Complex geometry of compact complex solvmanifolds

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Abstract

In this paper, we consider several representations of a complex solvmanifold, and we also consider complex structures on a real solvmanifold. The two considerations are sometimes related by one solvable Lie group. Namely, there exists a solvable Lie group G with left-invariant complex structures J_1, J_2 and lattices Γ_1, Γ_2 in G which have the following properties (i)(G, J_1) is a complex Lie group, but (G, J_2) is not a complex Lie group (ii)$(\Gamma_1\backslash G, J_1)$ and $(\Gamma_1\backslash G, J_2)$ are biholomorphic (iii)$(\Gamma_2\backslash G, J_1)$ and $(\Gamma_2\backslash G, J_2)$ are not biholomorphic. We use these to investigate complex geometric properties of complex solvmanifolds.

1. Introduction

By a solvmanifold we means a quotient space of a simply connected solvable Lie group G by a lattice Γ in G, where a lattice is its discrete co-compact subgroup. In the case that G is nilpotent, $\Gamma\backslash G$ is referred to as a nilmanifold. A complex structure J on $\Gamma\backslash G$ is called left-invariant if it is induced from a left-invariant complex structure on G. Similarly, a form on $\Gamma\backslash G$ is called left-invariant if it is induced from a left-invariant form on G.

In this paper, we consider solvmanifolds $\Gamma\backslash G$ with left-invariant complex structures. It is well known that nilpotent Lie groups have the following property: Let Γ_1 and Γ_2 be lattices in simply connected nilpotent Lie groups N_1 and N_2, respectively. If $\pi_1(\Gamma_1\backslash N_1)$ is isomorphic to $\pi_1(\Gamma_2\backslash N_2)$, then N_1 and N_2 are isomorphic as real Lie groups ([7, Theorem 2.11]). In the case of solvable Lie groups, the theorem holds in a weakened form. Let Γ_1 and Γ_2 be lattices in simply connected solvable Lie groups G_1 and G_2, respectively, and suppose that $\pi_1(\Gamma_1\backslash G_1) \cong \pi_1(\Gamma_2\backslash G_2)$. Then, G_1 and G_2 may not be isomorphic, but $\Gamma_1\backslash G_1$ and $\Gamma_2\backslash G_2$ are diffeomorphic ([7, Theorem 3.6]). Hence, to study a solvmanifold $\Gamma\backslash G$, one can exploit this flexibility; if there exists solvable Lie group $\tilde{G}$ such that $\Gamma\backslash G = \Gamma\backslash\tilde{G}$, then we can investigate $\Gamma\backslash G$ using $\tilde{G}$ (e.g. [2],[12, pp.118]).

Although we take advantage of this flexibility for the study of complex solvmanifolds, one of the main ideas of this paper is to represent a solvmanifold with a left-invariant complex structure in several forms.

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More precisely, there exists a solvable Lie group G with left-invariant complex structures J_1, J_2 and lattices Γ_1, Γ_2 in G which have the following properties (i)(G, J_1) is a complex Lie group, but (G, J_2) is not a complex Lie group (ii)$(\Gamma_1\backslash G, J_1)$ and $(\Gamma_1\backslash G, J_2)$ are biholomorphic (iii)$(\Gamma_2\backslash G, J_1)$ and $(\Gamma_2\backslash G, J_2)$ are not biholomorphic. Therefore in this paper we consider the following questions:

(1) Let G be a solvable Lie group and J a left-invariant complex structure on G. What complex geometric properties does a complex solvmanifold $(\Gamma\backslash G, J)$ have depending on a lattice Γ in G? What are relations between these complex solvmanifolds $(\Gamma_1\backslash G, J)$ and $(\Gamma_2\backslash G, J)$?
(2) Let G be a solvable Lie group and Γ a lattice in G. We construct complex solvmanifolds by considering left-invariant complex structures J on this solvable Lie group. What differences in the complex geometric properties can these complex solvmanifolds have depending on how a left-invariant complex structure J is taken?

For example, in this paper, we prove the following results:

Theorem 1.1. *There exist a complex solvable Lie group G_1, and real solvable Lie groups G_2 and G_3 with left-invariant complex structures satisfying the following conditions:*

(1) *There exist common lattices Γ_i $(i = 1, 2, 3)$ in G_1 and G_3 which satisfy the following properties:*
   (a) $\Gamma_1\backslash G_1 = \Gamma_1\backslash G_3 \longrightarrow \Gamma_2\backslash G_3 \longrightarrow \Gamma_3\backslash G_3$ *is a sequence of finite coverings of complex solvmanifolds,*
   (b) $\Gamma_2\backslash G_3$ *and* $\Gamma_3\backslash G_3$ *are not complex parallelizable, but each of* $\Gamma_i\backslash G_1$ $(i = 1, 2, 3)$ *is complex parallelizable,*
   (c) *each of* $\Gamma_i\backslash G_3$ $(i = 1, 2, 3)$ *admits a pseudo-Kähler structure, but* $\Gamma_3\backslash G_1$ *has no pseudo-Kähler structures.*
(2) *There exist lattices L_i $(i = 1, 2)$ in G_2 which satisfy the following properties:*
   (a) $\pi_1(L_1\backslash G_2) = L_1 \cong L_2 = \pi_1(L_2\backslash G_2)$,
   (b) *each of the complex solvmanifolds* $L_i\backslash G_2 \times T^1_{\mathbb{C}}$ $(i = 1, 2)$ *admits a holomorphic symplectic structure, where* $T^1_{\mathbb{C}}$ *is a* 1*-dimensional complex torus,*
   (c) $L_1\backslash G_1 = L_1\backslash G_2 \longrightarrow L_2\backslash G_2$ *is a double covering,*
   (d) $L_2\backslash G_2$ *is not complex parallelizable, but* $L_1\backslash G_2$ *is complex parallelizable.*

By a complex parallelizable manifold we mean a compact complex manifold with the trivial holomorphic tangent bundle. A complex parallelizable manifold can be written in the form of $\Gamma\backslash G$, where G is a complex Lie group and Γ is a discrete subgroup of G ([10]).

2. **Prelimaries**

Hopf manifolds do not have a Kähler metric, but have properties that are superior to those of general complex manifolds. Indeed, a Hopf manifold has a locally conformal Kähler structure. Similarly, complex solvmanifolds exhibit improved properties compared to other general complex manifolds. Hasegawa [4] has proved that a compact complex solvmanifold admits a Kähler metric if and only if it is a finite quotient of a complex torus, which has a structure of a complex torus bundle over a complex torus. In this paper, we consider the following non-degenerate closed 2-forms, which can be regarded as good symplectic structures.

Definition 2.1. Let (M, J) be a complex manifold.

(1) A *holomorphic symplectic structure* on (M, J) is a closed non-degenerate holomorphic 2-form on M.
(2) A *pseudo-Kähler structure* on (M, J) is a closed non-degenerate $(1, 1)$-form on M.

For example, the Kodaira-Thurston manifold, which is a nilmanifold, is the first example of a compact symplectic manifold which is not Kählerian. It has several different structures: a locally conformal Kähler structure, a holomorphic symplectic structure, a pseudo-Kähler structure and more.

3. **Solvable Lie groups and its homogeneous spaces**

In this section, we present a representative example of solvable Lie groups and their homogeneous spaces in this paper. The argument from here is not limited to this representative example.

A real solvable Lie group G is said to be *completely solvable*, if $\mathrm{ad}(X) : \mathfrak{g} \longrightarrow \mathfrak{g}$ has only real eigenvalues for each $X \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G.

We begin with the following solvable Lie groups with a left-invariant complex structure defined by

(1)
$$G_1 = \left\{ \left. \begin{pmatrix} e^z & 0 & 0 & w_1 \\ 0 & e^{-z} & 0 & w_2 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| z, w_1, w_2 \in \mathbb{C} \right\},$$
(2)
$$G_2 = \left\{ \left. \begin{pmatrix} e^{\frac{1}{2}(z+\bar{z})} & 0 & 0 & w_1 \\ 0 & e^{-\frac{1}{2}(z+\bar{z})} & 0 & w_2 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| z, w_1, w_2 \in \mathbb{C} \right\},$$
(3)
$$G_3 = \left\{ \left. \begin{pmatrix} e^z & 0 & 0 & w_1 \\ 0 & e^{-\bar{z}} & 0 & w_2 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| z, w_1, w_2 \in \mathbb{C} \right\}.$$

Then, G_1 is a complex Lie group, and G_2 is a completely solvable Lie group as a real Lie group. Put

$$\varphi_1(z) = \begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix}, \varphi_2(z) = \begin{pmatrix} e^{\frac{1}{2}(z+\bar{z})} & 0 \\ 0 & e^{-\frac{1}{2}(z+\bar{z})} \end{pmatrix}, \varphi_3(z) = \begin{pmatrix} e^z & 0 \\ 0 & e^{-\bar{z}} \end{pmatrix}.$$

Then we have $G_i \cong \mathbb{C} \ltimes_{\varphi_i} \mathbb{C}^2$ $(i = 1, 2, 3)$.

These Lie groups G_1, G_2, G_3 have a same subgroup as a lattice. Let $B \in SL(2, \mathbb{Z})$ be a unimodular matrix with distinct real eigenvalues, e.g., λ, $1/\lambda$. Take $t_0 = \log \lambda$, i.e., $e^{t_0} = \lambda$. Then there exists $P \in GL(2, \mathbb{R})$ such that

$$PBP^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

Let

$$L_{k\pi} = \{t_0 m + \sqrt{-1} k\pi \cdot n \mid m, n \in \mathbb{Z}\} \ (k = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1, 2)$$

and

$$\mathbb{Z}_P[\tau] = \left\{ P \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \middle| \ \mu_1, \mu_2 \in \mathbb{Z} + \mathbb{Z}\tau \right\} \ (\tau = \sqrt{-1}, \frac{-1 \pm \sqrt{-3}}{2}).$$

We write $\omega = \frac{-1+\sqrt{-3}}{2}$. Then, we have the following:

Lemma 3.1 (cf.[9])**.** *For each $i = 2, 3$ and $(k, \tau) = (1/2, \sqrt{-1}), (1, \sqrt{-1})$, $(2, \sqrt{-1}), (1/3, \omega), (2/3, \omega), (1, \omega), (2, \omega)$, $L_{k\pi} \ltimes_{\varphi_i} \mathbb{Z}_P[\tau]$ is a lattice in G_i. For $(k, \tau) = (1, \sqrt{-1})$, $(2, \sqrt{-1}), (1, \omega), (2, \omega)$, $L_{k\pi} \ltimes_{\varphi_1} \mathbb{Z}_P[\tau]$ is a lattice in G_1.*

Put

$$\Gamma^P_{k\pi,\tau} = L_{k\pi} \ltimes_{\varphi_3} \mathbb{Z}_P[\tau].$$

Note that $\varphi_1(\gamma) = \varphi_2(\gamma) = \varphi_3(\gamma)$ for $\gamma \in L_{2\pi}$. Thus, we have $L_{2\pi} \ltimes_{\varphi_1} \mathbb{Z}_P[\tau] = L_{2\pi} \ltimes_{\varphi_2} \mathbb{Z}_P[\tau] = L_{2\pi} \ltimes_{\varphi_3} \mathbb{Z}_P[\tau] = \Gamma^P_{2\pi,\sqrt{-1}}$.

Because $G_1 \cong G_2 \cong G_3 \cong \mathbb{C}^3$ as complex manifolds, and each product of G_1, G_2, G_3 induces the same action of $\Gamma^P_{2\pi,\sqrt{-1}}$ on $\mathbb{C}^3$, we have the following:

Proposition 3.2 ([12]).

$$\Gamma^P_{2\pi,\sqrt{-1}}\backslash G_1 \cong \Gamma^P_{2\pi,\sqrt{-1}}\backslash G_2 \cong \Gamma^P_{2\pi,\sqrt{-1}}\backslash G_3$$

as complex manifolds.

Although G_2 and G_3 are not complex Lie groups, $\Gamma^P_{2\pi,\sqrt{-1}}\backslash G_2 \cong \Gamma^P_{2\pi,\sqrt{-1}}\backslash G_3$ is complex parallelizable. In honor of Nakamura's research ([6]), this manifold $\Gamma^P_{2\pi,\sqrt{-1}}\backslash G_1$ is sometimes called the Nakamura manifold (e.g.,see [2]). Similarly, we have

Lemma 3.3.

$$\Gamma^P_{\pi,\sqrt{-1}}\backslash G_1 \cong \Gamma^P_{\pi,\sqrt{-1}}\backslash G_3, \Gamma^P_{2\pi,\omega}\backslash G_1 \cong \Gamma^P_{2\pi,\omega}\backslash G_3, \Gamma^P_{\pi,\omega}\backslash G_1 \cong \Gamma^P_{\pi,\omega}\backslash G_3$$

as complex manifolds.

We have used Proposition 3.2 to study this complex solvmanifold $\Gamma^P_{2\pi,\sqrt{-1}}\backslash G_1$. The following theorem is well-known (cf. [7, Corollary 7.29]):

Theorem 3.4. *Let G be a real completely solvable Lie group and $\mathfrak{g}$ the Lie algebra of G. Let Γ be a lattice in G. Then*

$$H^q_{dR}(\Gamma\backslash G, \mathbb{R}) \cong H^q(\mathfrak{g}, \mathbb{R})$$

for each q.

Because G_1 is a complex Lie group, $\Gamma^P_{2\pi,\sqrt{-1}}\backslash G_1$ is a complex parallelizable manifold. Because G_2 is a completely solvable Lie group as a real Lie group, we can compute the de Rham cohomology groups of $\Gamma^P_{2\pi,\sqrt{-1}}\backslash G_1$ by computing the cohomology groups $H^q(\mathfrak{g}_2, \mathbb{R})$ of the Lie algebra $\mathfrak{g}_2$ of G_2. Thus, the fact that a manifold M is complex parallelizable does not necessarily imply that the form of $M = \Gamma\backslash G$, where G is a complex Lie group, is always the optimal choice.

In the previous paper [12], we have directly shown that $\Gamma^P_{2\pi,\sqrt{-1}}\backslash G_1$ has a pseudo-Kähler structure and have only remarked that $\Gamma^P_{2\pi,\sqrt{-1}}\backslash G_3$ has a left G_3-invariant pseudo-Kähler structure. In other words, we have not sufficiently studied this solvmanifold $\Gamma^P_{2\pi,\sqrt{-1}}\backslash G_1$ as a homogeneous manifold of G_3.

4. **The relationship between G_1 and G_3**

In this section, we examine the relationship between G_1 and G_3 from the perspective of a complex coordinate transformation, which is not holomorphic.

Let us consider complex Lie group G_1 as a real Lie group. We denote this real Lie group by ${}_{\mathbb{R}}(G_1)$. Consider the following two global complex coordinates of ${}_{\mathbb{R}}(G_1)$:

$$\psi_1 : \begin{pmatrix} e^{z_1} & 0 & 0 & z_2 \\ 0 & e^{-z_1} & 0 & z_3 \\ 0 & 0 & 1 & z_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto (z_1, z_2, z_3) \in \mathbb{C}^3,$$

$$\psi_2 : \begin{pmatrix} e^{z_1} & 0 & 0 & z_2 \\ 0 & e^{-z_1} & 0 & z_3 \\ 0 & 0 & 1 & z_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto (\zeta_1, \zeta_2, \zeta_3) = (z_1, z_2, \bar{z}_3) \in \mathbb{C}^3.$$

We introduce products $*_1$, $*_2$ on $\mathbb{C}^3$ so that ψ_1, ψ_2 become isomorphisms as real Lie groups, respectively. Hence we have

$$(a_1, a_2, a_3) *_1 (z_1, z_2, z_3) = (z_1 + a_1, e^{a_1} z_2 + a_2, e^{-a_1} z_3 + a_3), \tag{1}$$

$$(b_1, b_2, b_3) *_2 (\zeta_1, \zeta_2, \zeta_3) = (\zeta_1 + b_1, e^{b_1} \zeta_2 + b_2, e^{-\bar{b}_1} \zeta_3 + b_3), \tag{2}$$

respectively, because

$$\psi_2 \left(\begin{pmatrix} e^{\zeta_1} & 0 & 0 & \zeta_2 \\ 0 & e^{-\zeta_1} & 0 & \bar{\zeta}_3 \\ 0 & 0 & 1 & \zeta_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) = (\zeta_1, \zeta_2, \zeta_3).$$

Thus

$$\psi_1 : {}_{\mathbb{R}}(G_1) \longrightarrow (\mathbb{C}^3, *_1), \qquad \psi_2 : {}_{\mathbb{R}}(G_1) \longrightarrow (\mathbb{C}^3, *_2)$$

are isomorphisms as real Lie groups, respectively. Hence we have an isomorphism

$$\psi_2 \circ \psi_1^{-1} : (\mathbb{C}^3, *_1) \longrightarrow (\mathbb{C}^3, *_2).$$

We can easily have that $G_1 = (\mathbb{C}^3, *_1)$ and $G_3 = (\mathbb{C}^3, *_2)$ by noting equations (1), (2). Let us consider holomorphic coordinate neighborhood systems $\mathcal{S}_1 = \{({}_{\mathbb{R}}(G_1), \psi_1)\}$, $\mathcal{S}_2 = \{({}_{\mathbb{R}}(G_1), \psi_2)\}$ of ${}_{\mathbb{R}}(G_1)$. By the above argument we have $G_1 = ({}_{\mathbb{R}}(G_1), \mathcal{S}_1)$ and $G_3 = ({}_{\mathbb{R}}(G_1), \mathcal{S}_2)$ as complex manifolds. Thus, we have the following:

Theorem 4.1. (1) *G_1 and G_3 are isomorphic as real Lie groups.*

(2) *The difference between G_1 and G_3 is how holomorphic coordinate neighborhood systems on ${}_{\mathbb{R}}(G_1)$ are taken.*

Corollary 4.2. *For each lattice in G_3, there exists only one lattice in G_1, and vice versa.*

We can directly construct lattices in G_3 in Section 5. It is important to reiterate that a variety of notations is useful. Whereas the notation in the theorem above clarifies the sharing of the lattices, the notation of G_3 in Section 3 easily shows that the complex structure of $G_3 = ({}_{\mathbb{R}}(G_1), \mathcal{S}_2)$ is left-invariant.

Remark 4.3. From this theorem, we see that previous studies [14], [15] of nilmanifolds serve as a fundamental framework for the study of solvmanifolds.

Let N be a real nilpotent Lie group defined by

$$N = \left\{ \left. \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \right| z_i \in \mathbb{C} \right\}.$$

Consider the following two global complex coordinate systems of N:

$$\varphi_1 : \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \mapsto (z_1, z_2, z_3) \in \mathbb{C}^3,$$

$$\varphi_2 : \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \mapsto (\bar{z}_1, z_2, z_3) \in \mathbb{C}^3.$$

Put $\mathcal{S}_1 = \{(N, \varphi_1)\}$, $\mathcal{S}_2 = \{(N, \varphi_2)\}$, and consider $N_1 = (N, \mathcal{S}_1)$ and $N_2 = (N, \mathcal{S}_2)$. Then $\Gamma \backslash N_1$ and $\Gamma \backslash N_2$ are diffeomorphic as C^∞-manifolds. In the previous papers [14], [15], we consider non-degenerate closed 2-forms and Hodge numbers to investigate differences between $\Gamma \backslash N_1$ and $\Gamma \backslash N_2$ as complex manifolds.

The following results should also be noted; Let $S_M^{(+)} = \mathbb{H} \times \mathbb{C}/G_M^{(+)}$, $S_M^{(-)} = \mathbb{H} \times \mathbb{C}/G_M^{(-)}$ be the simplest Inoue surfaces. Iku Nakamura pointed out that these complex manifolds are not biholomorphic. Then a diffeomorphism from $S_M^{(+)}$ to $S_M^{(-)}$ is induced by the diffeomorphism of $\mathbb{H} \times \mathbb{C}$ defined by

$$(w, z) \longrightarrow (w, \bar{z}).$$

See [5] for details.

5. **Lattices and finite coverings**

In this section, we investigate finite coverings over solvmanifolds $\Gamma \backslash G_3$ and $\Gamma \backslash G_2$ by considering lattices.

Because
$$\begin{pmatrix} e^{z} & 0 \\ 0 & e^{-\bar{z}} \end{pmatrix} = e^{\sqrt{-1}y} \begin{pmatrix} e^{x} & 0 \\ 0 & e^{-x} \end{pmatrix}$$
and $\omega^2+\omega+1=0$ (note that $e^{\sqrt{-1}\frac{2}{3}\pi}=\omega$), we have the following by the natural manner:

Theorem 5.1. *There exist following sequences of finite coverings:*

(i) $\Gamma^P_{2\pi,\sqrt{-1}}\backslash G_3 \overset{2\text{-}covering}{\longrightarrow} \Gamma^P_{\pi,\sqrt{-1}}\backslash G_3 \overset{2\text{-}covering}{\longrightarrow} \Gamma^P_{\frac{1}{2}\pi,\sqrt{-1}}\backslash G_3$,

(ii) $\Gamma^P_{2\pi,\omega}\backslash G_3 \overset{3\text{-}covering}{\longrightarrow} \Gamma^P_{\frac{2}{3}\pi,\omega}\backslash G_3 \overset{2\text{-}covering}{\longrightarrow} \Gamma^P_{\frac{1}{3}\pi,\omega}\backslash G_3$,

(iii) $\Gamma^P_{2\pi,\omega}\backslash G_3 \overset{2\text{-}covering}{\longrightarrow} \Gamma^P_{\pi,\omega}\backslash G_3 \overset{3\text{-}covering}{\longrightarrow} \Gamma^P_{\frac{1}{3}\pi,\omega}\backslash G_3$.

Because G_1 and G_3 are isomorphic as real Lie groups, each lattice $\Gamma^P_{k\pi,\tau}$ in G_3 induces a lattice $\tilde{\Gamma}^P_{k\pi,\tau}$ in G_1 for each k,τ. Hence, we have the following:

Corollary 5.2. *There exist following sequences of finite coverings:*

(i) $\tilde{\Gamma}^P_{2\pi,\sqrt{-1}}\backslash G_1 \overset{2\text{-}covering}{\longrightarrow} \tilde{\Gamma}^P_{\pi,\sqrt{-1}}\backslash G_1 \overset{2\text{-}covering}{\longrightarrow} \tilde{\Gamma}^P_{\frac{1}{2}\pi,\sqrt{-1}}\backslash G_1$,

(ii) $\tilde{\Gamma}^P_{2\pi,\omega}\backslash G_1 \overset{3\text{-}covering}{\longrightarrow} \tilde{\Gamma}^P_{\frac{2}{3}\pi,\omega}\backslash G_1 \overset{2\text{-}covering}{\longrightarrow} \tilde{\Gamma}^P_{\frac{1}{3}\pi,\omega}\backslash G_1$,

(iii) $\tilde{\Gamma}^P_{2\pi,\omega}\backslash G_1 \overset{2\text{-}covering}{\longrightarrow} \tilde{\Gamma}^P_{\pi,\omega}\backslash G_1 \overset{3\text{-}covering}{\longrightarrow} \tilde{\Gamma}^P_{\frac{1}{3}\pi,\omega}\backslash G_1$.

We also have the following:

Lemma 5.3. *There exists a sequence of double coverings*

$$\Gamma^P_{2\pi,\sqrt{-1}}\backslash G_2 \overset{2\text{-}covering}{\longrightarrow} L_\pi \ltimes_{\varphi_2} \mathbb{Z}_P[\sqrt{-1}]\backslash G_2 \overset{2\text{-}covering}{\longrightarrow} L_{\frac{1}{2}\pi} \ltimes_{\varphi_2} \mathbb{Z}_P[\sqrt{-1}]\backslash G_2.$$

Because $L_{2\pi} \ltimes_{\varphi_2} \mathbb{Z}_P[\sqrt{-1}] \cong L_\pi \ltimes_{\varphi_2} \mathbb{Z}_P[\sqrt{-1}]$, we have
$$L_{2\pi} \ltimes_{\varphi_2} \mathbb{Z}_P[\sqrt{-1}]\backslash G_2 \cong L_\pi \ltimes_{\varphi_2} \mathbb{Z}_P[\sqrt{-1}]\backslash G_2$$
as C^∞ manifolds. In next section, we show that $L_{2\pi} \ltimes_{\varphi_2} \mathbb{Z}_P[\sqrt{-1}]\backslash G_2$ and $L_\pi \ltimes_{\varphi_2} \mathbb{Z}_P[\sqrt{-1}]\backslash G_2$ are not biholomorphic, although they are diffeomorphic and have similar complex geometric properties.

We can consider that $\Gamma^P_{k\pi,\tau}\backslash G_3$ is a modification of a hyperelliptic manifold, that is, a finite covering of a complex torus. Indeed, as a simple generalization of hyperelliptic surfaces, let us consider the solvable Lie group defined by
$$\tilde{G} = \left\{ \left. \begin{pmatrix} e^{\sqrt{-1}\mathrm{Im} z} & 0 & 0 & w_1 \\ 0 & e^{\sqrt{-1}\mathrm{Im} z} & 0 & w_2 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| z, w_i \in \mathbb{C} \right\} = \mathbb{C} \ltimes_{\varphi_0} \mathbb{C}^2,$$

and subgroups

$$\tilde{\Gamma}^P_{k\pi,\tau} = L_{k\pi} \ltimes_{\varphi_0} \mathbb{Z}_P[\tau],$$

where

$$\varphi_0(z) = \begin{pmatrix} e^{\sqrt{-1}\mathrm{Im} z} & 0 \\ 0 & e^{\sqrt{-1}\mathrm{Im} z} \end{pmatrix}.$$

Then each $\tilde{\Gamma}^P_{k\pi,\tau}\backslash\tilde{G}$ is a compact complex manifold of which a finite covering is a complex torus. On the Lie group $G_3 \cong \mathbb{C} \ltimes_{\varphi_3} \mathbb{C}^2$,

$$\varphi_3(z) = \begin{pmatrix} e^{z} & 0 \\ 0 & e^{-\bar{z}} \end{pmatrix} = \begin{pmatrix} e^{x} & 0 \\ 0 & e^{-x} \end{pmatrix} \begin{pmatrix} e^{\sqrt{-1}\mathrm{Im} z} & 0 \\ 0 & e^{\sqrt{-1}\mathrm{Im} z} \end{pmatrix}.$$

Therefore, we can consider $\Gamma^P_{k\pi,\tau}\backslash G_3$ as a modification of this hyperelliptic manifolds $\tilde{\Gamma}^P_{k\pi,\tau}\backslash\tilde{G}$ twisted by $\varphi(x) = \begin{pmatrix} e^{x} & 0 \\ 0 & e^{-x} \end{pmatrix}$.

6. Non-degenerate 2-forms on $\Gamma\backslash G_3$ and $\Gamma\backslash G_2$

In this section, we consider non-degenerate 2-forms on $\Gamma\backslash G_3$ and $\Gamma\backslash G_2$. By considering these 2-forms, we can discern differences between $\Gamma\backslash G_3$ and $\Gamma\backslash G_1$ as complex manifolds.

Proposition 6.1 (cf. [12])**.** *G_3 has a left G_3-invariant pseudo-Kähler structure. In particular, for any lattice Γ in G_3, $\Gamma\backslash G_3$ also has a pseudo-Kähler structure.*

Proof. By a straightforward computation, we have that

$$\tau_0 = dz, \tau_1 = e^{-z}dw_1, \tau_2 = e^{\bar{z}}dw_2$$

consists of a basis of the set of the left G_3-invariant $(1,0)$-forms. Then,

$$\omega = \sqrt{-1}\tau_0\wedge\bar{\tau}_0 + \tau_1\wedge\bar{\tau}_2 + \bar{\tau}_1\wedge\tau_2 = \sqrt{-1}dz\wedge d\bar{z} + dw_1\wedge d\bar{w}_2 + d\bar{w}_1\wedge dw_2$$

is a left G_3-invariant pseudo-Kähler structure on G_3. □

Proposition 6.2 (cf.[12])**.** *$\Gamma^P_{k\pi,\sqrt{-1}}\backslash G_3 \times T^1_{\mathbb{C}}$ has a holomorphic symplectic structure for $k = 1, 2$.*

Proof. By a straightforward computation, we have that

$$\omega_0 = dz, \omega_1 = e^{-z}dw_1, \omega_2 = e^{z}dw_2, \omega_3 = dw_3$$

consists of a basis of the set of the left $G_1 \times \mathbb{C}$-invariant $(1,0)$-forms, where w_3 is the canonical coordinate on the complex Lie group $\mathbb{C}$. Then,

$$\Omega = \omega_0 \wedge \omega_3 + \omega_1 \wedge \omega_2 = dz \wedge dw_3 + dw_1 \wedge dw_2$$

is a left G_1-invariant holomorphic structure on G_1. Thus, Ω induces a holomorphic symplectic structure on $\Gamma^P_{k\pi,\sqrt{-1}}\backslash G_3 \times T^1_{\mathbb{C}} = \Gamma^P_{k\pi,\sqrt{-1}}\backslash G_1 \times T^1_{\mathbb{C}}$ $(k = 1, 2)$. □

Lemma 6.3 (cf.[12]). (1) *G_2 has a left G_2-invariant pseudo-Kähler structure. In particular, for any lattice Γ in G_2, $\Gamma\backslash G_2$ has a pseudo-Kähler structure.*

(2) *$\Gamma\backslash G_2 \times T^1_{\mathbb{C}}$ has a holomorphic symplectic structure for each lattice Γ in G_2.*

Proof. By a straightforward computation, we have that

$$\mu_0 = dz, \mu_1 = e^{-\frac{1}{2}(z+\bar{z})}dw_1, \mu_2 = e^{\frac{1}{2}(z+\bar{z})}dw_2$$

consists of a basis of the set of the left G_2-invariant $(1,0)$-forms. Put $\mu_3 = dw_3$, where w_3 is the canonical coordinate on the complex Lie group $\mathbb{C}$. Then,

$$\omega = \sqrt{-1}\mu_0\wedge\bar{\mu}_0+\mu_1\wedge\bar{\mu}_2+\bar{\mu}_1\wedge\mu_2 = \sqrt{-1}dz\wedge d\bar{z}+dw_1\wedge d\bar{w}_2+d\bar{w}_1\wedge dw_2$$

is a left G_2-invariant pseudo-Kähler structure on G_2, and

$$\Omega = \mu_0 \wedge \mu_3 + \mu_1 \wedge \mu_2 = dz \wedge dw_3 + dw_1 \wedge dw_2$$

is a left $G_2 \times \mathbb{C}$-invariant holomorphic structure on $G_2 \times \mathbb{C}$. □

Remark 6.4. The construction method of lattices in G_3 is based on the construction when a hyperelliptic surface is thought of as a solvable manifold. This construction of a pseudo-Kähler structure on $\Gamma\backslash G_3$ is analogous to a Kähler structure on a hyperelliptic surface. Indeed, let us consider the following solvable Lie group:

$$G = \left\{ \left. \begin{pmatrix} e^{\sqrt{-1}\mathrm{Im} z} & 0 & w \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \right| z, w \in \mathbb{C} \right\} = \mathbb{C}(z) \ltimes \mathbb{C}(w).$$

Put

(i) $\Gamma_{k\pi,\sqrt{-1}} = L_{k\pi} \ltimes \mathbb{Z}[\sqrt{-1}]$ $(k = 1, \frac{1}{2})$,

(ii) $\Gamma_{k\pi,\omega} = L_{k\pi} \ltimes \mathbb{Z}[\omega]$ $(k = \frac{2}{3}, \frac{1}{3})$,

where $\mathbb{Z}[\tau] = \mathbb{Z} + \mathbb{Z}\tau$ $(\tau = \sqrt{-1}, \omega)$. Then $\Gamma_{k\pi,\sqrt{-1}}\backslash G$ and $\Gamma_{k\pi,\omega}\backslash G$ are hyperelliptic surfaces. Because $\omega_0 = dz$, $\omega_1 = e^{-\sqrt{-1}\mathrm{Im} z}dw$ are left-invariant, a left-invariant $(1,1)$-form

$$\omega = \sqrt{-1}(\omega_0 \wedge \bar{\omega}_0 + \omega_1 \wedge \bar{\omega}_1) = \sqrt{-1}(dz \wedge d\bar{z} + dw \wedge d\bar{w})$$

induces a Kähler metric on $\Gamma\backslash G$, where Γ is a lattice in G.

7. **Canonical bundle of** $\Gamma\backslash G_3$

In this section, we consider properties of $\Gamma\backslash G_3$.

Let M be a compact complex manifold of dimension n. Let $g_q(M) = h^0(M, \Omega^q_M)$, $P_m(M) = h^0(M, (\Omega^n_M)^{\otimes m})$, where Ω^q_M is the sheaf of germs of holomorphic q-forms.

Proposition 7.1. *For a complex manifold* $\Gamma^P_{k\pi,\tau}\backslash G_3$, g_1, g_2, g_3, P_m *are given by the following table:*

	g_1	g_2	g_3	P_m
$k = 1, 2,\ \tau = \sqrt{-1}, \omega$	3	3	1	1
$k = \frac{1}{2},\ \tau = \sqrt{-1}$	2	1	0	$0(m = 2s - 1),\ 1(m = 2s)$
$k = \frac{1}{3}, \frac{2}{3},\ \tau = \omega$	2	1	0	$0(m = 3s - 1, 3s - 2),\ 1(m = 3s)$

Proof. **(The case of $k = 1, 2$, $\tau = \sqrt{-1}, \omega$)**
Because $\Gamma^P_{k\pi,\sqrt{-1}}\backslash G_3$, $\Gamma^P_{k\pi,\omega}\backslash G_3$ are complex parallelizable manifolds by Lemma 3.3, we have our claim.
(The case of $k = \frac{1}{2}$, $\tau = \sqrt{-1}$)
Let us consider a basis of the set of the left-invariant $(1,0)$-forms on G_3 consisting of $\tau_0 = dz, \tau_1 = e^{-z}dw_1, \tau_2 = e^{\bar{z}}dw_2$. Let $\varpi : G_3 \longrightarrow \Gamma^P_{\frac{1}{2}\pi,\sqrt{-1}}\backslash G_3$ be the natural projection. Let Ω be a holomorphic section on $K_{\Gamma^P_{\frac{1}{2}\pi,\sqrt{-1}}\backslash G_3}$. Then,

$$\varpi^*\Omega = f(z, w_1, w_2)dz \wedge dw_1 \wedge dw_2 = f(z, w_1, w_2)e^{z-\bar{z}}\tau_0 \wedge \tau_1 \wedge \tau_2,$$

where $f(z, w_1, w_2)$ is a holomorphic function on $G_3 \cong \mathbb{C}^3$. Because $\varpi^*\Omega$ is left $\Gamma^P_{\frac{1}{2}\pi,\sqrt{-1}}$-invariant, we see

$$f\left(z + \frac{\pi}{2}\sqrt{-1}, w_1, w_2\right)e^{(z+\frac{\pi}{2}\sqrt{-1})-(\bar{z}-\frac{\pi}{2}\sqrt{-1})} = f(z, w_1, w_2)e^{z-\bar{z}},$$

which implies $f(z+\frac{\pi}{2}\sqrt{-1}, w_1, w_2) = -f(z, w_1, w_2)$. Similarly, we have

$$f(z + t_0, w_1, w_2) = f(z, w_1, w_2).$$

Thus, $f(z, w_1, w_2)$ is a bounded holomorphic function with the variable z. Hence, $f(z, w_1, w_2)$ is a constant function with z. It is obvious that $f(z, w_1, w_2)$ is a bounded holomorphic function with variables w_1, w_2. Hence, $f(z, w_1, w_2)$ is a constant function. However,

$$\varpi^*\Omega = cdz \wedge dw_1 \wedge dw_2 = ce^{z-\bar{z}}\tau_0 \wedge \tau_1 \wedge \tau_2,$$

where c is constant, is not left $\Gamma^P_{\frac{1}{2}\pi,\sqrt{-1}}$-invariant except in the case of $c = 0$. Therefore, we see that $P_1 = 0$. In addition, because

$$(dz \wedge dw_1 \wedge dw_2)^{\otimes 2} = e^{2(z-\bar{z})}(\tau_0 \wedge \tau_1 \wedge \tau_2)^{\otimes 2},$$

we have that $P_2 = 1$.

Let α be a holomorphic 1-form on $\Gamma^P_{\frac{1}{2}\pi,\sqrt{-1}}\backslash G_3$. Then,

$$\varpi^*\alpha = f_0dz + f_1dw_1 + f_2dw_2 = f_0\tau_0 + f_1e^{z}\tau_1 + f_2e^{-\bar{z}}\tau_2,$$

where $f(z, w_1, w_2)$ is a holomorphic function on G_3. Because $\varpi^*\alpha$ is left $\Gamma^P_{\frac{1}{2}\pi,\sqrt{-1}}$-invariant, we see that f_0 and f_1e^{z} are constant functions.

Thus, $f_2dw_2 = f_2e^{-\bar{z}}\tau_2$ induces a holomorphic 1-form on $\Gamma^P_{\frac{1}{2}\pi,\sqrt{-1}}\backslash G_3$. Assume that $f_2dw_2 = f_2e^{-\bar{z}}\tau_2 \neq 0$. Then, $\tau_0 \wedge \tau_1 \wedge f_2dw_2$ induces a non-zero holomorphic 3-form on $\Gamma^P_{\frac{1}{2}\pi,\sqrt{-1}}\backslash G_3$. It is a contradiction to $P_1 = 0$. Thus, $g_1 = 2$.

Similarly, we see that if $f_{01}dz \wedge dw_1 + f_{02}dz \wedge dw_2 + f_{12}dw_1 \wedge dw_2$ is a left $\Gamma^P_{\frac{1}{2}\pi,\sqrt{-1}}$-invariant holomorphic 2-form on G_3, then $f_{02}dz \wedge dw_2 + f_{12}dw_1 \wedge dw_2$ is also left $\Gamma^P_{\frac{1}{2}\pi,\sqrt{-1}}$-invariant. Consider a left $\Gamma^P_{\frac{1}{2}\pi,\sqrt{-1}}$-invariant form on G_3 as a form on $\Gamma^P_{\frac{1}{2}\pi,\sqrt{-1}}\backslash G_3$. Assume that there exists a non-zero left $\Gamma^P_{\frac{1}{2}\pi,\sqrt{-1}}$-invariant holomorphic 2-form $\beta = f_{02}dz \wedge dw_2 + f_{12}dw_1 \wedge dw_2$. Then, $\tau_0 \wedge \beta$ or $\tau_1 \wedge \beta$ is a non-zero holomorphic 3-form on $\Gamma^P_{\frac{1}{2}\pi,\sqrt{-1}}\backslash G_3$. It is a contradiction to $P_1 = 0$. Thus, $g_2 = 1$.

(The case of $k = \frac{1}{3}, \frac{2}{3}$, $\tau = \omega$)
By the same argument as above, we have the table. □

Corollary 7.2. (1) *For $(k,\tau) = (\frac{1}{2}, \sqrt{-1}), (\frac{1}{3}, \omega), (\frac{2}{3}, \omega)$, $\Gamma^P_{k\pi,\tau}\backslash G_3$ is not complex parallelizable.*
(2) *For $(k,\tau) = (\frac{1}{2}, \sqrt{-1}), (\frac{1}{3}, \omega), (\frac{2}{3}, \omega)$, $\Gamma^P_{k\pi,\tau}\backslash G_3 \times T^1_{\mathbb{C}}$ has no holomorphic symplectic structures.*

Proof. Because a compact complex parallelizable manifold has the trivial canonical bundle, and a holomorphic symplectic structure induces a non-vanishing section on the canonical bundle, we have our corollary. □

Recall that G_1 and G_3 are isomorphic as real Lie groups. Thus, we have that each lattice $\Gamma^P_{k\pi,\tau}$ in G_3 induces a lattice $\tilde{\Gamma}^P_{k\pi,\tau}$ in G_1 for each k, τ. Then, because G_1 is a complex Lie group, we have $g_1(\tilde{\Gamma}^P_{k\pi,\tau}\backslash G_1) = g_2(\tilde{\Gamma}^P_{k\pi,\tau}\backslash G_1) = 3$, $g_3(\tilde{\Gamma}^P_{k\pi,\tau}\backslash G_1) = 1$, and $P_m(\tilde{\Gamma}^P_{k\pi,\tau}\backslash G_1) = 1$ for each k, τ, m.

Remark 7.3. Salamon [8, Theorem 1.3 and pp.326] proved that the canonical line bundle $K_{\Gamma\backslash N}$ on a compact nilmanifold $\Gamma\backslash N$ with a left-invariant complex structure is trivial as a holomorphic line bundle. Thus, $P_m(\Gamma\backslash N) = 1$ for each m.

8. **Properties of $\Gamma\backslash G_2$**

In this section, we consider properties of $\Gamma\backslash G_2$. Because

$$\mu_0 \wedge \mu_1 \wedge \mu_2 = dz \wedge dw_1 \wedge dw_2,$$

where $\mu_0 = dz, \mu_1 = e^{-\frac{1}{2}(z+\bar{z})}dw_1, \mu_2 = e^{\frac{1}{2}(z+\bar{z})}dw_2$, for each lattice Γ in G_2, the canonical bundle of $\Gamma\backslash G_2$ is trivial as a holomorphic line bundle.

Lemma 8.1. *Let f be a holomorphic function on $\mathbb{C}$ which satisfies*

$$f(z+t_0)=f(z)e^{-t_0}, \qquad f(z+2\pi\sqrt{-1})=f(z).$$

for some $t_0\in\mathbb{R}$. Then, $f(z)=ce^{-z}$, where c is constant. Similarly, if a holomorphic function f on $\mathbb{C}$ satisfies $f(z+t_0)=f(z)e^{t_0}$ and $f(z+2\pi\sqrt{-1})=f(z)$, then $f(z)=ce^{z}$.

Proof. Let $g(z)=e^{-z}$. Let us consider a holomorphic function $F(z)=\frac{f(z)}{g(z)}$ on $\mathbb{C}$. Then $F(z)$ satisfies

$$F(z+t_0)=F(z), F(z+2\pi\sqrt{-1})=F(z).$$

Because F is a double periodic holomorphic function, we have $F(z)=c$, which implies $f(z)=ce^{-z}$. $\square$

As in the case of $k=\frac{1}{2}$, $\tau=\sqrt{-1}$ we can prove the following:

Proposition 8.2. *Let $\Gamma_\pi=L_\pi\ltimes_{\varphi_2}\mathbb{Z}_P[\sqrt{-1}]$. Then $\Gamma_\pi\backslash G_2$ is not complex parallelizable .*

Proof. Let $\varpi: G_2\times\mathbb{C}\longrightarrow\Gamma_\pi\backslash G_2\times T^1_{\mathbb{C}}$ be the natural projection. Let $T^1_{\mathbb{C}}=L\backslash\mathbb{C}$. Assume that $\Gamma_\pi\backslash G_2$ is complex parallelizable . Then the vector bundle $\Omega^1(\Gamma_\pi\backslash G_2\times T^1_{\mathbb{C}})$ consisting of all holomorphic 1-forms at each point is trivial as a holomorphic vector bundle, because $\Gamma_\pi\backslash G_2\times T^1_{\mathbb{C}}$ has a holomorphic symplectic structure by Lemma 6.3. Let α be a holomorphic 1-form on $\Gamma_\pi\backslash G_2\times T^1_{\mathbb{C}}$. Then,

$$\begin{aligned}\varpi^*\alpha=&f_0dz+f_1dw_1+f_2dw_2+f_3dw_3\\=&f_0\mu_0+f_1e^{\frac{1}{2}(z+\bar z)}\mu_1+f_2e^{-\frac{1}{2}(z+\bar z)}\mu_2+f_3\mu_3\end{aligned}$$

where f_j $(j=0,1,2,3)$ are holomorphic functions on $G_2\times\mathbb{C}\cong\mathbb{C}^4$. Because f_j $(j=0,1,2,3)$ are holomorphic and $\varpi^*\alpha$ is left $\Gamma_\pi\times L$-invariant, we have that f_0 and f_3 are constant functions and $f_j(z,w_1,w_2,w_3)=f_j(w_1,w_2)$ for $j=1,2$ as in proof of Proposition 7.1. Moreover, if $f(z+\sqrt{-1}\pi)=f(z)$, then $f(z+2\pi\sqrt{-1})=f(z)$. Because

$$f_1(z+t_0)e^{\frac{1}{2}(z+t_0+\overline{z+t_0})}=f_1(z)e^{\frac{1}{2}(z+\bar z)},$$
$$f_2(z+t_0)e^{-\frac{1}{2}(z+t_0+\overline{z+t_0})}=f_2(z)e^{-\frac{1}{2}(z+\bar z)},$$

we have $f_1(z+t_0)e^{t_0}=f_1(z)$ and $f_2(z+t_0)e^{-t_0}=f_2(z)$. Hence, by Lemma 8.1, we have

$$f_1(z)=c_1e^{-z}, f_2(z)=c_2e^{z},$$

where c_j $(j=1,2)$ are constant. Thus, we have

$$\varpi^*\alpha=c_0\mu_0+c_1e^{\frac{1}{2}(\bar z-z)}\mu_1+c_2e^{-\frac{1}{2}(\bar z-z)}\mu_2+c_3\mu_3$$

However, because $e^{\frac{1}{2}(\bar{z}-z)}$ and $e^{-\frac{1}{2}(\bar{z}-z)}$ are not invariant by the translation of the vector $\sqrt{-1}\pi$, we have that $\varpi^*\alpha$ is not $\Gamma_\pi \times L$ except in the case of $c_1 = c_2 = 0$. It is a contradiction to that $\Omega^1(\Gamma_\pi\backslash G_2 \times T^1_{\mathbb{C}})$ is trivial as a holomorphic vector bundle. $\square$

Similarly, we see that $L_{\frac{1}{2}\pi} \ltimes_{\varphi_2} \mathbb{Z}_P[\sqrt{-1}]\backslash G_2$ is not complex parallelizable.

9. Non-degenerate 2-forms on $\Gamma\backslash G_1$

In this section, we consider non-degenerate 2-forms on $\Gamma\backslash G_1$.

Let G be a complex Lie group and $\mathfrak{g}$ its complex Lie algebra. Consider $\mathfrak{g}$ as a real Lie algebra with a complex structure J. Then, we denote by $\mathfrak{g}^+ \subset \mathfrak{g}^{\mathbb{C}}$ the vector space of the $\sqrt{-1}$ eigenvectors of this complex structure J. In the previous paper [12] we prove the following:

Proposition 9.1. *Let $(\Gamma\backslash G, J, \omega)$ be a compact complex parallelizable pseudo-Kähler manifold, and $\mathfrak{g}^+ = span\{X_1^+, \ldots, X_n^+\}$. Let $\bar{\tau}_i = i(X_i^+)\omega$. Then, $\bar{\tau}_i$ is $\bar{\partial}$-closed for each i, and $\bar{\tau}_1, \cdots \bar{\tau}_n$ are linearly independent on $\mathbb{C}$. In particular, for each $X^+ \in \mathfrak{g}^+$, $i(X^+)\omega$ is not $\bar{\partial}$-exact.*

Proof. See the proof of Theorem 3.3 in [12]. $\square$

Put

$$\Gamma_{\frac{\pi}{2}} = L_{\frac{\pi}{2}} \ltimes_{\varphi_3} \mathbb{Z}_P[\sqrt{-1}], \quad \Gamma_\pi = L_\pi \ltimes_{\varphi_3} \mathbb{Z}_P[\sqrt{-1}].$$

Then, $\Gamma_{\frac{\pi}{2}}$ and Γ_π are lattices in G_3. Because G_1 and G_3 are isomorphic as real Lie groups, $\Gamma_{\frac{\pi}{2}}$ and Γ_π induce lattices $\tilde{\Gamma}_{\frac{\pi}{2}}$ and $\tilde{\Gamma}_\pi$ in G_1. Because

$$\begin{aligned}\psi_1 \circ \psi_2^{-1} : G_3 =& (\mathbb{C}^3, *_2) \longrightarrow (\mathbb{C}^3, *_1) = G_1,\\ &(\zeta_1, \zeta_2, \zeta_3) \mapsto (z_1, z_2, z_3) = (\zeta_1, \zeta_2, \bar{\zeta_3})\end{aligned}$$

is an isomorphism, $\tilde{\Gamma}_{\frac{\pi}{2}}$ and $\tilde{\Gamma}_\pi$ can be written in the following forms:

$$\tilde{\Gamma}_{\frac{\pi}{2}} = L_{\frac{\pi}{2}} \ltimes_{\varphi_1} L_2, \tilde{\Gamma}_\pi = L_\pi \ltimes_{\varphi_1} L_2.$$

Let $p : \tilde{\Gamma}_\pi\backslash G_1 \longrightarrow \tilde{\Gamma}_{\frac{\pi}{2}}\backslash G_1$ be the natural two-covering, and $\varpi : G_1 \longrightarrow \tilde{\Gamma}_\pi\backslash G_1$ the natural projection. We consider a holomorphic transformation

$$\tau_{\frac{\pi}{2}} : \tilde{\Gamma}_\pi\backslash G_1 \longrightarrow \tilde{\Gamma}_\pi\backslash G_1, \quad [(z, w_1, w_2)] \mapsto [(z + \frac{\pi}{2}\sqrt{-1}, w_1, w_2)],$$

where $[(z, w_1, w_2)] = \varpi(z, w_1, w_2)$. This map is well-defined, because

$$(z + \frac{\pi}{2}\sqrt{-1}) + \sqrt{-1}k\pi = (z + \sqrt{-1}k\pi) + \frac{\pi}{2}\sqrt{-1}.$$

Then we have the following

Proposition 9.2. *$\tilde{\Gamma}_\pi \backslash G_1$ has a pseudo-Kähler structure, but $\tilde{\Gamma}_{\frac{\pi}{2}} \backslash G_1$ has no pseudo-Kähler structures.*

Proof. Let $\omega_0 = dz, \omega_1 = e^{-z} dw_1, \omega_2 = e^z dw_2$ as in proof of Proposition 6.2. Then,

$$\omega = \sqrt{-1} dz \wedge d\bar{z} + dw_1 \wedge d\bar{w}_2 + d\bar{w}_1 \wedge dw_2 = \sqrt{-1}\omega_0 \wedge \bar{\omega}_0 + e^{z-\bar{z}}\omega_1 \wedge \bar{\omega}_2 + e^{\bar{z}-z}\bar{\omega}_1 \wedge \omega_2$$

is a pseudo-Kähler structure on $\tilde{\Gamma}_\pi \backslash G_1$, because $e^{z-\bar{z}}$ and $e^{\bar{z}-z}$ are $\tilde{\Gamma}_\pi$-invariant.

Suppose that $\tilde{\Gamma}_{\frac{\pi}{2}} \backslash G_1$ has a pseudo-Kähler structure ω. Then, $\tilde{\omega} = p^*\omega$ is a pseudo-Kähler structure on $\tilde{\Gamma}_\pi \backslash G_1$. Then, we have

$$\tau_{\frac{\pi}{2}}^* \tilde{\omega} = \tau_{\frac{\pi}{2}}^* p^* \omega = (p \circ \tau_{\frac{\pi}{2}})^* \omega = \tilde{\omega}.$$

Note that

$$\bar{\omega}_0 = d\bar{z}, e^z d\bar{w}_1 = e^{\bar{z}-z}\bar{\omega}_1, e^{-z} d\bar{w}_2 = e^{z-\bar{z}}\bar{\omega}_2$$

are the representatives of a basis of $H^{0,1}_{\bar{\partial}}(\tilde{\Gamma}_\pi \backslash G_1)$,i.e., for each $[\tau] \in H^{0,1}_{\bar{\partial}}(\tilde{\Gamma}_\pi \backslash G_1)$, there exist $c_0, c_1, c_2 \in \mathbb{C}$ such that

$$[\tau] = c_0[\bar{\omega}_0] + c_1[e^{\bar{z}-z}\bar{\omega}_1] + c_2[e^{z-\bar{z}}\bar{\omega}_2].$$

Because $i(W^+)\tilde{\omega} = p^*(i(W^+)\omega)$ is not $\bar{\partial}$-exact for each $0 \neq W^+ \in \mathfrak{g}_1^+$ by Proposition 9.1, there exists a $W^+ \in \mathfrak{g}_1^+$ which satisfies for $(c_1, c_2) \neq (0, 0)$

$$[i(W^+)\tilde{\omega}] = c_0[\bar{\omega}_0] + c_1[e^{\bar{z}-z}\bar{\omega}_1] + c_2[e^{z-\bar{z}}\bar{\omega}_2]$$

on $H^{0,1}_{\bar{\partial}}(\tilde{\Gamma}_\pi \backslash G_1)$. Then, $i(W^+)\tilde{\omega}$ can be written as

$$i(W^+)\tilde{\omega} = c_0\bar{\omega}_0 + c_1 e^{\bar{z}-z}\bar{\omega}_1 + c_2 e^{z-\bar{z}}\bar{\omega}_2 + \bar{\partial} f,$$

where $f \in C^\infty(\tilde{\Gamma}_\pi \backslash G_1)$. Because $\tau_{\frac{\pi}{2}}^* p^*(i(W^+)\omega) = p^*(i(W^+)\omega)$, we have

$$c_0\bar{\omega}_0 + c_1 e^{\bar{z}-z}\bar{\omega}_1 + c_2 e^{z-\bar{z}}\bar{\omega}_2 + \bar{\partial} f = c_0\bar{\omega}_0 - c_1 e^{\bar{z}-z}\bar{\omega}_1 - c_2 e^{z-\bar{z}}\bar{\omega}_2 + \bar{\partial}\tau_{\frac{\pi}{2}}^* f.$$

Thus,

$$2c_1 e^{\bar{z}-z}\bar{\omega}_1 + 2c_2 e^{z-\bar{z}}\bar{\omega}_2 = \bar{\partial}(\tau_{\frac{\pi}{2}}^* f - f).$$

Because $\tau_{\frac{\pi}{2}}^* f, f \in C^\infty(\tilde{\Gamma}_\pi \backslash G_1)$, the left-hand side is $\bar{\partial}$-exact on $\tilde{\Gamma}_\pi \backslash G_1$. It is a contradiction to that $\{[\bar{\omega}_0], [e^{\bar{z}-z}\bar{\omega}_1], [e^{z-\bar{z}}\bar{\omega}_2]\}$ is a basis of $H^{0,1}_{\bar{\partial}}(\tilde{\Gamma}_\pi \backslash G_1)$. Hence, $\tilde{\Gamma}_{\frac{\pi}{2}} \backslash G_1$ has no pseudo-Kähler structures. $\square$

10. **General case**

The argument so far is not limited to the solvable Lie group associated with the Nakamura manifold. The results presented in the preceding sections can be readily generalized. To illustrate, let us consider the following solvable Lie group with a left-invariant complex structure:

$$G_1(\mathfrak{h}(3)) = \left\{ \left. \begin{pmatrix} e^z & 0 & w_0e^z & 0 & 0 & w_1 \\ 0 & e^{-z} & 0 & w_0e^{-z} & 0 & w_2 \\ 0 & 0 & e^z & 0 & 0 & w_3 \\ 0 & 0 & 0 & e^{-z} & 0 & w_4 \\ 0 & 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right| z, w_0, w_1, w_2, w_3, w_4 \in \mathbb{C} \right\}$$

By a straightforward computation, we have that

$$\begin{aligned} &\tau = dz, \mu_0 = dw_0 \\ &\mu_1 = e^{-z}dw_1 - w_0e^{-z}dw_3, \mu_2 = e^z dw_2 - w_0e^z dw_4 \\ &\mu_3 = e^{-z}dw_3, \mu_4 = e^z dw_4 \end{aligned}$$

consists of a basis of the set of the left $G_1(\mathfrak{h}(3))$-invariant $(1,0)$-forms. Let $Z, W_0, W_1, W_2, W_3, W_4$ be the dual fields of these 1-forms. Then, $\mathrm{span}\{W_0, W_1, W_3\}$ and $\mathrm{span}\{W_0, W_2, W_4\}$ are complex 3-dimensional Heisenberg algebras. This led to the use of the symbol $\mathfrak{h}(3)$ in $G_1(\mathfrak{h}(3))$ in this paper. Then, we have the following:

Lemma 10.1 (cf.[3])**.** *$G_1(\mathfrak{h}(3))$ has lattices and a left $G_1(\mathfrak{h}(3))$-invariant holomorphic symplectic structure.*

Proof. The holomorphic 2-form Ω defined by

$$\Omega = \tau \wedge \mu_0 + \tau_1 \wedge \tau_4 + \tau_2 \wedge \tau_3 = dz \wedge dw_0 + dw_1 \wedge dw_4 + dw_2 \wedge dw_3$$

is a left $G_1(\mathfrak{h}(3))$-invariant holomorphic symplectic structure. Let us consider $G_1(\mathfrak{h}(3))$ as a semi-direct product $G_1(\mathfrak{h}(3)) = \mathbb{C}^2(z, w_0) \ltimes_\phi \mathbb{C}^4(w_1, w_2, w_3, w_4)$, where $z, w_0, w_1, w_2, w_3, w_4$ mean complex coordinates and

$$\phi(z, w_0) = \begin{pmatrix} e^z & 0 & w_0e^z & 0 \\ 0 & e^{-z} & 0 & w_0e^{-z} \\ 0 & 0 & e^z & 0 \\ 0 & 0 & 0 & e^{-z} \end{pmatrix}.$$

Let $t_0 \in \mathbb{R}$ and $P \in GL(2, \mathbb{R})$ as in Section 3. Then,

$$\Gamma_k = L_k \ltimes_\phi \left\{ \left. \begin{pmatrix} P\boldsymbol{\mu}_1 \\ P\boldsymbol{\mu}_2 \end{pmatrix} \right| \boldsymbol{\mu}_1, \boldsymbol{\mu}_2 \in (\mathbb{Z}[\sqrt{-1}])^2 \right\},$$

where

$$L_k = \left\{ (t_0 m + \sqrt{-1}k\pi n, \mu_0) \,\middle|\, m, n \in \mathbb{Z}, \mu_0 \in \mathbb{Z}[\sqrt{-1}] \right\},$$

for $k = 1, 2$, are lattices in $G_1(\mathfrak{h}(3))$. $\square$

Let us consider the following solvable Lie group with a left-invariant complex structure

$$G_3(\mathfrak{h}(3)) = \left\{ \begin{pmatrix} e^z & 0 & \bar{w}_0 e^z & 0 & 0 & w_1 \\ 0 & e^{-\bar{z}} & 0 & w_0 e^{-\bar{z}} & 0 & w_2 \\ 0 & 0 & e^z & 0 & 0 & w_3 \\ 0 & 0 & 0 & e^{-\bar{z}} & 0 & w_4 \\ 0 & 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \,\middle|\, z, w_0, w_1, w_2, w_3, w_4 \in \mathbb{C} \right\}.$$

It is obvious that this complex structure is left-invariant from the form of $G_3(\mathfrak{h}(3))$. Let us consider the following global complex coordinate of a real Lie group ${}_{\mathbb{R}}(G_1(\mathfrak{h}(3)))$ defined by

$$\psi : \begin{pmatrix} e^z & 0 & w_0 e^z & 0 & 0 & w_1 \\ 0 & e^{-z} & 0 & w_0 e^{-z} & 0 & w_2 \\ 0 & 0 & e^z & 0 & 0 & w_3 \\ 0 & 0 & 0 & e^{-z} & 0 & w_4 \\ 0 & 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \mapsto (z, \bar{w}_0, w_1, \bar{w}_2, w_3, \bar{w}_4) \in \mathbb{C}^6.$$

Put $\mathcal{S} = \{{}_{\mathbb{R}}(G_1(\mathfrak{h}(3))), \psi\}$. Then, $G_3(\mathfrak{h}(3)) = ({}_{\mathbb{R}}(G_1(\mathfrak{h}(3))), \mathcal{S})$. Hence, a left-invariant holomorphic symplectic structure Ω defined in Lemma 10.1 on $G_1(\mathfrak{h}(3))$ induces a left-invariant pseudo-Kähler structure on $G_3(\mathfrak{h}(3))$. Indeed, a left-invariant 2-form

$$\Omega + \bar{\Omega} = dz \wedge dw_0 + dw_1 \wedge dw_4 + dw_2 \wedge dw_3 + d\bar{z} \wedge d\bar{w}_0 + d\bar{w}_1 \wedge d\bar{w}_4 + d\bar{w}_2 \wedge d\bar{w}_3$$

on $G_1(\mathfrak{h}(3))$ become a left-invariant pseudo-Kähler structure on $G_3(\mathfrak{h}(3))$ by changing the global complex coordinates (cf. [15]). Then, we have the following:

Proposition 10.2. (1) *$G_1(\mathfrak{h}(3))$ and $G_3(\mathfrak{h}(3))$ are isomorphic as real Lie groups. In particular, for each lattice in $G_1(\mathfrak{h}(3))$, there exists only one lattice in $G_3(\mathfrak{h}(3))$, and vice versa.*

(2) *$G_3(\mathfrak{h}(3))$ has a left $G_3(\mathfrak{h}(3))$-invariant pseudo-Kähler structure.*

Corollary 10.3. *For each a lattice Γ in $G_1(\mathfrak{h}(3))$, $\Gamma\backslash G_1(\mathfrak{h}(3))$ and $\Gamma\backslash G_3(\mathfrak{h}(3))$ are diffeomorphic, but not biholomorphic.*

Proof. Because the maximal nilpotent Lie subgroup of $G_1(\mathfrak{h}(3))$ is not abelian, $\Gamma\backslash G_1(\mathfrak{h}(3))$ has no pseudo-Kähler structures for each lattice Γ in $G_1(\mathfrak{h}(3))$ by Corollary 1.8 in [13]. On the other hand, $\Gamma\backslash G_3(\mathfrak{h}(3))$ has a pseudo-Kähler structure. $\square$

There exist several ways to take the conjugate(s) of the first complex coordinate (see the table in [16, pp. 228]). For example, we can consider the following solvable Lie group which is isomorphic to $G_1(\mathfrak{h}(3))$ as real Lie groups

$$G_3(\mathfrak{h}(3))_3 = \left\{ \left. \begin{pmatrix} e^z & 0 & w_0 e^z & 0 & 0 & w_1 \\ 0 & e^{-\bar{z}} & 0 & \bar{w}_0 e^{-\bar{z}} & 0 & w_2 \\ 0 & 0 & e^z & 0 & 0 & w_3 \\ 0 & 0 & 0 & e^{-\bar{z}} & 0 & w_4 \\ 0 & 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right| z, w_0, w_1, w_2, w_3, w_4 \in \mathbb{C} \right\},$$

where the subscript 3 comes from a Weyl chamber C_3 in the table of this paper [16].

We can also consider the following solvable Lie group

$$G_4(\mathfrak{h}(3)) = \left\{ \left. \begin{pmatrix} e^z & 0 & w_0 e^z & 0 & 0 & w_1 \\ 0 & e^{-\bar{z}} & 0 & w_0 e^{-\bar{z}} & 0 & w_2 \\ 0 & 0 & e^z & 0 & 0 & w_3 \\ 0 & 0 & 0 & e^{-\bar{z}} & 0 & w_4 \\ 0 & 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right| z, w_0, w_1, w_2, w_3, w_4 \in \mathbb{C} \right\}.$$

Then, there exists lattices Γ_1 and Γ_2 such that $\Gamma_1\backslash G_1(\mathfrak{h}(3))$ and $\Gamma_1\backslash G_4(\mathfrak{h}(3))$ are biholomorphic, but $\Gamma_2\backslash G_1(\mathfrak{h}(3))$ and $\Gamma_2\backslash G_4(\mathfrak{h}(3))$ are not biholomorphic as in the case of G_1 and G_3. We also see that $G_4(\mathfrak{h}(3))$ has lattices which induces a sequence of finite coverings

$$\tilde{\Gamma}_1\backslash G_4(\mathfrak{h}(3)) \longrightarrow \tilde{\Gamma}_2\backslash G_4(\mathfrak{h}(3)) \longrightarrow \tilde{\Gamma}_3\backslash G_4(\mathfrak{h}(3)),$$

where $\tilde{\Gamma}_1, \tilde{\Gamma}_2, \tilde{\Gamma}_3$ are lattice in $G_4(\mathfrak{h}(3))$ as in the case of G_3.

As further examples, we can consider solvable Lie groups obtained by turning the real variables into complex variables in examples in [11].

Remark 10.4. If there exists a semi-simple matrix $A \in SL(n, \mathbb{Z})$ which has positive eigenvalues $\alpha_1, \ldots, \alpha_n$ such that $\frac{\log \alpha_i}{\log \alpha_1} \in \mathbb{Q}$ for each i, then we can consider what we have done in the previous sections. For an explanation, it suffices to consider the case $A \in SL(3, \mathbb{Z})$.

By this assumption, we can write $\alpha_1 = e^{a_1 t_0}, \alpha_2 = e^{a_2 t_0}, \alpha_3 = e^{a_3 t_0}$, where $a_1, a_2, a_3 \in \mathbb{Z}$, and $t_0 \in \mathbb{R}$. Let $P \in GL(3, \mathbb{R})$ be a matrix such that

$$PAP^{-1} = \begin{pmatrix} e^{a_1 t_0} & 0 & 0 \\ 0 & e^{a_2 t_0} & 0 \\ 0 & 0 & e^{a_3 t_0} \end{pmatrix}.$$

Then, we can consider the following solvable Lie group which is a generalization of G_1:

$$G = \left\{ \begin{pmatrix} e^{a_1 z} & 0 & 0 & 0 & w_1 \\ 0 & e^{a_2 z} & 0 & 0 & w_2 \\ 0 & 0 & e^{a_3 z} & 0 & w_3 \\ 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \middle| z, w_1, w_2, w_3 \in \mathbb{C} \right\}.$$

Indeed, G_1 corresponds to the case of $n = 2, a_1 = 1, a_2 = -1$.

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References

[1] S. Console and A. Fino: *Dolbeault cohomology of compact nilmanifolds*, Transform. Groups **6** (2001), 111–124.

[2] S. Console, A. Fino and H. Kasuya: *On de Rham and Dolbeault cohomology of solvmanifolds*, Transform. Groups **21** (2016), 653–680.

[3] M. Fernández, M. de León and M. Saralegui: *A six-dimensional compact symplectic solvmanifold without Kähler structures*, Osaka J. Math. **33** (1996), 19-35.

[4] K. Hasegwa: *A note on compact solvmanifolds with Kähler structures*, Osaka J. Math. **43** (2006), 131–135.

[5] M. Inoue: *An example of an analytic surface*, Sūgaku **27** (1975), 358–364 (in Japanese).

[6] I. Nakamura: *Complex parallelisable manifolds and their small deformations*, J. Differential Geom. **10** (1975), 85–112.

[7] M.S. Raghunathan: Discrete subgroups of Lie groups, Springer-Verlag, Berlin-Heidelberg-New York, 1972.

[8] S.M. Salamon: *Complex structures on nilpotent Lie algebras*, J. Pure Appl. Algebra **157** (2001), 311–333.

[9] H. Sawai and T. Yamada: *Lattices on Benson-Gordon type solvable Lie groups*, Topology. Appl. **149** (2005), 85–95.

[10] H. C. Wang: *Complex parallisable manifolds*, Proc. Amer. Math. Soc. **5** (1954), 771–776.

[11] T. Yamada: *Examples of compact Lefschetz solvmanifolds*, Tokyo J. Math.**25** (2002), 261–283.

[12] T. Yamada: *A pseudo-Kähler structure on a nontoral compact non-toral complex parallelizable solvmanifold*, Geom. Dedicata **112** (2005), 115–122.

[13] T. Yamada: *A structure theorem of compact complex parallelizable pseudo-Kähler solvmanifolds*, Osaka J. Math. **43** (2006), 923–933.

[14] T. Yamada: *Duality of Hodge numbers of compact complex nilmanifolds*, Complex Manifolds **2** (2015), 168–177.

[15] T. Yamada: *Complex structures and non-degenerate closed* 2*-forms of compact real parallelizable nillmanifolds*, Osaka J. Math. **54** (2017), 125–132.
[16] T. Yamada: *Some relations between complex structures on compact nilmanifolds and flag manifolds*, Hiroshima Math. J. **51** (2021), 227–246.

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