) to $uHvK$. Therefore $|H||K| = (\sum_{j\in J} t^{(j)})|K|$, that is, $|H| = \sum_{j\in J} t^{(j)}$. We formalize this as follows:

Corollary 13.4. *For a finite clan $a_1H_1a_2H_2$ of length 2, let $a_1H_1a_2H_2 = \coprod_{j\in J} b^{(j)}H_2$ be its coset decomposition, and for each $b^{(j)}H_2$ let $t^{(j)}$ be its satellite index. Then*

$$|H_1| = \sum_{j\in J} t^{(j)}. \tag{13.9}$$

For general $n \geq 2$, a similar formula holds. The proof is essentially the same as that for $n = 2$ modulo the involvement of the booster and the planet decomposition. First, relevant decompositions and maps are as follows:

(i) the coset decomposition:

$$a_1H_1a_2H_2 \cdots a_nH_n = \coprod_{j\in J} b^{(j)}H_n. \tag{13.10}$$

(ii) the satellite decomposition:

$$\big(a_1H_1a_2H_2 \cdots a_{n-1}H_{n-1}a_n\big) \times H_n = \coprod_{\beta \,\in\, a_1H_1a_2H_2\cdots a_{n-1}H_{n-1}a_n} \{\beta\} \times H_n, \tag{13.11}$$

where each $\{\beta\} \times H_n$ is a satellite.

(iii) the planet decomposition:

$$\big(a_1H_1 \times a_2H_2 \times \cdots \times a_{n-1}H_{n-1}a_n\big) \times H_n = \coprod_{\gamma \,\in\, a_1H_1\times a_2H_2\times\cdots\times a_{n-1}H_{n-1}a_n} \{\gamma\} \times H_n, \tag{13.12}$$

where each $\{\gamma\} \times H_n$ is a planet.

(iv) the clan synergy $\pi = \psi \circ \varphi$ with ψ the guild synergy and φ the booster:

$$\begin{aligned} \pi : \ \big(a_1 H_1 \times a_2 H_2 \times \cdots \times a_{n-1} H_{n-1} a_n\big) \times H_n \\ \xrightarrow{\varphi} \big(a_1 H_1 a_2 H_2 \cdots a_{n-1} H_{n-1} a_n\big) \times H_n \\ \xrightarrow{\psi} a_1 H_1 a_2 H_2 \cdots a_{n-1} H_{n-1} a_n H_n. \end{aligned} \tag{13.13}$$

For the sake of simplicity, we set $C := a_1 H_1 a_2 H_2 \cdots a_{n-1} H_{n-1} a_n$, $K := H_n$ and $\widetilde{C} := a_1 H_1 \times a_2 H_2 \times \cdots \times a_{n-1} H_{n-1} a_n$. Then (i)–(iv) are rewritten as follows:

(i)$'$ the coset decomposition: $CK = \coprod_{j \in J} b^{(j)} K$.

(ii)$'$ the satellite decomposition: $C \times K = \coprod_{\beta \in C} \{\beta\} \times K$.

(iii)$'$ the planet decomposition: $\widetilde{C} \times K = \coprod_{\gamma \in \widetilde{C}} \{\gamma\} \times K$.

(iv)$'$ the clan synergy $\pi = \psi \circ \varphi : \widetilde{C} \times K \xrightarrow{\varphi} C \times K \xrightarrow{\psi} CK$.

We explicitly give π in (iv)$'$. To that end, for an element

$$\gamma := (a_1 h_1, a_2 h_2, \ldots, a_{n-1} h_{n-1} a_n) \in \widetilde{C}, \tag{13.14}$$

we set $\overline{\gamma} := a_1 h_1 a_2 h_2 \cdots a_{n-1} h_{n-1} a_n \in C$. Then

$$\pi : \widetilde{C} \times K \xrightarrow{\varphi} C \times K \xrightarrow{\psi} CK, \quad (\gamma, k) \longmapsto (\overline{\gamma}, k) \longmapsto \overline{\gamma} k. \tag{13.15}$$

We use the following terms:

- A satellite $\{\beta\} \times K$ is *over* a coset $b^{(j)} K$ if the guild synergy ψ maps $\{\beta\} \times K$ to $b^{(j)} K$, i.e. $\beta K = b^{(j)} K$.
- A planet $\{\gamma\} \times K$ is *over* a satellite $\{\beta\} \times K$ if the booster φ maps $\{\gamma\} \times K$ to $\{\beta\} \times K$, i.e. $\overline{\gamma} = \beta$.
- A planet $\{\gamma\} \times K$ is *over* a coset $b^{(j)} K$ if the clan synergy $\pi = \psi \circ \varphi$ maps $\{\gamma\} \times K$ to $b^{(j)} K$, i.e. $\overline{\gamma} K = b^{(j)} K$.

For each coset $b^{(j)} K$, the number of planets over $b^{(j)} K$ is the planet index of $b^{(j)} K$, and denoted by $r^{(j)}$. The absolute multiplicity of $b^{(j)} K$ is then, by definition, given by $\nu^{(j)} := r^{(j)} |b^{(j)} K|$. Here $|b^{(j)} K| = |K|$ (regardless of $b^{(j)}$), so $\nu^{(j)} = r^{(j)} |K|$. Summation of this over j gives

$$\sum_{j \in J} \nu^{(j)} = \big(\sum_{j \in J} r^{(j)}\big) |K|. \tag{13.16}$$

On the other hand, $|H_1||H_2| \cdots |H_{n-1}||K| = \sum_{j \in J} \nu^{(j)}$ by (12.4). This combined with (13.16) yields $|H_1||H_2| \cdots |H_{n-1}||K| = (\sum_{j \in J} r^{(j)})|K|$, that is, $|H_1||H_2| \cdots |H_{n-1}| = \sum_{j \in J} r^{(j)}$. The following is thus obtained:

Theorem 13.5 (Index theorem II). *For a finite clan $a_1H_1a_2H_2\cdots a_nH_n$ $(n \geq 2)$, let $a_1H_1a_2H_2\cdots a_nH_n = \coprod_{j\in J} b^{(j)}H_n$ be its coset decomposition, and for each $b^{(j)}H_n$ let $r^{(j)}$ be its planet index. Then*

$$|H_1||H_2|\cdots|H_{n-1}| = \sum_{j\in J} r^{(j)}. \tag{13.17}$$

Here $|H_1||H_2|\cdots|H_{n-1}|$ may be alternatively given by the factorized intersection formula (11.26) in the case of $n-1$: we denote the relevant notations as I_{n-1}, $\mu_{n-1}^{(i)}$ and $a_{n-1}^{(i)}$, and write (11.26) as

$$|H_1||H_2|\cdots|H_{n-1}| = \sum_{i\in I_{n-1}} \mu_{n-1}^{(i)}|H_1 \cap H_{n-1}^{a_{n-1}^{(i)}}|. \tag{13.18}$$

By equating the right hand sides of equations (13.17) and (13.18), we obtain:

Theorem 13.6. *In the above notation,*

$$\sum_{j\in J} r^{(j)} = \sum_{i\in I_{n-1}} \mu_{n-1}^{(i)}|H_1 \cap H_{n-1}^{a_{n-1}^{(i)}}|. \tag{13.19}$$

We now give an example for Theorem 13.5 with $n = 2$ and $a_1 = e$, i.e. $H_1a_2H_2$; this is for brevity denoted as HaK.

Example 13.7. As in Example 12.10, take $G = \mathfrak{S}_4$, $H := \langle h\rangle$ (order 4, generated by a cyclic permutation $h := (1\,2\,3\,4)$), $a := (1\,2)$ (transposition) and $K := \langle k\rangle$ (order 2, generated by $k := (1\,4)(2\,3)$). Then consider the satellite decomposition $H \times \{a\} \times K = \coprod_{x\in H} \{x\} \times \{a\} \times K$. Here H consists of e, h, h^2, h^3, so the satellites are

$$\{e\} \times \{a\} \times K, \quad \{h\} \times \{a\} \times K, \quad \{h^2\} \times \{a\} \times K, \quad \{h^3\} \times \{a\} \times K,$$

which, under the guild synergy $\psi : H \times \{a\} \times K \to HaK$, are mapped to $eaK\,(= aK)$, haK, h^2aK, h^3aK respectively. Here $aK = h^2aK$, $haK = h^3aK$, and $HaK = aK \amalg haK$ (see (12.12)). Consequently:

(i) The satellites over $aK\,(= h^2aK)$ are two: $\{e\} \times \{a\} \times K$ and $\{h^2\} \times \{a\} \times K$. So the satellite index $s^{(1)}$ of aK is 2.

(ii) The satellites over $haK\,(= h^3aK)$ are two: $\{h\} \times \{a\} \times K$ and $\{h^3\} \times \{a\} \times K$. So the satellite index $s^{(2)}$ of haK is 2.

Thus $|H| = s^{(1)} + s^{(2)} = 4$ (cf. (13.9)).

14 Comparison with analogous formulas in geometry

Poincaré–Hopf index theorem [3] p.134 states: *For a vector field V on an oriented closed manifold M with isolated zeros, say $p_j \in M$ ($j \in J$), we denote by* $\mathrm{Ind}_{p_j}(V)$ *their indices and by $\chi(M)$ the Euler number of M. Then*

$$\chi(M) = \sum_{j \in J} \mathrm{Ind}_{p_j}(V). \tag{14.1}$$

Our index formula (12.4) for a clan $a_1H_1a_2H_2 \cdots a_nH_n$ has an analogous form:

$$|H_1||H_2| \cdots |H_n| = \sum_{j \in J} \nu^{(j)} \quad (\nu^{(j)}\text{: absolute multiplicity}). \tag{14.2}$$

Based on this analogy, we adopt the following:

Definition 14.1. For a clan $a_1H_1a_2H_2 \cdots a_nH_n$, the order product

$$|H_1||H_2| \cdots |H_n|$$

is called its *Euler number* and denoted by $\chi(a_1H_1a_2H_2 \cdots a_nH_n)$; accordingly (14.2) is rewritten as

$$\chi(a_1H_1a_2H_2 \cdots a_nH_n) = \sum_{j \in J} \nu^{(j)}. \tag{14.3}$$

Recall another index formula (13.17), i.e.

$$|H_1||H_2| \cdots |H_{n-1}| = \sum_{j \in J} r^{(j)} \quad (r^{(j)}\text{: planet index}), \tag{14.4}$$

which is also analogous to Poincaré–Hopf index formula (14.1). Although our two index formulas are *in their forms* analogous to Poincaré–Hopf index formula, there is a subtle difference: $\mathrm{Ind}_{p_j}(V)$ in Poincaré–Hopf index formula may be positive or negative (see [3] p.133), and accordingly *in the sum* $\sum_{j \in J} \mathrm{Ind}_{p_j}(V)$*, there may be a cancellation of terms* (so that the total is always equal to $\chi(M)$). In contrast, in the right hand sides of our index formulas (14.2) and (14.4), no cancellation of terms occurs, as $\nu^{(j)}$ and $r^{(j)}$ are positive. Actually our index formulas are *in spirit* (though not in their forms) close to Riemann–Hurwitz formula (14.5) below, in that these formulas involve "upper structures" — synergy and covering —, whose "ramification data" appear in these formulas, and moreover no cancellation of terms occurs.

Riemann–Hurwitz formula ([6] p.569) *Let $\pi : T \to S$ be a finite covering between closed orientable surfaces of covering degree d with branch points $q_j \in S$ ($j \in J$) and e_j (≥ 2) their branch indices. Then*

$$\chi(T) = d\chi(S) + \sum_{j \in J} (e_j - 1). \tag{14.5}$$

Note that in the sum $\sum_{j\in J}(e_j-1)$, there is no cancellation of terms as $e_j-1\geq 1$.

finite covering $\pi: T\to S$	clan synergy (13.13) $\pi: a_1H_1\times a_2H_2\times\cdots\times a_{n-1}H_{n-1}a_n\times H_n$ $\to a_1H_1a_2H_2\cdots a_nH_n$
Riemann–Hurwitz formula (14.5)	index formula (14.2) (resp. index formula (14.4))
branch point q_j	coset $b^{(j)}H_n$ (in the decomposition (13.10))
branch index e_j	absolute multiplicity $\nu^{(j)}$ (resp. planet index $r^{(j)}$)
Euler number $\chi(T)$	order product $\lvert H_1\rvert\lvert H_2\rvert\cdots\lvert H_n\rvert$ (resp. $\lvert H_1\rvert\lvert H_2\rvert\cdots\lvert H_{n-1}\rvert$)

In deducing our intersection formulas, a coset decomposition and a double coset decomposition appeared, as in the following flowcharts:

(a) a monic clan $H_1a_2H_2\cdots a_nH_n$

$\rightsquigarrow$ its (H_1,H_n)-*double coset decomposition*

$\rightsquigarrow$ intersection formula (6.16) for $|H_1||a_2H_2a_3H_3\cdots a_{n-1}H_{n-1}a_n||H_n|$ and the factorized version (11.26) for $|H_1||H_2|\cdots|H_n|$.

(b) a clan $a_1H_1a_2H_2\cdots a_nH_n$

$\rightsquigarrow$ its H_n-*coset decomposition*

$\rightsquigarrow$ index formula II (13.17) for $|H_1||H_2|\cdots|H_{n-1}|$.

From the viewpoint of higher group theory, a coset and a double coset are clans of lengths 1 and 2 respectively, and a coset decomposition is of first order, while a double coset decomposition is of second order; accordingly the coset decomposition carries informations only up to first order, while the double coset decomposition carries informations up to second order (more informative). In fact, as in (a) and (b) above, the double coset decomposition gives a formula for $|H_1||H_2|\cdots|H_n|$, while the coset decomposition gives a formula for $|H_1||H_2|\cdots|H_{n-1}|$ (one length shorter — information is reduced). Such a reduction of informations already appears in more fundamental contexts:

(i) For a finite coset aH, we always have $|aH|=|H|$ (regardless of $a\in G$). In contrast, for a finite double coset HaK, its order $|HaK|$ generally varies with a as $|HaK|=|H||K|/|H\cap K^a|$ (Lemma 3.1 (B.3)).

(ii) For a finite coset aH, we always have $|aH|=|Ha|$ (regardless of $a\in G$). In contrast, for a finite double coset HaK, in general $|HaK|\neq|KaH|$; indeed by Lemma 3.1 (B.3),

$$\begin{cases} |HaK|=|H||K|/|H\cap K^a|=|H||K|/|H^{a^{-1}}\cap K|, \\ |KaH|=|K||H|/|K\cap H^a|, \end{cases} \tag{14.6}$$

where in general $|H^{a^{-1}}\cap K|\neq|K\cap H^a|$ (see Example 14.3 below).

Remark 14.2. While in general $|HaK| \neq |KaH|$, we always have $|Ha^{-1}K| = |KaH|$, indeed

$$|Ha^{-1}K| = |H||K|/|H^a \cap K| = |KaH|, \tag{14.7}$$

or alternatively from the fact that KaH is bijective to $Ha^{-1}K$ under the inverse map:

$$\begin{aligned} &KaH \xrightarrow{\cong} (KaH)^{-1} = H^{-1}a^{-1}K^{-1} = Ha^{-1}K \ \ (\text{as } H^{-1} = H,\ K^{-1} = K), \\ &kah \in KaH \longmapsto (kah)^{-1} = h^{-1}a^{-1}k^{-1} \in Ha^{-1}K. \end{aligned}$$

We give an example of $|HaK| \neq |KaH|$:

Example 14.3. $G = \mathfrak{S}_3$ (the symmetric group of degree 3): Take $H = \langle (1\,2) \rangle$ (a cyclic subgroup of order 2 generated by a transposition $(1\,2)$), $a = (1\,2\,3)$ (a cyclic permutation of order 3), and $K = \langle (2\,3) \rangle$ (a cyclic subgroup of order 2 generated by a transposition $(2\,3)$). Then

$$\begin{cases} H^a = \langle (2\,3) \rangle, \text{ so } K \cap H^a = K, \text{ thus } |K \cap H^a| = |K| = 2, \\ H^{a^{-1}} = \langle (1\,3) \rangle, \text{ so } K \cap H^{a^{-1}} = \{e\}, \text{ thus } |K \cap H^{a^{-1}}| = 1. \end{cases} \tag{14.8}$$

Accordingly by Lemma 3.1 (B.3),

$$\begin{cases} |HaK| = |H||K|/|H^{a^{-1}} \cap K| = 2 \cdot 2/1 = 4, \\ |KaH| = |K||H|/|K \cap H^a| = 2 \cdot 2/2 = 2. \end{cases} \tag{14.9}$$

Remark 14.4. Concerning Example 14.3, the following peculiar phenomenon happens: If we replace $a = (1\,2\,3)$ with a larger set $A = \{e, a\}$, then the magnitude of orders is reversed as follows:

$$|HaK| = 4 > |KaH| = 2, \text{ whereas } |HAK| < |KAH|. \tag{14.10}$$

In fact:

Claim 14.5. (1) $|HAK| = 4$ *and* (2) $|KAH| = 6$.

Proof. (1): Since $A = \{e, a\}$, we may write $HAK = HeK \cup HaK$, i.e. $HAK = HK \cup HaK$. Here

$$HK = HaK = \{e, (1\,2), (2\,3), (1\,2)(2\,3)\,(= (1\,2\,3))\}. \tag{14.11}$$

Thus $HAK = HK$, so $|HAK| = |HK| = 4$ (from (14.11)).

(2): As in (1), we first write $KAH = KH \cup KaH$. Here $KH = \{e, (1\,2), (2\,3), (2\,3)(1\,2)\,(= (1\,3\,2))\}$ and

$$\begin{aligned} KaH = \big\{&e(1\,2\,3)e\,(= (1\,2\,3)),\ (2\,3)(1\,2\,3)\,(= (1\,3)), \\ &(1\,2\,3)(1\,2)\,(= (1\,3)),\ (2\,3)(1\,2\,3)(1\,2)\,(= (1\,2\,3))\big\} \\ = \{&(1\,2\,3), (1\,3)\}. \end{aligned}$$

Hence $KAH = \big\{e, (1\,2), (2\,3), (1\,3), (1\,2\,3), (1\,3\,2)\big\}\,(= \mathfrak{S}_3)$, so $|KAH| = 6$. $\square$

15 Bases, frames, and isogeny of clans

The Euler number of a clan $a_1H_1a_2H_2\cdots a_nH_n$ is given by $|H_1||H_2|\cdots|H_n|$ (Definition 14.1). Actually, as we see later, a clumsy phenomenon happens to this (see Fact 15.7); but it will be resolved by introducing "words of clans" (Definition 16.1). This and the next sections are devoted to its explanation, examples, and related topics.

For a clan $a_1H_1a_2H_2\cdots a_nH_n$, its *coefficients* are $a_1, a_2, \ldots, a_n$ and its *basis* is $H_1, H_2, \ldots, H_n$. Here $H_1, H_2, \ldots, H_n$ are possibly not distinct. Moreover the basis does *not* care about the order of $H_1, H_2, \ldots, H_n$. e.g. the bases of $a_1H_1a_2H_2$ and $b_1H_2b_2H_1$ are the same — both are H_1, H_2 (possibly $H_1 = H_2$). When we are concerned with the order of $H_1, H_2, \ldots, H_n$, the "ordered basis" is called the *frame* of the clan.

Definition 15.1. Two clans of a group G are *isogenic* if they have the same basis and differ only by their coefficients.

Explicitly, two clans $a_1H_1a_2H_2\cdots a_nH_n$ and $b_1K_1b_2K_2\cdots b_mK_m$ are isogenic if and only if $n = m$ and $K_1, K_2, \ldots, K_n$ are a permutation of $H_1, H_2, \ldots, H_n$. For a clan $a_1H_1a_2H_2\cdots a_nH_n$, the set of clans of G isogenic to it is given by

$$\left\{b_1H_{\sigma(1)}b_2H_{\sigma(2)}\cdots b_nH_{\sigma(n)} \,:\, b_1, b_2, \ldots, b_n \in G,\ \sigma \in \mathfrak{S}_n\right\}, \tag{15.1}$$

where $\mathfrak{S}_n$ denotes the symmetric group of degree n consisting of the permutations of $1, 2, \ldots, n$.

Definition 15.2. Two clans of a group G are *strictly isogenic* if they have the same frame and differ only by their coefficients.

Explicitly, two clans $a_1H_1a_2H_2\cdots a_nH_n$ and $b_1K_1b_2K_2\cdots b_mK_m$ are strictly isogenic if and only if $n = m$ and $H_1 = K_1, H_2 = K_2, \ldots, H_n = K_n$.

The sets of clans spanned by bases or frames

Fix subgroups $H_1, H_2, \ldots, H_n$ of a group G, and consider the set of clans of G *spanned by the basis* $H_1, H_2, \ldots, H_n$:

$$\begin{aligned}\mathcal{B}(H_1, H_2, \ldots, H_n) := \big\{&x_1H_{\sigma(1)}x_2H_{\sigma(2)}\cdots x_nH_{\sigma(n)} : \\ &x_i \in G\ (i = 1, 2, \ldots, n).\ \sigma \in \mathfrak{S}_n\big\}.\end{aligned} \tag{15.2}$$

Alternatively this is the set of clans of G isogenic to a clan $a_1H_1a_2H_2\cdots a_nH_n$. We say that $\mathcal{B}(H_1, H_2, \ldots, H_n)$ is an *isogeny set*. Note that for any clan

$$x_1H_{\sigma(1)}x_2H_{\sigma(2)}\cdots x_nH_{\sigma(n)} \in \mathcal{B}(H_1, H_2, \ldots, H_n),$$

its Euler number $|H_{\sigma(1)}||H_{\sigma(2)}|\cdots|H_{\sigma(n)}|$ (Definition 14.1) is always equal to $|H_1||H_2|\cdots|H_n|$, independent of σ and $x_1, x_2, \ldots, x_n$. Therefore:

Lemma 15.3. *The Euler numbers of clans in an isogeny set* $\mathcal{B}(H_1, H_2, \ldots, H_n)$ *are the same, equal to* $|H_1||H_2|\cdots|H_n|$.

We next consider the set of clans of G *spanned by the frame* (i.e. *ordered basis*) $H_1, H_2, \ldots, H_n$:

$$\mathcal{F}(H_1, H_2, \ldots, H_n) := \big\{x_1H_1x_2H_2\cdots x_nH_n \,:\, x_i \in G\ (i = 1, 2, \ldots, n)\big\}, \tag{15.3}$$

which is called a *strict isogeny set* — a subset of $\mathcal{B}(H_1, H_2, \ldots, H_n)$ consisting of clans of G strictly isogenic to a clan $a_1H_1a_2H_2\cdots a_nH_n$.

Definition 15.4. The strictly isogenic set $\mathcal{F} := \mathcal{F}(H_1, H_2, \ldots, H_n)$ forms a *clan semigroup* with multiplication of two elements given by

$$x_1H_1x_2H_2\cdots x_nH_n \cdot y_1H_1y_2H_2\cdots y_nH_n := x_1y_1H_1x_2y_2H_2\cdots x_ny_nH_n,$$

and with *an* identity $H_1H_2\cdots H_n$ — possibly there may be another identity.

Note the following:

Fact 15.5. Take any $h_i \in H_i$ $(i = 1, 2, \ldots, n)$. Then we have $h_iH_i = H_i$ $(i = 1, 2, \ldots, n)$, so

$$h_1H_1h_2H_2\cdots h_nH_n = H_1H_2\cdots H_n. \tag{15.4}$$

However, in spite of (15.4) *with the right hand side being an identity of* $\mathcal{F}$, *the left hand side is generally* not *an identity of* $\mathcal{F}$: possibly

$$h_1H_1h_2H_2\cdots h_nH_n \cdot y_1H_1y_2H_2\cdots y_nH_n \neq y_1H_1y_2H_2\cdots y_nH_n, \tag{15.5}$$

that is, $h_1y_1H_1h_2y_2H_2\cdots h_ny_nH_n \neq y_1H_1y_2H_2\cdots y_nH_n$, where note that in general $h_iy_i \neq y_i$ or even $h_iy_iH_i \neq y_iH_i$.

Remark 15.6. In the above, while $h_1H_1h_2H_2\cdots h_nH_n$ is generally not a left identity of the semigroup $\mathcal{F}$ in (15.5), it is always a right identity of $\mathcal{F}$; indeed

$$\begin{aligned} y_1H_1y_2H_2\cdots y_nH_n \cdot h_1H_1h_2H_2\cdots h_nH_n &= y_1h_1H_1y_2h_2H_2\cdots y_nh_nH_n \\ &= y_1H_1y_2H_2\cdots y_nH_n \ \text{ as } h_iH_i = H_i. \end{aligned}$$

Clumsy phenomenon for the Euler numbers of clans

Besides the clumsy phenomenon for clan semigroups in Fact 15.5, there occurs another clumsy phenomenon:

Fact 15.7. Let $a_1H_1a_2H_2\cdots a_nH_n$ and $b_1K_1b_2K_2\cdots b_mK_m$ be finite clans of a group G such that

$$a_1H_1a_2H_2\cdots a_nH_n = b_1K_1b_2K_2\cdots b_mK_m \ \text{ (as subsets of } G). \tag{15.6}$$

Then in spite of this equation, their Euler numbers

$$|H_1||H_2|\cdots|H_n| \quad \text{and} \quad |K_1||K_2|\cdots|K_m|$$

may be different.

Actually this clumsy phenomenon occurs already for sects. We give examples for the cases $n \neq m$ and $n = m$ separately:

Example 15.8 (Case $n \neq m$). $G = \mathfrak{S}_3$: Let $H_1 = \langle(1\,2)\rangle$, $H_2 = \langle(2\,3)\rangle$ and $H_3 = \langle(3\,1)\rangle$ be cyclic subgroups of order 2 generated by transpositions $(1\,2)$, $(2\,3)$ and $(3\,1)$ respectively. Take K_1 as H_1, and let $K_2 = \langle(1\,2\,3)\rangle$ be a cyclic subgroup of order 3 generated by a cyclic permutation $(1\,2\,3)$. Then $H_1H_2H_3 = K_1K_2$ (see Claim 15.9 below), but their Euler numbers are different: $|H_1||H_2||H_3| = 2 \cdot 2 \cdot 2 = 8$ and $|K_1||K_2| = 2 \cdot 3 = 6$.

Claim 15.9. *In the notation of Example 15.8, $H_1H_2H_3 = K_1K_2 = \mathfrak{S}_3$.*

Proof. We first show $K_1K_2 = \mathfrak{S}_3$. Since $|\mathfrak{S}_3| = 6$, it suffices to show $|K_1K_2| = 6$. Note that any nontrivial element of K_1 is of even order (two), whereas any nontrivial element of K_2 is of odd order (three). Thus $K_1 \cap K_2 = \{e\}$, so $|K_1 \cap K_2| = 1$. Then by Formula 2.11, $|K_1K_2| = |K_1||K_2|/|K_1 \cap K_2| = 2 \cdot 3/1 = 6$.

We next show $H_1H_2H_3 = \mathfrak{S}_3$. Note first that $H_1H_2H_3$ contains H_1H_2e, eH_2H_3 and H_1eH_3, that is, H_1H_2, H_2H_3 and H_1H_3. Here

$$\begin{cases} H_1H_2 = \bigl\{e, (1\,2), (2\,3), (1\,2)(2\,3)\,(= (1\,2\,3))\bigr\}, \\ H_2H_3 = \bigl\{e, (2\,3), (3\,1), (2\,3)(3\,1)\,(= (1\,2\,3))\bigr\}, \\ H_1H_3 = \bigl\{e, (1\,2), (3\,1), (1\,2)(3\,1)\,(= (1\,3\,2))\bigr\}. \end{cases} \tag{15.7}$$

In particular $H_1H_2H_3 \supset \{e, (1\,2), (2\,3), (3\,1), (1\,2\,3), (1\,3\,2)\} = \mathfrak{S}_3$. Hence $H_1H_2H_3 = \mathfrak{S}_3$. □

Caution 15.10. In the proof of Claim 15.9, to show $H_1H_2H_3 = \mathfrak{S}_3$, one might attempt to, instead of the argument therein, apply the following argument using $K_1K_2 = \mathfrak{S}_3$:

> Noting that $H_1\,(= K_1) \subset H_1H_2H_3$ (Lemma 2.2), once we show $K_2 \subset H_1H_2H_3$, we have $K_1K_2 \subset H_1H_2H_3$, i.e. $\mathfrak{S}_3 \subset H_1H_2H_3$, so $\mathfrak{S}_3 = H_1H_2H_3$.

This argument is wrong: A priori, $K_1 \subset H_1H_2H_3$ and $K_2 \subset H_1H_2H_3$ do *not* imply $K_1K_2 \subset H_1H_2H_3$, because initially $H_1H_2H_3$ is not known to be a subgroup and as such, $H_1H_2H_3$ may not be closed under multiplication — for a subset S of a group G and subgroups A and B of G, even if $A \subset S$ and $B \subset S$, in general $AB \not\subset S$.

Example 15.11 (Case $n = m$). $G = \mathfrak{S}_4$: Take subgroups

$H_1 = \langle(1\ 2\ 3\ 4)\rangle\ (\cong \mathbb{Z}_4)$,

$H_2 = \langle(1\ 2)(3\ 4),\ (1\ 3)(2\ 4),\ (1\ 2\ 3)\rangle$

$\cong \mathfrak{A}_4$ (the alternating group of degree 4),

$K_1 = \langle(1\ 2),\ (1\ 2\ 3\ 4)\rangle \cong D_4$ (the dihedral group of degree 4),

$K_2 = \langle(1\ 2\ 3)\rangle\ (\cong \mathbb{Z}_3)$.

Then $H_1H_2 = K_1K_2$ (see Claim 15.12 below), but their Euler numbers are different: $|H_1||H_2| = 4 \cdot 12 = 48$ and $|K_1||K_2| = 8 \cdot 3 = 24$.

Claim 15.12. *In the notation of Example 15.11, $H_1H_2 = K_1K_2 = \mathfrak{S}_4$.*

Proof. Since $|\mathfrak{S}_4| = 24$, it suffices to show that $|H_1H_2| = |K_1K_2| = 24$. We first show $|H_1H_2| = 24$. Note that $H_1 \cap H_2 = \langle(1\,3)(2\,4)\rangle$. Here $(1\,3)(2\,4)$ is of order 2, so $|H_1 \cap H_2| = 2$. Then by Formula 2.11,

$$|H_1H_2| = |H_1||H_2|/|H_1 \cap H_2| = 4 \cdot 12/2 = 24.$$

We next show $|K_1K_2| = 24$. Note that any nontrivial element of K_1 is of even order — $K_1 \cong D_4$ is a 2-group of order 2^3 —, whereas any nontrivial element of K_2 is of odd order (three). Thus $K_1 \cap K_2 = \{e\}$, so $|K_1 \cap K_2| = 1$. Then by Formula 2.11, $|K_1K_2| = |K_1||K_2|/|K_1 \cap K_2| = 8 \cdot 3/1 = 24$. □

Example 15.13. For a subgroup H of a group G, we have $H = HH$ (closed under multiplication). Inductively

$$H = HH = HHH = \cdots = \underbrace{HH \cdots H}_{n} = \cdots . \tag{15.8}$$

In particular for any distinct positive integers n and m,

$$\underbrace{HH \cdots H}_{n} = \underbrace{HH \cdots H}_{m}. \tag{15.9}$$

Now suppose that H is a finite non-identity subgroup; so $|H| \geq 2$. Then in spite of (15.9), the Euler numbers of these two sects are distinct: $|H|^n \neq |H|^m$.

Definition 15.14. A sect $H_1H_2 \cdots H_n$ is *redundant* if for some i $(1 \leq i \leq n)$,

$$H_1H_2 \cdots \check{H}_i \cdots H_n = H_1H_2 \cdots H_n \quad \text{(as subsets of } G\text{)}, \tag{15.10}$$

where $\check{H}_i$ means the removal of H_i. The opposite case is *irredundant*: for *any* i $(1 \leq i \leq n)$,

$$H_1H_2 \cdots \check{H}_i \cdots H_n \subsetneq H_1H_2 \cdots H_n \quad \text{(as subsets of } G\text{)}. \tag{15.11}$$

Example 15.15. The sect $\underbrace{HH \cdots H}_{n}$ $(n \geq 2)$ in Example 15.13 is redundant, because $\underbrace{HH \cdots H}_{n-1} = \underbrace{HH \cdots H}_{n}$. In contrast, the sects in Examples 15.8 and 15.11 are irredundant.

In addition to the clumsy phenomena for clan semigroups and the Euler numbers of clans (Facts 15.5 and 15.7), there are still other clumsy phenomena for clans:

Fact 15.16. For clans of equal lengths

$$a_1H_1a_2H_2\cdots a_nH_n \quad \text{and} \quad b_1K_1b_2K_2\cdots b_nK_n$$

such that

$$a_1H_1a_2H_2\cdots a_nH_n = b_1K_1b_2K_2\cdots b_nK_n \ \ \text{(as subsets of } G\text{)}, \tag{15.12}$$

the following clumsy phenomena may occur:

(C1) In spite of (15.12), if $H_i \neq K_i$ for some i $(1 \leq i \leq n)$, then these two clans are *not* strictly isogenic.

(C2) In spite of (15.12), if $K_1, K_2, \ldots, K_n$ are not a permutation of $H_1, H_2, \ldots, H_n$, then these two clans are *even not* isogenic.

Remark 15.17. When two clans A and B are isogenic, we temporarily write $A \sim B$. Then however (C2) means that $A = B$ does *not* imply $A \sim B$; that is, in general $A \not\sim A$ (not reflexive). Thus $\sim$ is not an equivalence relation on the set of clans of G. Similarly (C1) means that strict isogeny of clans is not an equivalence relation on the set of clans of G.

Even for sects, (C1) and (C2) may occur:

Example 15.18 (C1). Take the subgroups H_1, H_2 and H_3 of $\mathfrak{S}_3$ in Example 15.8. Then $H_1H_2H_3 = \mathfrak{S}_3$ (see Claim 15.9). We can also show $H_2H_3H_1 = \mathfrak{S}_3$ (see Claim 15.19 below). In particular

$$H_1H_2H_3 = H_2H_3H_1. \tag{15.13}$$

Here the factors H_2, H_3, H_1 on the right hand side are a permutation of H_1, H_2, H_3 on the left hand side, thus $H_1H_2H_3$ and $H_2H_3H_1$ are isogenic. However in spite of (15.13), they are *not* strictly isogenic, as $H_1 \neq H_2$, $H_2 \neq H_3$ and $H_3 \neq H_1$.

Claim 15.19. *In the notation of Example 15.18, $H_2H_3H_1 = \mathfrak{S}_3$.*

Proof. Let e be the identity. Note that $H_2H_3H_1$ contains H_2H_3e, eH_3H_1 and H_2eH_1, that is, H_2H_3, H_3H_1 and H_2H_1. Here

$$\begin{cases} H_2H_3 = \big\{e, (2\,3), (3\,1), (2\,3)(3\,1)\,(= (1\,2\,3))\big\}, \\ H_3H_1 = \big\{e, (3\,1), (1\,2), (3\,1)(1\,2)\,(= (1\,2\,3))\big\}, \\ H_2H_1 = \big\{e, (2\,3), (1\,2), (2\,3)(1\,2)\,(= (1\,3\,2))\big\}. \end{cases} \tag{15.14}$$

In particular $H_2H_3H_1 \supset \{e, (1\,2), (2\,3), (3\,1), (1\,2\,3), (1\,3\,2)\} = \mathfrak{S}_3$. Hence $H_2H_3H_1 = \mathfrak{S}_3$. □

Example 15.20 (C2). Take the subgroups H_1, H_2, K_1, K_2 of $\mathfrak{S}_4$ in Example 15.11. Then $H_1H_2 = K_1K_2$ (Claim 15.12), but they are not isogenic, as K_1, K_2 are not a permutation of H_1, H_2; indeed $(K_1, K_2) \neq (H_1, H_2)$, (H_2, H_1).

16 Words of clans

As we saw in the above section, there occur clumsy phenomena for clans (Facts 15.5, 15.7 and 15.16). To avoid them, we introduce "words of clans":

Definition 16.1. For a clan $a_1H_1a_2H_2\cdots a_nH_n$, its *word* is given by its 'plasma form' $(a_1, H_1, a_2, H_2, \ldots, a_n, H_n)$.

Remark 16.2. Note that while $a_1H_1a_2H_2\cdots a_nH_n$ is a subset of G, its word $(a_1, H_1, a_2, H_2, \ldots, a_n, H_n)$ is not a subset of G.

The clumsiness for clans disappears for their words. To explain this, we adopt the following term:

Definition 16.3. Two clans $a_1H_1a_2H_2\cdots a_nH_n$ and $b_1K_1b_2K_2\cdots b_mK_m$ are *identical* if $n = m$, $a_i = b_i$ and $H_i = K_i$ for $i = 1, 2, \ldots, n$. Alternatively: They are *not identical* if either of the following holds:

(i) $n \neq m$. (ii) $n = m$ and for some i, $a_i \neq b_i$ or $H_i \neq K_i$.

The advantage of using words lies in that even if

$$a_1H_1a_2H_2\cdots a_nH_n = b_1K_1b_2K_2\cdots b_mK_m \quad \text{(as subsets of } G\text{)},$$

as long as these clans are not identical, their words are different:

$$(a_1, H_1, a_2, H_2, \ldots, a_n, H_n) \neq (b_1, K_1, b_2, K_2, \ldots, b_m, K_m). \tag{16.1}$$

Recall that the Euler number of a clan $a_1H_1a_2H_2\cdots a_nH_n$ is given by $\chi(a_1H_1a_2H_2\cdots a_nH_n) := |H_1||H_2|\cdots|H_n|$. We also define the Euler number of its word $(a_1, H_1, a_2, H_2, \ldots, a_n, H_n)$ by

$$\chi(a_1, H_1, a_2, H_2, \ldots, a_n, H_n) := |H_1||H_2|\cdots|H_n|. \tag{16.2}$$

Recall the clumsiness for the Euler numbers of clans — the following may happen (Fact 15.7):

Even if $a_1H_1a_2H_2\cdots a_nH_n = b_1K_1b_2K_2\cdots b_mK_m$,
possibly $\chi(a_1H_1a_2H_2\cdots a_nH_n) \neq \chi(b_1K_1b_2K_2\cdots b_mK_m)$.

This however does *not* occur for words of clans, because

$(a_1, H_1, a_2, H_2, \ldots, a_n, H_n) = (b_1, K_1, b_2, K_2, \ldots, b_m, K_m)$
implies $\chi(a_1, H_1, a_2, H_2, \ldots, a_n, H_n) = \chi(b_1, K_1, b_2, K_2, \ldots, b_m, K_m)$.

Indeed the equation on the first line implies, by definition, that $n = m$, $a_i = b_i$ and $H_i = K_i$ $(i = 1, 2, \ldots, n)$, so trivially

$$|H_1||H_2|\cdots|H_n| = |K_1||K_2|\cdots|K_m|. \tag{16.3}$$

Other concepts for words

Besides the Euler number of a clan, it is straightforward to generalize other concepts for clans to their words:

- For a word $(a_1, H_1, a_2, H_2, \ldots, a_n, H_n)$, its *coefficients* are $a_1, a_2, \ldots, a_n$, its *basis* is $H_1, H_2, \ldots, H_n$, and its *frame* is the "ordered" basis $H_1, H_2, \ldots, H_n$.
- Two words $(a_1, H_1, a_2, H_2, \ldots, a_n, H_n)$ and $(b_1, K_1, b_2, K_2, \ldots, b_m, K_m)$ are *isogenic* if $n = m$ and $K_1, K_2, \ldots, K_n$ are a permutation of $H_1, H_2, \ldots, H_n$. Note: This defines an equivalence relation on the set of words of clans of G (cf. Remark 15.17).
- Two words $(a_1, H_1, a_2, H_2, \ldots, a_n, H_n)$ and $(b_1, K_1, b_2, K_2, \ldots, b_m, K_m)$ are *strictly isogenic* if $n = m$ and $H_i = K_i$ $(i = 1, 2, \ldots, n)$. Note: This defines an equivalence relation on the set of words of clans of G (cf. Remark 15.17).

For words of clans, clumsy phenomena like (C1) and (C2) in Fact 15.16 *never* occur, because

$$(a_1, H_1, a_2, H_2, \ldots, a_n, H_n) = (b_1, K_1, b_2, K_2, \ldots, b_m, K_m)$$

implies $n = m$, $a_i = b_i$ and $H_i = K_i$ $(i = 1, 2, \ldots, n)$, so these two words are *trivially* strictly isogenic (in particular isogenic).

Word groups

Fix subgroups $H_1, H_2, \ldots, H_n$ of a group G, and consider the strictly isogenic set with frame $H_1, H_2, \ldots, H_n$:

$$\mathcal{F} := \big\{ x_1 H_1 x_2 H_2 \cdots x_n H_n \ : \ x_i \in G \ (i = 1, 2, \ldots, n) \big\}. \tag{16.4}$$

This forms a semigroup — a clan semigroup (Definition 15.4) — but generally not a group. In contrast, we show that the corresponding object of words, i.e. *the set of words with frame* $H_1, H_2, \ldots, H_n$ forms a group. First, this set is explicitly given by

$$\widetilde{\mathcal{F}} := \big\{ (x_1, H_1, x_2, H_2, \ldots, x_n, H_n) \ : \ x_i \in G \ (i = 1, 2, \ldots, n) \big\}. \tag{16.5}$$

Noting that $\widetilde{\mathcal{F}}$ is bijective to G^n under

$$(x_1, H_1, x_2, H_2, \ldots, x_n, H_n) \in \widetilde{\mathcal{F}} \longmapsto (x_1, x_2, \ldots, x_n) \in G^n, \tag{16.6}$$

we give a group structure to $\widetilde{\mathcal{F}}$ such that (16.6) is a group isomorphism to the direct product group G^n as follows: multiplication of two elements of $\widetilde{\mathcal{F}}$ is given by

$$\begin{aligned} &(x_1, H_1, x_2, H_2, \ldots, x_n, H_n) \cdot (y_1, H_1, y_2, H_2, \ldots, y_n, H_n) \\ &\quad := (x_1 y_1, H_1, x_2 y_2, H_2, \ldots, x_n y_n, H_n), \end{aligned} \tag{16.7}$$

the inverse of an element $(x_1, H_1, x_2, H_2, \ldots, x_n, H_n)$ of $\widetilde{\mathcal{F}}$ is given by

$$(x_1^{-1}, H_1, x_2^{-1}, H_2, \ldots, x_n^{-1}, H_n), \tag{16.8}$$

and the identity of $\widetilde{\mathcal{F}}$ is $(e, H_1, e, H_2, \ldots, e, H_n)$.

Definition 16.4. We say that $\widetilde{\mathcal{F}}$ is the *word group* of clans of G. By construction, $\widetilde{\mathcal{F}} \cong G^n$ (isomorphic as groups).

Expressions of clans

Words of clans are related to *expressions of clans*. First for sects, we introduce their expressions. To that end, we *formally* regard a sect of a group G as a subset $\mathcal{S}$ of G for which there exist subgroups $H_1, H_2, \ldots, H_n$ of G such that

$$\mathcal{S} = H_1 H_2 \cdots H_n. \tag{16.9}$$

We say that (16.9) is an *expression* of $\mathcal{S}$. In general $\mathcal{S}$ may have more than one expression. As such, a synergy on $\mathcal{S}$ is, in this formalism, generally not unique. In fact if $\mathcal{S}$ has more than one expression, say

$$\mathcal{S} = H_1 H_2 \cdots H_n = K_1 K_2 \cdots K_m = L_1 L_2 \cdots L_l, \tag{16.10}$$

then synergies on $\mathcal{S}$ are

$$\begin{aligned}
\pi &: H_1 \times H_2 \times \cdots \times H_n \longrightarrow H_1 H_2 \cdots H_n = \mathcal{S}, \\
\pi' &: K_1 \times K_2 \times \cdots \times K_m \longrightarrow K_1 K_2 \cdots K_m = \mathcal{S}, \\
\pi'' &: L_1 \times L_2 \times \cdots \times L_l \longrightarrow L_1 L_2 \cdots L_l = \mathcal{S}.
\end{aligned} \tag{16.11}$$

Next, as for sects, we formally regard a clan of a group G as a subset $\mathcal{C}$ of G for which there exist $a_1, a_2, \ldots, a_n \in G$ and subgroups $H_1, H_2, \ldots, H_n$ of G such that

$$\mathcal{C} = a_1 H_1 a_2 H_2 \cdots a_n H_n. \tag{16.12}$$

This is called an *expression* of $\mathcal{C}$. In general $\mathcal{C}$ may have more than one expression, and a clan synergy on $\mathcal{C}$ is, in this formalism, generally not unique: if $\mathcal{C}$ has more than one expression, say

$$\mathcal{C} = a_1 H_1 a_2 H_2 \cdots a_n H_n = b_1 K_1 b_2 K_2 \cdots b_m K_m = c_1 L_1 c_2 L_2 \cdots c_l L_l, \tag{16.13}$$

then clan synergies on $\mathcal{C}$ are

$$\begin{aligned}
\pi &: a_1 H_1 \times a_2 H_2 \times \cdots \times a_n H_n \longrightarrow a_1 H_1 a_2 H_2 \cdots a_n H_n = \mathcal{C}, \\
\pi' &: b_1 K_1 \times b_2 K_2 \times \cdots \times b_m K_m \longrightarrow b_1 K_1 b_2 K_2 \cdots b_m K_m = \mathcal{C}, \\
\pi'' &: c_1 L_1 \times c_2 L_2 \times \cdots \times c_l L_l \longrightarrow c_1 L_1 c_2 L_2 \cdots c_l L_l = \mathcal{C}.
\end{aligned} \tag{16.14}$$

Note moreover that the different expressions of $\mathcal{C}$ correspond to different words

$$\begin{aligned}
&(a_1, H_1, a_2, H_2, \ldots, a_n, H_n), \\
&(b_1, K_1, b_2, K_2, \ldots, b_m, K_m), \\
&(c_1, L_1, c_2, L_2, \ldots, c_l, L_l),
\end{aligned} \tag{16.15}$$

which further correspond to different telergies on $\mathcal{C}$:

$$\eta: \{a_1\} \times H_1 \times \{a_2\} \times H_2 \times \cdots \times \{a_n\} \times H_n \to a_1 H_1 a_2 H_2 \cdots a_n H_n = \mathcal{C},$$
$$\eta': \{b_1\} \times K_1 \times \{b_2\} \times K_2 \times \cdots \times \{b_m\} \times K_m \to b_1 K_1 b_2 K_2 \cdots b_m K_m = \mathcal{C},$$
$$\eta'': \{c_1\} \times L_1 \times \{c_2\} \times L_2 \times \cdots \times \{c_l\} \times L_l \to c_1 L_1 c_2 L_2 \cdots c_l L_l = \mathcal{C}.$$

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